TWMS J. App. Eng. Math. V.7, N.1, 2017, pp. 154-164

PARTITIONING A GRAPH INTO MONOPOLY SETS

AHMED MOHAMMED NAJI[1], SONER NANDAPPA D[1], §

Abstract. In a graph $G = (V, E)$, a set $M \subseteq V(G)$ is said to be a monopoly set of G if every vertex $v \in V - M$ has, at least, $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G, denoted by $mo(G)$, is the minimum cardinality of a monopoly set. In this paper, we study the problem of partitioning $V(G)$ into monopoly sets. An M-partition of a graph G is the partition of $V(G)$ into k disjoint monopoly sets. The monatic number of G, denoted by $\mu(G)$, is the maximum number of sets in M-partition of G. It is shown that $2 \leq \mu(G) \leq 3$ for every graph G without isolated vertices. The properties of each monopoly partite set of G are presented. Moreover, the properties of all graphs G having $\mu(G) = 3$, are presented. It is shown that every graph G having $\mu(G) = 3$ is Eulerian and have $\chi(G) \leq 3$. Finally, it is shown that for every integer $k \notin \{1, 2, 4\}$, there exists a graph G of order $n = k$ having $\mu(G) = 3$.

Keywords: Vertex degrees, distance in graphs, graph polynomials.

AMS Subject Classification: 05C70, 05C07, 05C69.

1. Introduction

The concept of monopoly in a graph was introduced by Khoshkhak K. et al. [10]. Some mathematical properties of monopoly in graphs have been studied in [12], other types of monopoly in graphs have been subsequently proposed by the authors ([13]-[16]). In particular, the monopoly in graphs is a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg D. [17]. For more details in monopoly and dynamos in graphs, we refer the reader to [2, 3, 7, 11, 19]. In this paper, we focus our attention on the problem of partitioning of the vertex set of a graph G into disjoint monopoly sets. We denote by M-partition to the partition of $V(G)$ into k disjoint monopoly sets. The idea of M-partition of G closely related to unfriendly partition [5, 1], and an offensive k-alliances partition [18].

We begin by stating the terminology and notations used through this article. A graph $G = (V, E)$ is a simple graph, that is finite, having no loops no multiple and directed edges. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. For a vertex $v \in V$, the open neighborhood of v in a graph G, denoted $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of vertex v in G is $d(v) = |N(v)|$, and the degree of a vertex v with respect to a subset $S \subset V(G)$ is $d_S(v) = |N(v) \cap S|$. We denote by

[1] Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru, INDIA.
e-mail: ama.moohsen78@gmail.com, ndsoner@yahoo.com;
§ Manuscript received: May 21, 2016; accepted: September 22, 2016.

$\Delta(G)$ and $\delta(G)$ to maximum and minimum degree among the vertices of G, respectively. $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (smallest) integer number less (greater) than or equal to x. An isolated vertex in G is a vertex with degree zero. As usual, $\overline{G}$ denotes the complement of G, for a subset $S \subseteq V$, $\overline{S} = V - S$ and kG denotes the k disjoint copies of G. A k-partite graph is a graph G whose vertex set $V(G)$ can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent. A k-partite graph in which each partite set has the same number of vertices is said to be a balanced k-partite graph. The Friendship graph F_n, for $n \geq 2$, is the graph constructed by joining n copies of K_3 graph with a common vertex. A set $I \subseteq V$ is independent if no two vertices in I are adjacent. The independent sets of maximum cardinality are called maximum independent sets. The number of vertices in a maximum independent set is the independence number (or vertex independence number) of G and is denoted by $\alpha(G)$. For more terminologies and notations in graph theory, we refer the reader to the books [4, 8].

A set $D \subseteq V(G)$ is called a dominating set of a graph G if every vertex $v \in V(G) - D$ adjacent to some vertex in D. The minimum cardinality of such set is called the domination number of G and denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ-set for G. A thorough treatment of domination in graphs can be found in the book by Haynes at el. [9]. The domatic number $d(G)$ of a graph G is the maximum positive integer k such that $V(G)$ can be partitioned into k pairwise disjoint dominating sets. A partition V into pairwise disjoint dominating sets is called a domatic partition. The concept of a domatic number was introduced by E. J. Cockayne at el. [6]. A proper coloring of a graph G is a k-coloring in which each color class is an independent set. The minimum k for which a graph is k-colorable is called its chromatic number and denoted by $\chi(G)$ [4].

A pigeonhole principle states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item. The pigeonhole principle has several generalizations and can be stated in various ways. In a more quantified version: for natural numbers k and m, if $n = km + 1$ objects are distributed among m sets, then the pigeonhole principle asserts that at least one of the sets will contain at least $k + 1$ objects. For arbitrary n and m this generalizes to $k + 1 = \lfloor \frac{n-1}{m} \rfloor + 1$ [20].

A set $M \subseteq V(G)$ is called a monopoly set of G if for every vertex $v \in V(G) - M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G, denoted by $mo(G)$, is the minimum cardinality of a monopoly set in G. An M-partition of a graph G is the partition of $V(G)$ into k disjoint monopoly sets. The monatic number of G, denoted by $\mu(G)$, is the maximum number of sets in M-partition of G. The word "monatic" was created from monopoly and chromatic in the same way the word "domatic" which created from domination and chromatic. It is shown that $2 \leq \mu(G) \leq 3$ for every graph G without isolated vertices. The properties of each monopoly partite set of G are presented. Moreover, the properties of all graphs G having $\mu(G) = 3$, are presented. It is shown that every graph G having $\mu(G) = 3$ is Eulerian and have $\chi(G) \leq 3$. Finally, it is shown that for every integer $k \notin \{1, 2, 4\}$, there exists a graph G of order $n = k$ with $\mu(G) = 3$.

The following are some fundamental results which will be required for many of our arguments in this paper:

Theorem 1.1. [8] *A graph G is eulerian if and only if every vertex of G is of even degree.*

The following results appear in paper [6].

Proposition 1.1. **(a):** *For any graph G, $d(G) \leq \delta + 1$.*
(b): *$d(G) \geq 2$, if and only if G has no isolated vertices.*
(c): *For any tree T with $n \geq 2$ vertices, $d(T) = 2$.*

2. Partitioning Vertex Set of a Graph into Monopoly sets

Theorem 2.1. *Any non-trivial graph G without isolated vertices has an M-partition.*

Proof. Let $\{X, Y\}$ be a partition of $V(G)$ such that the edge-cut between X and Y has maximum cardinality. Then X and Y are dominating sets. Moreover, for every vertex $x \in X$, has at least $\frac{d(v)}{2}$ neighbors in Y, then we have that Y is a monopoly set in G. Analogously, we obtain that X is a monopoly set in G. Hence, $\{X, Y\}$ is a partition of $V(G)$ into two monopoly sets in G. This complete the proof. □

Since any monopoly set M of a graph G must be contain every isolated vertices in G, then we have the following result.

Proposition 2.1. *Let G be a graph of order n. Then $\mu(G) = 1$, if and only if G having an isolated vertex.*

Accordingly to Theorem 2.1 and Proposition 2.1, we obtain the following fundamental result.

Theorem 2.2. *For any graph G without isolated vertices,*

$$2 \leq \mu(G) \leq 3.$$

Proof. By Theorem 2.1 and Proposition 2.1, we have $\mu(G) \geq 2$. For the upper bound, since, the M-partition of G is a partition of $V(G)$ into k monopoly subset, it follows by the definition of a monopoly set, every vertex $v \in V(G)$ must be adjacent to, at least, $\frac{d(v)}{2}$ vertices in every subset other then its own. If a graph G has $\mu(G) = k$, then every vertex $v \in V(G)$ must be adjacent to, at least, $(k-1)\frac{d(v)}{2}$ vertices, $\frac{d(v)}{2}$ vertices in each partite set of an M-partition. Hence, we have $(k-1)\frac{d(v)}{2} \leq d(v)$, this implies that $(k-3)d(v) \leq 0$. But since $d(v) > 0$, for every $v \in V(G)$, it follows that $k-3 \leq 0$. Therefore, $\mu(G) \leq 3$. □

Corollary 2.1. *For any graph G, $1 \leq \mu(G) \leq 3$.*

Theorem 2.3. *For any graph G without isolated vertices. If G has a vertex of odd degree, then $\mu(G) = 2$.*

Proof. Let $v \in V(G)$ be a vertex with odd degree. i.e., $d(v) = 2k+1$, for any $k \geq 0$. Suppose, to the contrary, that $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Assume, without loss of generality, that $v \in M_1$. Then by the definition of a monopoly set, $d_{M_2}(v) \geq \frac{d(v)}{2} \geq k+1$ and also, $d_{M_3}(v) \geq k+1$. Hence, $d(v) \geq d_{M_2}(v) + d_{M_3}(v) \geq 2k+2$, a contradiction. Therefore, by Theorem 2.2, $\mu(G) = 2$. □

Corollary 2.2. *For any graph G. If G has $\mu(G) = 3$, then every vertex of G is of even degree.*

Theorem 2.4. *Let G be a graph without isolated vertices and every vertex of G is of even degree. If G has a cycle, of order $k \equiv 1, 2 \pmod 3$, as an endblock. Then $\mu(G) = 2$.*

Proof. Let G be a graph with a cycle endblock C_k, for $k \equiv 1, 2 \pmod 3$, and let $\{v_1, v_2, ..., v_k\}$ be the vertex set of C_k, such that v_1 is the cut vertex of G on C_k. Clearly, $d(v_1) \geq 4$ and $d(v_i) = 2$, for every $2 \leq i \leq k$. Suppose, to the contrary, that $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Assume, without loss of generality, that $v_1 \in M_1$

and $v_2 \in M_2$, then $v_3 \in M_3$. Furthermore, $M_1 = \{v_i : i \equiv 1 \pmod 3\}$, $M_2 = \{v_i : i \equiv 2 \pmod 3\}$ and $M_3 = \{v_i : i \equiv 0 \pmod 3\}$. Then we have the following two cases.

Case 1: If $k \equiv 1 \pmod 3$, then $v_k \in M_1$ and hence, $d_{M_3}(v_1) = 0$, a contradiction.

Case 2: If $k \equiv 2 \pmod 3$, then $v_k \in M_2$ and hence a gain, $d_{M_3}(v_1) = 0$, a contradiction.

Therefore, $\mu(G) \neq 3$. □

Theorem 2.5. *For any graph G, $\mu(G) \leq d(G)$. Furthermore, if $\Delta(G) \leq 2$, then $\mu(G) = d(G)$.*

Proof. Clearly, from the definition of the monopoly set that any monopoly set of a graph G is a dominating set. Then, $\mu(G) \leq d(G)$. Now, let$G$ be a graph with $\Delta(G) \leq 2$. and let $D_1, D_2, ..., D_k$ be the partition of G into k dominating set. Since, $d_{D_i}(v) \geq 1 \geq \frac{\Delta}{2} \geq \frac{d(v)}{2}$, for every vertex $v \notin D_i$, and for every $i = 1, 2, .., k$, it follows that D_i is a monopoly set of G, for every $i = 1, 2, .., k$. Hence, $d(G) \leq \mu(G)$, but we have $\mu(G) \leq d(G)$. Then $\mu(G) = d(G)$. □

The converse of Theorem 2.5, is not true. For example, the star graph $K_{1,n}$, for every $n \geq 3$, has $d(K_{1,n}) = \mu(K_{1,n}) = 2$, but $\Delta(K_{1,n}) \geq 3$. The following result immediate consequences of Proposition 1.1 and Theorem 2.5.

Corollary 2.3. *For any tree T with $n \geq 2$ vertices, $\mu(G) = d(G) = 2$.*

In the following result, the exact values of the monatic number $\mu(G)$ for some standard graphs G are determined.

Proposition 2.2. .

(1) $\mu(P_n) = 2$, *for every* $n \geq 2$.

(2) $\mu(C_n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3; \\ 2, & \text{otherwise.} \end{cases}$

(3) $\mu(\overline{K_n}) = 1$, *for every* $n \geq 2$.

(4) $\mu(K_n) = \begin{cases} 1, & \text{if } n = 1; \\ 3, & \text{if } n = 3; \\ 2, & \text{otherwise.} \end{cases}$

(5) $\mu(K_{r,s}) = 2$, *for* $1 \leq r \leq s$.

(6) $\mu(F_n) = 3$, *for every* $n \geq 2$.

There are Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|V_1| = |V_2|$, $|E_1| = |E_2|$ and the sequence degrees $S_d(G_1) = S_d(G_2)$, where $S_d(G) = \{d_1, d_2, ..., d_n\}$ and d_i is the degree of vertex in G. But $\mu(G_1) \neq \mu(G_2)$. Figure 1, shows two graphs G_1 and G_2 with $n_1 = n_2 = 7$, $m_1 = m_2 = 9$ and $S_d(G_1) = S_d(G_2) = \{4, 4, 2, 2, 2, 2, 2\}$. But $\mu(G_1) = 3$ and $\mu(G_2) = 2$.

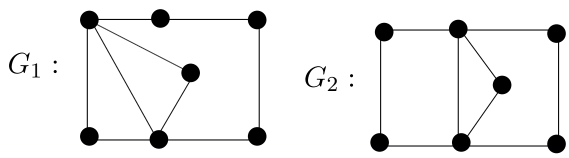

Figure 1

A bipartition (V_1, V_2) of a vertex set $V(G)$ of a graph G is called an unfriendly partition; if every vertex $u \in V_1$ has at least as many neighbors in V_2 as it does in V_1, and every vertex $v \in V_2$ has at least as many neighbors in V_1 as it does in V_2. This type of partition were defined and studied by Borodin et al. [5] and Aharoni et al. [1]. Clearly, for any graph G, if $\mu(G) = 2$, then the idea of M-partitions of a graph G is closely related to unfriendly partitions. Hence, in the following section, we shall focus our attention on the problem of partitioning a graph G into three monopoly sets.

3. Properties of the Monopoly Partite sets of Graphs G having $\mu(G) = 3$

In this section, we study the properties of every monopoly partite set of a graph G having $\mu(G) = 3$, number of edges which incident with every partite set.

Theorem 3.1. *For any graph G, if $\mu(G) = 3$, then every partite set in M-partition of G is an independent set.*

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. On the contrary, suppose, without loss of generality, that M_1 is not an independent. Then there exists, at least, a vertex $v \in M_1$ such that $|N(v) \cap M_1| \geq 1$. Since M_2 is a monopoly set in G and $v \notin M_2$, it follows by definition of a monopoly set that

$$d_{M_2}(v) = |N(v) \cap M_2| \geq \frac{d(v)}{2}. \tag{1}$$

Similarly,

$$d_{M_3}(v) = |N(v) \cap M_3| \geq \frac{d(v)}{2}. \tag{2}$$

Hence, by the definition of the degree of a vertex in a graph G and by equations 1 and 2, we obtain $d(v) = d_{M_1}(v) + d_{M_2}(v) + d_{M_3}(v) \geq d(v) + 1$, a contradiction. Therefore, M_1 must be an independent set. For M_2 and M_3 the proof is similar to the proof of M_1. $\square$

In the following two results, we investigate the sum of the degrees of vertices in every monopoly partite set of a graph G with $\mu(G) = 3$ and the edges which connected between any two monopoly partite sets in M-partition of G.

Proposition 3.1. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G. Then*

$$d_{M_i}(v) = d_{M_j}(v) = \frac{d(v)}{2}$$

for every $v \in M_k$, where i, j and $k \in \{1, 2, 3\}$ and $k \neq i \neq j$.

Theorem 3.2. *Let $\{M_1, M_2, M_3\}$ be the M-partition of G. Then*

$$\sum_{v \in M_i} d(v) = \frac{2m}{3}, \ for\ every\ 1 \leq i \leq 3.$$

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of a graph G. By Theorem 3.1, every partite set M_i and for $1 \leq i \leq 3$, in M-partition of G is an independent. Then $d(v) = |N(v) \cap (V - M_i)| = d_{\overline{M_i}}(v)$, for every $v \in M_i$, $1 \leq i \leq 3$. Also, by Observation 3.1, we have $d_{M_i}(v) = d_{\overline{M_i}}(v)$, for every $v \in \overline{M_i}$.

now, for every $1 \le i \le 3$,

$$\begin{aligned} 2m &= \sum_{v\in V(G)} d(v) = \sum_{v\in M_i} d(v) + \sum_{v\in \overline{M_i}} d(v); \\ &= \sum_{v\in M_i} d_{M_i}(v) + \sum_{v\in M_i} d_{\overline{M_i}}(v) + \sum_{v\in \overline{M_i}} d_{M_i}(v) + \sum_{v\in \overline{M_i}} d_{\overline{M_i}}(v); \\ &= 0 + \sum_{v\in M_i} d_{\overline{M_i}}(v) + 2\sum_{v\in \overline{M_i}} d_{M_i}(v); \\ &= 3\sum_{v\in M_i} d_{\overline{M_i}}(v) = 3\sum_{v\in M_i} d(v). \end{aligned}$$

Therefore, $\sum_{v\in M_i} d(v) = \frac{2m}{3}$, for every $i = 1, 2, 3$. □

For any graph G with $\mu(G) = 3$, Theorem 3.2 shows that the number of edges between any partite set and both the others partite sets in M-partition of G is equal to $\frac{2m}{3}$. In the following result, $m(M_i, M_j)$ denotes the number of edges between M_i and M_j, $i, j \in \{1, 2, 3\}$.

Corollary 3.1. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G. Then*

$$m(M_i, M_j) = \frac{m}{3}, \ \textit{for every } i, j \in \{1,2,3\} \textit{ and } i \ne j.$$

Theorem 3.3. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G such that $|M_1| \le |M_2| \le |M_3|$. Then*

(1) $mo(G) \le |M_1| \le \lfloor \frac{n}{3} \rfloor$;
(2) $|M_1| \le |M_2| \le \frac{n-mo(G)}{2}$;
(3) $\lceil \frac{n}{3} \rceil \le |M_3| \le |M_1||M_2|$.

Proof. Let G be a graph of order n and let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G such that $|M_1| \le |M_2| \le |M_3|$. Then

(1) Clearly that $|M_1| \ge mo(G)$. For the upper bound of $|M_1|$, assume, to the contrary, that $|M_1| \ge \lfloor \frac{n}{3} \rfloor + 1$. Since, $|M_1| \le |M_2| \le |M_3|$, then by the pigeonhole principle, $|M_3| \ge \lceil \frac{n}{3} \rceil$. We have the following Cases.
   Case 1: If $n \equiv 0 \pmod 3$, then $|M_1| \ge \frac{n}{3} + 1$. Hence, by the hypothesis, $n = |M_1| + |M_2| + |M_3| \ge n + 3$, a contradiction.
   Case 2: If $n \equiv 1 \pmod 3$, then $|M_1| \ge \frac{n-1}{3} + 1$ and $|M_3| \ge \frac{n+2}{3}$. Hence, we obtain, $n \ge 2(\frac{n-1}{3} + 1) + \frac{n+2}{3} = n + 2$, a contradiction.
   Case 3: if $n \equiv 2 \pmod 3$, then $|M_1| \ge \frac{n-2}{3} + 1$ and $|M_3| \ge \frac{n+1}{3}$. Hence, we obtain, $n \ge 2(\frac{n-2}{3} + 1) + \frac{n+1}{3} = n + 1$, a contradiction.
   Therefore, $|M_1| \le \lfloor \frac{n}{2} \rfloor$.
(2) Form the hypothesis, we have $|M_1| \le |M_2|$ and the cardinality of M_2 is maximum if and only if $|M_2| = |M_3|$. Since, $|M_2| \le n - (|M_1| + |M_3|)$, it follows that and by the maximality of $|M_2|$,

$$|M_2| \le \frac{n - |M_1|}{2} \le \frac{n - mo(G)}{2}.$$

(3) By the hypothesis and the pigeonhole principle, we get $|M_3| \ge \lceil \frac{n}{3} \rceil$. Since $d_{M_1}(v) \ge 1$, for every $v \in M_3$, it follows that $\sum_{v\in M_3} d_{M_1}(v) \ge |M_3|$ and by Observation 3.1,

$\sum_{v \in M_3} d_{M_1}(v) = \sum_{v \in M_2} d_{M_1}(v)$. Hence,

$$|M_3| \leq \sum_{v \in M_3} d_{M_1}(v) = \sum_{v \in M_2} d_{M_1}(v) \leq \sum_{v \in M_2} |M_1| \leq |M_2||M_1|.$$

□

Corollary 3.2. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G, such that $|M_1| \leq |M_2| \leq |M_3|$. If $|M_1| = 1$, then $|M_2| = |M_3| = \frac{n-1}{2}$. Furthermore, $G = K_3$ or $G \cong F_n$.*

4. Properties of Graphs G having $\mu(G) = 3$

In this section, we investigate the properties of the graphs G having $\mu(G) = 3$ and the relationships between the monatic number of G and some other parameters of G.

Theorem 4.1. *For any graph G, if $\mu(G) = 3$, then G is eulerian.*

Proof. The result is an immediate consequences of Theorem 1.1 and Corollary 2.2. □

Theorem 3.1, shows that for every graph G with $\mu(G) = 3$, every partite set in M-partition of G is independent set. Then we have the following result.

Corollary 4.1. *Every graph G having $\mu(G) = 3$ is a 3-partite graph.*

The converse of the Corollary 4.1, in general, is not true. For example, the complete 3-partite graph $K_{1,2,3}$ has a vertex of odd degree, then by Theorem 2.3, $\mu(K_{1,2,3}) = 2$. In the following result, we characterize each complete 3-partite graph G with $\mu(G) = 3$.

Theorem 4.2. *Let $G = K_{n_1,n_2,n_3}$ a complete 3-partite graph. Then $\mu(G) = 3$, if and only if $n_1 = n_2 = n_3$.*

Proof. Let $G = K_{n_1,n_2,n_3}$ a complete 3-partite graph with partite sets (V_1, V_2, V_3) such that $|V_1| \leq |V_2| \leq |V_3|$. Certainly, If $n_1 = n_2 = n_3$, then every partite set is a monopoly set of G. Thus, $\mu(G) = 3$.
Conversely, let $G = K_{n_1,n_2,n_3}$ a complete 3-partite with $\mu(G) = 3$, and let $\{M_1, M_2, M_3\}$ be the M-partition of G such that $|M_1| \leq |M_2| \leq |M_3|$. We claim that $|M_i| = |V_i|$ for every $i = 1, 2, 3$. Otherwise, there is at least a monopoly partite set $|M_i|$ form M-partition of G, for $i = 1, 2, 3$, such that $M_i \cup V_j$ and $M_i \cap V - V_j$ are not empty sets, for some $j = 1, 2, 3$. Hence, M_i is not independent set, a contradiction. Then the claim is true. Now, assume, without loss the generality, that $n_1 < n_2$. Then, there exists at least a vertex $v \in M_3$ such that $d_{M_2}(v) = |M_2| > |M_1| = d_{M_1}(v)$. Hence, either v of odd degree, a contradiction to Corollary 2.2, or a set M_1 is not a monopoly set of G, once again a contradiction to assumption. This complete a proof. □

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Theorem 4.3. *For any graph G of order n, if $\mu(G) = 3$, then*

$$n \leq m \leq \frac{n^2}{3}.$$

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Then by Corollary 2.2, every vertex in G is of even degree that means $\delta \geq 2$. Then the minimum number of edges in G, if G is a cycle graph hence $m \geq n$. For the upper bound, we denote $m(M_1, M_2)$ to the number of edges between M_1 and M_2. Since $m(M_1, M_2) \leq |M_1||M_2|$, it follows that the maximum value of $m(M_1, M_2)$ is $|M_1||M_2|$. Using calculus we can deduce that $m(M_1, M_2)$ is maximal when $|M_1| = |M_2|$ and Theorem 3.3, M_1 is maximal when $|M_1| = \frac{n}{2}$. Then by Corollary 3.1, $\frac{m}{3} = m(M_1, M_2) \leq \frac{n^2}{9}$. Therefore,, $m = \dfrac{n^2}{3}$. □

These bounds in Theorem 4.3 are sharp. The cycle C_n, for $n \equiv 0 \pmod 3$, gives the lower bound and the complete 3-partite $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$ gives the upper bound.

Proposition 4.1. *For any graph G, if $\mu(G) = 3$, then $\chi(G) \leq 3$.*

Proof. The result is the consequence of Theorem 3.1. □

The bound in Proposition 4.1, is sharp, the cycle graphs C_{3n}, for every n is odd, and the complete 3-partite graphs K_{n_1,n_2,n_3} attending it. The example of graphs G with $\mu(G) = 3$ and $\chi(G) = 2$ is the graphs $G = C_{3n}$, for every n is even. The converse of the Proposition 4.1, in general, is not true. For example, $\chi(C_5) = 2$ but $\mu(C_5) = 2$.

Corollary 4.2. *For any non-bipartite graph G without isolated vertices. If $\mu(G) = 3$, then $\chi(G) = 3$.*

Theorem 4.4. *Let G be a graph with a clique number $\omega(G)$. If $\mu(G) = 3$, then $\omega(G) \leq 3$.*

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Suppose, on the contrary, that $\omega(G) \geq 4$. Then there exists a clique $C \subseteq V(G)$ with vertex set $V(C) = \{v_1, v_2, ..., v_k\}$, $k \geq 4$. Hence, by the pigeonhole principle, there is at least on set from M-partition of G contains at least $\lfloor \frac{k-1}{3} \rfloor + 1$ vertices from $V(C)$. Since, $k \geq 4$ then $\lfloor \frac{k-1}{3} \rfloor + 1 \geq 2$. Hence, there is at least one set form M-partition of G is not independent, a contradiction to Theorem 3.1. Therefore, $\omega(G) \leq 3$. □

The converse of Theorem 4.4, in general, is not true. For example, the Path graph P_n with $\omega(P_n) = 2$, but $\mu(P_n) = 2$.

Theorem 4.5. *For any graph G of order n, if $\mu(G) = 3$, then $\alpha(G) \geq \lceil \frac{n}{3} \rceil$.*

Proof. Let G be a graph of order n and $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Then by the pigeonhole principle, there is at least one set from M-partitions of G contains at least $\lfloor \frac{n-1}{3} \rfloor + 1$ vertices from $V(G)$. Since, by Theorem 3.1, every set in M-partitions of G is an independent set, it follows that $\alpha(G) \geq \lfloor \frac{n-1}{3} \rfloor + 1 = \lceil \frac{n}{3} \rceil$. □

Corollary 4.3. *For any graph G of order n, if $\mu(G) = 3$, then the independence monopoly size, $imo(G)$, of G is defined. Furthermore, $imo(G) \leq \lceil \frac{n}{3} \rceil$.*

The bound in Corollary 4.3, is sharp. The cycle graphs C_{3n}, for every n, is attending it. The converse of Corollary 4.3, in general, is not true. For example, the star graph $K_{1,n}$ has $imo(K_{1,n}) = 1$ but $\mu(K_{1,n}) = 2$. For more details in the independence monopoly size of a graph, we refer the reader to [15].

Theorem 4.6. *Let G be a graph of order n and maximum degree $\Delta(G) = n - 1$. Then $\mu(G) = 3$, if and only if $G = K_3$ or $G \cong F_n$.*

Proof. Certainly, if $G = K_3$ or $G = F_n$, then $\Delta(G) = n - 1$ and $\mu(G) = 3$.
Conversely, Let G be a graph of order n, maximum degree $\Delta(G) = n-1$ and $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Now, let a vertex $v \in V(G)$ with $d(v) = n - 1$ and assume, with loss of generality, that $v \in M_1$. Then by Theorem 3.1, $M_1 = \{v\}$ and by Observation 3.1, $|N(v) \cap M_2| = |N(v) \cap M_3| = \frac{n-1}{2}$.
On the other hand, once again by the Observation 3.1, $|N(u) \cap M_1| = |N(u) \cap M_3| = 1$, for every $u \in M_2$. Hence, $d(u) = 2$ for every $u \in M_2$. Similarly, $d(w) = 2$, for every $w \in M_2$.

Hence, a graph G has only a vertex v with $d(v) = n-1$ and each other vertex with degree two. Therefore, If $n = 3$, then $G = K_3$ and if $n \geq 4$, then $G = F_{\frac{n-1}{2}}$. □

Theorem 4.7. *Let G be a graph having $\mu(G) = 3$. Then $mo(G) \leq \frac{n}{3}$.*

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Since $|M_i| \geq mo(G)$, for every $i \in \{1, 2, 3\}$, it follows that $n = |M_1| + |M_2| + |M_3| \geq 3mo(G)$. Therefore, $mo(G) \leq \frac{n}{3}$. □

This bound is sharp, The cycle graphs C_n, for every $n \equiv 0 \pmod 3$, and a complete 3-partite $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$, attending it.

Corollary 4.4. *For any graph G, $\mu(G) \leq \frac{n}{mo(G)}$.*

It is clear that every graph G of order $n \leq 4$, $G \neq K_3$ has $\mu(G) \leq 2$. In the following result, we study the existences graph G of order $n = k$ having $\mu(G) = 3$ for every positive integer number $k \notin \{1, 2, 4\}$.

Theorem 4.8. *For every positive integer $k \notin \{1, 2, 4\}$, there exists a graph G of order $n = k$ having $\mu(G) = 3$.*

Proof. For $k = 3$ and 5, the result is true, since $G_1 = K_3$ and $G_2 = F_2$ have the required property. Now, we may assume that $k \geq 6$. Then we consider the following cases.

Case 1: If $k \equiv 0 \pmod 3$, then the cycle graph $G_3 = C_k$ is holding the property, since $\mu(C_k) = 3$.

Case 2: If $k \equiv 1 \pmod 3$, let $v_1, v_2, ..., v_k$ be the vertex set of the cycle C_k. Then the graph G_4 which formed from C_k by firstly, removed the edge e_{k-1} which join the vertices v_{k-1} with v_k, then insert three new edges e'_1, e'_2 and e'_3, such that e'_1 join v_1 with v_{k-1}, e'_2 join v_1 with v_{k-2} and e'_3 join v_{k-2} with v_k. Figure 2, shows the graph G_4.

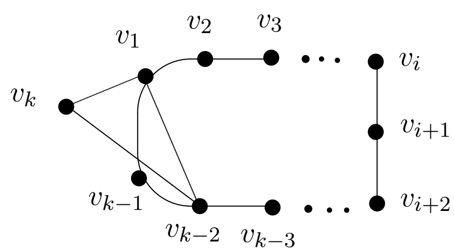

Figure 2: The graph G_4.

Then the partition $\{M_1, M_2, M_3\}$ where $M_1 = \{v_i : i \equiv 1 \pmod 3\} - \{v_k\}$, $M_2 = \{v_i : i \equiv 2 \pmod 3\}$ and $M_3 = \{v_i : i \equiv 0 \pmod 3\} \cup \{v_k\}$ is M-partition of G_4. Indeed, every partite set M_i for $i = 1, 2, 3$ is an independent monopoly set in G_4. Therefore, $\mu(G_4) = 3$.

Case 3: If $k \equiv 2 \pmod 3$, Then the graph G_5 which formed from the cycle C_k by removed the edge e which join the vertices v_{k-2} with v_{k-1} and then insert two new edges e'_1 join v_1 with v_{k-1} and e'_2 join v_1 with v_{k-2}. Figure 3, shows the graph G_5.

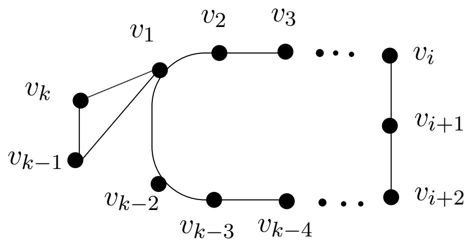

Figure 3: The graph G_5.

Then the partition $\{M_1, M_2, M_3\}$ where $M_1 = \{v_i : i \equiv 1 \pmod 3\} - \{v_{k-1}\}$, $M_2 = \{v_i : i \equiv 2 \pmod 3\}$ and $M_3 = \{v_i : i \equiv 0 \pmod 3\} \cup \{v_{k-1}\}$ is M-partition of G_5. Indeed, every partite set M_i for $i = 1, 2, 3$ is an independent monopoly set in G_5. Therefore, $\mu(G_5) = 3$.

□

Acknowledgement

Our thanks are due to the anonymous referee for careful reading and constructive suggestions for the improvement in the first draft of this paper.

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Ahmed Mohammed Naji was born in Yemen. He got his B.Sc. degree in mathematics and physics (2002) from Taiz University, Taiz, Yemen. He got his M.Sc. degree in mathematics (2013) from King Faisal University, Saudia Arabia. He is a research scholar and a Ph.D candidate in the field of graph theory, DOS in Mathematics, at the University of Mysore, Manasagongatri, Mysuru- 570006, India. He has published 9 papers in the field of graph theory.

Soner Nandappa D was born in India. He received his Ph.D from Gulbarga University, Gulbarga, India in 1999. He has overall teaching experience for more 25 year at PG level. His major research interests are graph theory and fuzzy graph theory. He has successfully guided 14 Ph.D candidates and presently 5 candidates who are working in field of graph theory. He has more than 105 research papers in scientic journals. He is currently working as professor, DOS in Mathematics, University of Mysore, Mysuru-570006, India.