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STUDY OF THE FIRST BOUNDARY VALUE PROBLEM FOR A FOURTH ORDER PARABOLIC EQUATION IN A NONREGULAR DOMAIN OF \mathbb{R}^{N+1}

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ABSTRACT. This paper is concerned with the extension of solvability results obtained for a fourth order parabolic equation, set in a nonregular domain of \mathbb{R}^3 obtained in [1], to the case where the domain is cylindrical, not with respect to the time variable, but with respect to N space variables, $N > 1$. More precisely, we determine optimal conditions on the shape of the boundary of a $(N + 1)$ -dimensional domain, $N > 1$, under which the solution is regular.

Keywords: Fourth order parabolic equations, Nonregular domains, Anisotropic weighted Sobolev spaces.

AMS Subject Classification: 35K05, 35K55

1. INTRODUCTION

Let Ω be an open set of \mathbb{R}^2 defined by

$$\Omega = \{(t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\}$$

where T is a finite positive number, while φ_1 and φ_2 are continuous real-valued functions defined on $[0, T]$, Lipschitz continuous on $[0, T]$, and such that

$$\varphi_2(t) - \varphi_1(t) > 0, \text{ for } t \in]0, T]$$

and

$$\varphi_2(0) = \varphi_1(0) = 0.$$

The lateral boundary of Ω is defined by

$$\Gamma_i = \{(t, \varphi_i(t)) \in \mathbb{R}^2 : 0 < t < T\}, \quad i = 1, 2.$$

For fixed positive numbers $b_i, i = 1, \dots, N - 1$, with $N > 1$, let Q be the $(N + 1)$ -dimensional domain defined by

$$Q = \Omega \times \prod_{i=1}^{N-1}]0, b_i[.$$

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In this work, we study the existence and the regularity of the solution of the fourth order parabolic equation with Cauchy-Dirichlet boundary conditions

$$\begin{cases} \partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u = f & \text{in } Q, \\ u|_{t=0} = 0, \\ u|_{\Sigma_i} = \partial_{x_1} u|_{\Sigma_i} = 0, \quad i = 1, 2, \\ u|_{\Sigma_0 \cup \Sigma_b} = \partial_{x_2} u|_{\Sigma_0 \cup \Sigma_b} = \dots = \partial_{x_N} u|_{\Sigma_0 \cup \Sigma_b} = 0, \end{cases} \quad (1)$$

where $\Sigma_i = \Gamma_i \times \prod_{k=1}^{N-1}]0, b_k[$, $i = 1, 2$, Σ_0 is the part of the boundary of Q where $x_k = 0, k = 2, \dots, N$ and Σ_b is the part of the boundary of Q where $x_k = b_{k-1}, k = 2, \dots, N$. The right-hand side term f of the equation lies in $L_\omega^2(Q)$ the space of square-integrable functions on Q with the measure $\omega dt dx_1 \dots dx_N$. Here the weight ω is a real-valued differentiable function on $[0, T]$.

We are especially interested in the question of what sufficient conditions, as weak as possible, the functions φ_1 , φ_2 and ω must verify in order that Problem (1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic weighted Sobolev space

$$H_{0,\omega}^{1,4}(Q) = \left\{ u \in H_\omega^{1,4}(Q) : u|_{\partial_p Q} = 0 \right\}$$

with

$$H_\omega^{1,4}(Q) = \left\{ u \in L_\omega^2(Q) : \partial_t u, \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \dots \partial_{x_N}^{i_N} u \in L_\omega^2(Q), 1 \leq i_1 + \dots + i_N \leq 4 \right\}$$

and $u|_{\partial_p Q} = 0$ means that

$$u|_{t=0} = u|_{\Sigma_i} = \partial_{x_1} u|_{\Sigma_i} = u|_{\Sigma_0 \cup \Sigma_b} = \partial_{x_2} u|_{\Sigma_0 \cup \Sigma_b} = \dots = \partial_{x_N} u|_{\Sigma_0 \cup \Sigma_b} = 0, \quad i = 1, 2.$$

Observe that the domain Q considered here is nonstandard since it shrinks at $t = 0$, $\varphi_2(0) = \varphi_1(0)$. This prevents the nonregular domain Q to be transformed into a usual cylindrical domain by means of a smooth transformation. On the other hand, the semi group generating the solution cannot be defined since the initial condition is defined on a set measure zero.

In Sadallah [2] a similar result has been obtained for a 2m-parabolic operator in the case of one space variable. The solvability of boundary value problems for a 2m-th order parabolic equation in Hölder spaces for noncylindrical domains (of the same kind but which cannot include our domain) with a nonsmooth (in t) lateral boundary was established in [3], [4] and [5]. Further references on the analysis of parabolic problems in noncylindrical domains are: Galaktionov [6], Baderko [7], Mikhailov [8], Savaré [9], Hoffmann and Lewis [10], Labbas, Medeghri and Sadallah [11], [12] and Kheloufi et al. [13], [14], [15], [16] and [17].

The organization of this paper is as follows. In Section 2, we prove that Problem (1) admits a (unique) solution in the case of a truncated domain. In Section 3 we approximate Q by a sequence (Q_n) of such domains and we establish (for T small enough) a uniform estimate of the type

$$\|u_n\|_{H_\omega^{1,4}(Q_n)} \leq K \|f\|_{L_\omega^2(Q_n)},$$

where u_n is the solution of Problem (1) in Q_n and K is a constant independent of n . Finally, in Section 4 we prove the two main results of this paper.

The main assumptions on the functions φ_1 , φ_2 and ω are

$$\varphi_i'(t) (\varphi_2 - \varphi_1)^2(t) \rightarrow 0 \quad \text{as } t \rightarrow 0, \quad i = 1, 2, \quad (2)$$

$$\forall t \in [0, T] : \omega(t) > 0, \quad (3)$$

and

$$\omega \text{ is a decreasing function on }]0, T]. \quad (4)$$

Note that this work may be extended at least in the following directions:

1. The function f on the right-hand side of the equation of Problem (1), may be taken in $L^p_\omega(Q)$, $p \in]1, \infty[$. The domain decomposition method used here does not seem to be appropriate for the space $L^p_\omega(Q)$ when $p \neq 2$.

2. The nonregular domain Q may be replaced by a noncylindrical conical type domain, such as, for example, the following domain

$$Q = \left\{ (t, x_1, x_2, \dots, x_N) \in \mathbb{R}^{N+1} : 0 \leq \sqrt{x_1^2 + x_2^2 + \dots + x_N^2} < \varphi(t), 0 < t < T \right\}$$

where φ is similar to $\varphi_i, i = 1, 2$. These questions will be developed in forthcoming works.

2. RESOLUTION OF PROBLEM (1) IN A TRUNCATED DOMAIN Q_n

In this section, we replace Q by $Q_n, n \in \mathbb{N}^*$ and $\frac{1}{n} < T$:

$$Q_n = \left\{ (t, x_1, \dots, x_N) \in Q : \frac{1}{n} < t < T \right\}.$$

Theorem 2.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the problem*

$$\begin{cases} \partial_t u_n + \sum_{k=1}^N \partial_{x_k}^4 u_n = f_n \in L^2_\omega(Q_n), \\ u_n|_{t=\frac{1}{n}} = u_n|_{\Sigma_{i,n}} = \partial_{x_1} u_n|_{\Sigma_{i,n}} = 0, i = 1, 2, \\ u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = \partial_{x_2} u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = \dots = \partial_{x_N} u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = 0, \end{cases} \quad (5)$$

admits a (unique) solution $u_n \in H^{1,4}_\omega(Q_n)$. Here,

$\Sigma_{i,n} = \{(t, \varphi_i(t)) \in \mathbb{R}^2 : \frac{1}{n} < t < T\} \times \prod_{k=1}^{N-1}]0, b_k[$, $i = 1, 2$, $\Sigma_{0,n}$ is the part of the boundary of Q_n where $x_k = 0, k = 2, \dots, N$ and $\Sigma_{b,n}$ is the part of the boundary of Q_n where $x_k = b_{k-1}, k = 2, \dots, N$.

Proof of Theorem 2.1: The change of variables

$$(t, x_1, x_2, \dots, x_N) \longmapsto (t, y_1, y_2, \dots, y_N) = \left(t, \frac{x_1 - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)}, x_2, \dots, x_N \right),$$

transforms Q_n into the cylindrical domain $P_n =]\frac{1}{n}, T[\times]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[$. Putting

$$v_n(t, y_1, y_2, \dots, y_N) = u_n(t, x_1, x_2, \dots, x_N)$$

and

$$g_n(t, y_1, y_2, \dots, y_N) = f_n(t, x_1, x_2, \dots, x_N),$$

then Problem (5) becomes

$$\begin{cases} \partial_t v_n + a(t, y_1) \partial_{y_1} v_n + c(t) \partial_{y_1}^4 v_n + \sum_{k=2}^N \partial_{y_k}^4 v_n = g_n \in L^2_\omega(P_n) \\ v_n|_{t=\frac{1}{n}} = v_n|_{\Sigma_{i,P_n}} = \partial_{y_1} v_n|_{\Sigma_{i,P_n}} = 0, i = 1, 2, \\ v_n|_{\Sigma_{0,P_n} \cup \Sigma_{b,P_n}} = \partial_{y_2} v_n|_{\Sigma_{0,P_n} \cup \Sigma_{b,P_n}} = \dots = \partial_{y_N} v_n|_{\Sigma_{0,P_n} \cup \Sigma_{b,P_n}} = 0, \end{cases}$$

where $\Sigma_{1,P_n} =]\frac{1}{n}, T[\times \{0\} \times \prod_{i=1}^{N-1}]0, b_i[$, $\Sigma_{2,P_n} =]\frac{1}{n}, T[\times \{1\} \times \prod_{i=1}^{N-1}]0, b_i[$, Σ_{0,P_n} is the part of the boundary of P_n where $x_k = 0, k = 2, \dots, N$, Σ_{b,P_n} is the part of the boundary of P_n where $x_k = b_{k-1}, k = 2, \dots, N$, $c(t) = \frac{1}{[\varphi_2(t) - \varphi_1(t)]^4}$ and $a(t, y_1) = -\frac{y_1(\varphi_2'(t) - \varphi_1'(t)) + \varphi_1'(t)}{\varphi_2(t) - \varphi_1(t)}$.

Since the functions a, c and $(\varphi_2 - \varphi_1)$ are bounded when $t \in]\frac{1}{n}, T[$, then the above change of variable which is $(N + 1)$ -Lipschitz preserves the spaces L^2_ω and $H^{1,4}_\omega$. In other words

$$f_n \in L^2_\omega(Q_n) \iff g_n \in L^2_\omega(P_n), u_n \in H^{1,4}_\omega(Q_n) \iff v_n \in H^{1,4}_\omega(P_n).$$

Proposition 2.1. For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the following operator is compact

$$a(t, y_1) \partial_{y_1} : H_{0,\omega}^{1,4}(P_n) \longrightarrow L_\omega^2(P_n).$$

Proof. P_n has the "horn property" of Besov [19], so

$$\partial_{y_1} : H_{0,\omega}^{1,4}(P_n) \longrightarrow H_\omega^{\frac{3}{4},3}(P_n), \quad v_n \longmapsto \partial_{y_1} v_n,$$

is continuous. Since P_n is bounded, the canonical injection is compact from $H_\omega^{\frac{3}{4},3}(P_n)$ into $L_\omega^2(P_n)$, where

$$H_\omega^{\frac{3}{4},3}(P_n) = L^2\left(\frac{1}{n}, T; H^3\left(]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[\right)\right) \cap H_\omega^{\frac{3}{4}}\left(\frac{1}{n}, T; L^2\left(]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[\right)\right).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces see for instance [20]. Consider the composition

$$\partial_{y_1} : H_{0,\omega}^{1,4}(P_n) \rightarrow H_\omega^{\frac{3}{4},3}(P_n) \rightarrow L_\omega^2(P_n), \quad v_n \mapsto \partial_{y_1} v_n \mapsto \partial_{y_1} v_n,$$

then ∂_{y_1} is a compact operator from $H_{0,\omega}^{1,4}(P_n)$ into $L_\omega^2(P_n)$. Since $a(\cdot, \cdot)$ is a bounded function for $\frac{1}{n} < t < T$, the operator $a\partial_{y_1}$ is also compact from $H_{0,\omega}^{1,4}(P_n)$ into $L_\omega^2(P_n)$. \square

So, thanks to Proposition 2.1, to complete the proof of Theorem 2.1, it is sufficient to show that the operator

$$\partial_t + c(t) \partial_{y_1}^4 + \sum_{k=2}^N \partial_{y_k}^4$$

is an isomorphism from $H_{0,\omega}^{1,4}(P_n)$ into $L_\omega^2(P_n)$.

Lemma 2.1. For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the operator

$$\partial_t + c(t) \partial_{y_1}^4 + \sum_{k=2}^N \partial_{y_k}^4$$

is an isomorphism from $H_{0,\omega}^{1,4}(P_n)$ into $L_\omega^2(P_n)$.

Proof. Since the coefficient $\frac{1}{[\varphi_2(t) - \varphi_1(t)]^4}$ is continuous in $\overline{P_n}$, the optimal regularity is given by Ladyzhenskaya-Solonnikov-Ural'tseva [18]. \square

We shall need the following result in order to justify some calculations in the next section, see [1].

Lemma 2.2. For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the space

$$\left\{ u_n \in H^4(P_n); \quad u_n|_{\partial P_n - \Gamma_T} = 0 \right\}$$

is dense in the space

$$\left\{ u_n \in H^{1,4}(P_n); \quad u_n|_{\partial P_n - \Gamma_T} = 0 \right\}.$$

Here Γ_T be the part of the boundary of P_n where $t = T$.

Remark 2.1. In Lemma 2.2, we can replace P_n by Q_n with the help of the change of variable defined above.

3. AN "ENERGY" TYPE ESTIMATE

For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we denote by $u_n \in H_\omega^{1,4}(Q_n)$ the solution of Problem (5) corresponding to the right-hand side $f_n = f|_{Q_n} \in L_\omega^2(Q_n)$. Such a solution exists by Theorem 2.1.

Proposition 3.1. *Assume that φ_1 and φ_2 fulfil condition (2) and the weight function ω verifies assumptions (3) and (4). Then, for T small enough, there exists a constant M independent of n such that*

$$\|u_n\|_{H_\omega^{1,4}(Q_n)} \leq M \|f_n\|_{L_\omega^2(Q_n)} \leq M \|f\|_{L_\omega^2(Q)},$$

where

$$\|u_n\|_{H_\omega^{1,4}(Q_n)} = \left(\|u_n\|_{L_\omega^2(Q_n)}^2 + \|\partial_t u_n\|_{L_\omega^2(Q_n)}^2 + \sum_{\substack{i_1, i_2, \dots, i_N=0 \\ 1 \leq i_1 + i_2 + \dots + i_N \leq 4}}^4 \|\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \dots \partial_{x_N}^{i_N} u_n\|_{L_\omega^2(Q_n)}^2 \right)^{1/2}.$$

Remark 3.1. *Let $\epsilon > 0$ be a real which we will choose small enough. The hypothesis (2) implies the existence of a real number $T > 0$ small enough such that*

$$\forall t \in (0, T), |\varphi'_i(t) (\varphi_2 - \varphi_1)^2(t)| \leq \epsilon, \quad i = 1, 2. \tag{6}$$

To derive the basic inequality of Proposition (3.1), we need the following lemmas.

Lemma 3.1. *Let $]\gamma, \delta[\subset \mathbb{R}$. There exists a positive constant K_2 (independent of γ and δ) such that for each $v \in H^4(]\gamma, \delta[) \cap H_0^2(]\gamma, \delta[)$*

$$\|v^{(l)}\|_{L^2(]\gamma, \delta[)}^2 \leq (\delta - \gamma)^{2(4-l)} K_2 \|v^{(4)}\|_{L^2(]\gamma, \delta[)}^2, \quad l = 0, 1, 2, 3.$$

The proof of the previous Lemma can be found in [1].

Lemma 3.2. *For every $\epsilon > 0$, chosen such that $(\varphi_2(t) - \varphi_1(t)) \leq \epsilon$, there exists a constant C_1 independent of n such that*

$$\|\partial_{x_1}^l u_n\|_{L_\omega^2(Q_n)}^2 \leq C_1 \epsilon^{2(4-l)} \|\partial_{x_1}^4 u_n\|_{L_\omega^2(Q_n)}^2, \quad l = 0, 1, 2, 3.$$

Proof. Replacing in Lemma 3.1 v by u_n and $]\gamma, \delta[$ by $]\varphi_1(t), \varphi_2(t)[$, for a fixed t , we obtain

$$\begin{aligned} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_{x_1}^l u_n)^2 dx_1 &\leq K_2 (\varphi_2(t) - \varphi_1(t))^{2(4-l)} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_{x_1}^4 u_n)^2 dx_1 \\ &\leq K_2 \epsilon^{2(4-l)} \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_{x_1}^4 u_n)^2 dx_1. \end{aligned}$$

Multiplying the previous inequality by $\omega(t)$ (which is positive) and integrating with respect to t , then with respect to x_2, x_3, \dots, x_N , we get the desired result with $C_1 = K_2$. \square

Lemma 3.3. *Let us denote the inner product in $L_\omega^2(Q_n)$ by $\langle \cdot, \cdot \rangle$. Under the assumptions of Proposition (3.1), we have*

- i) $2\langle \partial_t u_n, \partial_{x_1}^4 u_n \rangle \geq -K\epsilon \|\partial_{x_1}^4 u_n\|_{L_\omega^2(Q_n)}^2$ (for T small enough).
- ii) $2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle \geq 0, \quad k = 2, \dots, N.$
- iii) $2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n \rangle = 2 \|\partial_{x_j}^2 \partial_{x_k}^2 u_n\|_{L_\omega^2(Q_n)}^2, \quad j = 1, \dots, N-1, \quad k = j+1, \dots, N.$

Proof. 1) **Estimation of $2\langle \partial_t u_n, \partial_{x_1}^4 u_n \rangle$:** We have

$$\partial_t u_n \cdot \partial_{x_1}^4 u_n = \partial_{x_1} (\partial_t u_n \cdot \partial_{x_1}^3 u_n) - \partial_{x_1} (\partial_{x_1} \partial_t u_n \cdot \partial_{x_1}^2 u_n) + \frac{1}{2} \partial_t (\partial_{x_1}^2 u_n)^2.$$

Then

$$\begin{aligned} 2\langle \partial_t u_n, \partial_{x_1}^4 u_n \rangle &= 2 \int_{Q_n} \partial_t u_n \cdot \partial_{x_1}^4 u_n \cdot \omega(t) \, dt dx_1 \dots dx_N \\ &= \int_{\partial Q_n} \left[(\partial_{x_1}^2 u_n)^2 \nu_t + 2 (\partial_t u_n \cdot \partial_{x_1}^3 u_n - \partial_{x_1} \partial_t u_n \cdot \partial_{x_1}^2 u_n) \nu_{x_1} \right] \cdot \omega(t) \, d\sigma \\ &\quad - \int_{Q_n} (\partial_{x_1}^2 u_n)^2 \cdot \omega'(t) \, dt dx_1 \dots dx_N. \end{aligned}$$

We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_n where $t = \frac{1}{n}$, $x_k = 0$, $k = 2, \dots, N$ and $x_k = b_{k-1}$, $k = 2, \dots, N$ we have $\partial_{x_1} u_n = 0$ and consequently $\partial_{x_1}^2 u_n = \partial_{x_1}^3 u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_1} = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_0^{b_{N-1}} \dots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} [\partial_{x_1}^2 u_n(T, x_1, \dots, x_N)]^2 \omega(T) \, dx_1 \dots dx_N$$

is nonnegative. On the part of the boundary where $x_1 = \varphi_i(t)$, $i = 1, 2$, we have $\nu_{x_1} = \frac{(-1)^i}{\sqrt{1+(\varphi_i')^2(t)}}$, $\nu_t = \frac{(-1)^{i+1} \varphi_i'(t)}{\sqrt{1+(\varphi_i')^2(t)}}$ and $u = \partial_{x_1} u_n = 0$. Differentiating with respect to t we obtain

$$\begin{aligned} \partial_t u_n(t, \varphi_i(t), \dots, x_N) &= -\varphi_i'(t) \partial_{x_1} u_n(t, \varphi_i(t), \dots, x_N), \\ \partial_t \partial_{x_1} u_n(t, \varphi_i(t), \dots, x_N) &= -\varphi_i'(t) \partial_{x_1}^2 u_n(t, \varphi_i(t), \dots, x_N). \end{aligned}$$

Consequently, the corresponding boundary integrals I_1 and I_2 are the following:

$$\begin{aligned} I_1 &= - \int_0^{b_{N-1}} \dots \int_0^{b_1} \int_{\frac{1}{n}}^T \varphi_1'(t) [\partial_{x_1}^2 u_n(t, \varphi_1(t), \dots, x_N)]^2 \omega(t) \, dt dx_2 \dots dx_N \\ I_2 &= \int_0^{b_{N-1}} \dots \int_0^{b_1} \int_{\frac{1}{n}}^T \varphi_2'(t) [\partial_{x_1}^2 u_n(t, \varphi_2(t), \dots, x_N)]^2 \omega(t) \, dt dx_2 \dots dx_N. \end{aligned}$$

In virtue of (3) and (4), we have

$$2\langle \partial_t u_n, \partial_{x_1}^4 u_n \rangle \geq -|I_1| - |I_2|. \tag{7}$$

□

Lemma 3.4. *There exists a constant K_3 independent of n such that*

$$|I_i| \leq K_3 \epsilon \|\partial_{x_1}^4 u_n\|_{L^2_\omega(Q_n)}^2, \quad i = 1, 2.$$

Proof. We convert the boundary integral I_1 into a surface integral by setting

$$\begin{aligned} [\partial_{x_1}^2 u_n(t, \varphi_1(t), x_2, \dots, x_N)]^2 &= - \frac{\varphi_2(t) - x_1}{\varphi_2(t) - \varphi_1(t)} [\partial_{x_1}^2 u_n(t, x_1, x_2, \dots, x_N)]^2 \Big|_{x_1=\varphi_1(t)}^{x_1=\varphi_2(t)} \\ &= - \int_{\varphi_1(t)}^{\varphi_2(t)} \partial_{x_1} \left\{ \frac{\varphi_2(t) - x_1}{\varphi_2(t) - \varphi_1(t)} [\partial_{x_1}^2 u_n]^2 \right\} dx_1 \\ &= -2 \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\varphi_2(t) - x_1}{\varphi_2(t) - \varphi_1(t)} \partial_{x_1}^2 u_n \cdot \partial_{x_1}^3 u_n dx_1 \\ &\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{1}{\varphi_2(t) - \varphi_1(t)} [\partial_{x_1}^2 u_n]^2 dx_1. \end{aligned}$$

Then, we have

$$\begin{aligned} I_1 &= - \int_0^{b_{N-1}} \dots \int_0^{b_1} \int_{\frac{1}{n}}^T \varphi_1'(t) [\partial_{x_1}^2 u_n(t, \varphi_1(t), x_2, \dots, x_N)]^2 \omega(t) \, dt dx_2 \dots dx_N \\ &= - \int_{Q_n} \frac{\varphi_1'(t)}{\varphi_2(t) - \varphi_1(t)} [\partial_{x_1}^2 u_n(t, x_1, \dots, x_N)]^2 \omega(t) \, dt dx_1 \dots dx_N \\ &\quad + 2 \int_{Q_n} \frac{\varphi_2(t) - x_1}{\varphi_2(t) - \varphi_1(t)} \varphi_1'(t) (\partial_{x_1}^2 u_n) (\partial_{x_1}^3 u_n) \omega(t) \, dt dx_1 \dots dx_N. \end{aligned}$$

Thanks to Lemma 3.1, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_{x_1}^2 u_n]^2 dx_1 \leq K_2 [\varphi_2(t) - \varphi_1(t)]^4 \int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_{x_1}^4 u_n]^2 dx_1.$$

Therefore

$$\int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_{x_1}^2 u_n]^2 \frac{|\varphi_1'|}{\varphi_2(t) - \varphi_1(t)} \omega(t) dx_1 \leq K_2 |\varphi_1'| [\varphi_2(t) - \varphi_1(t)]^3 \int_{\varphi_1(t)}^{\varphi_2(t)} [\partial_{x_1}^4 u_n]^2 \omega(t) dx_1,$$

consequently

$$|I_1| \leq K_2 \int_{Q_n} |\varphi_1'| [\varphi_2(t) - \varphi_1(t)]^3 (\partial_{x_1}^4 u_n)^2 \omega(t) dt dx_1 \dots dx_N \\ + 2 \int_{Q_n} |\varphi_1'| |\partial_{x_1}^2 u_n| |\partial_{x_1}^3 u_n| \omega(t) dt dx_1 \dots dx_N,$$

since $\left| \frac{\varphi_2(t) - x_1}{\varphi_2(t) - \varphi_1(t)} \right| \leq 1$. Using the inequality

$$2 |\varphi_1' \partial_{x_1}^2 u_n| |\partial_{x_1}^3 u_n| \leq \epsilon (\partial_{x_1}^3 u_n)^2 + \frac{1}{\epsilon} (\varphi_1')^2 (\partial_{x_1}^2 u_n)^2$$

for all $\epsilon > 0$, we obtain

$$|I_1| \leq K_2 \int_{Q_n} |\varphi_1'| [\varphi_2(t) - \varphi_1(t)]^3 (\partial_{x_1}^4 u_n)^2 \omega(t) dt dx_1 \dots dx_N \\ + \int_{Q_n} \epsilon (\partial_{x_1}^3 u_n)^2 \omega(t) dt dx_1 \dots dx_N + \frac{1}{\epsilon} \int_{Q_n} (\varphi_1')^2 (\partial_{x_1}^2 u_n)^2 \omega(t) dt dx_1 \dots dx_N.$$

Lemma 3.2 yields

$$\frac{1}{\epsilon} \int_{Q_n} (\varphi_1')^2 (\partial_{x_1}^2 u_n)^2 \omega(t) dt dx_1 \dots dx_N \\ \leq K_2 \frac{1}{\epsilon} \int_{Q_n} (\varphi_1')^2 [\varphi_2(t) - \varphi_1(t)]^4 (\partial_{x_1}^4 u_n)^2 \omega(t) dt dx_1 \dots dx_N.$$

Thus,

$$|I_1| \leq K_2 \int_{Q_n} \left[|\varphi_1'| [\varphi_2(t) - \varphi_1(t)]^3 + \frac{1}{\epsilon} (\varphi_1')^2 [\varphi_2(t) - \varphi_1(t)]^4 \right] (\partial_{x_1}^4 u_n)^2 \omega(t) dt \dots dx_N \\ + \int_{Q_n} \epsilon (\partial_{x_1}^3 u_n)^2 \omega(t) dt dx_1 \dots dx_N \\ \leq (K_2 + 1) \epsilon \int_{Q_n} (\partial_{x_1}^4 u_n)^2 \omega(t) dt dx_1 \dots dx_N,$$

since $|\varphi_1'(\varphi_2(t) - \varphi_1(t))^2 [(\varphi_2(t) - \varphi_1(t)) - \varphi_1'(\varphi_2(t) - \varphi_1(t))^2]| \leq \epsilon$ thanks to the condition (6). Finally, taking $K_3 = (K_2 + 1)$, we obtain

$$|I_1| \leq K_3 \epsilon \|\partial_{x_1}^4 u_n\|_{L^2_\omega(Q_n)}.$$

The inequality

$$|I_2| \leq K_3 \epsilon \|\partial_{x_1}^4 u_n\|_{L^2_\omega(Q_n)},$$

can be proved by a similar argument.

2) Estimation of $2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle, k = 2, \dots, N$: We have

$$\partial_t u_n \cdot \partial_{x_k}^4 u_n = \partial_{x_k} (\partial_t u_n \cdot \partial_{x_k}^3 u_n) - \partial_{x_k} (\partial_{x_k} \partial_t u_n \cdot \partial_{x_k}^2 u_n) + \frac{1}{2} \partial_t (\partial_{x_k}^2 u_n)^2.$$

Then

$$2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle = 2 \int_{Q_n} \partial_t u_n \cdot \partial_{x_k}^4 u_n \cdot \omega(t) dt dx_1 \dots dx_N \\ = \int_{\partial Q_n} \left[(\partial_{x_k}^2 u_n)^2 \nu_t + 2 (\partial_t u_n \cdot \partial_{x_k}^3 u_n - \partial_{x_k} \partial_t u_n \cdot \partial_{x_k}^2 u_n) \nu_{x_k} \right] \cdot \omega(t) d\sigma \\ - \int_{Q_n} (\partial_{x_k}^2 u_n)^2 \cdot \omega'(t) dt dx_1 \dots dx_N.$$

Using the Cauchy-Dirichlet boundary conditions, we see that the above boundary integral is nonnegative. Consequently in virtue of (4), we have

$$2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle \geq 0. \tag{8}$$

3) Estimation of $2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n \rangle, j = 1, \dots, N - 1, k = j + 1, \dots, N$: We have

$$\partial_{x_j}^4 u_n \cdot \partial_{x_k}^4 u_n = \partial_{x_j} (\partial_{x_j}^3 u_n \cdot \partial_{x_k}^4 u_n) - \partial_{x_k} (\partial_{x_j}^3 u_n \cdot \partial_{x_j} \partial_{x_k}^3 u_n) + \partial_{x_j} \partial_{x_k}^3 u_n \cdot \partial_{x_k} \partial_{x_j}^3 u_n.$$

Then

$$\begin{aligned}
2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n \rangle &= 2 \int_{Q_n} \partial_{x_j}^4 u_n \cdot \partial_{x_k}^4 u_n \cdot \omega(t) dt dx_1 \dots dx_N \\
&= 2 \int_{Q_n} \partial_{x_j} \left(\partial_{x_j}^3 u_n \cdot \partial_{x_k}^4 u_n \right) \cdot \omega(t) dt dx_1 \dots dx_N \\
&\quad - 2 \int_{Q_n} \partial_{x_k} \left(\partial_{x_j}^3 u_n \cdot \partial_{x_j} \partial_{x_k}^3 u_n \right) \cdot \omega(t) dt dx_1 \dots dx_N \\
&\quad + 2 \int_{Q_n} \partial_{x_j} \partial_{x_k}^3 u_n \cdot \partial_{x_k} \partial_{x_j}^3 u_n \cdot \omega(t) dt dx_1 \dots dx_N \\
&= 2 \int_{Q_n} \partial_{x_j} \partial_{x_k}^3 u_n \cdot \partial_{x_k} \partial_{x_j}^3 u_n \cdot \omega(t) dt dx_1 \dots dx_N \\
&\quad + 2 \int_{\partial Q_n} \left[\partial_{x_j}^3 u_n \cdot \partial_{x_k}^4 u_n \nu_{x_j} - \partial_{x_j}^3 u_n \cdot \partial_{x_j} \partial_{x_k}^3 u_n \nu_{x_k} \right] \omega(t) d\sigma.
\end{aligned}$$

We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_n where $t = \frac{1}{n}$, $x_k = 0, k = 2, \dots, N$ and $x_k = b_{k-1}, k = 2, \dots, N$, we have $\partial_{x_j} u_n = 0$ and consequently $\partial_{x_j}^3 u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_k} = 0$. Accordingly the corresponding boundary integral vanishes. By using again Green formula and the Cauchy-Dirichlet boundary conditions, we obtain

$$2 \int_{Q_n} \partial_{x_j} \partial_{x_k}^3 u_n \cdot \partial_{x_k} \partial_{x_j}^3 u_n \cdot \omega(t) dt dx_1 \dots dx_N = 2 \left\| \partial_{x_j}^2 \partial_{x_k}^2 u_n \right\|_{L_\omega^2(Q_n)}^2.$$

Finally,

$$2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n \rangle = 2 \left\| \partial_{x_j}^2 \partial_{x_k}^2 u_n \right\|_{L_\omega^2(Q_n)}^2, j = 1, \dots, N-1, k = j+1, \dots, N. \quad (9)$$

□

Proof of Proposition (3.1): We have

$$\begin{aligned}
\|f_n\|_{L_\omega^2(Q_n)}^2 &= \langle \partial_t u_n + \sum_{k=1}^N \partial_{x_k}^4 u_n, \partial_t u_n + \sum_{k=1}^N \partial_{x_k}^4 u_n \rangle \\
&= \|\partial_t u_n\|_{L_\omega^2(Q_n)}^2 + \sum_{k=1}^N \|\partial_{x_k}^4 u_n\|_{L_\omega^2(Q_n)}^2 \\
&\quad + 2 \sum_{k=1}^N \langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle + 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^N \langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n \rangle.
\end{aligned}$$

Summing up the estimates (7), (8) and (9) of the inner products and making use of Lemma 3.4, we then obtain

$$\begin{aligned}
\|f_n\|_{L_\omega^2(Q_n)}^2 &\geq \|\partial_t u_n\|_{L_\omega^2(Q_n)}^2 + \sum_{k=1}^N \|\partial_{x_k}^4 u_n\|_{L_\omega^2(Q_n)}^2 \\
&\quad - |I_1| - |I_2| \\
&\quad + 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^N \left\| \partial_{x_j}^2 \partial_{x_k}^2 u_n \right\|_{L_\omega^2(Q_n)}^2 \\
&\geq \|\partial_t u_n\|_{L_\omega^2(Q_n)}^2 + (1 - 2K_3\epsilon) \|\partial_{x_1}^4 u_n\|_{L_\omega^2(Q_n)}^2 \\
&\quad + \sum_{k=2}^N \|\partial_{x_k}^4 u_n\|_{L_\omega^2(Q_n)}^2 + 2 \sum_{j=1}^{N-1} \sum_{k=j+1}^N \left\| \partial_{x_j}^2 \partial_{x_k}^2 u_n \right\|_{L_\omega^2(Q_n)}^2.
\end{aligned}$$

Then, it is sufficient to choose ϵ such that $(1 - 2K_3\epsilon) > 0$ to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|_{L_\omega^2(Q_n)} \geq K_0 \|u_n\|_{H_\omega^{1,4}(Q_n)},$$

and since

$$\|f_n\|_{L_\omega^2(Q_n)} \leq \|f\|_{L_\omega^2(Q)},$$

there exists a constant $M > 0$, independent of n satisfying

$$\|u_n\|_{H_\omega^{1,4}(Q_n)} \leq M \|f_n\|_{L_\omega^2(Q_n)} \leq M \|f\|_{L_\omega^2(Q)}.$$

This completes the proof of Proposition (3.1).

4. MAIN RESULTS

We are now able to prove the main results of the paper.

4.1. Local in time result.

Theorem 4.1. *Assume that φ_1 and φ_2 fulfil condition (2) and the weight function ω verifies assumptions (3) and (4). Then for T small enough, the fourth order parabolic operator*

$$L = \partial_t + \sum_{k=1}^N \partial_{x_k}^4$$

is an isomorphism from $H_{0,\omega}^{1,4}(Q)$ into $L_{\omega}^2(Q)$.

Proof. 1) Injectivity of the operator L: Let us consider $u \in H_{0,\omega}^{1,4}(Q)$ a solution of the problem (1) with a null right-hand side term. So,

$$\partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u = 0 \text{ in } Q.$$

In addition u fulfils the boundary condions

$$u|_{t=0} = u|_{\Sigma_i} = \partial_{x_1} u|_{\Sigma_i} = u|_{\Sigma_0 \cup \Sigma_b} = \partial_{x_2} u|_{\Sigma_0 \cup \Sigma_b} = \dots = \partial_{x_N} u|_{\Sigma_0 \cup \Sigma_b} = 0, \quad i = 1, 2.$$

Using Green formula, we have

$$\begin{aligned} & \int_Q \left(\partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u \right) u \cdot \omega(t) \, dt \, dx_1 \dots dx_N \\ &= \int_{\partial Q} \left[\frac{1}{2} |u|^2 \nu_t + \sum_{k=1}^N \left(\partial_{x_k}^3 u \cdot u - \partial_{x_k}^2 u \cdot \partial_{x_k} u \right) \nu_{x_k} \right] \omega(t) \, d\sigma \\ &+ \int_Q \left(\sum_{k=1}^N |\partial_{x_k}^2 u|^2 \right) \omega(t) \, dt \, dx_1 \dots dx_N - \int_Q \frac{1}{2} |u|^2 \omega'(t) \, dt \, dx_1 \dots dx_N \end{aligned}$$

where $\nu_t, \nu_{x_1}, \dots, \nu_{x_N}$ are the components of the unit outward normal vector at ∂Q . Taking into account the boundary conditions, all the boundary integrals vanish except $\int_{\partial Q} |u|^2 \omega(t) \nu_t \, d\sigma$. We have

$$\int_{\partial Q} |u|^2 \omega(t) \nu_t \, d\sigma = \int_0^{b_{N-1}} \dots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} |u|^2 \omega(T) \, dx_1 dx_2 \dots dx_N.$$

Then

$$\begin{aligned} & \int_Q \left(\partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u \right) \cdot u \, \omega(t) \, dt \, dx_1 \dots dx_N \\ &= \int_0^{b_{N-1}} \dots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \frac{1}{2} |u|^2 \omega(T) \, dx_1 dx_2 \dots dx_N - \int_Q \frac{1}{2} |u|^2 \omega'(t) \, dt \, dx_1 \dots dx_N \\ &+ \int_Q \left(\sum_{k=1}^N |\partial_{x_k}^2 u|^2 \right) \omega(t) \, dt \, dx_1 \dots dx_N. \end{aligned}$$

Consequently

$$\int_Q \left(\partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u \right) \cdot u \, \omega(t) \, dt \, dx_1 \dots dx_N = 0$$

yields

$$\int_Q \left(\sum_{k=1}^N |\partial_{x_k}^2 u|^2 \right) \omega(t) \, dt \, dx_1 \dots dx_N = 0,$$

because

$$\int_0^{b_{N-1}} \dots \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \frac{1}{2} |u|^2 \omega(T) \, dx_1 dx_2 \dots dx_N - \int_Q \frac{1}{2} |u|^2 \omega'(t) \, dt \, dx_1 \dots dx_N \geq 0$$

thanks to the conditions (3) and (4). This implies that $\sum_{k=1}^N |\partial_{x_k}^2 u|^2 = 0$ and consequently $\partial_{x_1}^4 u = \dots = \partial_{x_N}^4 u = 0$. Then, the hypothesis $\partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u = 0$ gives $\partial_t u = 0$. Thus, u is constant. The boundary conditions imply that $u = 0$ in Q . This proves the uniqueness of the solution of Problem (1).

2) Surjectivity of the operator L : Choose a sequence $(Q_n)_{n \in \mathbb{N}^*}$ of the domains defined above (see Section 2), such that $Q_n \subseteq Q$. Then, we have $Q_n \rightarrow Q$, as $n \rightarrow \infty$. Consider the solution $u_n \in H_{\omega}^{1,4}(Q_n)$ of the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_n + \sum_{k=1}^N \partial_{x_k}^4 u_n = f_n \text{ in } Q_n \\ u_n|_{t=\frac{1}{n}} = u_n|_{\Sigma_{i,n}} = \partial_{x_1} u_n|_{\Sigma_{i,n}} = 0, \quad i = 1, 2, \\ u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = \partial_{x_2} u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = \dots = \partial_{x_N} u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = 0, \end{cases}$$

where $\Sigma_{i,n} = \{(t, \varphi_i(t)) \in \mathbb{R}^2 : \frac{1}{n} < t < T\} \times \prod_{k=1}^{N-1}]0, b_k[$, $i = 1, 2$, $\Sigma_{0,n}$ is the part of the boundary of Q_n where $x_k = 0$, $k = 2, \dots, N$, and $\Sigma_{b,n}$ is the part of the boundary of Q_n where $x_k = b_{k-1}$, $k = 2, \dots, N$. Such a solution u_n exists by Theorem 2.1. Let \widetilde{u}_n the 0-extension of u_n to Q . In virtue of Proposition 3.1, we know that there exists a constant C such that

$$\|\widetilde{u}_n\|_{L_{\omega}^2(Q)} + \|\partial_t \widetilde{u}_n\|_{L_{\omega}^2(Q)} + \sum_{\substack{i_1, i_2, \dots, i_N = 0 \\ 1 \leq i_1 + i_2 + \dots + i_N \leq 4}} \left\| \widetilde{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \dots \partial_{x_N}^{i_N} u_n} \right\|_{L_{\omega}^2(Q)} \leq C \|f\|_{L_{\omega}^2(Q)}.$$

This means that $\widetilde{u}_n, \partial_t \widetilde{u}_n, \widetilde{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \dots \partial_{x_N}^{i_N} u_n}$ for $1 \leq i_1 + i_2 + \dots + i_N \leq 4$ are bounded functions in $L_{\omega}^2(Q)$. The following compactness result is well known: A bounded sequence in a reflexive Banach space (and in particular in a Hilbert space) is weakly convergent. So for a suitable increasing sequence of integers n_k , $k = 1, 2, \dots$, there exist functions u, v and v_{i_1, i_2, \dots, i_N} $1 \leq i_1 + i_2 + \dots + i_N \leq 4$ in $L_{\omega}^2(Q)$ such that

$$\widetilde{u}_{n_k} \rightharpoonup u, \quad \partial_t \widetilde{u}_{n_k} \rightharpoonup v, \quad \widetilde{\partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \dots \partial_{x_N}^{i_N} u_{n_k}} \rightharpoonup v_{i_1, i_2, \dots, i_N}, \quad 1 \leq i_1 + i_2 + \dots + i_N \leq 4$$

weakly in $L_{\omega}^2(Q)$ as $k \rightarrow \infty$. Clearly,

$$v = \partial_t u, \quad v_{i_1, i_2, \dots, i_N} = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \dots \partial_{x_N}^{i_N} u, \quad 1 \leq i_1 + i_2 + \dots + i_N \leq 4$$

in the sense of distributions in Q and so in $L_{\omega}^2(Q)$. So, $u \in H_{\omega}^{1,4}(Q)$ and

$$\partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u = f \text{ in } Q.$$

On the other hand, the solution u satisfies the boundary conditions

$$u|_{t=0} = u|_{\Sigma_i} = \partial_{x_1} u|_{\Sigma_i} = 0, \quad i = 1, 2$$

and

$$u|_{\Sigma_0 \cup \Sigma_b} = \partial_{x_2} u|_{\Sigma_0 \cup \Sigma_b} = \dots = \partial_{x_N} u|_{\Sigma_0 \cup \Sigma_b} = 0,$$

since

$$\forall n \in \mathbb{N}^*, \quad u|_{Q_n} = u_n.$$

This proves the existence of solution to Problem (1). This ends the proof of Theorem 4.1. \square

4.2. Global in time result. In the case where T is not in the neighborhood of zero, we set $Q = D_1 \cup D_2 \cup \Sigma_{T_1}$ where

$$D_1 = \{(t, x_1, \dots, x_N) \in Q : 0 < t < T_1\},$$

$$D_2 = \{(t, x_1, \dots, x_N) \in Q : T_1 < t < T\},$$

$$\Sigma_{T_1} = \{(T_1, x_1) \in \mathbb{R}^2 : \varphi_1(T_1) < x_1 < \varphi_2(T_1)\} \times \prod_{i=1}^{N-1}]0, b_i[$$

with T_1 small enough. In the sequel, f stands for an arbitrary fixed element of $L^2_\omega(Q)$ and $f_i = f|_{D_i}$, $i = 1, 2$.

Theorem 4.1 applied to the non-regular domain D_1 , shows that there exists a unique solution $v_1 \in H^{1,4}_\omega(D_1)$ of the problem

$$\begin{cases} \partial_t v_1 + \sum_{k=1}^N \partial_{x_k}^4 v_1 = f_1 \in L^2_\omega(D_1), \\ v_1|_{t=0} = 0, \\ v_1|_{\Sigma_{i,1}} = \partial_{x_1} v_1|_{\Sigma_{i,1}} = 0, \quad i = 1, 2, \\ v_1|_{\Sigma_{0,1} \cup \Sigma_{b,1}} = \partial_{x_2} v_1|_{\Sigma_{0,1} \cup \Sigma_{b,1}} = \dots = \partial_{x_N} v_1|_{\Sigma_{0,1} \cup \Sigma_{b,1}} = 0, \end{cases} \quad (10)$$

$\Sigma_{i,1}$ are the parts of the boundary of D_1 where $x_1 = \varphi_i(t)$, $i = 1, 2$, $\Sigma_{0,1}$ is the part of the boundary of D_1 where $x_k = 0$, $k = 2, \dots, N$ and $\Sigma_{b,1}$ is the part of the boundary of D_1 where $x_k = b_{k-1}$, $k = 2, \dots, N$.

Hereafter, we denote the trace $v_1|_{\Sigma_{T_1}}$ by ψ which is in the Sobolev space $H^2_\omega(\Sigma_{T_1})$ because $v_1 \in H^{1,4}_\omega(D_1)$ (see [20]). Now, consider the following problem in D_2

$$\begin{cases} \partial_t v_2 + \sum_{k=1}^N \partial_{x_k}^4 v_2 = f_2 \in L^2_\omega(D_2), \\ v_2|_{\Sigma_{T_1}} = \psi, \\ v_2|_{\Sigma_{i,2}} = \partial_{x_1} v_2|_{\Sigma_{i,2}} = 0, \quad i = 1, 2, \\ v_2|_{\Sigma_{0,2} \cup \Sigma_{b,2}} = \partial_{x_2} v_2|_{\Sigma_{0,2} \cup \Sigma_{b,2}} = \dots = \partial_{x_N} v_2|_{\Sigma_{0,2} \cup \Sigma_{b,2}} = 0, \end{cases} \quad (11)$$

$\Sigma_{i,2}$ are the parts of the boundary of D_2 where $x_1 = \varphi_i(t)$, $i = 1, 2$, $\Sigma_{0,2}$ is the part of the boundary of D_2 where $x_k = 0$, $k = 2, \dots, N$ and $\Sigma_{b,2}$ is the part of the boundary of D_2 where $x_k = b_{k-1}$, $k = 2, \dots, N$.

We use the following result, which is a consequence of [20, Theorem 4.3, Vol.2] to solve Problem (11).

Proposition 4.1. *Let R be the cylinder $]0, T[\times]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[$, $f \in L^2_\omega(R)$ and $u_0 \in H^2_\omega(\gamma_0)$. Then, the problem*

$$\begin{cases} \partial_t u + \sum_{k=1}^N \partial_{x_k}^4 u = f \text{ in } R, \\ u|_{\gamma_0} = u_0, \\ u|_{\gamma_i} = \partial_{x_1} u|_{\gamma_i} = 0, \quad i = 1, 2, \\ u|_{\partial R - (\gamma_0 \cup \gamma_i)} = \partial_{x_2} u|_{\partial R - (\gamma_0 \cup \gamma_i)} = \dots = \partial_{x_N} u|_{\partial R - (\gamma_0 \cup \gamma_i)} = 0, \quad i = 1, 2, \end{cases}$$

where $\gamma_0 = \{0\} \times]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[$, $\gamma_1 =]0, T[\times \{0\} \times \prod_{i=1}^{N-1}]0, b_i[$ and $\gamma_2 =]0, T[\times \{1\} \times \prod_{i=1}^{N-1}]0, b_i[$, admits a (unique) solution $u \in H^{1,4}_\omega(R)$ if and only if the following compatibility conditions are fulfilled

$$\partial_{x_j}^k u_0 \Big|_{\partial \gamma_0} = 0, \quad k = 0, 1; \quad j = 1, \dots, N.$$

The transformation

$$(t, x_1, x_2, \dots, x_N) \longmapsto (t, y_1, y_2, \dots, y_N) = (t, (\varphi_2(t) - \varphi_1(t))x_1 + \varphi_1(t), x_2, \dots, x_N)$$

leads to the following result:

Proposition 4.2. *Problem (11) admits a (unique) solution $v_2 \in H_\omega^{1,4}(D_2)$ if and only if the following compatibility conditions are fulfilled*

$$\partial_{x_j}^k \psi \Big|_{\partial \Sigma_{T_1}} = 0, \quad k = 0, 1; \quad j = 1, \dots, N.$$

Remark 4.1. *We can observe that the boundary conditions of Problems (10) and (11) yield*

$$v_1|_{\Sigma_{T_1}} = v_2|_{\Sigma_{T_1}}$$

and $\partial_{x_j}^k v_i \Big|_{\Sigma_{T_1}} \in H_\omega^{\frac{3}{4}}(\Sigma_{T_1}); \quad k = 0, 1; \quad j = 1, \dots, N..$ Then the compatibility conditions

$$\partial_{x_j}^k \psi \Big|_{\partial \Sigma_{T_1}} = 0, \quad k = 0, 1; \quad j = 1, \dots, N$$

are satisfied since $v_1|_{\Sigma_{T_1}} = \psi$.

Now, consider the function u in Q defined by

$$u := \begin{cases} v_1 & \text{in } D_1 \\ v_2 & \text{in } D_2 \end{cases}$$

where v_1 and v_2 are the solutions of Problem (10) and Problem (11) respectively. Observe that $v_1|_{\Sigma_{T_1}} = v_2|_{\Sigma_{T_1}}$, see Remark 4.1, so

$$\partial_{x_j}^k v_1 \Big|_{\Sigma_{T_1}} = \partial_{x_j}^k v_2 \Big|_{\Sigma_{T_1}}, \quad k = 0, 1; \quad j = 1, \dots, N.$$

This implies that $u \in H_\omega^{1,4}(Q)$ and u is the (unique) solution of Problem (1) for an arbitrary T .

Our second main result is as follows.

Theorem 4.2. *Under the assumptions (2), (3) and (4) on the functions φ_1 , φ_2 and ω , Problem (1) admits a (unique) solution $u \in H_\omega^{1,4}(Q)$.*

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