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INTEGRITY AND DOMINATION INTEGRITY OF GEAR GRAPHS

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Abstract. C.A. Barefoot, et. al. [4] introduced the concept of the integrity of a graph. It is an useful measure of vulnerability and it is defined as follows. $I(G) = min\{|S| + m(G - S) : S \subset V(G)\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. A subset S of $V(G)$ is said to be an I-set if $I(G) = |S| + m(G - S)$. We introduced a new vulnerability parameter in [4],namely domination integrity of a graph G. It is a defined as $DI(G) = min\{|S| + m(G - S)\}$, where S is a dominating set of G and $m(G - S)$ denotes the order of the largest component in $G - S$. K.S. Bagga,et. al. [2] gave a formula for $I(K_2 \times C_n)$. In this paper, we give a correct formula for $I(K_2 \times C_n)$. We find some results on the integrity and domination integrity of gear graphs.

Keywords: Connectivity, Network Design and Communication, vulnerability, Integrity , Domination Integrity, Gear Graph.

Mathematics Subject Classification: 05C40, 68M10, 68R10, 05C76, 05C40.

1. Introduction

The communication network can be represented as an undirected and unweighted graph, where a processor(station) is represented as a node and a communication link as an edge between corresponding nodes. Communication network can be represented as a graph, there are many graph theoretical parameters used to describe the vulnerability of communication networks.

The stability of a communication network is of prime importance for network designers. In an analysis of the vulnerability of a communication network to disruption, two quantities that come to our mind are

(1) the number of elements that are not functioning and

(2) the size of the largest remaining sub network within which mutual communications can still occur. In adverse relationship, it would be desirable for an opponent's network to be such that the two quantities can be made simultaneously small. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the amount of damage that has been caused.

For convenience, we recall some graph parameters [3]. Let $G = (V, E)$ be a simple, undirected and finite graph with vertex set $V(G)$ and edge set $E(G)$. $\kappa(G), \alpha(G), \beta(G)$ denote the connectivity , vertex covering and independence number of G respectively. For a subset S of $V(G)$, $m(G - S)$ and $\omega(G - S)$ denote the number of components and the

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order of the largest component in $G-S$ respectively. A cut-set or vertex cut of G is a set $S \subset V(G)$ such that $G-S$ has more than one component. An edge is said to be subdivided when it is replaced by a path of length two connecting its ends and the internal vertex in this path is a new vertex.

Vertex connectivity
$\kappa(G) = \min\{|S| : S \subset V$ is a cut set of $G\}$

The vertex connectivity and edge connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently , a number of other parameters has been introduced that attempt to cope with this difficulty. Those parameters are listed below.

The toughness (Chvtal, 1973, [6])
$t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subset V(G)$ is a vertex cut of $G\}$

The scattering number (Jung, 1978, [12])
$s(G) = \max\{\omega(G-S) - |S| : S \subset V(G)$ is a vertex cut of $G\}$

The integrity (Barefoot, et al., 1987, [4])
$I(G) = \min\{|S| + m(G-S) : S \subset V(G)$ is a vertex cut of $G\}$

Definition 1.1. *A set S for which the minimum is attained is called an I-set and an I-set with smallest cardinality is called a minimum I-set.*

The tenacity (Cozzens, et al., 1995, [7])
$T(G) = \min\{\frac{|S|+m(G-S)}{\omega(G-S)} : S \subset V(G)$ is a vertex cut of $G\}$

The rupture degree (Li. Zhang and Li [14])
$r(G) = \max\{\omega(G-S) - |S| - m(G-S) : S \subset V(G)$ is a vertex cut of $G\}$
Edge analogues of these parameters are defined similarly.
Integrity of paths was found by Barefoot, Entringer and Swart in [9].

Theorem 1.1. *$I(C_n) = \lceil 2\sqrt{n} \rceil - 1$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.*

Theorem 1.2. [11] *For any graphs G and H, if $G \subseteq H$ then $I(G) \leq I(H)$.*

Theorem 1.3. [2]
(a) For $n = 3$ or $4, I(K_2 \times C_n) = 2I(C_n) - 1 = 5$.
(b) For $n \geq 5$, if $n = r^2 + k$ with $0 \leq k \leq 2r$, then

$$I(K_2 \times C_n) = \begin{cases} 2I(C_n) - 1 & , \textit{if } 1 \leq k \leq \frac{r}{2} \textit{ or } r < k \leq \frac{3r}{2}, \\ 2I(C_n) & , \textit{otherwise} \end{cases}$$

Theorem 1.4. [2]
For $n \geq 2$, if $n = r^2 + k$ with $1 \leq k \leq 2r$, then

$$I(K_2 \times P_n) = \begin{cases} 2I(P_n) - 1 & , \textit{if } 0 \leq k \leq \frac{r}{2} \textit{ or } r \leq k < \frac{3r}{2}, \\ 2I(P_n) & , \textit{otherwise} \end{cases}$$

P.D. Chawathe and S.A. Shende [8] proved the correct formula for $I(K_2 \times P_n)$, stated below.

Theorem 1.5. $I(K_2 \times P_n) = \begin{cases} 4r-3, & r^2 \leq n < r^2 + \lfloor \frac{r}{2} \rfloor \\ 4r-2, & r^2 + \lfloor \frac{r}{2} \rfloor \leq n < r^2 + r \\ 4r-1, & r^2 + r \leq n < r^2 + \lceil \frac{3r+1}{2} \rceil \\ 4r, & r^2 + \lceil \frac{3r+1}{2} \rceil + \leq n < (r+1)^2 \end{cases}$

Similarly, In [2], a formula for $I(K_2 \times C_n)$ is given. According to this formula, $I(K_2 \times C_{14}) = 14, I(K_2 \times C_{60}) = 30$. However it can be easily proved that $I(K_2 \times C_{14}) = 13, I(K_2 \times C_{60}) = 29$.

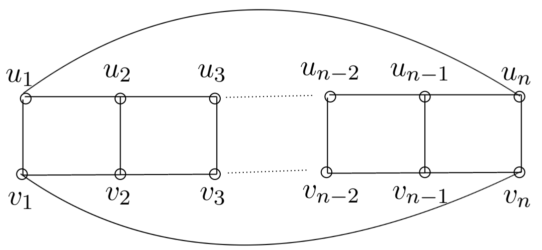
In section 2, we give a correct formula for $I(K_2 \times C_n)$.

2. Integrity of $K_2 \times C_n$

Definition 2.1. *The Cartesian product of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as follows: $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.*

Example 2.1.

$K_2 \times C_n$:



The following lemmas and remark are proved in [8].

Lemma 2.1. *If S is a minimum I-set with smallest number of components of order $m = m(G-S)$ in $G-S$, then $\{u_i, u_{i+1}\} \not\subseteq S$ and $\{v_i, v_{i+1}\} \not\subseteq S$ for $i = 1, 2, \cdots, n-1$.*

Remark 2.1. *Let S be a minimum I-set for $G = P_2 \times P_n, n \geq 4$. If $m = 1$, then by minimality of S, exactly one of $\{u_1, v_1\}$ and exactly one $\{u_n, v_n\}$ is in S. Suppose $u_1, v_n \in S$. Then $S' = S - \{u_1, v_1\}$is an I -set , a contradiction. Hence $m \geq 2$.*

Lemma 2.2. *If S is a minimum I-set with smallest number of components of order $m = m(G-S)$, then there does not exist any component of order one in $G-S$.*

In a similar way, we prove the following.

Lemma 2.3. *If S is a minimum I-set with smallest number of components of order $m = m(G-S)$ in $G-S$, then $\{u_i, u_{i+1}\} \not\subseteq S$ and $\{v_i, v_{i+1}\} \not\subseteq S$ for $i = 1, 2, , \cdots, n-1$.*

Remark 2.2. *Let S be a minimum I-set for $G = K_2 \times C_n, n \geq 9$. If $m = 1$, then by minimality of S, exactly one of $\{u_1, v_1\}$ and exactly one $\{u_n, v_n\}$ is in S. Suppose $u_1, v_n \in S$. Then $S' = S - \{u_1, v_1\}$is an I -set , a contradiction. Hence $m \geq 2$.*

Lemma 2.4. *If S is a minimum I-set with smallest number of components of order $m = m(G - S)$, then there does not exist any component of order one in $G - S$.*

Remark 2.3. *By using the above lemmas 2.3 , 2.4 and remark 2.2, $|S| = 2t$ for $t \in \mathbb{N}$ and $G - S$ has t components. Hence, $I(S) \geq 2t + \left\lceil \frac{2n-2t}{t} \right\rceil = 2t - 2 + \left\lceil \frac{2n}{t} \right\rceil$.*

Lemma 2.5. *For $t, n \in \mathbb{N}$, let $f(t) = 2t - 2 + \left\lceil \frac{2n}{t} \right\rceil$. Then the minimum value of $f(t)$ is given by* $minf(t) = \begin{cases} 4r-3, & r^2 - r + 1 \leq n < r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \\ 4r-2, & r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \leq n < r^2 + 1 \\ 4r-1, & r^2 + 1 \leq n < r^2 + \left\lceil \frac{r+1}{2} \right\rceil \\ 4r, & r^2 + \left\lceil \frac{r+1}{2} \right\rceil + \leq n \leq r^2 + r \end{cases}$

Proof:

Let $g(x) = 2x - 2 + \frac{2n}{x}$, where x is positive real number. Then $g(x)$ achieves its minimum at $x = \sqrt{n}$. Moreover, $g(x)$ is increasing when $x > \sqrt{n}$ and decreasing when $0 < x < \sqrt{n}$. Let $l = \lceil \sqrt{n} \rceil$ and $l' = \lfloor \sqrt{n} \rfloor$.Then

$g(s) \geq g(l)$ for all $s \in \mathbb{N}, s \geq l$

and

$g(s) \geq g(l')$ for all $s \in \mathbb{N}, s \geq l'$

It follows that

$f(s) \geq f(l)$ for all $s \in \mathbb{N}, s \geq l$

$f(s) \geq f(l)$ for all $s \in \mathbb{N}, s \geq l$

Hence the minimum of $f(t)$ is minimum of $\{f(l), f(l')\}$ and it is stated above. ■

Theorem 2.1. $I(K_2 \times C_n) = \begin{cases} 4r-3, & r^2 - r + 1 \leq n < r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \\ 4r-2, & r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \leq n < r^2 + 1 \\ 4r-1, & r^2 + 1 \leq n < r^2 + \left\lceil \frac{r+1}{2} \right\rceil \\ 4r, & r^2 + \left\lceil \frac{r+1}{2} \right\rceil + \leq n \leq r^2 + r \end{cases}$

Proof:

It can be easily verified that

$I(K_2 \times C_3) = 4$; $I(K_2 \times C_4) = 5$; $I(K_2 \times C_5) = 6$;

$I(K_2 \times C_6) = 7$; $I(K_2 \times C_7) = 8$; $I(K_2 \times C_8) = 9$

Let $n \geq 9$ and S be a minimum I-set with smallest number of components of order $m = m(G - S)$. Then by Remark 2.2, it follows that $I(S) \geq \min\{f(g) | t \in N\}$. Hence by Lemma 2.5 , we have

$$I(K_2 \times C_n) \geq \begin{cases} 4r-3, & r^2 - r + 1 \leq n < r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \\ 4r-2, & r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \leq n < r^2 + 1 \\ 4r-1, & r^2 + 1 \leq n < r^2 + \left\lceil \frac{r+1}{2} \right\rceil \\ 4r, & r^2 + \left\lceil \frac{r+1}{2} \right\rceil + \leq n \leq r^2 + r \end{cases}$$

We now construct a suitable S with $I(S) \geq \min\{f(g) | t \in \mathbb{N}\}$ to prove the reverse inequality.

Case 1. $r^2 - r + 1 \leq n < r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor$.

Case 1a. Let r be odd.

Define $S = \{u_2, v_2, u_{r+2}, v_{r+2}, u_{2r+1}, v_{2r+1}, \cdots, u_{r^2 - (\frac{3r+2}{5})}, v_{r^2 - (\frac{3r+2}{5})}\}$.

$|S| = 2r, m(G - S) = 2r - 3$ and $I(C_n) = 4r - 3$.

Case 1b. Let r be even.

Define $S = \{u_2, v_2, u_{r+2}, v_{r+2}, u_{2r+1}, v_{2r+1}, \cdots, u_{r^2 - (\frac{3r-6}{2})}, v_{r^2 - (\frac{3r-4}{2})}\}$.

$|S| = 2r, m(G - S) = 2r - 3$ and $I(C_n) = 4r - 3$.

Case 2. $r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \le n < r^2 + 1$.
Define $S = \{u_2, v_2, u_{r+2}, v_{r+2}, u_{2r+2}, v_{2r+2}, \cdots, u_{r^2-(r-2)}, v_{r^2-(r-2)}\}$.
$|S| = 2r, m(G - S) = 2r - 2$ and $I(C_n) = 4r - 2$.
Case 3. $r^2 + 1 \le n < r^2 + \lceil \frac{r+1}{2} \rceil$.
Case 3a. Let r be odd.
Define $S = \{u_2, v_2, u_{r+2}, v_{r+3}, u_{2r+3}, v_{2r+3}, \cdots, u_{r^2-\lfloor \frac{r}{2} \rfloor -1}, v_{r^2-\lfloor \frac{r}{2} \rfloor -1}\}$.
$|S| = 2r, m(G - S) = 2r - 1$ and $I(C_n) = 4r - 1$.
Case 3b. Let r be even.
Define $S = \{u_2, v_2, u_{r+2}, v_{r+3}, u_{2r+3}, v_{2r+3}, \cdots, u_{r^2-\frac{r}{2}+1}, v_{r^2-\frac{r}{2}+2}\}$.
Case 4. $r^2 + \lceil \frac{r+1}{2} \rceil + \le n \le r^2 + r$.
Define $S = \{u_2, v_2, u_{r+2}, v_{r+2}, u_{2r+1}, v_{2r+1}, \cdots, u_{r^2-(\frac{3r-6}{2})}, v_{r^2-(\frac{3r-4}{2})}\}$.
$|S| = 2r, m(G - S) = 2r - 3$ and $I(C_n) = 4r - 3$.

Thus ,we get $I(K_2 \times C_n) \ge \begin{cases} 4r-3, & r^2 - r + 1 \le n < r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \\ 4r-2, & r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \le n < r^2 + 1 \\ 4r-1, & r^2 + 1 \le n < r^2 + \lceil \frac{r+1}{2} \rceil \\ 4r, & r^2 + \lceil \frac{r+1}{2} \rceil + \le n \le r^2 + r \end{cases}$. Hence the result.

3. Integrity of Gear Graphs

Geared systems are used in dynamic modeling. These are graph theoretic models that are obtained by using gear graphs. Similarly the complement of a gear graph, the Cartesian product of gear graphs and the sequential join of gear graphs can be used to design a gear network. Consequently these considerations motivated us to investigate the vulnerability of gear graphs by using the integrity and the domination integrity in this section.

Example 3.1.

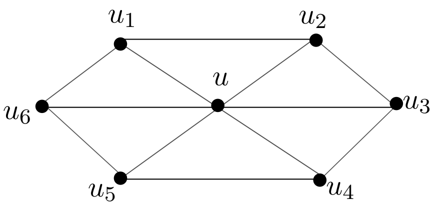


Figure 1(a): W_6 Wheel graph

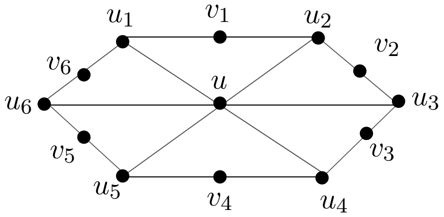


Figure 1(b): G_6 Gear graph

Definition 3.1. [3] *The wheel graph with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle. In Figure 1(a) we display W_6.*

Definition 3.2. [5] *The gear graph is a wheel graph with a vertex added between each pair of adjacent vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges. In Figure 1(b) we display G_6.*

The following table gives a survey of results on Gear graphs [9, 10, 13] for various vulnerability parameters.

Parameter	G_n	$\overline{G_n}$	$L(G_n)$	$K_2 \times G_n$
Toughness	$\frac{n}{n+1}$	$\frac{n}{2}$	1	-
Tenacity	1	n	$\frac{2n+1}{n}$	$\frac{2n+2}{2n+1}$
Rupture Degree	0	2 - 2n	$\leq n+2-2\sqrt{6n}$	-
Scattering number	1	2 - n	0	-
Integrity	-	-	-	-
Domination Integrity	-	-	-	-

Theorem 3.1. *Let G_n be a gear graph of order $n \geq 3$. Then $I(G_n) = \lceil 2\sqrt{2n} \rceil$.*

Proof:

Let S be a cut set of G_n with $|S| = x, (2 \leq x \leq \sqrt{2n})$, then the remaining graph $G_n - S$ has atmost $x-1$ components, and so,

$m(G_n - S) \geq \frac{2n+1-x}{x-1}$. Since $\frac{2n+1-x}{x-1} \geq 1$, So x must be at most $\sqrt{2n}$.

By the definition of Integrity, $I(G_n) \geq \min_x \{x + \frac{2n+1-x}{x-1}\}$.

Now, for $x \geq 0$, the function $f(x) = x + \frac{2n+1-x}{x-1}$ has a minimum value of $2\sqrt{2n}$. Since the integrity is integer valued, we round this up to get a lower bound. Thus , $I(G_n) \geq \lceil 2\sqrt{2n} \rceil$.

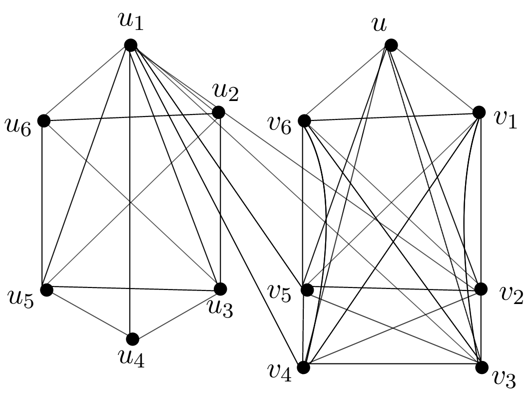
On the other hand, let $S = \{u\} \cup S_1$ is a cutset of $V(G_n)$ and S_1 is a cut set of $V(C_{2n})$ with $I(C_{2n}) = |S_1| + m(C_{2n} - S_1) = \lceil 2\sqrt{2n} \rceil - 1$ (by Theorem 1.1). Thus $I(G_n) \leq |S_1| + 1 + m(G_n - S_1) = \lceil 2\sqrt{2n} \rceil$.

Definition 3.3. [3]

The complement of a graph G denoted by $\overline{G}$, has the same vertex set of G such that two vertices of G are adjacent if and only if they are not adjacent in $\overline{G}$.

The complement of G_n has two complete subgraphs K_n which consists of all vertices of the outer cycle ,say $\{u_1, u_2, \ldots, u_n\}$ and K_{n+1} consists of the subdivided vertices of the outer cycle along with the center vertex u, say $\{u, v_1, v_2, \ldots, v_n\}$. Also, $\overline{G_n}$ contains some edges joining K_{n+1} to K_n.

Example 3.2.



$\overline{G_6}$: The complement of G_6

Similarly, $u_i's, 2 \leq i \leq 6$ are adjacent with respective $v_i's$ except the vertices v_{i-1} and v_{i+1}.

Theorem 3.2. *Let G_n be a gear graph of order $n \geq 3$. Then $I(\overline{G_n}) = 2n - 1$.*

Proof:

Let $S_1 = \{u_1, u_2, \ldots, u_n\}$ be a set of vertices of the outer n-cycle in W_n, and let $S_2 = \{u, v_1, v_2, \ldots, v_n\}$ be a set of vertices which are added to the outer n-cycle in G_n. Let u be a center vertex. Since S_1 is an independent set of G_n, these vertices form a complete graph with order n in G_n. Similarly, since $S_2 \cup \{u\}$ is an independent set of G_n, these vertices form a complete graph with order $n+1$ in G_n. Moreover, the graph G_n contains some edges joining K_{n+1} to K_n. It is obvious that the vertex u in G_n is not adjacent to any vertex in K_n. So we have three cases:

Case 1. If we remove $n-1$ vertices from K_n and $n-1$ vertices from K_{n+1} except u, we get exactly two components of order 2, say $\{u, v_1\}$ and $\{u_1, u_3\}$. Clearly , v_1 is not adjacent with u_1 and u_3 in $\overline{G_n}$. So, we get $|S| = 2n - 2$ and $m(G_n - S) = 2$. Therefore, $I(\overline{G_n}) = 2n - 1$. Since, any vertex cut other S produces higher value than $I(\overline{G_n})$.

Case 2. If we remove the vertices of S_1 in G_n, then we have only one component which is graph K_{n+1}. Then

$m(G_n - S_1) = |V(K_{n+1})| = n + 1$ and so $|S_1| + m(G_n - S_1) = 2n + 1$

Case 3. If we remove the vertices of S_2 in G_n, then we have two components which are graphs K_n and K_1. Then

$m(G_n - S_2) = |V(K_n)| = n$ and so $|S_2| + m(G_n - S_2) = n$

Hence, we have $I(\overline{G_n}) = \min\{2n - 1, n, 2n + 1\} = 2n - 1$.

Definition 3.4. [3] *The line graph $L(G)$ of a graph G is a graph such that each vertex of $L(G)$ represents an edge of G, and any two vertices of $L(G)$ are adjacent if and only if their edges are incident, meaning they share a common end vertex in G.*

The line graph of G_n has $3n$ vertices and $\frac{n^2+7n}{2}$ edges.

Theorem 3.3. *Let G_n be a gear graph of order $n \geq 3$. Then $I(L(G_n)) = n + \lceil 2\sqrt{2n} \rceil - 1$.*

Proof:

Let S be a cut set of G_n with $|S| = n + x, (2 \leq x \leq \sqrt{2n})$, then the remaining graph $G_n - S$ has atmost x components, and so,

$m(G_n - S) \geq \frac{3n-(n+x)}{x}$. Since $\frac{3n-(n+x)}{x} \geq 1$, by the definition of integrity, $I(G_n) \geq \min\limits_{x}\{(n+x) + \frac{3n-(n+x)}{x}\}$.

Now, for $x \geq 0$, the function $f(x) = (n+x) + \frac{3n-(n+x)}{x}$ has a minimum value of $2\sqrt{2n}$. Since the integrity is integer valued, we round this up to get a lower bound.

Thus , $I(L(G_n)) \geq n + \lceil 2\sqrt{2n} \rceil - 1. \rightarrow (1)$

On the other hand, let $S = \{u_1, u_2, \cdots, u_n\} \cup S_1$ is a cut set of $V(L(G_n))$ and S_1 is a cut set of $V(C_{2n})$ with $I(C_{2n}) = |S_1| + m(C_{2n} - S_1) = \lceil 2\sqrt{2n} \rceil - 1$ (by Theorem 1.1).

Thus $I(L(G_n)) \leq |S_1| + n + m(C_{2n} - S_1) = n + \lceil 2\sqrt{2n} \rceil - 1 \rightarrow (2)$.

From (1) and (2), we get the result.

Theorem 3.4. *Let $n \geq 3$ be a positive integer.*

$$\text{Then } I(K_2 \times G_n) = I(K_2 \times C_n) = 2 + \begin{cases} 4r - 3, & r^2 - r + 1 \leq n < r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \\ 4r - 2, & r^2 - \left\lfloor \frac{r-1}{2} \right\rfloor \leq n < r^2 + 1 \\ 4r - 1, & r^2 + 1 \leq n < r^2 + \left\lceil \frac{r+1}{2} \right\rceil \\ 4r, & r^2 + \left\lceil \frac{r+1}{2} \right\rceil + \leq n \leq r^2 + r \end{cases}$$

Proof:
Clearly, $(K_2 \times G_n) - \{u, v\} \cong K_2 \times C_n$. Clearly $S = S_1 \cup \{u, v\}$ is a I- set of $K_2 \times G_n$ where S_1 is an I-set of $K_2 \times C_n$. Hence the result.

4. Domination Integrity

In an administrative set up, decisions are taken by a small group whose members have effective communication links with other members of the organization. Domination in graphs provides a model for such a concept. A subset D of $V(G)$ of a graph is a dominating set if for every $u \in V - D$, there exists a $v \in D$ such that $uv \in E(G)$. In a network, a minimum dominating set of nodes provides a link with the rest of the nodes. If D is a minimum dominating set and if the order of the largest component of $G - D$ is small, then the removal of D results in a chaos in the network because not only the decision making process is paralyzed but also the communication between the remaining members is minimized. In the case of disruption of a network, the damage will be more when vital nodes are under siege. This motivated the study of domination integrity when the sets of nodes disturbed are dominating sets. So, we introduce the concept of Domination Integrity of a graph [15] as another measure of vulnerability of a graph which is defined as follows.

$DI(G) = min\{|S| + m(G - S)\}$, where S is a dominating set of G and $m(G - S)$ denotes the order of the largest component in $G - S$ and is denoted by $DI(G)$.

Theorem 4.1. [15] $DI(C_n) = \begin{cases} 3 & n = 3,4 \\ \lceil \frac{n}{3} \rceil & n \geq 5 \end{cases}$

In this section, we find some results on the domination integrity of gear graphs.

Theorem 4.2. *Let G_n be a gear graph of order $n \geq 3$. Then $DI(G_n) = \lceil \frac{2n}{3} \rceil + 3$.*

Proof:
Clearly, $\gamma(C_{2n}) = \lceil \frac{2n}{3} \rceil = \gamma(G_n)$. Also $D = S \cup \{u\}, m(G_n - D) = 2$, where S is a minimum dominating set of C_{2n} and u is the center vertex of G_n. Therefore, $DI(G_n) \leq |D| + 2 = \lceil \frac{2n}{3} \rceil + 3$. If S is any dominating set other than D of G_n, then $|D| + m(G_n - D) = \lceil \frac{2n}{3} \rceil + 3$.

Theorem 4.3. *Let G_n be a gear graph of order $n \geq 3$. Then $DI(\overline{G_n}) = 2n - 1$.*

Proof:
It is easily seen that $I(\overline{G_n}) = 2n - 1 \leq DI(\overline{G_n})$. On the other hand, remove $n-1$ vertices from K_n and $n-1$ from K_{n+1} except u,we get exactly two components of order 2 say $\{u, v_1\}$ and $\{u_1, v_3\}$. Clearly, v_1 is not adjacent with u_1 and u_3 in $\overline{G_n}$. So, $|S| - 2n - 2, m(\overline{G_n} - S) = 2$. Therefore, $DI(\overline{G_n}) \leq 2n - 1$. Hence $DI(\overline{G_n}) = 2n - 1$.

Theorem 4.4. *Let G_n be a gear graph of order $n \geq 3$. Then $DI(L(G_n)) = 2n - 1$.*

Proof:
It is easily seen that $I(L(G_n)) = 2n - 1 \leq DI(L(G_n))$. On the other hand, we let S denote a cut set of $L(G_n)$ with $I(L(G_n) = 2n - 1$, which is also a dominating set of G_n. Therefore, $DI(L(G_n)) \leq 2n - 1$. Clearly, any other dominating set S' of $L(G_n), m(L(G_n) - S') \geq m(L(G_n)) - S)$. Hence $DI(L(G_n)) = 2n - 1$.

5. Concluding Remarks

A communication network can considered as a graph model with nodes and links. There are many graph theoretical parameters used to describe the vulnerability of communication networks including connectivity, integrity, toughness, tenacity , rupture degree and scattering number. In order to measure the performance, we are interested in the following performance metrics, the number of elements that are not functioning, the number of remaining connected sub-networks and the size of a largest remaining group within which mutual communication can still occur. The vertex-connectivity and edge- connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have been introduced that attempt to cope with this difficulty, including toughness and edge-toughness , integrity and edge-integrity, tenacity and edge-tenacity, rupture degree and scattering number. Unlike the connectivity measures, each of these parameters shows not only the difficulty to break down the network but also the damage that has been caused. The domination and vulnerability of network are two important concepts for the network security. In this paper, We have studied two important measures of vulnerability known as integrity and domination integrity and investigate domination integrity of gear graphs.

References

[1] K.S.Bagga, L.W. Beineke, W.D. Goddard, M.J. Lipman and R.E.Pippert, *A surety of Integrity*, Discrete Applied Mathematics, 37/38, (1992), 13-28.

[2] K.S. Bagga, L.w. Beineke and R.E. Pippert, *The integrity of Prisms* (Preliminary report), Abstracts Amer. Math. Soc. 10, 12, 1989.

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, Newyork , 1976.

[4] C.A. Barefoot, R. Entringer and Henda C. Swart,*Vulnerability in graphs A comparative survey*, J. Combin. Math. Combin. Comput. 1 (1987),1322.

[5] A. Brandstadt, V. B. Le and J. P. Spinrad, *Graph classes: a survey*, SIAM, Philadelphia, PA, 1999.

[6] J. A. Bondy, and U. S. R. Murty, *Graph Theory with Applications*, The McMilan Press Ltd., 1976.

[7] M. Cozzens, D. Moazzami, S. Stueckle, *The tenacity of a graph*, Proc. Seventh International Conf. on the Theory and Application of Graphs, Wiley, New York, 1995, pp. 1111-1122.

[8] P.D.Chawathe and S.A.Shede , *Integrity of* $P_2 \times P_n$, Number Theory and Discrete Mathematics, Trends in Mathematics 2002, pp 149-155.

[9] Ersin Aslan and Alpay Kirlangic, *Computing The Scattering Number and The Toughness for Gear Graphs*, Bulletin of International Mathematical Virtual Institute, ISSN 1840-4367 , Vol. 1(2011), 1-11.

[10] Fengwei Li, *Some Results on Tenacity of Graphs*, WSEAS Transcations on Mathematics, Issue 9, Volume 11, September 2012, 760 -772.

[11] Goddard, W., Swart, H.C.,*Integrity in graphs : Bounds and Basics*, J. Combin. Math. Combin. Comput. 7(1990), 139-151.

[12] H.A. Jung, *On a class of posets and the corresponding comparability graphs*, J. Combin. Theory Ser. B 24 (1978) 125-133.

[13] A. Kirlangic, *The Rupture Degree and Gear Graphs*, Bull. Malays. Math. Sci. Soc., Vol.32, 2009, (1), pp. 31-36.

[14] Y. K. Li , S. G. Zhang and X. L. Li, *Rupture degree of graphs*, Int. J. Comput. Math., Vo. 82, 2005 (7), 793 - 803.

[15] R. Sundareswaran and V. Swaminathan, *Domination Integrity in Graphs*, Proceedings of International Conference on Mathematical and Experimental Physics, Prague, 3-8 August 2009, pp. 46-57.

[16] R. Sundareswaran and V. Swaminathan, *Domination Integrity of Middle Graphs*, Algebra, Graph Theory and Their Applications, T. Chelvam, S. Somasundaram and R. Kala, Eds.,, Narosa Publishing House, New Delhi, 2010, pp. 88-92.

[17] R. Sundareswaran and V. Swaminathan, *Domination Integrity in Trees*, Bulletin of International Mathematical Virtual Institute,ISSN 1840-4367, Vol. 2 (2012), 153-161.

[18] R. Sundareswaran and V. Swaminathan, *Domination Integrity of Powers of Cycles*, International Journal of Mathematical Research, 3(3),2011, 257-265.

R.Sundareswaran for the photography and short autobiography, see TWMS J. App. Eng. Math., V.5, N.1.

V.Swaminathan for the photography and short autobiography, see TWMS J. App. Eng. Math., V.5, N.1.