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MODIFIED DIFFERENTIAL TRANSFORM METHOD FOR SINGULAR LANE-EMDEN EQUATIONS IN INTEGER AND FRACTIONAL ORDER

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Abstract. In the present work the modified differential transform method, incorporating the Adomian polynomials into the differential transform method(DTM), is used to solve the nonlinear and singular Lane-Emden equations in integer and fractional order. Numerical examples with different types are solved. The results show that this method is very effective and simple.

Keywords: Differential transform method, Adomian polynomials, singular Lane-Emden equation.

AMS Subject Classification: 35A15, 35A20

1. Introduction

In recent years, the study of singular initial value problems modelled by second order nonlinear ordinary differential equations[1, 2] has attracted many mathematicians and physicists. One of the important equations in this literature is the Lane-Emden type equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{2}{t}\frac{\partial u}{\partial t} + f(u) = 0 \tag{1}$$

with initial conditions

$$u(0) = a, \ u'(0) = b, \tag{2}$$

where f is a given function of u and a,b are constants. Lane-Emden type equations, first published by Janathan Homer Lane in 1870 [3] and further explored in detail by Emden[4]. Lane-Emden type equation was used the model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical lows of thermodynamics. So, a substantial amount of work has been done on this type of problems. In astrophysics, the Lane-Emden equation is Poisson's equation for the gravitational potential of a self-gravitating spherically symmetric polytropic fluid. Also, this is a type of equations have many applications in the fields of radioactiviely colling and in the mean-field treatment of a phase transition in critical adsorption or in the modelling of clusters of galaxies.

In most differential equations with variable coefficients it is impossible to obtain an exact solution, so one must resort to various approximation methods of solution, such as asymptotic techniques[11, 12, 13], analytical[14, 15] and numerical methods[16]. Analytical

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techniques have been dominated by perturbation methods and have found many applications in science, engineering and technology. However, like other analytical techniques, perturbation methods require the presence of a small parameter in the nonlinear equation. Selection of small parameter requires a special skill and very important. Therefore, analytical methods which do not require a small parameter are welcome. Because of singularity behavior at the origin and nonlinearity, the Lane-Emden equations may not be solved by standard semi-analytical methods, such as the homotopy perturbation method[17], variational iteration method[7, 8] and Adomian decomposition method[6, 5]. The basic idea of differential transform method was initially introduced by Zhou[18]. The DTM is an alternative procedure for getting Taylor series solution of the equation. This method reduces the size of computations of taylor coefficients. The motivation for presenting this work comes from the aim of introducing a reliable framework that combines the powerful differential transform method and Adomian decomposition method.
Many physical problems can be described by mathematical models that involves fractional differential equations. There are several mathematical definitions about the generalizing of the notion of differential to fractional orders e.g. Riemann-Lioville, Grunuald-Letinikow, Caputo and generalized functions approach.

Definition 1.1. *A real function $f(t), t > 0$, is said to be in the space C_μ, $\mu \in R$, if there exist a real number $p \geq \mu$ such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C(0, \infty)$ and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in N$.*

Definition 1.2. *For a continuous function $f : [0, \infty) \to R$, the Caputo derivative of fractional order α is defined by*

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n-1 < \alpha < n$, $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 1.3. *The Riemann- Liouville fractional derivative of order α for a continuous function f is defined by*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n-1}} ds, \ n = [\alpha]+1,$$

where the right-hand side is pointwise defined on $(0, \infty)$.

Definition 1.4. *Let $[a, b]$ be an interval in R and $\alpha > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a, b], R)$ is defined by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

whenever the integral exists.

2. The fractional differential transform method(FDTM)

In the FDTM the analytic function $u(t)$ is expanded in terms of a fractional power series in the form

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^{\frac{k}{\theta}},$$

where θ is the order of the fraction to be selected and U(k) is the kth fractional differential transform of $u(t)$. Since the initial conditions are implemented by integer-order derivatives for practical applications, the transformation of the initial conditions is defined as follows

$$U(k) = \begin{cases} \frac{1}{(\frac{k}{\theta})!}[\frac{\partial^{\frac{k}{\theta}}}{\partial t^{\frac{k}{\theta}}} u(t)]_{t=t_0}, & \frac{k}{\theta} \in Z^+, \\ 0, & \frac{k}{\theta} \notin Z^+, \end{cases} \tag{3}$$

where $k = 0, 1, \cdots, (\alpha\theta - 1)$ and α is the order of the fractional diffential equation being considered. Thus θ should be chosen such that $\alpha\theta$ is a positive integer. Here we present some basic properties of the FDTM. Let $u(t)$, $v(t)$ and $w(t)$ be functions of time t and $U(k)$, $V(k)$ and $W(k)$ are their corresponding fractional differential transforms with order of fraction θ. Then for constant c and p the followings hold,

i: If $u(t) = v(t) \pm w(t)$, then $U(k) = V(k) \pm W(k)$.

ii: If $u(t) = cv(t)$, then $U(k) = cV(k)$.

iii: If $u(t) = D_t^\alpha v(t)$, then $U(k) = \frac{\Gamma(\alpha+1+\frac{k}{\theta})}{\Gamma(1+\frac{k}{\theta})} V(k + \alpha\theta)$.

We consider the case of a nonlinear function $f(u)$ that is approximated by the series

$$f(u) = \sum_{n=0}^{\infty} A_n,$$

where the A_n are the Adomian polynomials determined by the definitional formula

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n}[f(\sum_{i=0}^{\infty} \lambda^i u_i)]]_{\lambda=0}, \qquad n = 0, 1, \cdots.$$

For the nonlinear fractional differential equation of the form

$$D^\alpha u = f(u),$$

where $f(u)$ denotes a nonlinear function, we apply recurrence scheme of the form

$$\frac{\Gamma(\alpha + 1 + \frac{k}{\theta})}{\gamma(1 + \frac{k}{\theta})} U(k + \alpha\theta) = \tilde{A}_k,$$

where $\tilde{A}_k$ are obtained from the Adomian polynomials of $f(u)$ by replacing each $u(k)$ and $D^\beta u_k$ by $U(k)$ and $\frac{\Gamma(\alpha+1+\frac{k}{\theta})}{\gamma(1+\frac{k}{\theta})} U(k + \alpha\theta)$, respectively [20].

3. Numerical Applications

The Lane-Emden equation has been used to formulate several phenomena in mathematical physics and astrophysics. This equation encounters wide applications in modeling of the thermal behaviour of a spherical cloud of gas acting under a mutual attraction of its molecules and subject to the classical laws of thermodynamics. In this section, we solve three differential equations, the so-called Lane-Emden equation, to demonstrate the effectiveness and the validity of the present method.

Example 3.1. *Consider the fractional Lane-Emden equation*

$$D^\beta u + \frac{2}{t} u_t = -u, \qquad 1 < \beta \leq 2, \tag{4}$$

subject to the initial conditions

$$u(0) = 1, \ u_t(0) = 0. \tag{5}$$

Taking the fractional differential transform and choosing $\theta = \frac{2}{\beta}$, we write the recurrence scheme

$$\frac{\Gamma(1+(k+1)\frac{\beta}{2})}{\Gamma(1+(k-1)\frac{\beta}{2}}U(k+1)+2(k+1)U(k+1) = -U(k-1), \qquad k = 1, 2, \cdots. \tag{6}$$

So, we will get the approximations

$$U(0) = 1, \quad U(1) = 0, \quad U(2) = -\frac{1}{4+\Gamma(1+\beta)}, \quad U(3) = 0,$$

$$U(4) = \frac{1}{4+\Gamma(1+\beta)} \cdot \frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)+8\Gamma(1+\beta)}, \quad U(5) = 0, \cdots.$$

Here, if we put $\beta = 2$ then we will get the following solution

$$u(t) = 1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 - \frac{1}{7!}t^6 + \cdots$$

which is the Taylor expansion of the exact solution $u(t) = \frac{\sin t}{t}$ of the problem in this case.

Example 3.2. *Consider the fractional non-linear Lane-Emden equation*

$$D^\beta u + \frac{2}{t}u_t = -u^5, \qquad 1 < \beta \le 2, \tag{7}$$

subject to the initial conditions

$$u(0) = 1, \qquad u_t(0) = 0. \tag{8}$$

By taking the fractional differential transform and choosing $\theta = \frac{2}{\beta}$, we obtain the recurrence scheme

$$\frac{\Gamma(1+(\frac{k+1}{2})\beta)}{\Gamma(1+\frac{k-1}{2}\beta)}U(k+1)+2(k+1)U(k+1) = -\tilde{A}_{k-1}, \qquad k = 1, 2, \cdots \tag{9}$$

where $\tilde{A}_k$ are DTM's of the Adomian polynomials, A_k, for the non-linear term, u^5, as follows

$A_0 = (u_0)^5$, $\tilde{A}_0 = (U(0))^5$,
$A_1 = 5(u_0)^4 u_1$, $\tilde{A}_1 = 5(U(0))^4 U(1)$,
$A_2 = 5(u_0)^4 u_2 + 10(u_0)^3(u_1)^2$, $\tilde{A}_2 = 5(U(0))^4 U(2) + 10(U(0))^3(U(1))^2$,
$A_3 = 5(u_0)^4 u_3 + 20(u_0)^3 u_1 u_2 + 10(u_0)^2(u_1)^3$, $\tilde{A}_3 = 5(U(0))^4 U(3) + 20(U(0))^3 U(1)U(2) + 10(U(0))^2(U(1))^3$,
$\vdots$

Then, we have the following approximations

$$U(0) = 1, \quad U(1) = 0, \quad U(2) = -\frac{1}{\Gamma(1+\beta)+4}, \quad U(3) = 0,$$

$$U(4) = \frac{5\Gamma(1+\beta)}{6(\Gamma(1+2\beta)+8\Gamma(1+\beta))}, \; U(5) = 0, \; U(6) = -\frac{420}{864}\frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)+12\Gamma(1+2\beta)}, \cdots$$

So

$$u(t) = 1 - \frac{1}{\Gamma(1+\beta)+4}t^2 + \frac{5}{6}\frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)+8\Gamma(1+\beta)}t^4 - \frac{420}{864}\frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)+12\Gamma(1+2\beta)}t^6 + \ldots$$

When the limit of the series solution as β approaches 2 is evaluated, we obtain

$$u(t) = 1 - \frac{1}{6}t^2 + \frac{1}{24}t^4 - \frac{5}{432}t^6 + \cdots = \frac{1}{\sqrt{1+\frac{t^2}{3}}},$$

which is the exact solution of the problem in this case.

Example 3.3. *Now, we consider the following fractional non-linear Lane-Emden equation*

$$D^{\beta}u + \frac{2}{t}u_t = -e^{u}, \qquad 1 < \beta \leq 2, \tag{10}$$

subject to conditions

$$u(0) = 0, \qquad u(1) = 0. \tag{11}$$

Taking the fractional differential transform and choosing $\theta = \frac{2}{\beta}$ we write the recurrence scheme

$$\frac{\Gamma(1 + (\frac{k+1}{2})\beta)}{\Gamma(1 + \frac{k-1}{2}\beta)}U(k+1) + 2(k+1)U(k+1) = -\tilde{A}_{k-1}, \qquad k = 1, 2, \cdots \tag{12}$$

where $\tilde{A}_k$ are obtained from the Adomian polynomials for the nonlinear term, e^u, as follows

$$\begin{aligned} A_0 &= e^{u_0} & \tilde{A}_0 &= e^{U(0)}, \\ A_1 &= u_1 e^{u_0}, & \tilde{A}_1 &= U(1)e^{U(0)}, \\ A_2 &= (u_2 + \tfrac{(u_1)^2}{2})e^{u}_0, & \tilde{A}_2 &= (U(2) + \tfrac{(U(1))^2}{2})e^{U(0)}, \\ A_3 &= (u_3 + u_1u_2 + \tfrac{(u_1)^3}{3!})e^{u_0} & \tilde{A}_3 &= (U(3) + U(1)U(2) + \tfrac{(U(1))^3}{3!})e^{U(0)}, \\ &\vdots \end{aligned}$$

Then, we can obtain the following values

$$U(0) = U(1) = U(3) = 0, \quad U(2) = -\frac{1}{\Gamma(1+\beta)+4}, \quad U(4) = \frac{1}{6}\frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)+8\Gamma(1+\beta)},$$

$$U(5) = 0, \quad U(6) = -\frac{8}{5! \times 3} \times \frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)+12\Gamma(1+2\beta)}, \cdots$$

This yields the series solution

$$u(t) = \frac{-1}{\Gamma(1+\beta)+4}t^2 + \frac{1}{6}\frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)+8\Gamma(1+\beta)}t^4 - \frac{8}{360}\frac{\Gamma(1+2\beta)}{\Gamma(1+3\beta)+12\Gamma(1+2\beta)}t^6 + ...$$

When the limit of the series solution as β approaches 2 is evaluated, we obtain

$$u(t) = -\frac{1}{3 \times 2!}t^2 + \frac{1}{5 \times 4!}x^4 - \frac{8}{21 \times 6!}t^6 + \frac{122}{81 \times 8!}t^8 - \cdots \tag{13}$$

which is the same solution obtained in [19] by modified homotopy perturbation method for the case $\beta = 2$. Using the Matlab symbolic code, we plot the Pade approximations of[2,2], [3,3] and [5,5] in fig.1. This shows the trend of convergence of the solution for these three approximants. Shown in fig.2 are the modified FDTM solution of the problem with 5 components as parametrized by β.

4. Conclusion

Instead of computing the differential transform of the nonlinear term, it is replaced in the recurrence relation by its Adomian polynomial of index k. The nonlinear partial differential equations related to the Lane-Emden problem are solved. The validity and the effectiveness of the method are shown in a systematic fashion. The proposed solutions of the three singular linear and nonlinear differential equations of Lane-Emden problem using the method have verified the physical properties of the equilibrium of Lane-Emden problem, as $x \to \infty$ the solution monotonically approaches a constant. This is indeed an important physical property of the Lane-Emden problem known as the equilibrium state.

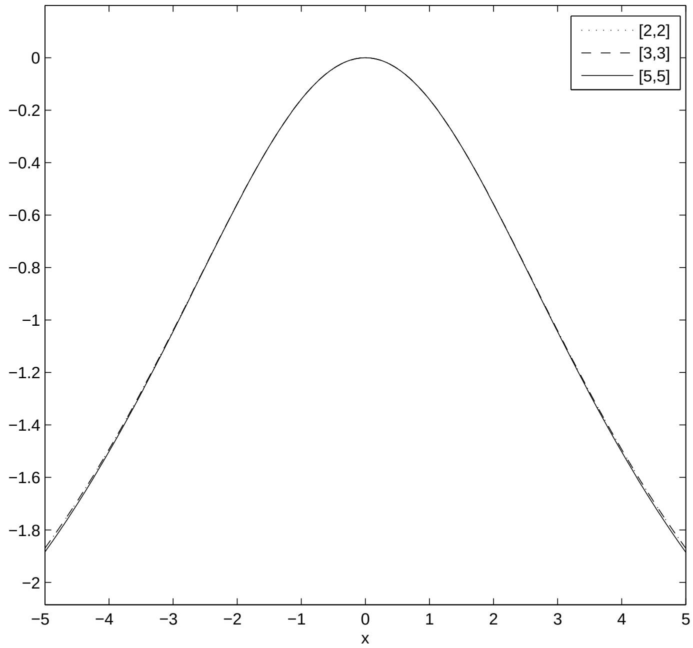

FIGURE 1. Plots of the [2,2], [3,3], [5,5] Pade approximants for Example(3.3).

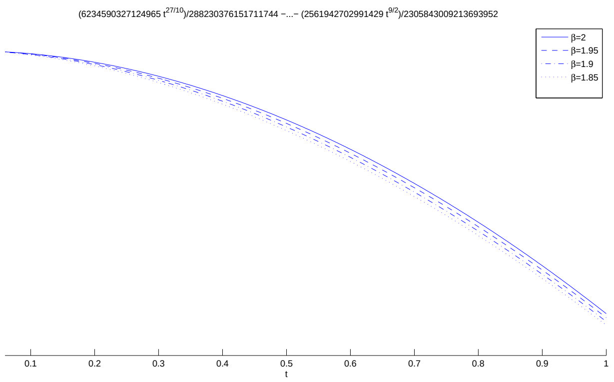

FIGURE 2. Plots of the FDTM Solution with 5 components for Example(3.3) as parametrized by β.

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