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STABILITY OF A FUNCTIONAL EQUATION IN COMPLEX BANACH SPACES

PRATAP MONDAL [1], T.K. SAMANTA[2], §

Abstract. Using fixed point technique, in the present paper, we wish to examine generalization of the Hyers-Ulam-Rassias stability theorem for the functional equations

$$f(2x+iy)+f(x+2iy)=4f(x+iy)+f(x)+f(y) \tag{0.1}$$

and

$$f(2x+iy)-f(ix-2y)=-4f(ix-y)+f(x)-f(-y) \tag{0.2}$$

in complex Banach spaces.

Keywords: fixed point, Hyers-Ulam-Rassias stability, functional equation, stability of functional equation, complex Banach space.

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1. Introduction

An intriguing and famous talk presented by Stanislaw M.Ulam [10] in 1940 triggering the study of stability problems for various functional equations. In his talk, Ulam discussed a number of important unsolved mathematical problems. Among them, a question concerning the stability of group homomorphisms seemed too abstract for anyone to reach any conclusion. In the following year, Donald H.Hyers [5] was able to give a partial solution to Ulam's question that was the first significant breakthrough and step toward more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem. In 1978, Themistocles M.Rassias [9] succeeded in extending the result of Hyers's theorem and his exciting result attracted a number of mathematicians who began to be stimulated to investigate the stability problems of several functional equations. Due to influence of S.M. Ulam, D.H. Hyers in the work of Th.M. Rassias regarding the study of stability problems of functional equations, the stability phenomenon proved by Th.M. Rassias is termed as the Hyers-Ulam-Rassias stability thereafter. For the last thirty five years many results concerning the Hyers-Ulam-Rassias stability of various functional equations have

[1] Department of Mathematics, Bijoy Krishna Girls' College, Howrah, West Bengal, India−711101.
e-mail: pratapmondal111@gmail.com;
[2] Department of Mathematics, Uluberia College, Uluberia, Howrah, West Bengal, India−711315.
e-mail:mumpu_tapas5@yahoo.co.in;
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been obtained and a number of definitions of stability have been introduced in various aspects [1, 3, 4, 6, 7, 8]. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is known as quadratic functional equation because, every solution of this functional equation is a quadratic function. Particularly in 2002, Chang and Kim [2] analyzed the functional equation

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y) \tag{1.2}$$

in Banach spaces, which is also a quadratic and equivalent to (1.1). Furthermore they investigated the general solution and generalized Hyers-Ulam stability of (1.2). Following the equation (1.2), in this paper, we have considered the equations (0.1) and (0.2) in complex plane. In fact, it is proved in this paper that both the equations (0.1) and (0.2) can be reduced to the equation (1.2) and the equation (1.2) can also be reduced to the equation (0.1) if we further assume that $f(ix)=f(x)$ and also the equation (1.2) can be reduced to the equation (0.2) on the assumption $f(ix)=-f(x)$. Accordingly, in the next, our goal is to present some of the most important developments in this area about the functional equation (0.1) and equation (0.2) in complex Banach spaces in the spirit of Hyers-Ulam-Rassias stability theorem via fixed point technique.

2. Preliminaries

We adopted a definition and a theorem which will be needed in the sequel.

Definition 2.1. *Let X be a non-empty set. A function $d: X\times X\to[0,\infty]$ is called a **generalized metric** on X if d satisfies*
(i) $d(x,y)=0$ if and only if $x=y$;
(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(iii) $d(x,y)\le d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Theorem 2.2. *(The fixed point alternative theorem, [11])*
Let (X,d) be a complete generalized metric space and let $J: X\to X$ be a strictly contractive mapping with Lipschitz constant $0<L<1$, that is,

$$d(Jx,Jy)\le L\,d(x,y),$$

for all $x,y\in X$.
Then for each $x\in X$, either

$$d(J^n x, J^{n+1}x)=\infty$$

or,

$$d(J^n x, J^{n+1}x)<\infty \quad \forall\, n\ge n_o$$

for some non-negative integers n_0. Moreover, if the second alternative holds then
(1) the sequence $\{J^n x\}$ converges to a fixed point $y^\star$ of J;
(2) $y^\star$ is the unique fixed point of J in the set

$$Y=\{y\in X: d(J^{n_0}x,y)<\infty\};$$

(3) $d(y,y^\star)\le(\frac{1}{1-L})\,d(y,Jy)$ for all $y\in Y$.

3. Properties the functions satisfying (0.1) and (0.2).

Throughout this section, we consider that X and Y are complex vector spaces.

Result 3.1. *A function* $f : X \to Y$ *satisfies*

$$f(2x + iy) + f(x + 2iy) = 4f(x + iy) + f(x) + f(y) \tag{3.1}$$

for all $x, y \in X$, *then the function* $f : X \to Y$ *is quadratic, i.e.,*

$$f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y)$$

for all $x, y \in X$. *If a mapping* $f : X \to Y$ *is quadratic and* $f(ix) = f(-x)$ *holds for all* $x \in X$, *then the mapping* $f : X \to Y$ *satisfies (3.1)*

Proof. Let $f : X \to Y$ satisfy (3.1).
Putting $y = ix$ in (3.1) we have

$$f(x) + f(-x) = 4f(0) + f(x) + f(ix)$$

,i.e., $f(-x) = f(ix)$
,i.e., $f(ix) = f(x)$ for all $x \in X$.
Hence

$$\begin{aligned} f(2x + iy) + f(x + 2iy) &= 4f(x + iy) + f(x) + f(y) \\ &= 4f(x + iy) + f(x) + f(iy) \end{aligned} \tag{3.2}$$

for all $x, y \in X$. Putting $iy = z$ in (3.2), we have

$$f(2x + z) + f(x + 2z) = 4f(x + z) + f(x) + f(z)$$

for all $x, z \in X$.
Conversely let a quadratic mapping $f : X \to Y$ satisfies $f(ix) = f(-x)$ for all $x \in X$. Then

$$\begin{aligned} f(2x + iy) + f(x + 2iy) &= 4f(x + iy) + f(x) + f(iy) \\ &= 4f(x + iy) + f(x) + f(y) \end{aligned}$$

for all $x, y \in X$. Hence $f : X \to Y$ satisfies (3.1) □

Result 3.2. *A function* $f : X \to Y$ *satisfies*

$$f(2x + iy) - f(ix - 2y) = -4f(ix - y) + f(x) - f(-y) \tag{3.3}$$

for all $x, y \in X$, *then the function* $f : X \to Y$ *is quadratic, i.e.,*

$$f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y)$$

for all $x, y \in X$. *If a mapping* $f : X \to Y$ *is quadratic and* $f(ix) = -f(x)$ *holds for all* $x \in X$, *then the mapping* $f : X \to Y$ *satisfies (3.3)*

Proof. Let $f : X \to Y$ satisfy (3.3).
Putting $y = 0$, in (3.3) we have

$$f(2x) - f(ix) = -4f(ix) + f(x)$$

,i.e.,

$$f(2x) = -3f(ix) + f(x)$$

Thus

$$f(-2x) = -3f(-ix) + f(-x) \tag{3.4}$$

Also, putting $x = 0$ and $y = x$ in (3.3) we have

$$f(ix) - f(-2x) = -4f(-x) - f(-x) = -5f(-x)$$

for all $x \in X$.
Therefore using (3.4) we have

$$f(ix) + 3f(-ix) - f(-x) = -5f(-x)$$

or,

$$f(ix) + 3f(-ix) = -4f(-x) \tag{3.5}$$

Now replacing i by $-i$ we have

$$f(-ix) + 3f(ix) = -4f(-x) \tag{3.6}$$

Solving (3.5) and (3.6) we get $f(ix) = -f(x)$ for all $x \in X$.
Hence

$$\begin{aligned} f(2x + iy) + f(x + 2iy) &= f(2x + iy) - f(ix - 2y) \\ &= -4f(ix - y) + f(x) - f(-y) \\ &= 4f(x + iy) + f(x) + f(iy), \end{aligned} \tag{3.7}$$

since $-f(-y) = -f(i^2y) = f(iy)$ for all $y \in X$.
Putting $iy = z$ in (3.7), we have

$$f(2x + z) + f(x + 2z) = 4f(x + z) + f(x) + f(z)$$

for all $x, z \in X$.
Conversely let a quadratic mapping $f : X \to Y$ satisfies $f(ix) = -f(x)$ for all $x \in X$. Then

$$\begin{aligned} f(2x + iy) - f(ix - 2y) &= f(2x + iy) + f(x + 2iy) \\ &= 4f(x + iy) + f(x) + f(iy) \\ &= -4f(ix - y) + f(x) - f(-y), \end{aligned}$$

since $f(iy) = f(i^5y) = -f(i^4y) = f(i^3y) = -f(i^2y) = -f(-y)$ for all $y \in X$.
Hence $f : X \to Y$ satisfies (3.3) □

4. The Hyers-Ulam-Rassias Stability of (0.1) and (0.2).

Throughout this section, we assume X is a normed vector space and Y is a Banach space with norm $\|.\|$.

Theorem 4.1. *Assume $\phi : X \times X \to [0, \infty)$ be a mapping such that, for which* $0 < L < 1$
$\phi(x, -ix) \leq 9L\phi(\frac{x}{3}, \frac{-ix}{3})$
and let $f : X \to Y$ be a mapping with $f(-ix) = f(x)$ satisfying

$$\begin{aligned} &\|(f(2x + iy) + f(x + 2iy) - 4f(x + iy) - f(x) - f(y)\| \\ &\leq \phi(x, y) \end{aligned} \tag{4.1}$$

for all $x, y \in X$.
Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{9 - 9L}\left(\frac{\phi(x, -ix)}{2} + 2\phi(x, 0)\right) \tag{4.2}$$

Proof. Step 1.
Consider a generalized metric on the set $S := \{ g / g : X \to Y \}$ such that $d(g, h) = inf \left\{ c \in (0, \infty) : \|g(x) - h(x)\| \leq c \left(\frac{\phi(x, -ix)}{2} + 2\phi(x, 0) \right) \right\}$ for all $x, y \in X$. It is easy to show that (S, d) is a complete metric space.
Now we consider the mapping
$J : S \to S$ such that $Jg(x) := \frac{1}{9} g(3x)$ for all $x \in X$.
For a fix $C \in (0, \infty)$ and $g, h \in S$ such that $d(g, h) < C$. Since

$$\begin{aligned} d(g, h) = inf \{ c \in (0, \infty) : \|g(x) - h(x)\| \\ \leq C \left(\frac{\phi(x, -ix)}{2} + 2\phi(x, 0) \right) \right\} \end{aligned}$$

therefore we find C_1 such that $d(g, h) \leq C_1 < C$.
and

$$\begin{aligned} \|g(x) - h(x)\| &\leq C_1 \left(\frac{\phi(x, -ix)}{2} + 2\phi(x, 0) \right) \\ &< C \left(\frac{\phi(x, -ix)}{2} + 2\phi(x, 0) \right) \\ \Rightarrow \|\frac{1}{9} g(3x) - \frac{1}{9} h(3x)\| &< \frac{C}{9} \left(\frac{\phi(3x, -i3x)}{2} + 2\phi(3x, 0) \right) \end{aligned}$$

$\Rightarrow \|Jg(x) - Jh(x)\| < CL \left(\frac{\phi(x, -ix)}{2} + 2\phi(x, 0) \right)$
This implies that $d(Jg, Jh) \leq L d(g, h)$ for all $g, h \in S$. ,i.e, J is a strictly contractive mapping of S with the Lipschitz constant L.
Step 2.
Also note that if we put $y = 0$ in(4.1) we have

$$\|f(2x) - 4f(x)\| \leq \phi(x, 0)$$

and putting $y = -ix$ in (4.1) we have

$$\|f(3x) - 2f(2x) - f(x)\| \leq \frac{\phi(x, -ix)}{2}$$

Therefore

$$\begin{aligned} \|f(3x) - 9f(x)\| &\leq \|f(3x) - 2f(2x) - f(x)\| + 2\|f(2x) - 4f(x)\| \\ &\leq \frac{\phi(x, -ix)}{2} + 2\phi(x, 0) \end{aligned}$$

Hence

$$\|f(3x) - 9f(x)\| \leq \frac{1}{9} \{ \frac{\phi(x, -ix)}{2} + 2\phi(x, 0) \}$$

Thus

$$d(f, Jf) \leq \frac{1}{9} \tag{4.3}$$

Also $d(Jf, J^2 f) \leq L d(f, Jf) < \infty$ using (4.3)
Therefore we find a natural number $n_0 = 1$ such that

$$d(J^n f, J^{n+1} f) = d(Jf, J^2 f) \leq L d(f, Jf) < \infty$$

for all $n \geq n_0 = 1$
Step 3.

So we can apply the fixed point alternative theorem and obtain the existence of a mapping $Q : X \to Y$ such that the following conditions hold

(i). Q is a point of J, i.e.,

$$Q(3x) = 9Q(x) \tag{4.4}$$

for all $x \in X$.
The mapping Q is a unique fixed point of J in the set

$Y = \{g \in X, d(f, g) < \infty\}$
Thus Q is the unique mapping with satisfying (4.4) and (4.5)
where $\exists C \in (0, \infty)$ such that

$$\|f(x) - Q(x)\| \leq C\{\frac{\phi(x, -ix)}{2} + 2\phi(x, 0)\} \tag{4.5}$$

for all $x \in X$.
(ii). $d(J^n f, Q) \leq L^n C \to 0$ as $n \to \infty$
where $J^n f(x) = \frac{f(3^n x)}{9^n}$
This implies

$$\lim_{n\to\infty}\left(\frac{f(3^n x)}{9^n}\right) = Q(x) \tag{4.6}$$

for all $x \in X$.
(iii). $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$,which implies the inequality
$d(f, Q) \leq \frac{1}{9-9L}$
Hence the (4.2) holds.
Also replacing x and y by $3^n x$ and $3^n y$ respectively in (4.1) we obtain

$$\begin{aligned}\Bigg\| \frac{f(2\cdot 3^n x + i\cdot 3^n y)}{9^n} + \frac{f(3^n x + i\cdot 2\cdot 3^n y)}{9^n} - \frac{4f(3^n x + i\cdot 3^n y)}{9^n} \\ - \frac{f(3^n x)}{9^n} - \frac{f(3^n y)}{9^n}\Bigg\| \\ \leq \frac{\phi(3^n x, 3^n y)}{9^n}\end{aligned} \tag{4.7}$$

taking limit as $n \to \infty$ the R. H. S. of (4.7) tends to zero as $n \to \infty$.
and using (4.6) we get
$Q(2x + iy) + Q(x + 2iy) = 4Q(x + iy) + Q(x) + Q(y)$
for all $x, y \in X$.
Also $Q(ix) = \lim_{n\to\infty}\left(\frac{f(3^n ix)}{9^n}\right) = \lim_{n\to\infty}\left(\frac{f(3^n x)}{9^n}\right) = Q(x)$
for all $x \in X$.
Again by the Result 3.1, the mapping $Q : X \to Y$ is a quadratic. □

Corollary 4.2. *Let $p < 2$ and θ be positive real numbers and $f : X \to Y$ be a mapping satisfying $f(ix) = f(x)$ and*

$$\|(f(2x + iy) + f(x + 2iy) - 4f(x + iy) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta\|x\|^p}{3 - 3^{p-1}}$$

Proof. Define $\phi (x , y) = \theta (\|x\|^p + \|y\|^p)$, and $L = 3^{p-2}$ then the proof is followed by Theorem 4.1 .

□

Corollary 4.3. *Let* $p < 1$ *and* θ *be positive real numbers and* $f : X \to Y$ *be a mapping satisfying* $f (i x) = f (x)$ *and*

$$\|(f (2x + i y) + f (x + 2 i y) - 4 f (x + i y) - f (x) - f (i y)\| \leq \theta (\|x\|^p . \|y\|^p)$$

for all $x , y \in X$. *Then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\| f (x) - Q (x) \| \leq \frac{\theta \|x\|^{2p}}{18 - 2.9^p}$$

Proof. It can be proved in the similar way as Theorem 4.1 by defined
$\phi (x , y) = \theta (\|x\|^p . \|y\|^p)$, and $L = 3^{2p-2}$ □

Theorem 4.4. *Assume* $\phi : X \times X \to [0 , \infty)$ *be a mapping such that, for which* $0 < L < 1$,
$\phi (x , - i x) \leq 9 L \phi(\frac{x}{3} , \frac{- i x}{3})$
and let $f : X \to Y$ *be a mapping with* $f (i x) = - f (x)$ *satisfying*

$$\| f (2 x + i y) - f (i x - 2 y) + 4 f (i x - y) - f (x) + f (- y)\| \leq \phi (x , y)$$

for all $x , y \in X$.
Then there exists a unique quadratic mapping $Q : X \to Y$ *such that*

$$\|f (x) - Q (x) \| \leq \frac{1}{9 - 9L} \left(\frac{\phi (x , - i x)}{2} + 2 \phi (x , 0) \right)$$

Proof. Proof of this theorem is same as that of Theorem 4.1 only using
$f (i x) = - f (x)$ for all $x \in X$. □

Conclusion : Though we have seen that both the equations (0.1) and (0.2) can be reduced to the equation (1.2), one can try to find particular solution of the equations (0.1) and (0.2). In the next, can we be able to reduce the equation (0.1) to the equation (0.2)) and vice - versa, that is, one can try to find a relation among the solutions of the equations (0.1) and (0.2).

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Pratap Mondal is an assistant professor in mathematics, in the Department of Mathematics, Bijoy Krishna Girls College, Howrah, under the University of Calcutta, India. He completed his M.Sc degree in 2006 from the Department of Applied Mathematics, University of Calcutta. He is currently doing his doctoral research work under the joint supervision of Professor Parbati Saha of Indian Institute of Engineering Science and Technology and Dr. T.K. Samanta, an associate professor at Uluberia College, India. His research interests include functional equations and its stability in fuzzy Banach spaces and complex Banach Spaces.

Tapas Kumar Samanta is an associate professor in mathematics, in the Department of Mathematics, Uluberia College, Howrah, India. He received his Ph.D. degree (2003) in complex analysis from the Department of Pure Mathematics, University of Calcutta. His present interests include fuzzy mathematical analysis and soft mathematical analysis.