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NON-SELF MAPPINGS UNDER COMMON LIMIT RANGE PROPERTY IN SYMMETRIC SPACES AND FIXED POINTS

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Abstract. In this paper, some sufficient conditions are provided for the existence of fixed points for two pairs of non-self mappings satisfying CLR property without the condition of continuity on mappings in the framework of symmetric spaces. Several interesting corollaries are also deduced. Some examples are also provided which illustrate the usability of the results obtained.

Keywords: weak compatible mappings, (ϕ, ψ)-weak rational type contraction mappings, metric space.

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1. Introduction

The classical Banach's contraction principle given by S. Banach proved to be an effective tool which assures the existence and uniqueness of fixed points in complete metric spaces. Besides offering a constructive procedure to compute the fixed points of the underlying mappings this principle has played a major role in the development of nonlinear analysis. A vast amount of mathematical activites has been carried out in various branches of mathematics and in economics, life sciences, physical sciences, engineering, computer science, and others by using Banach's contraction principle by the several authors. Fixed point theorems for various types of nonlinear contractive mappings in different abstract spaces have been scrutinized extensively by various researchers (see [1]-[17] and the references cited therein). Sometimes one may come across situations where the full force of metric requirements are not used in the proofs of certain metrical fixed point theorems. Motivated by this fact, several authors obtained fixed point and common fixed point results in symmetric and semi-metric space.

In this paper, we establish some theorems of common fixed point for two pairs of non-self weakly compatible mappings using CLR (common limit range) property satisfying a generalized rational type contractive condition in the setting of symmetric spaces. To discuss the importance of the results obtained some illustrative examples have been discussed at the last.

2. Preliminaries

In this section, definitions and notations are defined which are required to establish the results.

Definition 2.1. A symmetric function d_1 defined on a nonempty set X as $d_1 : X \times X \to [0, \infty)$ satisfies the following conditions:

(1) $d_1(s_1, t_1) = 0$ if and only if $s_1 = t_1$, for $s_1, t_1 \in X$;

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(2) $d_1(s_1, t_1) = d_1(t_1, s_1)$, for $s_1, t_1 \in X$.

A symmetric space (X, d_1) is a toplogical space whose topology is induced by symmetric d_1. The main difference between a metric and a symmetric space is of triangle inequality. Since a symmetric space may not be Hausdorff, therefore to prove fixed point theorems we required some additional axioms. which are discussed in [1, 3, 4, 5, 6, 17].

From now on (X, d_1) stands for a symmetric space, whereas X denotes an arbitrary non-empty set. Then

(W_3) [17] for given $\{t_n\}, s$ and t in X, $\lim_{n\to\infty} d_1(t_n, s) = 0$ and $\lim_{n\to\infty} d_1(t_n, t) = 0$ imply $s = t$;

(W_4) [17] for given $\{t_n\}, \{s_n\}$ and s in X, $\lim_{n\to\infty} d_1(t_n, s) = 0$ and $\lim_{n\to\infty} d_1(t_n, s_n) = 0$ imply $\lim_{n\to\infty} d_1(s_n, s) = 0$;

(HE) [1] for given $\{t_n\}, \{s_n\}$ and s in X, $\lim_{n\to\infty} d_1(t_n, s) = 0$ and $\lim_{n\to\infty} d_1(s_n, s) = 0$ imply $\lim_{n\to\infty} d_1(t_n, s_n) = 0$;

$(1C)$ [5] a symmetric d_1 is said to be 1-continuous if $\lim_{n\to\infty} d_1(t_n, s) = 0$ implies $\lim_{n\to\infty} d_1(t_n, s) = d_1(s, t)$ where $\{t_n\}$ is a sequence in X and $s, t \in X$;

(CC) [5] a symmetric d_1 is said to be continuous if $\lim_{n\to\infty} d_1(t_n, s) = 0$ and $\lim_{n\to\infty} d_1(s_n, t) = 0$ implies $\lim_{n\to\infty} d_1(t_n, s_n) = d_1(s, t)$ where $\{t_n\}$, $\{s_n\}$ are sequences in X and $s, t \in X$.

Here , it can be seen that $(1C) \Rightarrow (W_3)$ and $(W_4) \Rightarrow (W_3)$ but the converse implications i.e $(W_2) \Rightarrow (1C)$ and $(W_3) \Rightarrow (W_4)$ are not true. In general, all other possible implications amongst $(W_3), (1C)$ and (HE) are not true. However, (CC) implies $(W_3), (W_4), (HE)$ and $(1C)$. For detail, we refer to an interesting note written by Cho et al.[4] which contains few illustrative examples. Employing these axioms, several authors have proved some common fixed point theorems in the framework of semi-metric and symmetric spaces (see, e.g.,[1, 3, 4, 5, 6, 17]).

It is noted that if (X, d_1) is a cone metric space over a normal cone and $d_2 = ||d_1||$ then (X, d_2) is a symmetric space which satisfies axioms (CC) but not in general a metric space (see [10]). Some results on fixed point were obtained in the framework of cone symmetric spaces.[15].

Definition 2.2. Let (X, d_1) be symmetric space. The mappings P and Q: $X \to X$ are said to be

(1) commuting if $PQt = QPt$ for all $t \in X$;
(2) compatible [11] if $\lim_{n\to\infty} d_1(PQt_n, QPt_n) = 0$ for each sequence $\{t_n\}$ in X such that $\lim_{n\to\infty} Pt_n = \lim_{n\to\infty} Qt_n$;
(3) non-compatible [14] if a sequence $\{t_n\}$ exists in X such that $\lim_{n\to\infty} Pt_n = \lim_{n\to\infty} Qt_n$, but $\lim_{n\to\infty} d_1(PQt_n, QPt_n)$ is either nonzero or nonexistent;
(4) weakly compatible [12] if P and Q commute at their coincidence points, that is , $PQt = QPt$ whenever $Pt = Qt$, for some $t \in X$.

Definition 2.3. [9] Two families $\{P_i\}_{i=1}^m$ and $\{Q_k\}_{k=1}^n$ of self mappings are said to pairwise commuting if

(1) $P_iP_j = P_jP_i$ for all $i, j \in \{1, 2, ..., m\}$;
(2) $Q_kQ_l = Q_lQ_k$ for all $k, l \in \{1, 2, ..., n\}$;
(3) $P_iQ_k = Q_kP_i$ for all $i \in \{1, 2, ..., m\}$ and $k \in \{1, 2, ..., n\}$.

Some definitions for non-self mappings are given below;

Definition 2.4. Let Y be an arbitrary set, (X, d) be a symmetric space and let P, Q, R, S be mappings from Y into X. Then

(1) the pair (P, R) is said to satisfy (E.A) property [2] if there exists a sequence $\{t_n\}$ in Y such that $\lim_{n\to\infty} Pt_n = \lim_{n\to\infty} Rt_n = z_1$, for some $z_1 \in X$;
(2) the pairs (P, R) and (Q, S) are said to share the common (E.A) property [13], if there exists two sequences $\{t_n\}$ and $\{s_n\}$ in Y such that $\lim_{n\to\infty} Pt_n = \lim_{n\to\infty} Rt_n = \lim_{n\to\infty} Qs_n = \lim_{n\to\infty} Ss_n = z_1$, for some $z_1 \in X$;
(3) the pairs (P, R) is said to have $(CLR_R))$ property [16] if there exists a sequence $\{t_n\}$ in Y such that $\lim_{n\to\infty} Pt_n = \lim_{n\to\infty} Rt_n = z$, for some $z_1 \in R(Y)$;

(4) the pairs (P,R) and (Q,S) are said to share (CLR_{RS}) property [7] if there exit two sequences $\{t_n\}$ and $\{s_n\}$ in Y such that $\lim_{n\to\infty} Pt_n = \lim_{n\to\infty} Rt_n = \lim_{n\to\infty} Qs_n = \lim_{n\to\infty} Ss_n = z_1$, for some $z_1 \in R(Y) \cap S(Y)$.

Remark 2.1. Note that

(1) If we set $P = Q$ and $R = S$, then condition (4) reduces to condition (3).
(2) Evidently, (CLR_{RS}) property implies the common (E.A) property but converse is not true.

3. Main Results

To begin our main section, we have some notations which will be required in the sequel to establish the results.

Let Ψ be the collection of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is continuous as well as monotonically nondecreasing function on $[0,\infty)$;
(ii) $\psi(t_1) = 0$ if and only if $t_1 = 0$.

Let Θ be the collection of all functions $\alpha : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) α is continuous as well as monotonically non decreasing on $[0,\infty)$;
(ii) $\alpha(t_1) = 0$ if and only if $t_1 = 0$ and $\alpha(t_1) \le t_1$ for all $t_1 > 0$.

Let Φ be the collection of all functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) φ is continuous on $[0,\infty)$;
(ii) $\varphi(t_1) = 0$ if and only if $t_1 = 0$.

we begin with the following lemma.

Lemma 3.1. *Let (X, d_1) be symmetric space where d_1 satisfies the condition (CC) while Y be an arbitrary non- empty set with $P, Q, R, S : Y \to X$. Suppose that the following hypotheses hold:*

(1) *the pair (P,R) satisfies the (CLR_R) property [respectively the pair (Q,S) satisfies the (CLR_S) property];*
(2) *$P(Y) \subset S(Y)$ [respectively $Q(Y) \subset R(Y)$];*
(3) *$S(Y)$ [respectively, $R(Y)$] is a closed subset of X;*
(4) *$\{Qy_n\}$ converges for every sequence $\{y_n\}$ in Y such that $\{Sy_n\}$ converges [repectively $\{Px_n\}$ converges for every sequence $\{x_n\}$ in Y such that $\{Rx_n\}$ converges];*
(5) *for all $t \ge 0$, $\psi(t) - \varphi(t) \ge 0$ and $\psi(t) - \varphi(t) = 0 \Rightarrow t = 0$, for some $\psi \in \Psi$, $\varphi \in \Phi$, $\alpha \in \Theta$,*
(6) *the mappings P, Q, R and S satisfy, for some $\psi \in \Psi$, $\varphi \in \Phi$, $\alpha \in \Theta$,*

$$\psi(d_1(Px,Qy)) \le \alpha(M(x,y)) - \beta(\varphi(d_1(Px,Rx)), \varphi(d_1(Px,Sy))) \tag{3.1}$$

where

$$M(x,y) = \max\Big\{\varphi(d_1(Px,Qy)), \varphi(d_1(Px,Rx)), \varphi(d_1(Qy,Sy)), \varphi(\frac{d_1(Px,Sy) + d_1(Rx,Qy)}{2}), \tag{3.2}$$

$$\varphi(\frac{1+d_1(Px,Rx)}{1+d_1(Px,Qy)} d_1(Qy,Sy)), \varphi(\frac{1+d_1(Qy,Sy)}{1+d_1(Px,Qy)} d_1(Px,Rx)), \varphi(\frac{1+d_1(Px,Sy)+d_1(Rx,Qy)}{1+d_1(Px,Rx)+d_1(Qy,Sy)} d_1(Px,Rx))\Big\}$$

and $\beta : [0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that $\beta(x,y) = 0$ if and only if $x = 0, y = 0$ for all $x, y \in X$.

Then the pair (P,R) and (Q,S) share the (CLR_{RS}) property.

Proof. Suppose that (CLR_R) property for the pair (P,R) holds. So, there exists a sequence of $\{x_n\}$ in Y such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = z$, where $z \in R(Y)$. As $S(Y)$ is a closed subset of X, $P(Y) \subset S(Y)$ and for each $\{x_n\} \subset Y$ there corresponds a sequence $\{y_n\} \subset Y$ such that $Px_n = Sy_n$. Therefore, $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Px_n = z$, where $z \in R(Y) \cap S(Y)$. Thus, we have $\lim_{n\to\infty} d_1(Px_n, z) = \lim_{n\to\infty} d_1(Rx_n, z) = \lim_{n\to\infty} d_1(Sy_n, z) = 0$ Therefore, we have $\lim_{n\to\infty} d_1(Px_n, Rx_n) = 0$ and $\lim_{n\to\infty} d_1(Rx_n, Sy_n) = 0$.

By (4), sequence $\{Qy_n\}$ converges. Now, we need to show that $Qy_n \to z$ as $n \to \infty$. By (CC), we get $\lim_{n\to\infty} d_1(Px_n, Qy_n) = \lim_{n\to\infty} d_1(z, \lim_{n\to\infty} Qy_n)$, $\lim_{n\to\infty} d_1(Rx_n, Qy_n) = \lim_{n\to\infty} d_1(z, \lim_{n\to\infty} Qy_n)$ and $\lim_{n\to\infty} d_1(Qy_n, Sy_n) = d_1(\lim_{n\to\infty} Qy_n), z)$. By inserting $x = x_n$ and $y = y_n$ in equation (3.1) we get

$$\psi(d_1(Px_n, Qy_n)) \leq \alpha(M(x_n, y_n)) - \beta(\varphi(d_1(Px_n, Rx_n)), \varphi(d_1(Px_n, Sy_n))), \tag{3.3}$$

where

$$M(x_n, y_n) = \max\Big\{\varphi(d_1(Px_n, Qy_n)), \varphi(d_1(Px_n, Rx_n), d_1(Qy_n, Sy_n)), \varphi(\frac{d_1(Px_n, Sy_n) + d_1(Rx_n, Qy_n)}{2}),$$
$$\varphi(\frac{1 + d_1(Px_n, Rx_n)}{1 + d_1(Px_n, Qy_n)} d_1(Qy_n, Sy_n)), \varphi(\frac{1 + d_1(Qy_n, Sy_n)}{1 + d_1(Px_n, Qy_n)} d_1(Px_n, Rx_n)),$$
$$\varphi(\frac{1 + d_1(Px_n, Sy_n) + d_1(Rx_n, Qy_n)}{1 + d_1(Px_n, Rx_n) + d_1(Qy_n, Sy_n)} d_1(Px_n, Rx_n))\Big\}$$

Taking the limit $n \to \infty$ in equation (3.3), we have

$$\psi(d_1(z, \lim_{n\to\infty} Qy_n)) \leq \alpha(\lim_{n\to\infty} M(x_n, y_n)) - \beta(\varphi(d_1(z, z)), \varphi(d_1(z, z))), \tag{3.4}$$

where

$$\lim_{n\to\infty} M(x_n, y_n)$$
$$= \max\Big\{\varphi(d_1(z, \lim_{n\to\infty} Qy_n)), \varphi(d_1(z, z)), \varphi(d_1(\lim_{n\to\infty} Qy_n, z)), \varphi(\tfrac{d_1(z,z)+d_1(z,\lim_{n\to\infty} Qy_n)}{2}),$$
$$\varphi(\tfrac{1+d_1(z,z)}{1+d_1(z,\lim_{n\to\infty} Qy_n)} d_1(\lim_{n\to\infty} Qy_n, z)), \varphi(\tfrac{1+d_1(\lim_{n\to\infty} Qy_n,z)}{1+d_1(z,\lim_{n\to\infty} Qy_n)} d_1(z, z)), \varphi(\tfrac{1+d_1(z,z)+d_1(z,\lim_{n\to\infty} Qy_n)}{1+d_1(z,z)+d_1(\lim_{n\to\infty} Qy_n,z)} d_1(z, z))\Big\}$$
$$= \max\Big\{\varphi(d_1(z, \lim_{n\to\infty} Qy_n)), \varphi(0), \varphi(d_1(\lim_{n\to\infty} Qy_n, z)), \varphi(\frac{d_1(z, \lim_{n\to\infty} Qy_n)}{2}),$$
$$\varphi(\frac{1}{1 + d_1(z, \lim_{n\to\infty} Qy_n)} d_1(\lim_{n\to\infty} Qy_n, z)), \varphi(0), \varphi(0)\Big\}$$
$$= \varphi(d_1(z, \lim_{n\to\infty} Qy_n))$$

Hence equation (3.4) implies

$$\psi(d_1(z, \lim_{n\to\infty} Qy_n)) \leq \alpha(\varphi(d_1(z, \lim_{n\to\infty} Qy_n)) - \beta(\varphi(0), \varphi(0)),$$
$$\psi(d_1(z, \lim_{n\to\infty} Qy_n)) \leq \alpha(\varphi(d_1(z, \lim_{n\to\infty} Qy_n)),$$

Using property of Θ, we get

$$\psi(d_1(z, \lim_{n\to\infty} Qy_n)) \leq \varphi(d_1(z, \lim_{n\to\infty} Qy_n))$$
$$\psi(d_1(z, \lim_{n\to\infty} Qy_n)) - \varphi(d_1(z, \lim_{n\to\infty} Qy_n)) \leq 0$$

Using condition (5), we have

$$\psi(d_1(z, \lim_{n\to\infty} Qy_n)) - \varphi(d_1(z, \lim_{n\to\infty} Qy_n)) \geq 0$$

hence

$$\psi(d_1(z, \lim_{n\to\infty} Qy_n)) - \varphi(d_1(z, \lim_{n\to\infty} Qy_n)) = 0$$

Again by using condition (5), we have $d_1(z, \lim_{n\to\infty} Qy_n) = 0$, Hence $\lim_{n\to\infty} Qy_n = z$ which shows that the pair (P, R) and (Q, S) share the (CLR_{RS}) property. □

Theorem 3.2. *Let (X, d_1) be a symmetric space where the symmetric d_1 satisfies the condition (HE) and (1C) and Y be a non-empty set with $P, Q, R, S : Y \to X$. Suppose that the conditions (5) and (6) of Lemma 3.1 holds. If the pair (P, R) and (Q, S) share the (CLR_{RS}) property, then (P, R) and (Q, S) have a coincidence point each. Moreover if $Y = X$ both the pair (P, R) and (Q, S) are weakly compatible then P, Q, R and S have a unique common fixed point.*

Proof. Since the pair (P, R) and (Q, S) share (CLR_{RS}) property. Therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Qy_n = z$, where $z \in R(Y) \cap S(Y)$. Since $z \in R(Y)$, there exists a pair $u \in Y$ such that $Ru = z$. Putting $x = u$ and $y = y_n$ in equation (3.1) we get

$$\psi(d_1(Pu, Qy_n)) \leq \alpha(M(u, y_n)) - \beta(\varphi(d_1(Pu, Ru)), \varphi(d_1(Pu, Sy_n))) \tag{3.5}$$

where

$$\begin{aligned} M(u, y_n) \;=\; \max\Big\{ & \varphi(d_1(Pu, Qy_n)), \varphi(d_1(Pu, Ru)), \varphi(d_1(Qy_n, Sy_n)), \varphi(\frac{d_1(Pu, Sy_n) + d_1(Ru, Qy_n)}{2}), \\ & \varphi(\frac{1 + d_1(Pu, Ru)}{1 + d_1(Pu, Qy_n)} d_1(Qy_n, Sy_n)), \varphi(\frac{1 + d_1(Qy_n, Sy_n)}{1 + d_1(Pu, Qy_n)} d_1(Pu, Ru)), \\ & \varphi(\frac{1 + d_1(Pu, Sy_n) + d_1(Ru, Qy_n)}{1 + d_1(Pu, Ru) + d_1(Qy_n, Sy_n)} d_1(Pu, Ru)) \Big\} \end{aligned}$$

Letting $n \to \infty$ (taking the limit) in equation (3.5) and using the properties (1C) and (HE), we have

$$\psi(d_1(Pu, z)) \leq \alpha(\lim_{n\to\infty} M(u, y_n)) - \beta(\lim_{n\to\infty}(\varphi(d_1(Pu, Ru)), \varphi(d_1(Pu, Sy_n)))), \tag{3.6}$$

where

$$\begin{aligned} \lim_{n\to\infty} M(u, y_n) = \max\Big\{ & \varphi(d_1(Pu, z)), \varphi(d_1(Pu, z)), \varphi(d_1(z, z)), \varphi(\frac{d_1(Pu, z) + d_1(z, z)}{2}), \\ & \varphi(\frac{1 + d_1(Pu, z)}{1 + d_1(Pu, z)} d_1(z, z)), \varphi(\frac{1 + d_1(z, z)}{1 + d_1(Pu, z)} d_1(Pu, z)), \varphi(\frac{1 + d_1(Pu, z) + d_1(z, z)}{1 + d_1(Pu, z) + d_1(z, z)} d_1(Pu, z)) \Big\} \\ = \max\Big\{ & \varphi(d_1(Pu, z)), \varphi(d_1(Pu, z)), \varphi(0), \varphi(\frac{d_1(Pu, z)}{2}), \varphi(0), \varphi(\frac{1}{1 + d_1(Pu, z)} d_1(Pu, z)), \varphi(d_1(Pu, z)) \Big\} \\ = \; & \varphi(d_1(Pu, z)) \end{aligned}$$

From equation (3.6), we obtain

$$\psi(d_1(Pu, z)) \leq \alpha(\varphi(d_1(Pu, z))) - \beta(\varphi(d_1(Pu, z)), \varphi(d_1(Pu, z)))$$

$$\psi(d_1(Pu, z)) \leq \alpha(\varphi(d_1(Pu, z)))$$

Using property of Θ, we get

$$\psi(d_1(Pu, z)) \leq \varphi(d_1(Pu, z))$$

$$\psi(d_1(Pu, z)) - \varphi(d_1(Pu, z)) \leq 0$$

Using condition 5 of lemma, we have

$$\psi(d_1(Pu, z)) - \varphi(d_1(Pu, z)) \geq 0$$

Therefore

$$\psi(d_1(Pu, z)) - \varphi(d_1(Pu, z)) = 0$$

and hence it follows easily that $Pu = z$. Therefore $Pu = Ru = z$, which shows that u is a coincidence point of the pair (P, R).

As $z \in S(Y)$, there exists a point $v \in Y$ such that $Sv = z$. Putting $x = u$ and $y = v$ in equation (3.1), we have

$$\psi(d_1(z,Qv)) = \psi(d_1(Pu,Qv)) \leq \alpha(M(u,v)) - \beta(\varphi(d_1(z,z)),\varphi(d_1(z,z))), \tag{3.7}$$

where

$$M(u,v) = \max\Bigg\{\varphi(d_1(Pu,Qv)),\varphi(d_1(Pu,Ru)),\varphi(d_1(Qv,Sv)),\varphi(\frac{d_1(Pu,Sv)+d_1(Ru,Qv)}{2}),$$
$$\varphi(\frac{1+d_1(Pu,Ru)}{1+d_1(Pu,Qv)}d_1(Qv,Sv)),\varphi(\frac{1+d_1(Qv,Sv)}{1+d_1(Pu,Qv)}d_1(Pu,Ru)),\varphi(\frac{1+d_1(Pu,Sv)+d_1(Ru,Qv)}{1+d_1(Pu,Ru)+d_1(Qv,Sv)}d_1(Pu,Ru))\Bigg\}$$

$$= \max\Bigg\{\varphi(d_1(z,Qv)),\varphi(d_1(z,z)),\varphi(d_1(Qv,z)),\varphi(\frac{d_1(z,z)+d_1(z,Qv)}{2}),$$
$$\varphi(\frac{1+d_1(z,z)}{1+d_1(z,Qv)}d_1(Qv,z)),\varphi(\frac{1+d_1(Qv,z)}{1+d_1(z,Qv)}d_1(z,z))\varphi(\frac{1+d_1(z,z)+d_1(z,Qv)}{1+d_1(z,z)+d_1(Qv,z)}d_1(z,z))\Bigg\}$$

$$= \max\Bigg\{\varphi(d_1(z,Qv)),\varphi(0),\varphi(d_1(Qv,z)),\varphi(\frac{d_1(z,Qv)}{2}),\varphi(\frac{1}{1+d_1(z,Qv)}d_1(Qv,z)),\varphi(0),\varphi(0)\Bigg\}$$

$$= \varphi(d_1(z,Qv))$$

Hence the equation (3.7) implies

$$\psi(d_1(z,Qv)) \leq \alpha(\varphi(d_1(z,Qv))) - \beta(\varphi(0),\varphi(0)),$$

$$\psi(d_1(z,Qv)) \leq \alpha(\varphi(d_1(z,Qv)))$$

Using property of Θ, we get

$$\psi(d_1(z,Qv)) \leq \varphi(d_1(z,Qv))$$

$$\psi(d_1(z,Qv)) - \varphi(d_1(z,Qv)) \leq 0,$$

Using condition 5 of lemma, we have

$$\psi(d_1(z,Qv)) - \varphi(d_1(z,Qv)) \geq 0,$$

Hence

$$\psi(d_1(z,Qv)) - \varphi(d_1(z,Qv)) = 0,$$

and it follows easily that $z = Qv$. Thus $Qv = Sv = z$, which shows that v is a coincidence point of the pair (Q,S).
Suppose now that $Y = X$. Since the pair (P,R) and (Q,S) are weakly compatible, $Pu = Ru$ and $Qv = Sv$, therefore $Pz = PRu = RPu = Rz$ and $Qz = QSv = SQv = Sz$. Putting $x = z$ and $y = v$ in equation (3.1), we have

$$\psi(d_1(Pz,z)) = \psi(d_1(Pz,Qv)) \leq \alpha(M(z,v)) - \beta(\varphi(d_1(Pz,Rz)),\varphi(d_1(Pz,Sv))), \tag{3.8}$$

where

$$M(z,v) = \max\Bigg\{\varphi(d_1(Pz,Qv)),\varphi(d_1(Pz,Rz)),\varphi(d_1(Qv,Sv)),\varphi(\frac{d_1(Pz,Sv)+d_1(Rz,Qv)}{2}),$$
$$\varphi(\frac{1+d_1(Pz,Rz)}{1+d_1(Pz,Qv)}d_1(Qv,Sv)),\varphi(\frac{1+d_1(Qv,Sv)}{1+d_1(Pz,Qv)}d_1(Pz,Rz)),\varphi(\frac{1+d_1(Az,Tv)+d_1(Sz,Bv)}{1+d_1(Az,Sz)+d_1(Bv,Tv)}d_1(Az,Sz))\Bigg\}$$

$$= \max\Big\{\varphi(d_1(Pz,z)), \varphi(d_1(Pz,Pz)), \varphi(d_1(z,z)), \varphi(\frac{d_1(Pz,z)+d_1(Pz,z)}{2}),$$

$$\varphi(\frac{1+d_1(Pz,Pz)}{1+d_1(Pz,z)}d_1(z,z)), \varphi(\frac{1+d_1(z,z)}{1+d_1(Pz,z)}d_1(Pz,Pz)), \varphi(\frac{1+d_1(Pz,z)+d_1(Pz,z)}{1+d_1(Pz,z)+d_1(z,z)}d_1(Pz,Pz))\Big\}$$

$$= \max\Big\{\varphi(d_1(Pz,z)), \varphi(0), \varphi(0), \varphi(d_1(Pz,z)), \varphi(0), \varphi(0), \varphi(0)\Big\}$$

$$= \varphi(d_1(Pz,z))$$

From equation (3.8), we get

$$\psi(d_1(Pz,z)) \leq \alpha(\varphi(d_1(Pz,z))) - \beta(\varphi(d_1(Pz,Pz)), \varphi(d_1(Pz,z)))$$

$$\psi(d_1(Pz,z)) \leq \alpha(\varphi(d_1(Pz,z))) - \beta(\varphi(0), \varphi(d_1(Pz,z)))$$

$$\psi(d_1(Pz,z)) \leq \alpha(\varphi(d_1(Pz,z)))$$

Using property of Θ, we get

$$\psi(d_1(Pz,z)) \leq \varphi(d_1(Pz,z))$$

$$\psi(d_1(Pz,z)) - \varphi(d_1(Pz,z)) \leq 0$$

Using condition 5 of lemma, we have

$$\psi(d_1(Pz,z)) - \varphi(d_1(Pz,z)) \geq 0$$

Hence,

$$\psi(d_1(Pz,z)) - \varphi(d_1(Pz,z)) = 0$$

Using condition 5 of Lemma 3.1 it follows easily that $z = Pz = Rz$ and therefore z is a common fixed point of the pair (P,R). Putting $x = u$ and $y = z$ in equation (3.1), we have

$$\psi(d_1(z,Qz)) = \psi(d_1(Pu,Qz)) \leq \alpha(M(u,z)) - \beta(\varphi(d_1(Pu,Ru)), \varphi(d_1(Pu,Sz))) \tag{3.9}$$

where

$$M(u,z) = \max\Big\{\varphi(d_1(Pu,Qz)), \varphi(d_1(Pu,Ru)), \varphi(d_1(Qz,Sz)), \varphi(\frac{d_1(Pu,Sz)+d_1(Ru,Qz)}{2}),$$

$$\varphi(\frac{1+d_1(Pu,Ru)}{1+d_1(Pu,Qz)}d_1(Qz,Sz)), \varphi(\frac{1+d_1(Qz,Sz)}{1+d_1(Pu,Qz)}d_1(Pu,Ru)), \varphi(\frac{1+d_1(Pu,Sz)+d_1(Ru,Qz)}{1+d_1(Pu,Ru)+d_1(Qz,Sz)}d_1(Pu,Ru))\Big\}$$

$$= \max\Big\{\varphi(d_1(z,Qz)), \varphi(d_1(z,z)), \varphi(d_1(Qz,Qz)), \varphi(\frac{d_1(z,Qz)+d_1(z,Qz)}{2}),$$

$$\varphi(\frac{1+d_1(z,z)}{1+d_1(z,Qz)}d_1(Qz,Qz)), \varphi(\frac{1+d_1(Qz,Qz)}{1+d_1(z,Qz)}d_1(z,z)), \varphi(\frac{1+d_1(z,Qz)+d_1(z,Qz)}{1+d_1(z,z)+d_1(z,z)}d_1(z,z))\Big\}$$

$$= \max\Big\{\varphi(d_1(z,Qz)), \varphi(0), \varphi(0), \varphi(d_1(z,Qz)), \varphi(0), \varphi(0), \varphi(0)\Big\}$$

$$= \varphi(d_1(z,Qz))$$

From equation (3.9) we get

$$\psi(d_1(z,Qz)) \leq \alpha(\varphi(d_1(z,Qz))) - \beta(\varphi(d_1(z,z)), \varphi(d_1(z,Qz)))$$

$$\psi(d_1(z,Qz)) \leq \alpha(\varphi(d_1(z,Qz)))$$

Using property of Θ, we get

$$\psi(d_1(z, Qz)) \leq \varphi(d_1(z, Qz))$$

$$\psi(d_1(z, Qz)) - \varphi(d_1(z, Qz)) \leq 0$$

Using condition 5 of lemma, we have

$$\psi(d_1(z, Qz)) - \varphi(d_1(z, Qz)) \geq 0$$

Hence,

$$\psi(d_1(z, Qz)) - \varphi(d_1(z, Qz)) = 0$$

Using condition 5 of Lemma 3.1 it follows easily that $z = Qz$. Therefore $Qz = Sz = z$ and we can conclude that z is a common fixed point of P, Q, R and S. The uniqueness of a common fixed point can be easily checked using the equation (3.1). □

Theorem 3.3. *Let (X, d_1) be symmetric space, where the symmetric d_1 satisfies the condition (CC), and Y be an arbitrary non-empty set with $P, Q, R, S : Y \to X$. Suppose that the conditions $(1-6)$ of Lemma 3.1 hold. Then (P, R) and (Q, S) have coincidence point each. Moreover, if $Y = X$ and both the pairs (P, R) and (Q, S) are weakly compatible then P, Q, R and S have a unique common fixed point.*

Proof. In view of Lemma 3.1 , the pairs (P, R) and (Q, S) share the (CLR_{RS}) property, therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Qy_n = z$, where $z \in R(Y) \cap S(Y)$. The rest proof can be done on the same lines of Theorem 3.2. □

Theorem 3.4. *Let (X, d_1) be symmetric space, where the symmetirc d_1 satisfies the conditions (HE) and (1C), and let Y be an arbitrary non-empty set with $P, Q, R, S : Y \to X$. Suppose that the conditions (5) and (6) of Lemma 3.1 and the following hypotheses hold:*

(1) *the pairs (P, R) and (Q, S) satisfy the common property (E.A);*
(2) *$R(Y)$ and $S(Y)$ are closed subsets of X.*

Then (P, R) and (Q, S) have coincidence point each. Moreover, if $Y = X$ and both the pairs (P, R) and (Q, S) are weakly compatible then P, Q, R and S have a unique common fixed point.

Proof. If the pairs (P, R) and (Q, S) share the common property (E.A), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Qy_n = z$, for some $z \in X$. Since $R(Y)$ is closed, $\lim_{n\to\infty} Rx_n = z = Ru$ for some $u \in Y$. Also, since $S(Y)$ is closed, then $\lim_{n\to\infty} Sy_n = z = S\nu$ for some $\nu \in Y$. The rest proof can be done on the same lines of Theorem 3.2. □

Corollary 3.5. *The conclusion of Theorem 3.4 remain same on replacing condition* (2) *by the following:*

($2'$) *$\overline{P(Y)} \subset S(Y)$ and $\overline{Q(Y)} \subset R(Y)$, where $\overline{P(Y)}$ and $\overline{Q(Y)}$ denote the closure of ranges of the mappings P and Q.*

Corollary 3.6. *The conclusion of Theorem 3.4 remain same on replacing condition* (2) *by the following:*

($2''$) *$P(Y)$ and $Q(Y)$ are closed subsets of X, and $P(Y) \subset S(Y)$, $Q(Y) \subset R(Y)$.*

Corollary 3.7. *Let (X, d_1) be symmetric space where the symmetric d_1 satisfies the conditions (HE) and (1C), and Y be an arbitrary non- empty set with $P, R : Y \to X$. Suppose that the given hypotheses hold:*

(1) *the pair (P, R) satisfies the (CLR_R) property;*

(2) *for some $\psi \in \Psi$, $\varphi \in \Phi$, $\alpha \in \Theta$ and all $x, y \in Y$, satisfies*

$$\psi(d_1(Px, Py)) \leq \alpha(M(x, y)) - \beta(\varphi(d_1(Px, Rx)), \varphi(d_1(Px, Ry))) \tag{3.10}$$

where

$$M(x,y) = \max\Big\{\varphi(d_1(Px, Py)), \varphi(d_1(Px, Rx)), \varphi(d_1(Py, Ry)), \varphi(\frac{d_1(Px, Ry) + d_1(Rx, Py)}{2}),$$
$$\varphi(\frac{1 + d_1(Px, Rx)}{1 + d_1(Px, Py)} d_1(Py, Ry)), \varphi(\frac{1 + d_1(Py, Ry)}{1 + d_1(Px, Py)} d_1(Px, Rx)), \varphi(\frac{1 + d_1(Px, Ry) + d_1(Rx, Py)}{1 + d_1(Px, Rx) + d_1(Py, Ry)} d_1(Px, Rx))\Big\}$$

and $\beta : [0, \infty)^2 \to [0, \infty)$ is a continuous mapping such that $\beta(x, y) = 0$ if and only if $x = 0, y = 0$ for all $x, y \in Y$

Then the pair (P, R) has a coincidence point. Moreover, if $Y = X$ and the pair (P, R) is weakly compatible then P and R have a unique common fixed point.

Corollary 3.8. *Let (X, d_1) be symmetric space where the symmetric d_1 satisfies the conditions (HE) and (1C), and Y be an arbitrary non- empty set with $P, Q, T, R, S : Y \to X$. Suppose that the following hypotheses hold:*

(1) *the pairs (P, RT) and (Q, S) satisfies the $(CLR_{(RT)(S)})$ property.*

(2) *for some $\psi \in \Psi$, $\varphi \in \Phi$, $\alpha \in \Theta$ and all $x, y \in Y$, satisfies*

$$\psi(d_1(Px, Qy)) \leq \alpha(M(x, y)) - \beta(\varphi(d_1(Px, RTx)), \varphi(d_1(Px, Sy))) \tag{3.11}$$

where

$$M(x,y) = \max\Big\{\varphi(d_1(Px, Qy)), \varphi(d_1(Px, RTx)), \varphi(d_1(Qy, Sy)), \varphi(\frac{d_1(Px, Sy) + d_1(RTx, Qy)}{2}),$$
$$\varphi(\frac{1 + d_1(Px, RTx)}{1 + d_1(Px, Qy)} d_1(Qy, Sy)), \varphi(\frac{1 + d_1(Qy, Sy)}{1 + d_1(Px, Qy)} d_1(Px, RTx)), \varphi(\frac{1 + d_1(Ax, Ty) + d_1(SRx, By)}{1 + d_1(Ax, SRx) + d_1(By, Ty)} d_1(Ax, SRx))\Big\}$$

and $\beta : [0, \infty)^2 \to [0, \infty)$ is a continuous mapping such that $\beta(x, y) = 0$ if and only if $x = 0, y = 0$ for all $x, y \in Y$

Then the pairs (P, RT) and (Q, S) have a coincidence point each. Moreover, if $Y = X$ and both the pairs (P, RT) and (Q, S) commute pairwise, that is $PR = RP$, $PT = TP$, $RT = TR$, $QS = SQ$ then P, Q, T, R and S have a unique common point.

Corollary 3.9. *Let P, Q, R and S be self mappings of a symmetric space (X, d_1) satisfying the conditions (HE) and (1C). Suppose that, for fixed positive integer m, n, p, q.*

(1) *the pair (P^m, R^p) and (Q^n, S^q) share the $(CLR_{R^pS^q})$ property;*

(2) *for some $\psi \in \Psi$, $\varphi \in \Phi$, $\alpha \in \Theta$ and all $x, y \in Y$, satisfies*

$$\psi(d_1(P^m x, Q^n y)) \leq \alpha(M(x, y)) - \beta(\varphi(d_1(P^m x, R^p x)), \varphi(d_1(P^m x, S^q y))) \tag{3.12}$$

where

$$M(x,y) = \max\Big\{\varphi(d_1(P^m x, Q^n y)), \varphi(d_1(P^m x, R^p x)), \varphi(d_1(Q^n y, S^q y)),$$

$$\varphi(\frac{d_1(P^m x, S^q y) + d_1(R^p x, Q^n y)}{2}), \varphi(\frac{1 + d_1(P^m x, R^p x)}{1 + d_1(P^m x, Q^n y)} d_1(Q^n y, S^q y)), \varphi(\frac{1 + d_1(Q^n y, S^q y)}{1 + d_1(P^m x, Q^n y)} d_1(P^m x, R^p x))$$

$$, \varphi(\frac{1 + d_1(P^m x, S^q y) + d_1(R^p x, Q^n y)}{1 + d_1(P^m x, R^p x) + d_1(Q^n y, S^q y)} d_1(P^m x, R^p x))\Big\}$$

and $\beta : [0,\infty)^2 \to [0,\infty)$ *is a continuous mapping such that* $\beta(x,y) = 0$ *if and only if* $x = 0, y = 0$ *for all* $x, y \in Y$

If $PR = RP$ and $QS = SQ$ then P, Q, R and S have a unique common fixed point.

Example 3.10. *Let* $Y = [2, 11) \subset [1,\infty) = X$ *equipped with the symmetric* $d_1(s,t) = (s-t)^2$ *for all* $s, t \in Y$ *which also satisfies* (1C) *and* (HE). *Define the mappings* $P, Q, R, S : Y \to X$ *by*

$$P(s) = \begin{cases} 2 & \text{if } s \in \{2\} \cup (4, 11) \\ 12 & \text{if } s \in (2, 4] \end{cases}, Q(s) = \begin{cases} 2 & \text{if } s \in \{2\} \cup (4, 11) \\ 14 & \text{if } s \in (2, 4] \end{cases}$$

$$R(s) = \begin{cases} 2 & \text{if } s = 2 \\ 11 & \text{if } s \in (2, 4] \\ \frac{3s+2}{7} & \text{if } s \in (4, 11) \end{cases}, S(s) = \begin{cases} 2 & \text{if } s = 2 \\ 11 & \text{if } s \in (2, 4] \\ s-2 & \text{if } s \in (4, 11) \end{cases}$$

Then we have $P(Y) = \{2, 9\} \nsubseteq [2, 9) \cup \{11\} = S(Y)$ *and* $Q(Y) = \{2, 5\} \nsubseteq [2, 5) \cup \{11\} = R(Y)$. *Consider two sequences* $\{s_n\} = \{4 + \frac{1}{n}\}_{n\in N}$ *and* $\{t_n\} = \{2\}_{n\in N}$. *Then the pair* (P, R) *and* (Q, S) *satisfies the* (CLR_{RS}) *property. Indeed we have*

$$\lim_{n\to\infty} Ps_n = \lim_{n\to\infty} Rs_n = \lim_{n\to\infty} Qs_n = \lim_{n\to\infty} St_n = 2$$

where $2 \in R(Y) \cap S(Y)$; *however,* $R(Y)$ *and* $S(Y)$ *are not closed subsets of* X.
Now define $\phi \in \Phi$ *by* $\phi(t_1) = t_1$, $\psi \in \Phi$ *by* $\psi(t_1) = \frac{t_1}{2}$, $\alpha \in \Theta$ *by* $\alpha(t_1) = t_1$ *and* $\beta : [0,\infty)^2 \to [0,\infty)$ *by* $\beta(t_1, s_1) = \frac{t_1+s_1}{16}$. *In order to verify the inequality* (3.1), *we distinguish the following possible cases:*

(1) *If* $s = t = 2$, *then we get* $d_1(Ps, Qt) = 0$ *and hence* $\psi(0) = 0$ *and* (3.1) *is trivially satisfied;*

(2) *If* $s = 2, t \in (2, 4]$, *then we get* $d_1(Ps, Qt) = 144$ *and hence* $\psi(d_1(Ps, Qt)) = 72$, $M(s,t) = 144$, $\alpha(M(s,t)) = 144$, $\beta(\phi(d_1(Ps, Rs), \phi(d_1(Ps, St)))) = \frac{81}{16}$, *therefore R.H.S of the inequality* (3.1) *is* $\frac{2223}{16}$ *so* $L.H.S \le R.H.S$;

(3) *If* $s = 2, t \in (4, 11)$, $d_1(Ps, Qt) = 0$ *and hence* $\psi(d_1(Ps, Qt)) = 0$, $M(s,t) = (4-t)^2$, $\alpha(M(s,t)) = (4-t)^2$, $\beta(\phi(d_1(Ps, Rs), \phi(d_1(Ps, St)))) = \frac{(4-t)^2}{16}$, *therefore R.H.S of the inequality* (3.1) *is* $\frac{15(4-t)^2}{16}$ *so* $L.H.S \le R.H.S$;

(4) *If* $s \in (2, 4], t = 2$,*then* $d_1(Ps, Qt) = 100$ *and hence* $\psi(d_1(Ps, Qt)) = 50$, $M(s,t) = 100$, $\alpha(M(s,t)) = 100$, $\beta(\phi(d_1(Ps, Rs), \phi(d_1(Ps, St)))) = \frac{101}{16}$, *therefore R.H.S of the inequality* (3.1) *is* $\frac{1499}{16}$ *so* $L.H.S \le R.H.S$;

(5) *If* $s, t \in (2, 4]$, *then* $d_1(Ps, Qt) = 4$ *and hence* $\psi(d_1(Ps, Qt)) = 2$, $M(s,t) = 9$, $\alpha(M(s,t)) = 9$, $\beta(\phi(d_1(Ps, Rs), \phi(d_1(Ps, St)))) = \frac{1}{8}$, *therefore R.H.S of the inequality* (3.1) *is* $\frac{71}{8}$ *so* $L.H.S \le R.H.S$;

(6) *If* $s \in (2,4], t \in (4,11)$,*then* $d_1(Ps,Qt) = 100$ *and hence* $\psi(d_1(Ps,Qt)) = 50$, $M(s,t) = 100$ *and* $\beta(\phi(d_1(Ps,Rs),\phi(d_1(Ps,St)))) = \frac{1+(14-t)^2}{16}$, *therefore the R.H.S of the inequality* (3.1) *is* $\frac{1599-(4-s)^2}{16}$ *so L.H.S* $\leq$ *R.H.S;*

(7) *If* $s \in (4,11), t = 2$ *then,* $d_1(Ps,Qt) = 0$ *and hence* $\psi(d_1(Ps,Qt)) = 0$, $M(s,t) = \frac{(12-3s)^2}{49}$, $\alpha(M(s,t)) = \frac{(12-3s)^2}{49}$, $\beta(\phi(d_1(Ps,Rs),\phi(d_1(Ps,St)))) = \frac{(12-3s)^2}{98}$, *therefore R.H.S of the inequality* (3.1) *is* $\frac{(12-3s)^2}{98}$ *so L.H.S* $\leq$ *R.H.S;*

(8) *If* $s \in (4,11), t \in (2,4]$ *then,* $d_1(Ps,Qt) = 144$ *and hence* $\psi(d_1(Ps,Qt)) = 72$, $M(s,t) = 144$ *and* $\beta(\phi(d_1(Ps,Rs),\phi(d_1(Ps,St)))) = \frac{(12-3s)^2+3969}{784}$, *thereforeR.H.S of the inequality* (3.1) *is* $\frac{108927-(12-3s)^2}{784}$ *so L.H.S* $\leq$ *R.H.S;*

(9) *If* $s,t \in (4,11)$ *then,* $d_1(Ps,Qt) = 0$ *and hence* $\psi(d_1(Ps,Qt)) = 0$, $M(s,t) = \frac{(49+(12-3s)^2)(4-s)^2}{49}$ *and* $\beta(\phi(d_1(Ps,Rs),\phi(d_1(Ps,St)))) = \frac{(12-3s)^2+49(4-s)^2}{98}$, *therefore, in all the cases inequality* (3.1) *holds;*

Thus, the above example satisfies all the conditions of Theorem 3.2, except for $Y = X$, *but a unique common fixed point of the pair* (P,R) *and* (Q,S) *is* 2. *Here, one may check that all the involved mappings are discontinuous at their unique fixed point* $'2'$.

Example 3.11. *In the setting of Example 3.10, replace the mappings R and T by the following, beside retaining the rest:*

$$R(t) = S(t) = \begin{cases} 2 & if\ t = 2 \\ 16 & if\ t \in (2,4] \\ \frac{9t+6}{7} & if t \in (4,11) \end{cases}$$

Then we have $P(Y) = \{2,12\} \subset [2,16] = S(Y)$ *and* $Q(Y) = \{2,14\} \subset [2,16] = R(Y)$; *Now* $R(Y)$ *and* $S(Y)$ *are closed subsets of* X. *Thus, all the conditions of Theorem 3.3 are satisfied, except* $Y = X$; *however the unique common fixed point of the pairs* (P,R) *and* (Q,S) *is 2.*

Example 3.12. *In the setting of Example 3.10, replace the mappings R and S by the following, beside retaining the rest:*

$$R(t) = S(t) = \begin{cases} 2 & if\ t = 2 \\ 16 & if\ t \in (2,4] \\ \frac{8t+17}{7} & if t \in (4,11) \end{cases}$$

Then we have $P(Y) = \{2,12\} \subset [2,15) \cup \{16\} = S(Y)$ *and* $Q(Y) = \{2,14\} \subset [2,15) \cup \{16\} = R(Y)$. *Now,* $R(Y)$ *and* $S(Y)$ *are not closed subsets of* X, *but the conditions* $(2^{'})$ *and* $(2^{''})$ *of Corollaries 3.5 and 3.6 are satisfied, except* $Y = X$; *however the unique common fixed point of the pairs* (P,R) *and* (Q,S) *is 2.*

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