SOLVING FRACTIONAL-ORDER COMPETITIVE LOTKA-VOLTERRA MODEL BY NSFD SCHEMES

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ABSTRACT. In this paper, we introduce fractional-order into a model competitive Lotka-Volterra prey-predator system. We will discuss the stability analysis of this fractional system. The non-standard finite difference (NSFD) scheme is implemented to study the dynamic behaviors in the fractional-order Lotka-Volterra system. Proposed non-standard numerical scheme is compared with the forward Euler and fourth order Runge-Kutta methods. Numerical results show that the NSFD approach is easy and accurate for implementing when applied to fractional-order Lotka-Volterra model.

Keywords: fractional differential equations, Lotka-Volterra model, competitive prey-predator system, non-standard finite difference scheme, stability.

AMS Subject Classification: 37M05, 65L12, 65L20

1. INTRODUCTION

The study of biological systems has been developed through many years. In these systems it is common that state variables represent non-negative quantities, such as concentrations, physical properties, the size of populations and the amount of chemical compounds [11]. These biological models are commonly based on systems of ordinary differential equations (ODEs). Exact solutions of these systems are rarely accessible and usually complicated; hence good approximations are required. The interspecies interaction is among the most intensively explored fields of biology. The increasing amount of realistic mathematical models in that area helps in understanding the population dynamics of analyzed biological systems. Mathematical models of predator-prey systems, characterized by decreasing growth rate of one of the interacting populations and increasing growth rate of the other, consist of the ODEs systems. The current technological advance has made it possible for humans to disturb the environmental balance in nature that may cause immense damages, such as species extinction or starvation. Therefore, understanding the behaviour of the interaction between the species may help biologists and other related parties to prevent those events from happening. The real interaction of prey-predator in nature is complex and comprises both interspecies and external environmental factors. Therefore, several simplifications are usually assumed so that a basic model can be constructed and then developed or modified to approach the real system.

In most of the interactions modeled, all rate of changes are assumed to be time independent, which makes the corresponding systems autonomous. It is not always possible to find

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the exact solutions of the nonlinear models that have at least two ODEs. It is sometimes more useful to find numerical solutions of these type systems in order to programme easily and visualize the results. By applying a numerical method on a continuous differential equation system, it becomes a difference equation system, i.e., discrete time system. While applying these numerical methods, it is necessary that the new difference equation system provide the positivity conditions and exhibit the same quantitative behaviours of continuous system such as stability, bifurcation and chaos. It is well known that some traditional and explicit schemes such as forward Euler and Runge-Kutta are unsuccessful at generating oscillation, bifurcation, chaos and false steady states, despite using adaptive step size [10, 13]. Instead of classical methods, NSFD schemes can alternatively be used to obtain more qualitative results and remove numerical instabilities. These schemes are developed for compensating the weaknesses, such as numerical instabilities that may be caused by standard finite difference methods. The most important advantages of this scheme is that choosing a convenient denominator function instead of the step size h, better results can be obtained. If the step size h is chosen small enough, the obtained results do not change significantly but if the step size h gets larger this advantage comes into focus.

As is well known, in field of mathematical biology, the traditional Lotka-Volterra systems are very important mathematical models which describe multispecies population dynamics in a non-autonomous environment. The Lotka-Volterra equations are a system of differential equations of the following form:

$$\begin{cases} x' = ax - bxy, \\ y' = -cy + dxy, \\ x(0) = x_0, \qquad y(0) = y_0, \end{cases}$$

where x and y are prey and predator, respectively. The parameter a stands for the prey growth rate in the absence of the predators, b is the capture rate of prey per predator, d is the rate at which each predator converts captured prey into predator births and c is the constant rate at which death occurs in the absence of prey. They showed that ditrophic food chains (i.e., prey-predator systems) permanently oscillate for any initial condition if the prey growth rate is constant and the predator functional response is linear. Thus the basic Lotka-Volterra model is represented by the system of the two previous differential equations.

A slight variation of the Lotka-Volterra model, the competition model, is also quite useful in comparing the numerical approximation methods. Let $x(t)$ and $y(t)$ represent the number of individuals or population density in species x and y respectively in a food web at time t, a sensible model for the food web population at time t is competitive Lotka-Volterra system of the form

$$\begin{cases} x' = x(r_1 - a_{11}x - a_{12}y), \\ y' = y(r_2 - a_{21}x - a_{22}y), \\ x(0) = x_0, \qquad y(0) = y_0, \end{cases} \tag{1}$$

where r_1, r_2, a_{11}, $a_{22} > 0$ and a_{12}, $a_{21} \geq 0$. In this model the parameters $r_i, i = 1, 2$ are the intrinsic growth rates for the two species x and y. In addition, a_{12} and a_{21} present the interspecific acting coefficients.

The dynamics of the model is well-known [2] and we will briefly mention the main properties of the system.

In order to get a better analysis for the system, we reduce the number of parameters using the nondimensionalization method as in [11] as follows. Letting

$$u(T) = \frac{a_{11}}{r_1} x(t), \qquad v(T) = \frac{a_{12}}{r_1} y(t),$$

where $T = r_1 t$, consequently we get

$$x'(t) = \frac{r_1^2}{a_{11}} u'(T), \qquad y'(t) = \frac{r_1^2}{a_{12}} v'(T). \qquad (2)$$

Substituting (2) into (1) and renaming T to t, we have

$$\begin{cases} u' = u(1 - u - v), \\ v' = v(A - Bu - Cv), \\ u(0) = u_0, \qquad v(0) = v_0, \end{cases} \qquad (3)$$

where

$$A = \frac{r_2}{r_1}, \qquad B = \frac{a_{21}}{a_{11}}, \qquad C = \frac{a_{22}}{a_{12}},$$

with

$$u_0 = \frac{a_{11}}{r_1} x_0, \qquad v_0 = \frac{a_{12}}{r_1} y_0.$$

This paper is organized as follows: In the next section, we elaborate some basic definitions and properties of the Grünwald-Letnikov (GL) approximation and provides a brief overview of the important feature of the procedures for constructing NSFD schemes for systems of ODEs. In Section 3, fractional-order of the competitive Lotka-Volterra prey-predator model is introduced and also stability theorems are given for the local asymptotic stability of the fractional systems. In Section 4, we will discuss the stability analysis of fractional-order system. In Section 5, we present the idea of NSFD scheme for solving the fractional-order competitive Lotka-Volterra prey-predator system. Finally, the theoretical results obtained in former section are compared with the other numerical methods and the simulated results are given.

2. Preliminaries

Fractional differential equations (FDEs) have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry and engineering [12]. In the recent years, the dynamic behaviors of fractional-order differential systems have received increasing attention. Although the concept of the fractional calculus was discussed in the same time interval of integer-order calculus, the complexity and the lack of applications postponed its progress till a few decades ago. Recently, most of the dynamical systems based on the integer-order calculus have been modified into the fractional order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer-order modeling.

2.1. **Grünwald-Letnikov approximation.** Derivatives of fractional-order have been introduced in several ways. In this paper, we consider GL approach. The GL method for the one-dimensional fractional derivative takes the following form [12]:

$$D^\alpha x(t) = f(t, x(t)), \qquad x(0) = x_0, \qquad t \in [0, t_f], \qquad (4)$$

$$D^\alpha x(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t}{h}\right]} (-1)^j \binom{\alpha}{j} x(t - jh),$$

where $0 < \alpha \leq 1$, D^{α} denotes the fractional derivative and h is the step size and $[\frac{t}{h}]$ denotes the integer part of $\frac{t}{h}$. Therefore, Eq. (4) is discretized as follows:

$$\sum_{j=0}^{n} c_j^{\alpha} x_{n-j} = f(t_n, x_n), \qquad n = 1, 2, 3, ...$$

where $t_n = nh$, x_{n-j} is an approximation of $x(t_{n-j})$ and $c_1, c_2, \ldots, c_n$ are GL coefficients that defined as:

$$c_j^{\alpha} = (1 - \frac{1+\alpha}{j})c_{j-1}^{\alpha}, \qquad c_0^{\alpha} = h^{-\alpha}, \qquad j = 1, 2, 3, \ldots, n.$$

2.2. **Non-standard finite difference schemes.** The initial foundation of NSFD schemes came from the exact finite difference schemes. These schemes are well developed by Mickens [8, 9] in the past decades. These schemes are developed for compensating the weaknesses such as numerical instabilities that may be caused by standard finite difference methods. Regarding the positivity, boundedness and monotonicity of solutions, NSFD schemes have a better performance over the standard finite difference schemes, due to its flexibility to construct a NSFD scheme that can preserve certain properties and structures, which are obeyed by the original equations. The advantages of NSFD schemes have been shown in many numerical applications. Gonzalez-Parra et al. [4] and Arenas et al. [3] developed NSFD schemes to solve population and biological models. Jordan [5] constructed NSFD schemes for heat transfer problems.

To describe NSFD scheme, we consider an ODEs such as

$$x' = f(t, x, \lambda), \qquad x(0) = x_0, \qquad t \in [0, t_f],$$

where λ is a parameter and $f(t, x, \lambda)$ is a nonlinear function. For a discrete-time grid with step size, $\triangle t = h$, we replace the independent variable t by

$$t \approx t_n = nh, \qquad n = 0, 1, 2, \ldots, N$$

where $h = \frac{t_f}{N}$. The dependent variable $x(t)$ is replaced by

$$x(t) \approx x_n,$$

where x_n is the approximation of $x(t_n)$.

The first NSFD requirement is that the dependent functions should be modeled on the discrete-time computational grid. Particular examples of this include the following functions [8, 9].

$$\begin{cases} xy & \approx 2x_{n+1}y_n - x_{n+1}y_{n+1}, \\[2mm] x^2 & \approx x_{n+1}x_n, \\[2mm] x^3 & \approx (\frac{x_{n+1} + x_{n-1}}{2})x_n^2. \end{cases}$$

A standard way for representing a discrete first-derivative is given by

$$x' \cong \frac{x_{n+1} - x_n}{h}.$$

However, the NSFD scheme requires that x' has the more general representation

$$x' \cong \frac{x_{n+1} - x_n}{\phi},$$

where the denominator function, i.e., ϕ has the properties:

i: $\phi(h) = h + O(h^2)$,

ii: $\phi(h)$ is an increasing function of h,

iii: $\phi(h)$ may depends on the parameters appearing in the differential equations.

Examples of denominator function that satisfy condition (i) are

$$h, \qquad \sin(h), \qquad 1 - e^{-h}, \qquad \frac{1 - e^{-\lambda h}}{\lambda},$$

and so forth. The paper of Mickens [8] gives a general procedure for determining $\phi(h)$ for systems of ODEs. An example of the NSFD discretization process is its application to the decay equation

$$x' = -\lambda x,$$

where λ is a constant. The discretization scheme is as follows [8]:

$$\frac{x_{n+1} - x_n}{\phi} = -\lambda x_n, \qquad \phi(h, \lambda) = \frac{1 - e^{-\lambda h}}{\lambda}.$$

Another example is given by

$$x' = \lambda_1 x - \lambda_2 x^2,$$

where the NSFD scheme is

$$\frac{x_{n+1} - x_n}{\phi} = \lambda_1 x_n - \lambda_2 x_{n+1} x_n, \qquad \phi(h, \lambda_1) = \frac{e^{\lambda_1 h} - 1}{\lambda_1}.$$

It should be noted that the NSFD schemes for these two ODEs are exact in the sense that $x_n = x(t_n)$ for all values of $h > 0$. In general, for an ODE with polynomial term

$$x' = ax + (NL), \qquad NL \equiv \text{Nonlinear terms},$$

the NSFD discretization for the linear expression is given by Mickens [8]:

$$\frac{x_{n+1} - x_n}{\phi} = ax_n + (NL)_n,$$

where the denominator function is

$$\phi(h, a) = \frac{e^{ah} - 1}{a}.$$

It follows that if x' is to a function of x which does not have a linear term, then the denominator function is just h, i.e., $\phi(h) = h$.

By applying this technique and using the GL discretization method, it yields the following relations

$$x_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^\alpha x_{n+1-j} + f(t_{n+1}, x_{n+1})}{c_0^\alpha}, \qquad n = 0, 1, 2, \ldots$$

where $c_0^\alpha = \phi(h)^{-\alpha}$.

3. Fractional-Order Competitive Lotka-Volterra system

Now we introduce fractional-order form of system Eqs. (3). The new system is described by the following set of fractional ODEs of order α_1, α_2:

$$\begin{cases} D^{\alpha_1} u(t) = u(1 - u - v), \\\\ D^{\alpha_2} v(t) = v(A - Bu - Cv), \\\\ 0 < \alpha_i \leq 1, \qquad i = 1, 2 \end{cases} \tag{5}$$

with initial conditions

$$u(0) = u_0, \qquad v(0) = v_0.$$

In order to analyze the stability of the model, stability theorems on fractional-order systems and fractional Routh-Hurwitz stability conditions for fractional-order differential equations are introduced. The first stability theorem has been given for incommensurate fractional-order systems.

Theorem 3.1. ([7]) *Consider the incommensurate fractional-order system*

$$D^{\alpha} x(t) = f(x(t)), \qquad x(0) = x_0, \tag{6}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in (0, 1]$ for $i = 1, 2, \ldots, n$ and $x \in \mathbb{R}^n$. The equilibrium points of system Eqs. (6) are calculated by solving the equation:

$$f(x) = 0.$$

These points are locally asymptotically stable if all eigenvalues λ of the Jacobian matrix $J \equiv \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy:

$$|arg(\lambda)| > \alpha^{\star} \frac{\pi}{2}, \qquad \alpha^{\star} = \max(\alpha_1, \ldots, \alpha_n), \qquad i = 1, \ldots, n.$$

Theorem 3.2. ([6]) *Consider the commensurate fractional-order system Eqs. (6), i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha^{\star\star}$. If all eigenvalues of the Jacobian matrix of an equilibrium point satisfy:*

$$|arg(\lambda)| > \alpha^{\star\star} \frac{\pi}{2}, \tag{7}$$

then, the fractional system is locally asymptotically stable at the equilibrium point.

Proposition 3.1. ([6]) *Suppose that $P(\lambda) = \lambda^2 + a_1 \lambda + a_2$ is characteristic polynomial of the Jacobian matrix $\frac{\partial f}{\partial x}$, evaluated at the equilibrium point E. For $0 < \alpha \leq 1$, the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial x}$, satisfy condition (7) in Theorem 3.2 if*

$$a_1 > 0, \qquad a_2 > 0,$$

or

$$a_1 < 0, \qquad 4a_2 > a_1^2, \qquad |tan^{-1}(\frac{\sqrt{4a_2 - a_1^2}}{a_1})| > \alpha^{\star\star} \frac{\pi}{2}.$$

In addition $a_2 > 0$, is a necessary condition for $|arg(\lambda)| > \alpha^{\star\star} \frac{\pi}{2}$.

4. Stability analysis of the equilibrium points

In this section we deal with asymptotically stability of the equilibrium points of system Eqs. (5). The equilibrium points of this system are:

$$E_0 = (0,0), \quad E_1 = (1,0), \quad E_2 = (0, \frac{A}{C}), \quad E_3 = (\frac{A-C}{B-C}, -\frac{A-B}{B-C}).$$

Only the equilibrium points E_1, E_2 and E_3 have real biological meaning. Note that the equilibrium point E_3 has real biological meaning when $C > A > B$. Jacobian matrix of the system Eqs. (5) at the equilibrium point $E = (u,v)$ is

$$J(E) = \begin{pmatrix} 1 - 2u - v & -u \\ -Bv & A - Bu - 2Cv \end{pmatrix}, \tag{8}$$

with the characteristic polynomial P

$$P(\lambda) = \lambda^2 + a_1 \lambda + a_2. \tag{9}$$

The asymptotically stability conditions of these equilibrium points are as follows:

i: The Jacobian matrix of the system Eqs. (5) at the equilibrium point $E_1 = (1,0)$ is

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & A - B \end{pmatrix}. \tag{10}$$

In this case the characteristic equation is

$$P(\lambda) = \lambda^2 + a_1 \lambda + a_2 = 0,$$

where

$$a_1 = 1 - A + B, \qquad a_2 = -A + B.$$

Therefore, the eigenvalues of the Jacobian matrix (10) corresponding to the equilibrium point E_1 are

$$\lambda_1 = -1, \qquad \lambda_2 = A - B.$$

If $A < B$, then these eigenvalues are negative, hence by Theorem 3.2 and Proposition 3.1 a necessary and sufficient condition, for stability the equilibrium point E_1 is $A < B$.

ii: The Jacobian matrix of the system Eqs. (5) at the equilibrium point $E_2 = (0, \frac{A}{C})$ is

$$J(0, \frac{A}{C}) = \begin{pmatrix} \dfrac{C-A}{C} & 0 \\ -\dfrac{AB}{C} & -A \end{pmatrix}. \tag{11}$$

The characteristic equation of the Jacobian matrix J evaluated at the equilibrium point E_2 is

$$P(\lambda) = \lambda^2 + a_1 \lambda + a_2 = 0,$$

where

$$a_1 = \frac{AC + A - C}{C}, \qquad a_2 = \frac{A(A-C)}{C}.$$

Eigenvalues of the Jacobian matrix J at the equilibrium point E_2 are

$$\lambda_1 = -A, \qquad \lambda_2 = -\frac{A-C}{C}.$$

These eigenvalues are negative if $A > C$, hence by Theorem 3.2 and Proposition 3.1 a necessary and sufficient condition, for stability the equilibrium point E_2 is $A > C$.

iii: The Jacobian matrix (8) at the equilibrium point $E_3 = (\frac{A-C}{B-C}, -\frac{A-B}{B-C})$ is

$$J(\frac{A-C}{B-C}, -\frac{A-B}{B-C}) = \begin{pmatrix} -\dfrac{A-C}{B-C} & -\dfrac{(A-C)}{B-C} \\ \dfrac{(A-B)B}{B-C} & -\dfrac{(A-B)C}{B-C} \end{pmatrix},$$

with the characteristic equation

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0,$$

where

$$a_1 = -\frac{-BC - A + AC + C}{B-C}, \qquad a_2 = \frac{(A-B)(A-C)}{B-C}.$$

Note that inequalities $a_1, a_2 > 0$ hold, provided $C > A > B$. We now consider the following two cases:

i: If $4a_2 - a_1^2 \leq 0$ then

$$\lambda_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} < 0, \quad \lambda_2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} < 0.$$

ii: If $4a_2 - a_1^2 > 0$ then

$$\lambda_1 = \frac{-a_1 + i\sqrt{4a_2 - a_1^2}}{2}, \qquad \lambda_2 = \frac{-a_1 - i\sqrt{4a_2 - a_1^2}}{2}.$$

In both cases, since $a_1 > 0$, hence by Theorem 3.2 and Proposition 3.1 a necessary and sufficient condition for stability the equilibrium point E_3 is $C > A > B$.

Theorem 4.1. *Let $0 < \alpha_i \leq 1$ for $i = 1, 2$ then for the equilibrium points E_1, E_2 and E_3 of system Eqs. (5), the following statements hold.*

i: *If $A < B$, then the equilibrium point E_1 is stable.*
ii: *If $A > C$, then the equilibrium point E_2 is stable.*
iii: *If $C > A > B$, then the equilibrium point E_3 is stable.*

5. NSFD SCHEME FOR FRACTIONAL-ORDER COMPETITIVE LOTKA-VOLTERRA MODEL

In this section, we present numerical simulation to illustrate the results obtained in the previous section. By using definition of GL derivative and use NSFD for the model we have:

$$\sum_{j=0}^{n+1} c_j^{\alpha_1} u_{n+1-j} = u_n - u_{n+1}u_n - u_{n+1}v_n,$$

$$\sum_{j=0}^{n+1} c_j^{\alpha_2} v_{n+1-j} = Av_n - Bu_{n+1}v_{n+1} - Cv_{n+1}v_n. \tag{12}$$

Comparing Eqs. (12) with system Eqs. (5), we note that the following statements:
i: The linear and nonlinear terms on the right-hand side of the first equation in system (5) are in the form

$$u \to u_n, \qquad -u^2 \to -u_{n+1}u_n, \qquad -uv \to -u_{n+1}v_n.$$

ii: The linear and nonlinear terms on the right-hand side of the second equation in system (5) are

$$v \to v_n, \qquad -uv \to -u_{n+1}v_{n+1}, \qquad -v^2 \to -v_{n+1}v_n.$$

Manipulating previous discretization we get the following equations

$$u_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} u_{n+1-j} + u_n}{c_0^{\alpha_1} + u_n + v_n},$$

$$v_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} v_{n+1-j} + Av_n}{c_0^{\alpha_2} + Bu_{n+1} + Cv_n}, \tag{13}$$

where

$$c_0^{\alpha_1} = \phi_1(h)^{-\alpha_1}, \qquad\qquad c_0^{\alpha_2} = \phi_2(h)^{-\alpha_2},$$

with [14]

$$\phi_1(h) = e^h - 1, \qquad \phi_2(h) = \frac{e^{Ah} - 1}{A}.$$

Proposition 5.1. *The numerical solutions obtained from system (13) for case $0 < \alpha_i \le 1$, $i = 1, 2$ satisfy*

$$u_n > 0, \ v_n > 0 \Longrightarrow u_{n+1} > 0, \ v_{n+1} > 0, \tag{14}$$

for all relevant values of n.

Proof. Since $c_0^{\alpha_i} > 0$, for $i = 1, 2$ by recursive relation

$$c_j^{\alpha_i} = (1 - \frac{1 + \alpha_i}{j})c_{j-1}^{\alpha_i}, \qquad j = 1, 2, \dots, n$$

we have $c_j^{\alpha_i} < 0$ for $j > 0$, therefore the system Eqs. (13) show that relation (14) is established. For case $\alpha_i = 1, i = 1, 2$ by considering the following system

$$\frac{u_{n+1} - u_n}{\phi_1} = u_n - u_{n+1}u_n - u_{n+1}v_n,$$

$$\frac{v_{n+1} - v_n}{\phi_2} = Av_n - Bu_{n+1}v_{n+1} - Cv_{n+1}v_n,$$

and solving this system in terms of u_{n+1}, v_{n+1} we conclude the relation (14) holds. □

6. Numerical results

Analytical studies always remain in complete without numerical verification of the results. In this section, numerical results from the implementation of NSFD scheme for the fractional-order competitive Lotka-Volterra model are presented. The approximate solutions displayed in Figs. 1 and 2 for the step size $h = 0.5$ and different $0 < \alpha_i \le 1$, for $i = 1, 2$ and parameters $A = 1$, $B = 2$ and $C = 3$ with the initial conditions $u(0) = 2$ and $v(0) = 1$, for simulation time 30s.

The approximate solutions displayed in Figs. 3 and 4 for the step size $h = 0.5$ and different $0 < \alpha_i \le 1$, for $i = 1, 2$ and parameters $A = 2$, $B = 2$ and $C = 1$ with the initial conditions $u(0) = 1$ and $v(0) = 1$, for simulation time 30s.
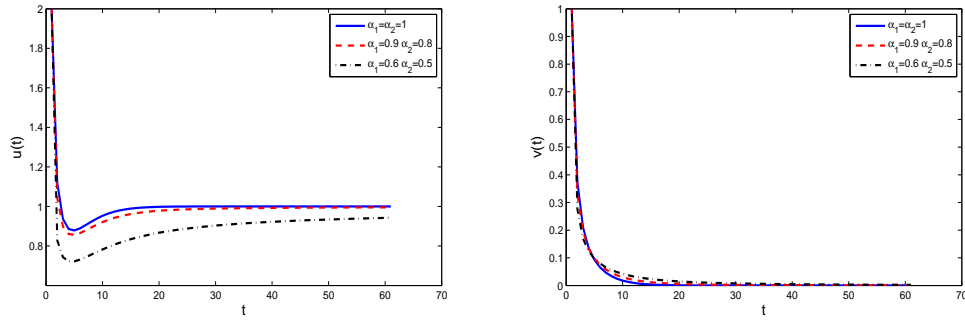
FIGURE 1. Plot of populations u, v over time for different α_1 and α_2 with step size $h = 0.5$.
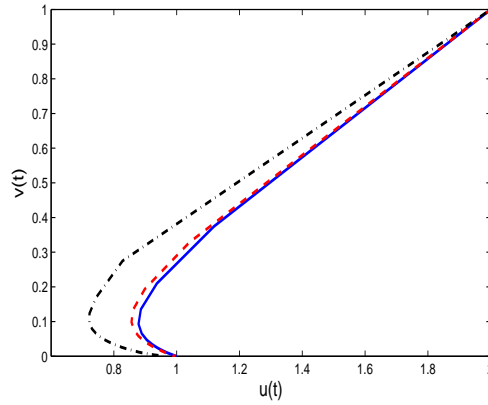

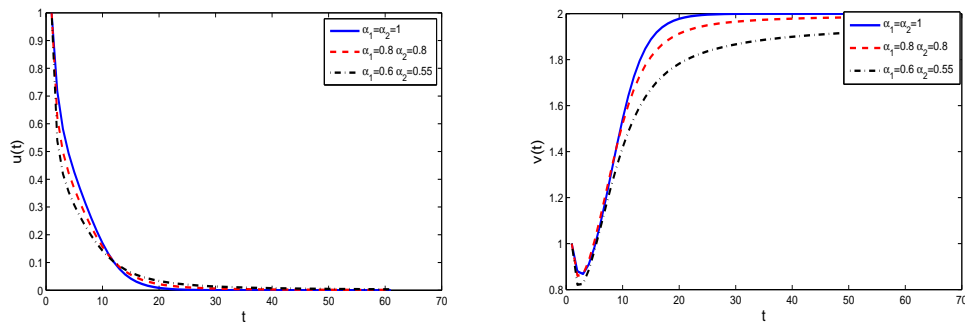FIGURE 2. Plot of populations u, v for different α_1 and α_2 with step size $h = 0.5$.

FIGURE 3. Plot of populations u, v over time for different α_1 and α_2 with step size $h = 0.5$.

The approximate solutions displayed in Fig. 5 and 6 for the step size $h = 1$ and different $0 < \alpha_i \leq 1$, for $i = 1, 2$ and parameters $A = 1$, $B = 0.5$ and $C = 2$ with the initial conditions $u(0) = 1$ and $v(0) = 1$, for simulation time 50s.
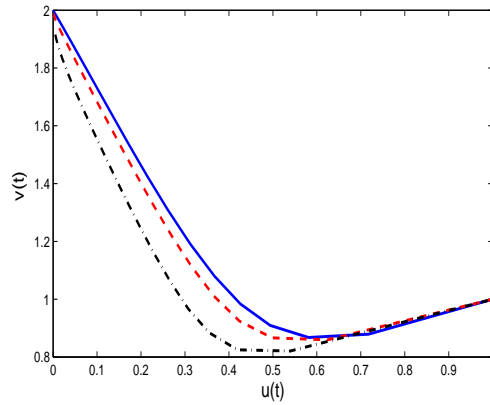
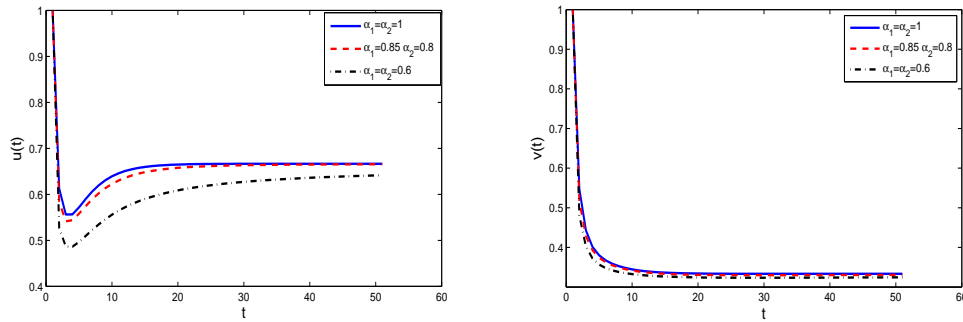FIGURE 4. Plot of populations u, v for different α_1 and α_2 with step size $h = 0.5$.

FIGURE 5. Plot of populations u, v over time for different α_1 and α_2 with step size $h = 1$.
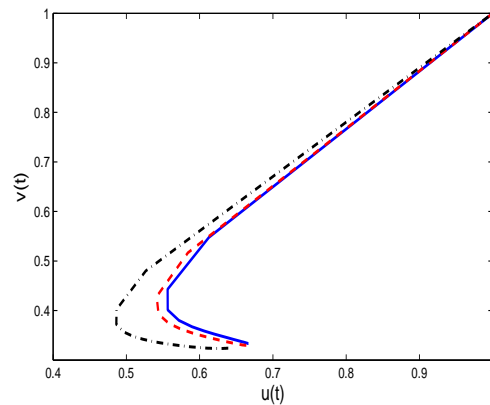

FIGURE 6. Plot of populations u, v for different α_1 and α_2 with step size $h = 1$.

In Tables 1, 2 and 3, for different step sizes h, the qualitative stability results, obtained by NSFD scheme, of the equilibrium points E_1, E_2 and E_3 are respectively compared to classical methods such as forward Euler and fourth order Runge-Kutta methods. If step

size h is chosen small enough, the results of the proposed NSFD scheme are similar with the results of the other two numerical methods, but if the step size h is chosen larger, the efficiency of NSFD scheme is clearly seen. As we observe, the CPU times of the NSFD method is less than the CPU times forward Euler and fourth order Runge-Kutta methods.

TABLE 1. Qualitative results of the equilibrium point E_1 for different time step sizes, $t= 0\text{-}60$.

h	Euler	CPU time	Runge-Kutta	CPU time	NSFD	CPU time
0.001	Convergence	0.011167	Convergence	0.027123	Convergence	0.001486
0.01	Convergence	0.010702	Convergence	0.026794	Convergence	0.001301
0.1	Convergence	0.010605	Convergence	0.025951	Convergence	0.000815
0.3	Divergence	–	Convergence	0.024713	Convergence	0.000302
1	Divergence	–	Divergence	–	Convergence	0.000153
10	Divergence	–	Divergence	–	Convergence	0.000020

TABLE 2. Qualitative results of the equilibrium point E_2 for different time step sizes, $t= 0\text{-}60$.

h	Euler	CPU time	Runge-Kutta	CPU time	NSFD	CPU time
0.001	Convergence	0.001586	Convergence	0.003324	Convergence	0.001820
0.01	Convergence	0.000922	Convergence	0.002973	Convergence	0.000153
0.1	Convergence	0.000868	Convergence	0.002926	Convergence	0.000079
1	Divergence	–	Convergence	0.002682	Convergence	0.000035
1.2	Divergence	–	Divergence	–	Convergence	0.000020
10	Divergence	–	Divergence	–	Convergence	0.000007

TABLE 3. Qualitative results of the equilibrium point E_3 for different time step sizes, $t= 0\text{-}50$.

h	Euler	CPU time	Runge-Kutta	CPU time	NSFD	CPU time
0.001	Convergence	0.001586	Convergence	0.003324	Convergence	0.001820
0.01	Convergence	0.000922	Convergence	0.002973	Convergence	0.000153
0.1	Convergence	0.000868	Convergence	0.002926	Convergence	0.000079
0.7	Divergence	–	Convergence	0.002682	Convergence	0.000035
1	Divergence	–	Divergence	–	Convergence	0.000020
10	Divergence	–	Divergence	–	Convergence	0.000007

In Figs. 7, 8 and 9, the NSFD solutions of u and v converge to equilibrium points E_1, E_2 and E_3 as simulated and also forward Euler and Runge-Kutta methods are compared with NSFD scheme graphically. All the numerical calculations and simulations are performed by using Matlab programme. In conclusion, the efficiency of the proposed NSFD scheme is investigated and compared with other numerical methods.

7. CONCLUSION

In this paper, the fractional form of the competitive Lotka-Volterra model is introduced. The stability of the equilibrium points are studied. Exploiting the NSFD scheme, we study the stability analysis as well as the dynamic behaviour of mentioned system. Moreover the obtained numerical results of NSFD scheme are compared with forward Euler and fourth order Runge-Kutta methods in integer-order case. Numerical results show the accuracy and efficiency of the NSFD scheme.
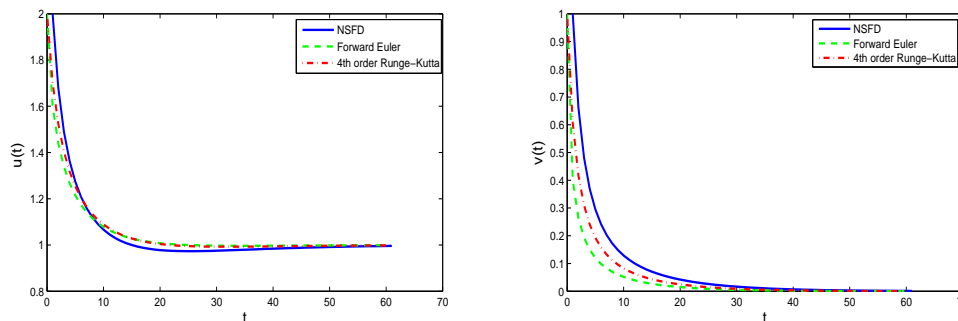
FIGURE 7. Numerical solutions of forward Euler and fourth order Runge-Kutta and NSFD methods converge to the equilibrium point E_1 with step size $h = 0.1$ for $\alpha_1 = \alpha_2 = 1$.
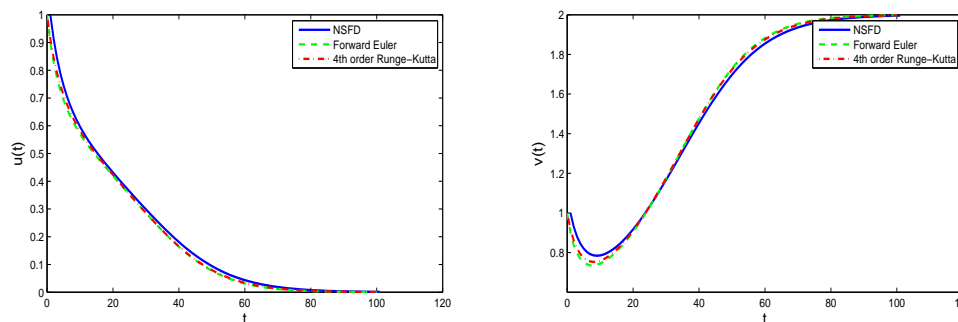

FIGURE 8. Numerical solutions of forward Euler and fourth order Runge-Kutta and NSFD methods converge to the equilibrium point E_2 with step size $h = 0.1$ for $\alpha_1 = \alpha_2 = 1$.
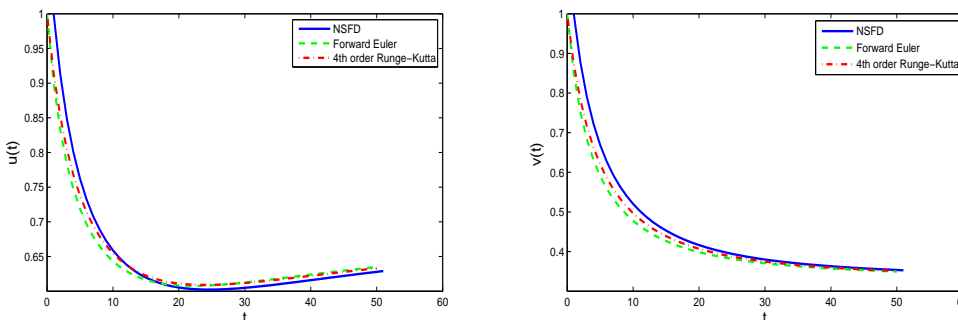

FIGURE 9. Numerical solutions of forward Euler and fourth order Runge-Kutta and NSFD methods converge to the equilibrium point E_3 with step size $h = 0.1$ for $\alpha_1 = \alpha_2 = 1$.

REFERENCES

[1] Ahmed,E., El-Sayed,A.M.A. and El-Saka,H.A.A., (2006), On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems, Phys. Lett. A, pp. 358.
[2] Allen,L.J.S., (2007), An Introduction to Mathematical Biology, Prentice Hall, New Jersey.

[3] Arenas,A.J., Gonzalez-Parra,G. and Chen-Charpentier,B.M., (2010), A non-standard numerical scheme of predictor-corrector type for epidemic models, Comput. Math. Appl., 59(12), pp. 3740-3749.

[4] Gonzalez-Parra,G., Arenas,A.J. and Chen-Charpentier,B.M., (2010), Combination of non-standard schemes and Richardsons extrapolation to improve the numerical solution of population models, Math. Comput. Modelling, 52(7-8), pp. 1030-1036.

[5] Jordan,P.M., (2003), A non-standard finite difference scheme for nonlinear heat transfer in a thin finite rod, J. Difference Equ. Appl., 9(11), pp. 1015-1021.

[6] Matignon,D., (1996), Stability result on fractional differential equations with applications to control processing, Computational Engineering in Systems Applications, pp. 963-968.

[7] Mehmat,A.A., Secer,A. and Bayram,M., (2014), Stability, synchronization control and numerical solution of fractional Shimizu-Morioka dynamical system, Appl. Math. Inf. Sci., 8(14), pp. 1699-1705.

[8] Mickens,R.E., (2007), Calculation of denominator functions for non-standard finite difference schemes for differential equations satisfying a positivity condition, Numer. Methods Partial Differential Equations, 23(3), pp. 672-691.

[9] Mickens,R.E., (2000), Applications of Nonstandard Finite Difference Schemes, Singapore.

[10] Mickens,R.E., (1999), Discretizations of nonlinear differential equations using explicit non-standard methods, J. Comput. Appl. Math., 110, pp. 181-185.

[11] Murray,J.D., (2003), Mathematical Biology I, II, Third edition, Springer.

[12] Podlubny,I., (1999), Fractional Differential Equations, Academic Press, New York.

[13] Roger,L.W., (2004), Local stability of Eulers and Kahans methods, J. Difference Equ. Appl., 10(6), pp. 601-614.

[14] Zibaei,S. and Namjoo,M., (2014), A NSFD scheme for Lotka-Volterra food web model, Iran. J. Sci. Technol. Trans. A Sci., 38(4), pp. 399-414.

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