TWMS J. App. Eng. Math. V.6, N.2, 2016, pp. 298-306

ON A SUM FORM FUNCTIONAL EQUATION

PREM NATH[1], DHIRAJ KUMAR SINGH[2], §

ABSTRACT. The general solutions of a sum form functional equation containing two unknown mappings, without imposing any regularity condition on them, have been obtained.

Keywords: functional equation, additive mapping, multiplicative mapping, logarithmic mapping.

Mathematics subject classification (2010): 39B22, 39B52.

1. INTRODUCTION

Functional equations appear in various branches of pure mathematics and applied mathematics, business mathematics, economics, information theory, thermodynamics, physics, engineering, and so on (see [1], [3], [4], [5])

For $n = 1, 2, \ldots$; let $\Gamma_n = \left\{ (p_1, \ldots, p_n) : p_i \geq 0,\ i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1 \right\}$ denote the set of all n-component complete discrete probability distributions with nonnegative elements. Let $\mathbb{R}$ denote the set of all real numbers; $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$; $]0,1[= \{x \in \mathbb{R} : 0 < x < 1\}$ and $]0,1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

By giving necessary motivations from statistics point of view, considering the first and second order moments of a specific random variable, Nath and Singh [7] derived the functional equation

$$\phi_2(pq) = q\phi_2(p) + p\phi_2(q) + 2\phi_1(p)\phi_1(q)$$

for all $p \in I$, $q \in I$; $\phi_2 : I \to \mathbb{R}$, $\phi_1 : I \to \mathbb{R}$ with $\phi_2(0) = 0$, $\phi_2(1) = 0$, $\phi_1(0) = 0$, $\phi_1(1) = 0$.

For all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, the authors [7] considered the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + c\sum_{i=1}^{n} g(p_i)\sum_{j=1}^{m} g(q_j) \tag{A}$$

[1] Department of Mathematics, University of Delhi, Delhi - 110007, India.
e-mail: pnathmaths@gmail.com;

[2] Department of Mathematics, Zakir Husain Delhi College (University of Delhi), Jawaharlal Nehru Marg, Delhi - 110002, India.
e-mail: dhiraj426@rediffmail.com;

§ Manuscript received: November 25, 2015; accepted: February 29, 2016.
TWMS Journal of Applied and Engineering Mathematics, Vol.6, No.2;

in which $c \neq 0$ is a given real constant; $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ are unknown mappings. Clearly $f = \phi_2$ and $g = \phi_1$ satisfy (A) with $c = 2$. Keeping in view $\phi_1(0) = 0$, $\phi_1(1) = 0$, $\phi_2(0) = 0$, $\phi_2(1) = 0$ and the fact that $f = \phi_2$ and $g = \phi_1$, we have

$$\text{(i)} \quad f(0) = 0, \qquad \text{(ii)} \quad g(0) = 0 \tag{1}$$

and

$$\text{(i)} \quad f(1) = 0, \qquad \text{(ii)} \quad g(1) = 0. \tag{2}$$

Nath and Singh [7] obtained the general solutions of (A) by assuming (1) and (2); and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers.

The object of this paper is to **obtain the general solutions of (A)** for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers; **without assuming (1) and (2)**.

2. Some Preliminary Results

In this section, we mention some known definitions and results.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if $A(x+y) = A(x)+A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is known (see Daróczy and Losonczi [2]) that if a mapping $a : I \to \mathbb{R}$ is additive on I, then there exists a unique mapping $A : \mathbb{R} \to \mathbb{R}$ which is additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative if $M(pq) = M(p)M(q)$ for all $p \in I$, $q \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic if $\ell(0) = 0$ and $\ell(pq) = \ell(p) + \ell(q)$ for all $p \in \,]0, 1]$, $q \in \,]0, 1]$.

Result 2.1 ([6])**.** *Let $\psi : I \to \mathbb{R}$ be a mapping which satisfies the equation $\sum_{i=1}^{k} \psi(x_i) = c$ for all $(x_1, \ldots, x_k) \in \Gamma_k$; c a given real constant and $k \geq 3$ a fixed integer. Then there exists an additive mapping $b : \mathbb{R} \to \mathbb{R}$ such that $\psi(x) = b(x) - \frac{1}{k}b(1) + \frac{c}{k}$ for all $x \in I$.*

Chaundy and Mcleod [1] considered the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \tag{B}$$

where $f : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; n and m being positive integers.

Result 2.2 ([6])**.** *If a mapping $f : I \to \mathbb{R}$ satisfies (B) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers, then f is of the form*

$$f(p) = \begin{cases} f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) & \text{if } 0 < p \leq 1 \\ f(0) & \text{if } p = 0, \end{cases}$$

where $f(0)$ is an arbitrary real constant; $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping; the mapping $D : \mathbb{R}\times \,]0, 1] \to \mathbb{R}$ is additive in the first variable; there exists a mapping $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ additive in both variables such that $a(1) = E(1, 1)$ and $D(pq, pq) = D(pq, p) + D(pq, q) + E(p, q)$ for all $p \in \,]0, 1]$, $q \in \,]0, 1]$.

Modified Form of Result 2.2. *If a mapping $f : I \to \mathbb{R}$ satisfies (B) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers, then f is of the form*

$$f(p) = f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) \tag{3}$$

for all $p \in I$; $f(0)$ is an arbitrary real constant; $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping; $D : \mathbb{R} \times I \to \mathbb{R}$ is additive in the first variable; there exists a mapping $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ additive in both variables such that $a(1) = E(1, 1)$ and

$$D(pq, pq) = D(pq, p) + D(pq, q) + E(p, q) \tag{4}$$

for all $p \in I$, $q \in I$.

Using the fact that $a(1) = E(1, 1)$, it can be easily deduced from (4) that

$$a(1) + D(1, 1) = 0. \tag{5}$$

3. On the Functional Equation (A)

The main result of this paper is the following:

Theorem. *Let c be a nonzero given constant and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be mappings which satisfy the equation (A) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, for all $p \in I$, any general solution (f, g) of (A) is of the form*

$$\begin{cases} \text{(i)} & f(p) = f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) \\ \text{(ii)} & g(p) = A_1(p) + g(0); \end{cases} \tag{β_1}$$

or

$$\begin{cases} \text{(i)} & f(p) = f(0) + f(0)(nm - n - m)p + a(p) \\ & \qquad + D(p, p) + \dfrac{1}{2}\, cp[\ell^*(p)]^2 \\ \text{(ii)} & g(p) = p\ell^*(p); \end{cases} \tag{β_2}$$

or

$$\begin{cases} \text{(i)} & f(p) = f(0) + f(0)(nm - n - m)p \\ & \qquad + c\lambda^2[M(p) - p] + a(p) + D(p, p) \\ \text{(ii)} & g(p) = \lambda[M(p) - p]; \end{cases} \tag{β_3}$$

or

$$\begin{cases} \text{(i)} & f(p) = f(0) + \{f(0)(nm - n - m) \\ & \qquad - c[g(1) + (n - 1)g(0)][g(1) + (m - 1)g(0)]\}p + a(p) + D(p, p) \\ \text{(ii)} & g(p) = A_2(p) + g(0); \quad g(1) + (m - 1)g(0) \neq 0 \end{cases} \tag{β_4}$$

where λ is an arbitrary nonzero real constant; $A_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ are additive mappings such that $A_1(1) = -mg(0)$ and $A_2(1) = g(1) - g(0)$; $\ell^* : I \to \mathbb{R}$ is a logarithmic mapping which does not vanish identically on the open interval $]0, 1[$; $M : I \to \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$; $a : \mathbb{R} \to \mathbb{R}$ and $D : \mathbb{R} \times I \to \mathbb{R}$ are as described in the Modified Form of Result 2.2.

Proof. Let us write (A) in the form

$$\sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} f(p_i q_j) - f(p_i) - p_i \sum_{j=1}^{m} f(q_j) - cg(p_i) \sum_{j=1}^{m} g(q_j) \right\} = 0.$$

By using Result 2.1 and proceeding as in [7], we can obtain

$$\left[\sum_{j=1}^{m} g(xq_j) - g(x) - (m-1)g(0)\right] \sum_{t=1}^{m} g(r_t)$$
$$= \left[\sum_{t=1}^{m} g(xr_t) - g(x) - (m-1)g(0)\right] \sum_{j=1}^{m} g(q_j) \tag{6}$$

as $c \neq 0$. Equation (6) is valid for all $x \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$, $(r_1, \ldots, r_m) \in \Gamma_m$; $m \geq 3$ being a fixed integer.

From now onwards, we divide our discussion into two cases.

Case 1. $\sum\limits_{t=1}^{m} g(r_t)$ vanishes identically on Γ_m, that is,

$$\sum_{t=1}^{m} g(r_t) = 0$$

for all $(r_1, \ldots, r_m) \in \Gamma_m$. By Result 2.1, there exists an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ such that

$$g(p) = A_1(p) - \frac{1}{m} A_1(1) \tag{7}$$

for all $p \in I$. The substitution $p = 0$ in (7) gives $A_1(1) = -\,mg(0)$. Now (7) gives (β_1)(ii) with $A_1(1) = -\,mg(0)$. Utilizing this form of g in (A), we obtain the functional equation (B) for $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$ and $m \geq 3$ being fixed integers.

By the Modified Form of Result 2.2, it follows that $f : I \to \mathbb{R}$ is of the form (β_1)(i). Thus, we have obtained the solution (β_1) of (A).

Case 2. $\sum\limits_{t=1}^{m} g(r_t)$ does not vanish identically on Γ_m.

In this case, there exists a probability distribution $(r_1^*, \ldots, r_m^*) \in \Gamma_m$ such that

$$\sum_{t=1}^{m} g(r_t^*) \neq 0. \tag{8}$$

Setting $r_t = r_t^*$, $t = 1, \ldots, m$ in (6) and using (8), we obtain

$$\sum_{j=1}^{m} g(xq_j) - g(x) - (m-1)g(0)$$
$$= \left[\sum_{t=1}^{m} g(r_t^*)\right]^{-1} \left[\sum_{t=1}^{m} g(xr_t^*) - g(x) - (m-1)g(0)\right] \sum_{j=1}^{m} g(q_j). \tag{9}$$

Define a mapping $M : I \to \mathbb{R}$ as

$$M(x) = \left[\sum_{t=1}^{m} g(r_t^*)\right]^{-1} \left[\sum_{t=1}^{m} g(xr_t^*) - g(x) - (m-1)g(0)\right] \tag{10}$$

for all $x \in I$. Now, from (9) and (10), it follows that

$$\sum_{j=1}^{m} g(xq_j) = M(x) \sum_{j=1}^{m} g(q_j) + g(x) + (m-1)g(0). \tag{11}$$

From (10), it is easy to conclude that

$$M(0) = 0. \tag{12}$$

The substitution $x = 1$, in (10), gives

$$1 - M(1) = [g(1) + (m-1)g(0)] \left[\sum_{t=1}^{m} g(r_t^*) \right]^{-1}. \tag{13}$$

Let us write (11) in the form

$$\sum_{j=1}^{m} \{g(xq_j) - M(x)g(q_j) - [g(x) + (m-1)g(0)]q_j\} = 0.$$

By Result 2.1, there exists a mapping $E : I \times \mathbb{R} \to \mathbb{R}$, additive in the second variable, such that

$$g(xq) - M(x)g(q) - [g(x) + (m-1)g(0)]q = E(x;q) - \frac{1}{m}E(x;1). \tag{14}$$

Equation (14) holds for all $x \in I$ and $q \in I$. The substitution $q = 0$ in it gives (using $E(x;0) = 0$)

$$E(x;1) = mg(0)[M(x) - 1] \tag{15}$$

for all $x \in I$. From (14) and (15), we obtain

$$g(xq) - M(x)[g(q) - g(0)] - [g(x) + (m-1)g(0)]q - g(0) = E(x;q). \tag{16}$$

Case 2.1. $E(x;q) \equiv 0$ on $I \times I$.
In this case, $E(x;1) = 0$. So, (15) gives

$$mg(0) = mg(0)M(x) \tag{17}$$

for all $x \in I$. Since the left hand side of (17) is independent of the variable x, $x \in I$, it follows that

$$mg(0)M(x) = mg(0)M(q) \tag{18}$$

for all $x \in I$ and $q \in I$. Also, from (16) and the fact that $E(x;q) \equiv 0$ on $I \times I$, we obtain

$$g(xq) - g(0) = M(x)[g(q) - g(0)] + [g(x) + (m-1)g(0)]q \tag{19}$$

for all $x \in I$ and $q \in I$. The left hand side of (19) is symmetric in x and q. Hence, so should be its right hand side. This fact gives rise to the equation

$$\begin{aligned} M(x)[g(q) - g(0)] + [g(x) + (m-1)g(0)]q \\ = M(q)[g(x) - g(0)] + [g(q) + (m-1)g(0)]x. \end{aligned} \tag{20}$$

Making use of (18), (20) gives rise to the equation

$$[g(q) + (m-1)g(0)][M(x) - x] = [g(x) + (m-1)g(0)][M(q) - q] \tag{21}$$

valid for all $x \in I$ and $q \in I$.

Case 2.1.1. $M(x) - x = 0$ for all $x \in I$.
In this case, $M(x) = x$ for all $x \in I$. Now, (17) gives $mg(0)(1 - x) = 0$ for all $x \in I$. Choosing $x = \frac{1}{2}$, we obtain $g(0) = 0$. Using $M(x) = x$ for all $x \in I$ and the fact that $g(0) = 0$, (19) gives the functional equation $g(xq) = xg(q) + qg(x)$ whose general solution is $g(x) = x\ell(x)$ for all $x \in I$; $\ell : I \to \mathbb{R}$ being any logarithmic mapping. If $\ell(x) = 0$ for all

$x \in I$, then $g(x) = 0$ for all $x \in I$. Consequently, $\sum_{t=1}^{m} g(r_t^*) = 0$ contradicting (8). So, g must be of the form (β_2)(ii). Making use of this form of g in (A), we obtain the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i q_j \ell^*(p_i)\ell^*(q_j).$$

The above equation can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{f(p_i q_j) - \frac{1}{2}\,c p_i q_j[\ell^*(p_i q_j)]^2\right\}$$
$$= \sum_{i=1}^{n}\left\{f(p_i) - \frac{1}{2}\,c p_i[\ell^*(p_i)]^2\right\} + \sum_{j=1}^{m}\left\{f(q_j) - \frac{1}{2}\,c q_j[\ell^*(q_j)]^2\right\}.$$

Define a mapping $f_1 : I \to \mathbb{R}$ as $f_1(p) = f(p) - \frac{1}{2}cp[\ell^*(p)]^2$ for all $p \in I$. Then making use of Modified Form of Result 2.2, it can be proved that f is of the form (β_2)(i). Thus, we have obtained the solution (β_2) of (A).

Case 2.1.2. $[M(x) - x] \not\equiv 0$ on I.

In this case, there exists an element $x_0 \in I$ such that $[M(x_0) - x_0] \neq 0$. Setting $x = x_0$ in (21), we obtain

$$g(q) = \lambda[M(q) - q] - (m-1)g(0) \tag{22}$$

where $\lambda = [M(x_0) - x_0]^{-1}[g(x_0) + (m-1)g(0)]$. If $\lambda = 0$, then (22) gives $g(q) = -(m-1)g(0)$ for all $q \in I$. From this, it follows that $g(0) = 0$ as $m \geq 3$. Now (22) gives $g(q) = 0$ for all $q \in I$. In particular, $\sum_{t=1}^{m} g(r_t^*) = 0$ contradicting (8). Hence, $\lambda \neq 0$. Putting $q = 0$ in (22) and using (12), it follows that $g(0) = 0$. Thus, (22) gives

$$g(q) = \lambda[M(q) - q], \quad \lambda \neq 0 \tag{23}$$

for all $q \in I$. From (19), and the fact that $g(0) = 0$, we obtain

$$g(xq) = M(x)g(q) + qg(x) \tag{24}$$

for all $x \in I$, $q \in I$. From (23) and (24), it follows that $M(xq) = M(x)M(q)$ for all $x \in I$, $q \in I$. Thus, M is a multiplicative mapping. But, we have to consider only those multiplicative mappings M which satisfy the condition (12). The possibility $M(x) \equiv 1$, $x \in I$, is ruled out as, in this case, $M(0) \neq 0$. Since $[M(x_0) - x_0] \neq 0$ for some $x_0 \in I$, it follows that $g(x_0) \neq 0$ for some $x_0 \in I$. Since $g(0) = 0$, the possibility $x_0 = 0$ is ruled out. So, $x_0 \in]0, 1]$. Consider $x_0 = 1$. This means $g(1) \neq 0$. Hence, by (23), $M(1) \neq 1$. But, M is multiplicative. So, $M(x)[M(1) - 1] = 0$. Since $M(1) \neq 1$, it follows that $M(x) = 0$ for all $x \in I$. Consequently, (23) gives $g(q) = -\lambda q$ for all $q \in I$ with $\lambda \neq 0$ which is included in (β_4)(ii) upon choosing $A_2(q) = -\lambda q$ (as $g(0) = 0$) with $A_2(1) = g(1) = -\lambda \neq 0$. Now proceeding as in the Case 2.1.1, the corresponding form of f is

$$f(p) = f(0) + \{f(0)(nm - n - m) - c[g(1)]^2\}p + a(p) + D(p, p)$$

which is included in (β_4)(i).

Now we consider the case when $x_0 \in]0, 1[$. In this case, we must have $g(0) = 0$ and also $g(1) = 0$. Now, from (23), it follows $M(1) = 1$.

Now we prove that M is not additive. To the contrary, suppose $M : I \to \mathbb{R}$ is additive. Then, for all $(r_1, \ldots, r_m) \in \Gamma_m$, using (23) and $M(1) = 1$, we have

$$\sum_{t=1}^{m} g(r_t) = \lambda \left[\sum_{t=1}^{m} M(r_t) - 1\right] = \lambda[M(1) - 1] = 0$$

contradicting (8). So, M is not additive. Thus, the solution (β_3)(ii) stands obtained in which M is a multiplicative mapping with $M(0) = 0$, $M(1) = 1$ and M is not additive.

Now, making use of (β_3)(ii) in (A) and proceeding as in the Case 2.1.1, we can obtain (β_3)(i). Thus the solution (β_3) follows.

Case 2.2. $E(x; q) \not\equiv 0$ on $I \times I$.

In this case, there exists an element $(x^*, q^*) \in I \times I$ such that $E(x^*; q^*) \neq 0$. Now we prove that

$$\begin{aligned} r = [E(x^*; q^*)]^{-1}\{&E(x^*; q^*r) + M(x^*)E(q^*; r) - E(x^*q^*; r) \\ &+ [M(x^*)M(q^*) - M(x^*q^*)][g(r) - g(0)] + rmg(0)[M(x^*) - 1]\} \end{aligned} \tag{25}$$

holds for all $r \in I$. Using (16), we have

$$\begin{aligned} &g((x^*q^*)r) - rq^*[g(x^*) + (m-1)g(0)] - rM(x^*)[g(q^*) - g(0)] - g(0) \\ &\quad = E(x^*q^*; r) + M(x^*q^*)[g(r) - g(0)] + rE(x^*; q^*) + rmg(0) \end{aligned} \tag{26}$$

and

$$\begin{aligned} &g(x^*(q^*r)) - q^*r[g(x^*) + (m-1)g(0)] - rM(x^*)[g(q^*) - g(0)] - g(0) \\ &\quad = E(x^*; q^*r) + M(x^*)E(q^*; r) + M(x^*)M(q^*)[g(r) - g(0)] + rmM(x^*)g(0). \end{aligned} \tag{27}$$

Since the left hand sides of (26) and (27) are same, we get

$$\begin{aligned} &E((x^*q^*); r) + M(x^*q^*)[g(r) - g(0)] + rE(x^*; q^*) + rmg(0) \\ &\quad = E(x^*; q^*r) + M(x^*)E(q^*; r) + M(x^*)M(q^*)[g(r) - g(0)] + rmM(x^*)g(0). \end{aligned} \tag{28}$$

Using the fact that $E(x^*; q^*) \neq 0$, (25) follows from (28).

Let us write (25) as

$$\begin{aligned} &r - [E(x^*; q^*)]^{-1}\{E(x^*; q^*r) + M(x^*)E(q^*; r) - E(x^*q^*; r) + rmg(0)[M(x^*) - 1]\} \\ &\quad = [E(x^*; q^*)]^{-1}[M(x^*)M(q^*) - M(x^*q^*)][g(r) - g(0)]. \end{aligned} \tag{29}$$

Case 2.2.1. $[M(x^*)M(q^*) - M(x^*q^*)] \neq 0$.

In this case, (29) gives

$$g(r) = A_1(r) + g(0), \quad 0 \leq r \leq 1, \tag{30}$$

where $A_1 : \mathbb{R} \to \mathbb{R}$ is a mapping defined as

$$\begin{aligned} A_1(t) = [M(x^*)M(q^*) - M(x^*q^*)]^{-1}\{&tE(x^*; q^*) - E(x^*; q^*t) \\ &- M(x^*)E(q^*; t) + E(x^*q^*; t) - tmg(0)[M(x^*) - 1]\} \end{aligned} \tag{31}$$

for all $t \in \mathbb{R}$. Since $E : I \times \mathbb{R} \to \mathbb{R}$ is additive in the second variable, it follows that $A_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping. Putting $r = 1$ in (31) and using (15), it turns out that $A_1(1) = -mg(0)$. From (8), (30) and the fact that $A_1(1) = -mg(0)$, we observe that

$$\begin{aligned} 0 \neq \sum_{t=1}^{m} g(r_t^*) &= \sum_{t=1}^{m} [A_1(r_t^*) + g(0)] \\ &= A_1(1) + mg(0) = -mg(0) + mg(0) = 0 \end{aligned}$$

a contradiction. So, this case is not possible.

Case 2.2.2. $[M(x^*)M(q^*) - M(x^*q^*)] = 0$.
The substitution $r = 1$, in (29), gives

$$mg(0)[M(x^*)M(q^*) - M(x^*q^*)] = 0\,.$$

Since $m \geq 3$ is a fixed integer and $[M(x^*)M(q^*) - M(x^*q^*)] = 0$, it follows that $g(0)$ is an arbitrary real number. Now, let us put $x = 1$ in (16). We obtain

$$[g(q) - g(0)][1 - M(1)] = E(1;q) + [g(1) + (m-1)g(0)]q \tag{32}$$

for all $q \in I$.

Case 2.2.2.1. $1 - M(1) \neq 0$.
In this case, (13) gives $[g(1) + (m-1)g(0)] \neq 0$. Consequently, $[g(1) - g(0)] \neq -mg(0)$. Also, from (32),

$$g(q) = [1 - M(1)]^{-1}\{E(1;q) + [g(1) + (m-1)g(0)]q\} + g(0). \tag{33}$$

Let us define a mapping $A_2 : \mathbb{R} \to \mathbb{R}$ as

$$A_2(t) = [1 - M(1)]^{-1}\{E(1;t) + [g(1) + (m-1)g(0)]t\} \tag{34}$$

for all $t \in \mathbb{R}$. Then, $A_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping. Now, it follows from (33) and (34) that g is of the form (β_4)(ii) with $A_2(1) = [g(1) - g(0)]$. From (β_4)(ii) and (A), it follows that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j)&\\ + c\,[g(1) + (n-1)g(0)][g(1) + (m-1)g(0)]&\end{aligned} \tag{35}$$

with $[g(1) + (m-1)g(0)] \neq 0$. Now, proceeding as in the Case 2.1.1, it can be proved that f is of the form (β_4)(i). Thus, we have obtained the solution (β_4).

Case 2.2.2.2. $1 - M(1) = 0$.
In this case, (13) gives

$$g(1) + (m-1)g(0) = 0. \tag{36}$$

The mapping $g : I \to \mathbb{R}$, mentioned in (β_1)(ii), (β_2)(ii) and (β_3)(ii), satisfies (36). But, we have to consider only those solutions of (A) which meet the requirement $[M(x^*)M(q^*) - M(x^*q^*)] = 0$ for some $x^* \in I$ and $q^* \in I$. There is only one such solution, namely β_3(ii), as in this solution, the mapping M is multiplicative and thus the condition $[M(x^*)M(q^*) - M(x^*q^*)] = 0$ for some $x^* \in I$, $q^* \in I$, is met with. Also $M(1) = 1$ and $M(0) = 0$. So, (β_3)(ii) gives $g(1) = 0$ and $g(0) = 0$. Now, from (16), $g(0) = 0$ and the fact that M is multiplicative, it follows that $E(x;q) = 0$ for all $x \in I$, $q \in I$, thereby, contradicting the fact that $E(x^*;q^*) \neq 0$ for some $x^* \in I$, $q^* \in I$. So, in this case we do not get any new solution. □

Remark. The solutions (β_1), (β_2) and (β_3) are respective **nontrivial generalizations** of solutions (3.1), (3.2) and (3.3) of the Theorem ([7], pp. 86–87). **The solution** (β_4) **is absolutely a new solution**. The solution (3.1) is included in (β_1) but not in (β_4) as $g(1) + (m-1)g(0) \neq 0$.

Acknowledgement. The authors are grateful to the referee for his critical comments.

References

[1] Chaundy,T.W. and Mcleod,J.B., (1960), On a functional equation, Edinburgh Mathematical Notes, 43, pp. 7-8.

[2] Daróczy,Z. and Losonczi,L., (1967), Über die erweiterung der auf einer punktmenge additiven funktionen, Publicationes Mathematicae Debrecen, 14, pp. 239-245.

[3] Deepmala and Pathak,H.K., (2013), Study on existence of solutions for some nonlinear functional-integral equations with applications, Mathematical Communications, 18 (1), pp. 97-107.

[4] Deepmala and Pathak,H.K., (2013), A study on some problems on existence of solutions for nonlinear functional-integral equations, Acta Mathematica Scientia, 33B (5), pp. 1305-1313.

[5] Deepmala and Pathak,H.K., (2013), Some common fixed point theorems for D-operator pair with applications to nonlinear integral equations, Nonlinear Functional Analysis and Applications, 18 (2), pp. 205-218.

[6] Losonczi,L. and Maksa,Gy., (1982), On some functional equations of the information theory, Acta Mathematica Academiae Scientiarum Hungaricae, 39, pp. 73-82.

[7] Nath,P. and Singh,D.K., (2009), On a sum form functional equation related to entropies and some moments of a discrete random variable, Demonstratio Mathematica, 42 (1), pp. 83-96.

Prem Nath is retired as Professor of Mathematics from Department of Mathematics, University of Delhi. He has published more than 70 research papers in information theory and functional equations.

Dhiraj Kumar Singh, received his Ph.D. degree from the Department of Mathematics, University of Delhi in February 2009. At present he is working as Assistant Professor of Mathematics in the Department of Mathematics, Zakir Husain Delhi College (University of Delhi).