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ON CURVES OF CONSTANT BREADTH IN $\mathbb{G}_3^1$

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Abstract. In this work, differential equations characterizing curves of constant breadth have been given in pseudo-Galilean space $\mathbb{G}_3^1$. The special cases related to these differential equations have been studied in $\mathbb{G}_3^1$.

Keywords: Pseudo-Galilean space, pseudo-Galilean norm, curves of constant breadth.

AMS Subject Classification: 53A35, 53Z05

1. Introduction

The works concerning shapes of constant breadth are based on the paper "De curvis triangularibus" by Euler in 1870 [6]. In [11], a method was given to obtain some curves of constant breadth to use their applications in the kinematics of machinery by Reuleaux. In [2, 3, 6], some properties of the plane curves of constant breadth were given. The curves of constant breadth on the sphere were given by Blaschke in [3]. Fujivara had put forward a problem based on determining whether there exist "space curves of constant breadth" or not, and as a solution of the problem, the "breadth" concept for space curves was defined and these curves were shown on a surface of constant breadth in [7].

Some geometric properties of the curves of constant breadth were given in the plane in [8]. Then these properties were extended to the Euclidean 3-space $\mathbb{E}^3$ in [9]. Also, these kind of curves were studied in four dimensional Euclidean space $\mathbb{E}^4$ in [10]. Also the curves of constant breadth were studied in Euclidean $n-$space in [1]. Furthermore, the results of the curves of constant breadth were also obtained in Galilean space in [12].

In this paper, we give the differential equations characterizing curves of constant breadth in pseudo-Galilean space $\mathbb{G}_3^1$.

2. Preliminaries

Let $\mathbb{G}_3^1$ be the pseudo-Galilean 3-space, that is, $\mathbb{G}_3^1$ is a Cayley–Klein space equipped with the projective metric of signature $(0, 0, +, -)$. The absolute figure of the pseudo-Galilean geometry consists of an ordered triple $\{\omega, f, I\}$, where ω is the real (absolute)

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plane, f the real line (absolute line) in ω and I the fixed hyperbolic involution of points of f. In [4], the hyperbolic involution is given by

$$(0:0:x_2:x_3) \to (0:0:x_3:x_2) \tag{2.1}$$

at homogeneous coordinates.

There are two types of plane in the pseudo-Galilean space. Pseudo-Euclidean planes are in the following form $x = k, k \in \mathbb{R}$. Other planes are isotropic. A vector $u = (u_1, u_2, u_3)$ is said to be non-isotropic if $u_1 \neq 0$. All unit non-isotropic vectors are of the form $u = (1, u_2, u_3)$. For isotropic vectors, u_1 vanishes [4].

Let $a = (x, y, z)$ and $b = (x_1, y_1, z_1)$ be vectors in the pseudo-Galilean space. The scalar product is defined by

$$< a, b >= x_1 x. \tag{2.2}$$

The norm of a is defined by $\|a\| = |x|$, and a is called a unit vector if $\|a\| = 1$.The scalar product of two isotropic vectors $p = (0, y, z)$ and $q = (0, y_1, z_1)$ is defined by

$$< p, q >_1 = y y_1 - z z_1. \tag{2.3}$$

The norm of p is defined by $\|p\|_1 = \sqrt{|y^2 - z^2|}$. An isotropic vector $p = (0, y, z)$ is said to be spacelike, timelike or lightlike if $y^2 - z^2 > 0$, $y^2 - z^2 < 0$ or $y = \pm z$, respectively [4].

The cross product in the pseudo-Galilean space can be defined analogously to the Minkowski case. The cross product of $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is defined as follows:

$$u \wedge v = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (0, u_3 v_1 - u_1 v_3, u_2 v_1 - u_1 v_2). \tag{2.4}$$

Given an admissible curve $r(u) = (u, y(u), z(u))$, the associated invariant moving trihedron is given by

$$\begin{aligned} T &= (1, y'(u), z'(u)), \\ N &= \frac{1}{\kappa}(0, y''(u), z''(u)), \\ B &= \frac{1}{\kappa}(0, \varepsilon z''(u), \varepsilon y''(u)) \end{aligned} \tag{2.5}$$

where $\kappa = \sqrt{|y''(u)^2 - z''(u)^2|}$ is the curvature and $\tau = \dfrac{1}{\kappa^2}\det[r'(u), r''(u), r'''(u)]$ is the torsion [5].

A curve is said to be timelike or spacelike if the principal normal vector N is spacelike or timelike vector, respectively. The principal normal vector or simply the normal is spacelike if $\varepsilon = +1$ and time-like if $\varepsilon = -1$.

For derivatives of the tangent (vector) t, the normal n and the binormal b, respectively, the following Serret-Frenet formulas hold

$$T' = \kappa N, \quad N' = \tau B, \quad B' = \tau N. \tag{2.6}$$

From (6), the following important relation

$$r(x) = \kappa(x)N(x) + \kappa(x)\tau(x)B(x)$$

is obtained as in [5].

3. Curves of constant breadth in $\mathbb{G}_3^1$

Let $\varphi = \varphi(s)$ and $\varphi^* = \varphi^*(s)$ be simple closed curves of constant breadth in pseudo-Galilean space $\mathbb{G}_3^1$. These curves will be denoted by C. The normal plane at every point P on the curve meets the curve in the class Γ as in [7] having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve. A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\varphi^* = \varphi + m_1T + m_2N + m_3B, \tag{3.1}$$

where $m_i(s)$, $1 \leq i \leq 3$ arbitrary functions of s and φ and φ^* are opposite points. The vector $d = \varphi^* - \varphi$ is called "the distance vector" of C. Differentiating both sides of (3.1) and considering Frenet equations, we have

$$\begin{aligned} \frac{d\varphi^*}{ds} = T^*\frac{ds^*}{ds} &= (1 + \frac{dm_1}{ds})T + (m_1\kappa + \frac{dm_2}{ds} + m_3\tau)N \\ &+ (\frac{dm_3}{ds} + m_2\tau)B. \end{aligned} \tag{3.2}$$

Thus, $T^* = -T$. Rewriting (3.2) we obtain following system of equations,

$$\begin{aligned} &\frac{dm_1}{ds} + 1 + \frac{ds^*}{ds} = 0, \\ &\frac{dm_2}{ds} + m_1\kappa + m_3\tau = 0, \\ &\frac{dm_3}{ds} + m_2\tau = 0. \end{aligned} \tag{3.3}$$

If we call θ as the angle between the tangent of the curve C at point φ with a given fixed direction and s as arc length parameter of $\varphi(s)$, consider $\frac{d\theta}{ds} = \kappa$, we have (3.3) as following;

$$\begin{aligned} \frac{dm_1}{d\theta} &= -f(\theta), \\ \frac{dm_2}{d\theta} &= -m_1 - m_3\rho\tau, \\ \frac{dm_3}{d\theta} &= -m_2\rho\tau, \end{aligned} \tag{3.4}$$

where $f(\theta) = \rho + \rho^*$; $\rho = \frac{1}{\kappa}$ and $\rho^* = \frac{1}{\kappa^*}$ denote the radius of curvature at φ and φ^*, respectively. If m_1, m_3 and their derivates are eliminated in equations (3.4), we obtain the following equation with respect to m_2:

$$\begin{aligned} &\frac{d^3m_2}{d\theta^3} + \frac{d}{d\theta}\left[\ln(\frac{\tau}{\kappa})\right]\frac{d^2m_2}{d\theta^2} + \left[(\frac{\tau}{\kappa})^2 + \frac{d^2}{d\theta^2}\left(\ln(\frac{\tau}{\kappa})\right)\right]\frac{dm_2}{d\theta} \\ &+2(\frac{\tau}{\kappa})\frac{d}{d\theta}(\frac{\tau}{\kappa})m_2 + \frac{df}{d\theta} - \frac{d}{d\theta}\left[\ln(\frac{\tau}{\kappa})\right]f - \int f d\theta \frac{d^2}{d\theta^2}\left[\left(\ln(\frac{\tau}{\kappa})\right)\right] = 0. \end{aligned} \tag{3.5}$$

The equation (3.5) is the characterization of the curve φ^*. If the distance between opposite points of C and C^* is constant, then we can write that

$$\|\varphi^* - \varphi\|^2 = \left\{ \begin{array}{ll} m_2^2 - m_3^2 & , \quad m_1 = 0 \\ m_1^2 & , \quad m_1 \neq 0 \end{array} \right\} \tag{3.6}$$

According to the conditions in (3.6), we shall study the following cases:

Case 1. If $m_1 = 0$, then we write that

$$m_2^2 - m_3^2 = k^2 = const., \tag{3.7}$$

hence, by differentiating (3.7) we have

$$m_2 \frac{dm_2}{d\theta} - m_3 \frac{dm_3}{d\theta} = 0, \tag{3.8}$$

considering system (3.4) in (3.8), we obtain

$$m_1 m_2 = 0, \tag{3.9}$$

therefore, we have

$$m_2 = 0 \text{ or } m_2 \neq 0. \tag{3.10}$$

Also, as a result of case 1, we find

$$f = 0$$

from (3.4), so the equation (3.5) turns to

$$\begin{aligned} &\frac{d^3 m_2}{d\theta^3} + \frac{d}{d\theta}\left[\ln(\frac{\tau}{\kappa})\right]\frac{d^2 m_2}{d\theta^2} + \left[(\frac{\tau}{\kappa})^2 + \frac{d^2}{d\theta^2}\left(\ln(\frac{\tau}{\kappa})\right)\right]\frac{dm_2}{d\theta} \\ &+ 2(\frac{\tau}{\kappa})\frac{d}{d\theta}(\frac{\tau}{\kappa})m_2 = 0. \end{aligned} \tag{3.11}$$

Based on the equation (3.9), we shall study the subcases:

Case 1.1. If $m_2 = 0$, then

$$m_3 = 0$$

from the equations (3.4). Therefore the curves $\varphi^*$and φ coincide.

Case 1.2: If $m_2 \neq 0$ and $\frac{\tau}{\kappa} = const.$, then we have

$$m_2 = m_3 = e^{-\frac{\tau}{\kappa}\theta}$$

from the equations (3.4). Thus again the curves $\varphi^*$and φ coincide.

Case 2. If $m_1 \neq 0$, then from the equation (3.6), we have

$$m_1^2 = k^2 \tag{3.12}$$

where $k \in \mathbb{R}$. Hence, differentiating (3.12) we obtain

$$m_1 \frac{dm_1}{d\theta} = 0. \tag{3.13}$$

Then $m_1 = k \in \mathbb{R}$, from the equation (3.5), we obtain the following differential equation of third order

$$\begin{aligned} &\frac{d^3 m_2}{d\theta^3} + \frac{d}{d\theta}\left[\ln(\frac{\tau}{\kappa})\right]\frac{d^2 m_2}{d\theta^2} + \left[(\frac{\tau}{\kappa})^2 + \frac{d^2}{d\theta^2}\left(\ln(\frac{\tau}{\kappa})\right)\right]\frac{dm_2}{d\theta} \\ &+2(\frac{\tau}{\kappa})\frac{d}{d\theta}(\frac{\tau}{\kappa})m_2 - \frac{d}{d\theta}\left[\ln(\frac{\tau}{\kappa})\right] f - \textstyle\int f d\theta \frac{d^2}{d\theta^2}\left[\left(\ln(\frac{\tau}{\kappa})\right)\right] = 0. \end{aligned} \tag{3.14}$$

4. Conclusion

In this work, differential equations characterizing curves of constant breadth have been given in pseudo-Galilean space $\mathbb{G}_3^1$. The special cases related to these differential equations have been studied in $\mathbb{G}_3^1$. It is an open problem to get the results of curves of constant breadth in higher dimensions of both Galilean and pseudo-Galilean spaces.

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