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Ruibin Qu
Department of Mathematices and Statistics,
Brunel University, Uxbridge, Middx., UB8 3PH, Britain

Abstract

In this paper, an interpolatory subdivision algorithm for surfaces over arbitrary triangulations is introduced and its convergence properties over nonuniform triangulations studied. The so called Butterfly Scheme (interpolatory) is a special case of this algorithm. In our analysis of the algorithm over uniform triangulations, a matrix approach is employed and the idea, of "Cross Difference of Directional Divided Difference" analysis is presented. This method is a generalization of the technique used by Dyn, Gregory and Levin etc. to analyse univariate subdivision algorithms. While for nonuniform data, an extraordinary point analysis is introduced and the local subdivision matrix analysis is presented. It is proved that the algorithm produces smooth surfaces over arbitrary triangular networks provided the shape parameters are kept within an appropriate range.

§1. Introduction

Although subdivision algorithms have been being studied intensively for many years, they have been used for scientists and technicians since long ago. For example, the so called Carpenter's Technique is a very simple algorithm of this type. While the de Rahm's "Trisection Algorithm" (1947), the de Casteljau's Algorithm (1959) for the Bernstein-Bézier curves and the Chaikin's Algorithm (1974) for curves are subdivision algorithms which contribute much to the rapid development and investigation of this type of algorithms. Recently, a lot of work has been done in this area to study subdivision algorithms systematically. This includes the works by Dyn, Gregory, Levin, Dahmen, Micchelli, Cavaretta, Daubechies and Largarias ... etc. And Interpolatory Subdivision algorithms play a very important role in these applications. Our work in this area is to investigate explicit conditions under which a subdivision algorithm could produce smooth

surfaces with certain prescribed properties such as interpolatory and monotonicity. That is, we try to generalize the Dyn-Gregory-Levin's uniform analysis (cf. [11]) for univariate subdivision algorithms to the surface case.

In this paper, we report briefly some of our work on this subject. We firstly introduce a general interpolatory subdivision algorithm for surfaces over arbitrary triangulations and then present a convergence analysis of a 10-point Interpolatory Scheme for surfaces over arbitrary triangulations. The Butterfly Scheme, which is a 8-point interpolatory scheme is a special case of the algorithm. In the analysis, we use a matrix approach and hence the idea of "Cross Difference of Directional Divided Difference" analysis is introduced. This method is a generalization of the "Diadic Parametrization" technique used by Dyn, Gregory and Levin (cf. [11,13,14]). which was firstly used to analyse uniform subdivision algorithms for curves. It is proved that the algorithm produces smooth surfaces provided the shape parameters are kept within an appropriate range and an explicit condition for this is also provided. From this condition, it can be seen clearly that the Butterfly Scheme cannot guarantee generating a smooth surface over an arbitrary triangulation which can also be shown by graphic examples. More details about the analysis can be found in [18]. Other analyses of uniform subdivision algorithms can also be found in [2,5,6,16,17,etc.].

This algorithm has wide practical applications. For example, it can be used to solve interpolatory-type surface fitting problem, or reversely, it can be employed to simplify problems like data reduction. It is also hoped that subdivision algorithms could be applied successfully in some optimization problems such as optimized data-transmission and wavelets processing etc.

§2. Mathematical Description of the Scheme and its Basic Properties

The construction of the scheme is, originally, motivated by the ideas described in papers by Dubec (cf. [10]), Dyn, Gregory and Levin (cf. [11,12]). The scheme is formulated in order to solve such problems as high accuracy surface fitting and fast surface representation. Thus, the aim is to generalize the "4-point interpolatory subdivision scheme" described in [10,11] for surfaces. The scheme is so constructed that it preserves the advantages of the "4-point scheme". The main property of the scheme, in addition to the properties of general uniform subdivision schemes, is its generation of smooth interpolatory surfaces and the reproductivity of cubic parametric polynomial surfaces provided that the shape parameters are chosen within an appropriate range.

A mathematical description of a uniform subdivision scheme over uniform triangulations, which is also called Binary Subdivision Algorithm, is as follows. Suppose that the initial "control points" of a uniform triangular net work are denoted by $P_\alpha^0, \alpha \in Z^2$, then, the refined control points $P_\alpha^{k+1}, \alpha \in Z^2, k \geq 0$, are obtained from $P_\alpha^k, \alpha \in Z^2$, recursively by the following formula ("Mask"):

$$P_{\alpha}^{k+1} = \sum_{\beta \in Z^2} \alpha_{\alpha-2\beta} P_{\beta}^{k}, \alpha \in Z^2. \tag{2.1}$$

An equivalent form of this expression is

$$P_{\gamma+2\alpha}^{k+1} = \sum_{\beta \in Z^2} \alpha_{\gamma} - 2_{\beta} P_{\alpha+\beta}^{k}, \alpha \in Z^2 \tag{2.2}$$

where, $\gamma =: (\gamma_1, \gamma_2)$ and $\gamma_i = 0$ or 1, i = 1,2. Thus, the scheme is interpolatory if and only if

$$a_{\alpha} = \delta_{\alpha,o,} \forall \alpha \in Z^2. \tag{2.3}$$

Equation (2.2) shows clearly that the scheme is a 4-step subdivision scheme which can be described by the following

$$\begin{cases} P_{2i,2j}^{k+1} & = \sum_{m,n} a_{m,n} P_{i+m,j+n}^{k} \\ P_{2i+1,2j}^{k+1} & = \sum_{m,n} b_{m,n} P_{i+m,j+n}^{k} \\ P_{2i,2j+1}^{k+1} & = \sum_{m,n} c_{m,n} P_{i+m,j+n}^{k} \\ P_{2i+1,2j+1}^{k+1} & = \sum_{m,n} d_{m,n} P_{i+m,j+n}^{k}. \end{cases} \tag{2.4}$$

The 10-point scheme is given by the following choice of the coefficients in (2.4):

$$\begin{cases} a_{0,0} = 1,\ b_{-1,0} = b_{2,0} = w_3, b_{0,-1} = b_{1,1} = w_2, \\ b_{0,0} = b_{1,0} = \frac{1}{2} - 2w_1 - w_2 - w_3, b_{-1,-1} = b_{1,-1} = b_{0,1} = b_{2,1} = w_1, \\ c_{0,-1} = c_{0,2} = w_3, c_{-1,0} = c_{1,1} = w_2, \\ c_{0.0} = c_{0,1} = \frac{1}{2} - 2w_1 - w_2 - w_3, c_{-1,-1} = c_{-1,1} = c_{1,0} = c_{1,2} = w_1, \\ d_{-1,-1} = d_{2,2} = w_{3,} d_{1,0} = d_{0,1} = w_2, \\ d_{0,0} = d_{1,1} = \frac{1}{2} - 2w_1 - w_2 - w_3, d_{0,-1} = d_{-1,0} = d_{1,2} = d_{2,1} = w_1 \end{cases} \tag{2,5}$$

where, w_i ,i= 1,2,3, are three (tension) parameters. This special choice of the coefficients comes from the 3-D symmetric structure of the scheme. In fact, there is a simpler way to describe the scheme which uses only a single formula (only one 'Mask') to characterize the scheme. The formula is given below (cf. Figure 1). This is due to the 3-direction-symmetry property of the scheme. Since the scheme is interpolatory, only the inserted values are to be evaluated. The formula for an inserted point, P_o, is given by

$$P_o = \frac{1}{2}(P_e + P_f) + w_1(P_a + P_c + P_h + P_j - 2P_e - 2P_f) \tag{2.6}$$

$$+ w_2(P_b + P_i - P_e - P_f) + w_3(P_d + P_g - P_e - P_f)$$

where, o is the midpoint of the edge joining the vertices e and f, see Figure 1. From this construction, it is obvious that the scheme can be used (possibly, with some modification at those so called *Extraordinary Points*) to produce surfaces over arbitrary triangulations (cf. [18]).

In the uniform subdivision process, formula (2.6) is used to evaluate all "midpoint" values to produce a refined uniform triangular control net in which the triangulation of the refined control nets is formed by the "standard 3-D meshing rule" which will be explained later in our convergence analysis. Repeated applications of this process will therefore result in finer and finer control nets. Moreover, further studies show that if the shape parameters $\{w_i\}$ are chosen appropriately, the scheme will produce smooth interpolatory surfaces. This will be discussed in the next section.

It can be shown that the scheme has the following properties.

1. *The scheme is interpolatory.*
2. *The parameters* $\{w_i\}$ *work as tension controls along the three mesh directions respectively.*
3. *The scheme reproduces linear surfaces for all* $\{w_i\}$.
4. *The scheme reproduces bivariate cubic parametric surfaces if* $\{w_i\}$ *satisfy*

$$w_1 = \mathrm{t} - \frac{9}{16}, w_2 = -2\mathrm{t} - \frac{18}{16} = 2w_1, w_3 = \frac{1}{2} - \mathrm{t} \tag{2.7}$$

where, t is any real number.

5. *The scheme reduces to the Butterfly Scheme [cf. 11] if the parameters satisfy*

$$w_1 =: w, w_2 = -2w, w_3 = 0. \tag{2.8}$$

6. *The scheme has certain data-dependent shape preserving properties.*
7. *The scheme produces smooth surfaces if the shape control parameters are chosen properly. This will be discussed later.*

§3. Some Covergence Results of the Scheme

To study the convergence property of the subdivision algorithm over arbitrary triangulations and the property of the surfaces produced by it, a definition of convergence of subdivision algorithms and a parametrization of the surfaces as well should be be introduced. By contrast to the univariate case, uniform convergence and the "dyadic parametrization" are natural choices for uniform triangulations. The "diadic parametrization" means that for uniform triangulations, the control points P^k_α, $\alpha \in Z^2$, $k \geq 0$, are parametrized at the "diadic points": $2^{-k}\alpha$, $\alpha \in Z^2$ in the parameter plane, e.g., the u-v plane. So, if we define

$$(u^k, v^k) := 2^{-k}\alpha,\ \forall \alpha \in Z^2, \tag{3.1}$$

then, the control net, which is defined by P^k_α, $\alpha \in Z^2$, $k \geq 0$, can be regarded as the unique piecewise linear interpolant $P^k(u,v)$ from the uniform 3-D meshed u-v plane, which is produced by mesh directions (0,1),(1, 0) and (1,1), to R^3 satisfying

$$P^k(u^k, v^k) = P^k_\alpha,\ \alpha \in Z^2. \tag{3.2}$$

Hence, the convergence of the scheme can be defined as the convergence of the continuous surface sequence $\{P^k(u,v)\}$. So we say the scheme is convergent if for any initial dada, there is a continuous surface $P(u,v)$ such that

$$\lim_{k\to\infty} P^k(u,v) = P(u,v),\ \forall u, v \in R. \tag{3.3}$$

If we assume here that the initial data are just real numbers and that they are function values on the uniform integer grid (i,j), $i,j \in Z$ in the u-v plane. Then at lever k, the control point values $P^k_\alpha, \alpha \in Z^2$, will be the function values at a refined grid $2^{-k}(i,j), (i,j) \in Z^2$ since the diadic parametrization is assumed. By meshing the control nets $P^k_{i,j}$ in the same way as the uniform grid 2^{-k} (i,j) in the u-v plane, the 10-point scheme can then be written in the following compact form:

$$\begin{cases} P^{k+1}_{2i,2j} & = P^k_{i,j} \\ P^{k+1}_{2i+1,2j} & = (\frac{1}{2} - 4w_1 - 2w_2 - 2w_3)(P^k_{i,j} + P^k_{i+1,j}) \\ & = +w_1(P^k_{i-1,j-1} + P^k_{i+1,j-1} + P^k_{i,j+1} + P^k_{i+2,j+1}) \\ & = +w_2(P^k_{i,j-1} + P^k_{i+1,j+1}) + w_3(P^k_{i-1,j} + P^k_{i+2,j}) \end{cases} \tag{3.4}$$

with $P^{k+1}_{2i,2i+1}$ and $P^{k+1}_{2i+1,2i+1}$ being duals of the second equation. Now the forward difference operators $\{\Delta_i\}$, $i = 1, 2, 3$, along the mesh directions can be defined:

$$\begin{cases} \Delta_1 & =: P^k_{i+1,j} - P^k_{i,j} \\ \Delta_2 & =: P^k_{i,j+1} - P^k_{i,j} \\ \Delta_3 & =: P^k_{i+1,j+1} - P^k_{i,j}. \end{cases} \tag{3.5}$$

Using the above notion, the following convergence results are obtained (cf.[18,19j):

Theorem 3.1. *The scheme produces C° surfaces if the parameters $\{w_i\}$ satisfy*

$$\begin{cases} |\frac{1}{2} - 2w_1 - w_2| + 2|w_2| + 2|w_3| + |w_1 - w_3| & < 1 \\ 4|w_1| + 2|w_2| + 2|w_3| & < \frac{1}{2} \end{cases} \tag{3.6}$$

A simple symmetric solution to (3.6) is given by

$$5|w_1| + 3|w_2| + 3|w_3| \;<\; \tfrac{1}{2}. \tag{3.7}$$

Remark 3.2. *For the cubic precision scheme, (3.6) becomes*

$$\frac{1}{2} < t < \frac{37}{64}. \tag{3.8}$$

Remark 3.3. *Other conditions for C° convergence can also be obtained (cf. [*18*]).*

In order to prove that the scheme produces C^1 surfaces, the Cross Differences of *the Directional Divided Differences, CDD*, of the control nets are introduced and studied. This process is similar to the *Divided Difference Analysis* of univariate subdivision schemes described in [2,3,10,ll,etc.].

The *CDD* at lever k along mesh direction m and n, m,n = 1,2, 3, $m \neq$ n, is defined as follows:

$$C^k_{i,j,m,n} =: 2^k \Delta_m \Delta_n P^k_{i,j}, \qquad \forall i,j \in Z. \tag{3.9}$$

Since the scheme is symmetric, we only need to study one type of *CDD*. Hence, without loss of generality, we assume that in (3.9), m = 1, n = 2 and the subscripts m and n will be omitted in our future discussion.

From the subdivision process (3.4) and definition (3.9), one can show that if

$$w_2 = -2\, w_1, \tag{3.10}$$

then all these *CDD* terms will satisfy the following refinement equations:

$$\begin{cases}
C^{k+1}_{2i,2j} & = 2w_1 C^k_{i-1,j} - (4w_1 - 2w_3) C^k_{i-1,j-1} + 2w_3 C^k_{i,j+1} \\
& \quad + (1+8w_1) C^k_{i,j} + 2w_1 C^k_{i,j-1} \\
& \quad + 2w_3 C^k_{i+1,j+1} + 2w_3 C^k_{i+1,j} \\
C^{k+1}_{2i+1,2j} & = (2w_1 - 2w_3) C^k_{i-1,j-1} - 8w_1 C^k_{i,j} \\
& \quad - (2w_1 - 2w_3) C^k_{i,j+1} + (2w_1 - 2w_3) C^k_{i+1,j+1} \\
& \quad - (2w_1 - 2w_3) C^k_{i+1,j} \\
C^{k+1}_{2i+1,2j+1} & = (2w_1 + 2w_3) C^k_{i-1,j} + (2w_1 - 2w_3) C^k_{i-1,j-1} \\
& \quad - (2w_1 - 2w_3) C^k_{i-1,j+1} - 8w_1 C^k_{i,j} \\
& \quad + (2w_1 - 2w_3) C^k_{i+1,j+1} \\
C^{k+1}_{2i+1,2j+1} & = 2w_3 C^k_{i-1,j} - 2w_3 C^k_{i-1,j-1} + 2w_1 C^k_{i,j+1} \\
& \quad + (1+8w_1) C^k_{i,j} + 2w_3 C^k_{i,j+1} \\
& \quad - (4w_1 - 2w_3) C^k_{i+1,j+1} + 2w_1 C^k_{i+1,j}
\end{cases} \tag{3.10}$$

By applying this recursive relation repeatedly, the following result is obtained:

Theorem 3.4. *There exists a constant $B(w_1, w_3)$, which is a piecewise quadratic function of w_1 and w_3, such that*

$$C_d^{k+2} \le B(w_1, w_3) C_d^k, \qquad \forall k \ge 0, \tag{3.12}$$

where

$$C_d^k =: \max_{i,j,m,n,m\neq n} \left| C_{i,j,m,n}^k \right| \tag{3.13}$$

and $B(w_1, w_3) < 1$ provided that the shape parameters w_1 and w_2 satisfy

$$\begin{cases} w_2 + 2w_1 & = 0 \\ w_1 & \neq 0 \\ w_1 + 7w_3 & \le 0 \\ 8(w_1 + 0.07) - 3(w_3 - 0.01) & \ge 0 \\ (w_1 + 0.10) + (w_3 + 0.07) & \ge 0 \\ 10w_1 - 7w_3 & \le 0 \end{cases} \tag{3.14}$$

Remark 3.5. *Condition (3.14) means that the parameters $\{w_i\}$ should lie in a polygonal region Ω in the plane $\pi : w_2 = -2\, w_1$. The region $\Omega \in R^3$* is *depicted in Figure* 2.

Theorem 3.6. *The 10-point scheme produces C^1 surfaces if the shape parameters satisfy* (3.14). ■

Corollary 3.7. *The cubic precision scheme produces smooth surfaces if the tension parameter t satisfy*

$$\frac{49}{100} \le t \le \frac{54}{100}. \ ■ \tag{3.15}$$

Remark 3.8. *Condition* (3.14) *is a simple one. Other C^1 convergence conditions may also be obtained* (cf. [18]).

§4. The Scheme over Arbitrary Triangulations

In this section, we study the 10-point scheme over nonuniform triangulations. Our main result is that the limit surface is smooth even at the extraordinary points provided that the scheme is modified properly at these points. In particular, these results are valid for the butterfly scheme. The analyses of the scheme here are different from the previous analyses of the scheme over uniform data. In fact, the analysis to be presented here is an extraordinary point analysis. The Block-Circulant Matrix theory is used here. This technique is quite suitable for the nonuniform analysis.

4.1. Generalization of the Scheme to Arbitrary Triangulations

Since nonuniform triangular control polyhedrons often arise in practice, it is significant to investigate the behaviour of the scheme over nonuniform triangular networks. From its construction, we know that the scheme can be easily ajusted to refine triangular networks which leads to the generation of surfaces on arbitrary triangular networks. Depending on the local topology (more explicitly, the valances of the vertices), the 10-point scheme can be easily generalized to arbitrary triangular networks in the following way.

Before describing the modified scheme, we introduce some conventions. In the following formulae, the index i is a cyclic integer in the range: $i = 0, 1, 2, \ldots, n-1, n$. Here, n is the valance of the vertex. For simplicity, it is also assumed in scheme that the cubic precision parameters are used. That is, the parameters $\{w_i\}$ satisfy (2.7).

For simplicity, we assume also, without loss of generality, that the initial data is locally uniform except one extraordinary point V. (In fact, this situation can be achieved locally after the first subdivision.) Let V, P_i^k, Q_i^k, R_i^k denote the corresponding refined control points near the vertex at lever k, then, the local scheme can be described as follows.

Case I, $n = 2$.

In this case, there are several alternative choices that can be used. One of them is described by the following (Figure 3). For i = 0, 1, 2, ..., n-1, n, we have the following subdivision process:

$$\begin{cases} P_i^{k+1} = w_4 V + (w_1 + w_4)P_i^k + (w_1 + w_3)Q_i^k + w_2 R_i^k + w_1 P_{i+1}^k \\ \qquad + w_3 R_{i+1}^k + w_1 P_{i-1}^k + w_2 R_{i-1}^k, \\ Q_i^{k+1} = P_i^k, \\ R_i^{k+1} = w_2 V + w_4 P_i^k + w_1 Q_i^k + w_2 R_i^k + w_4 P_{i+1}^k + w_1 Q_{i+1}^k \\ \qquad + w_3 P_{i+1}^k + 2w_1 P_{i-1}^k + w_3 R_{i-1}^k \end{cases} \tag{4.1}$$

where t is the local tension control, and $\{w_i\}$ are defined by

$$\begin{cases} w_1 = t - \dfrac{9}{16}, \\ w_2 = -2t + \dfrac{9}{8} = -2w_1, \\ w_3 = \dfrac{1}{2} - t, \\ w_4 = t. \end{cases} \tag{4.2}$$

Case II, $n \geq 3$

In this case, the scheme is just the utterfly scheme That is, using the butterfly

formula everywhere. Since in this case the scheme also produces surfaces (to be proved later), it is not necessary to construct more complicated schemes at this vertex although some other schemes may also be used. In fact, a cubic precision scheme can be constructed but the the coefficients of the formulae are quite complicated. The scheme is like this. Applying the butterfly scheme near the extraordinary point we obtain the following subdivision formulae (Figure 4)

$$\begin{cases} P_i^{k+1} = \frac{1}{2}V + \frac{1}{2}P_i^k + wR_i^k - 2wP_{i+1}^k + wP_{i+2}^k - 2wP_{i-1}^k \\ \qquad + wR_{i-1}^k + wP_{i-2}^k, \\ Q_i^{k+1} = P_i^k, \\ R_i^{k+1} = -2wV + \frac{1}{2}P_i^k + wQ_i^k - 2wR_i^k + \frac{1}{2}P_{i+1}^k + wQ_{i+1}^k \\ \qquad + wP_{i+2}^k + wP_{i-1}^k \end{cases} \tag{4.3}$$

where w is the local tension control.

4.2. The. Subdivision Matrix at the Extraordinary Point

Writing (4.1) and (4.3) in a matrix form, we obtain:

$$\begin{pmatrix} P_i^{k+1} \\ Q_i^{k+1} \\ R_i^{k+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & w \\ 1 & 0 & 0 \\ \frac{1}{2} & 0 & w_2 \end{pmatrix} . \begin{pmatrix} P_i^k \\ Q_i^k \\ R_i^k \end{pmatrix} + \begin{pmatrix} w_2 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & w & 0 \end{pmatrix} . \begin{pmatrix} P_{i+1}^k \\ Q_{i+1}^k \\ R_{i+1}^k \end{pmatrix}$$

$$+ \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & 0 \\ w & 0 & 0 \end{pmatrix} . \begin{pmatrix} P_{i+2}^k \\ Q_{i+2}^k \\ R_{i+2}^k \end{pmatrix} + \begin{pmatrix} w_2 & 0 & w \\ 0 & 0 & 0 \\ w & 0 & 0 \end{pmatrix} . \begin{pmatrix} P_{i-2}^k \\ Q_{i-2}^k \\ R_{i-2}^k \end{pmatrix} \tag{4.3}$$

$$+ \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} P_{i-1}^k \\ Q_{i-1}^k \\ R_{i-1}^k \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ w_2 \end{pmatrix} . V.$$

From this expression, we introduce the following basic matrices:

$$C_0 := \begin{pmatrix} \frac{1}{2} & 0 & w \\ 1 & 0 & 0 \\ \frac{1}{2} & w & w_2 \end{pmatrix}, \quad C_1 := \begin{pmatrix} w_2 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & w & 0 \end{pmatrix}, \quad C_2 := \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & 0 \\ w & 0 & 0 \end{pmatrix}, \tag{4.4}$$

$$C_3 := \begin{pmatrix} w_2 & 0 & w \\ 0 & 0 & 0 \\ w & 0 & 0 \end{pmatrix}, \quad C_4 := \begin{pmatrix} w & 0 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the control point vector:

$$\mathrm{F}^k := (v, P_0^k, Q_0^k, R_0^k, P_1^k, Q_1^k, R_1^k, P_2^k, Q_2^k, R_2^k, ..., P_n^k, Q_n^k, R_n^k)^t \tag{4.4}$$

Here, F^k is a vector of length 3(n+1)+1. Thus, the subdivision process (4.1) and (4.3) at V can be written in a more compact form:

$$F^{k+1} = A.F^k, \qquad k = 0,1,2,..... \tag{4.6}$$

where, A is called the local subdivision matrix. More explicitly the matrix is given in the form

$$A := \begin{pmatrix} 1 & 0 \\ a & A' \end{pmatrix}. \tag{4.7}$$

Here, a is a vector of length 3(n+l) and A' is a *Block Circulant Matrix* defined by

$$A' = Bcirc(A_0, A_1, A_2, ..., A_{n-1}, A_n)$$

$$:= \begin{pmatrix} A_0 & A_1 & A_2 & \dots & A_{n-1} & A_n \\ A_n & A_0 & A_1 & \dots & A_{n-2} & A_{n-1} \\ A_{n-1} & A_n & A_0 & \dots & A_{n-3} & A_{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_1 & A_2 & A_3 & \dots & A_n & A_0 \end{pmatrix} \tag{4.8}$$

and $\{A_i\}$ are some 3 by 3 matrices defined by $\{C_i\}$ and $\{w_i\}$. In fact, for n=3, we have

$$\begin{cases} A_0 := C_0, \\ A_1 := C_1, \\ A_2 := C_2 + C_4, \\ A_3 := C_3 \end{cases} \tag{4.9}$$

and for n=4, we have

$$\begin{cases} A_0 := C_0, \\ A_1 := C_1, \\ A_2 := C_2, \\ A_i := 0, \quad i = 3,4,.....,n-2, \\ A_{n-1} := C_3, \\ A_n := C_4. \end{cases} \tag{4.10}$$

Now, we have constructed the subdivision matrix upon which the properties of the limit surfaces depend. Next we will study the convergent properties of the modified schemes at the extraordinary point.

4.3. The Spectrum Analysis of the Subdivision Matrix

In order to study the C^0 and C^1 properties of the scheme over arbitrary triangulations, it is sufficient to prove that the limit surface of the schemes C^0 or C^1 is at the extraordinary point since the limit surface is C^1 everywhere else provided that the tension parameter satisfies (3.14). Since the eigen-properties of the subdivision matrix play a very important role in the convergence analysis (cf. [1,12,20]), we first study the eigen-properties of the subdivision matrix A. It should be stressed that eigenvalues and their corresponding eigenvectors of A can be evaluated analytically since the matrix is a *Block-Circulant Matrix* composed of 3 by 3 sub-matrices, therefore these eigenvalues are roots of cubic polynomials.

Let the eigenvalues and their corresponding (generalized) eigenvectors of A be denoted by $\{\lambda_i, v_i\}$, where $|\lambda_{i,}| \geq |\lambda_{i+1}|$ for all $i \geq 1$. Then, we can obtain the following result (cf. [20]).

Theorem 4.1. *The subdivision matrix* A has the following properties:

$$\begin{cases} \lambda_1 = 1, & v_1(1,1,1,....,1)^t \\ |\lambda_i| < 1, & i = 2,3,.....,3n+3,3n+3,3n+4 \end{cases} \tag{4.11}$$

Provided that

$$\begin{cases} 0.3125 < t < 0.6000, & for \quad n = 2 \\ -\dfrac{1}{12} < w < 0, & for \quad n \geq 3. \end{cases} \tag{4.12}$$

Furthermore, We have

$$\begin{cases} 0 < \lambda_2 = \lambda_3 < \lambda_1, \quad |\lambda_i| < \lambda_2 \quad i \geq 4, \\ \qquad \text{dim span}\,\{v_2, v_3\} = 2 \end{cases} \tag{4.13}$$

if

$$\begin{cases} 0.5275 < t < 0.5500, & for \quad n = 2, \\ -\dfrac{1}{12} < w < 0, & for \quad \mathrm{n} \geq 3. \end{cases} \tag{4.14}$$

Remark 4.2. Tlae *eigenvalue* λ_2 *is a double root* of A *and has two linearly independent eigenvectors. This can be shown explicitly by using the Block Circulant Matrix theory (cf. [20]) or the Fourier Transform Technique (*cf. *[1]).*

4.4. The Convergence Analysis

In this section, we will prove that the limit surface has tangent plane continuity at the extraordinary point. Thus, the surface is smooth everywhere. Firstly, we have the following C^0 and C^1 convergence results that can be proved in a similar way as in in the uniform case. For details see reference [20].

Theorem 4.3. *The limit* surface is C^0 *if (4.12) holds.* ■

Theorem 4.4. *The limit* surface is C^1 *if (4.14) holds.* ■

Remark 4.5(cf. *[20]). The limit surface of the interpolatory scheme has a unique tangent plane at the extraordinary point, that is the surface is* C^1 if *the subdivision matrix has following properties:*

$$\begin{cases} (i).\ \ \lambda_1 = 1, \quad v_1(1,1,....,1)^t; \\ (ii)\ \ 0 < \lambda_2 = \lambda_3 < 1, \text{ dim span } \{v_2,\ v_3\} = 2; \\ (iii)\ \ |\lambda_i| < \lambda_2, \quad i = 4,5,....,3n+4. \end{cases} \tag{4.15}$$

It can also be shown that a necessary condition for the limit surface to have a unique tangent plane at the extraordinary point is:

$$\begin{cases} (i).\quad \lambda_1 = 1, \quad v_1 = (1,1,....,1)^t; \\ (ii).\quad \text{There exists } \mathrm{N}_0 \geq 3, \text{ such that} \\ \qquad 0 < \lambda_2 = \lambda_3 = \lambda_4 = ... = \lambda_{N_0} < 1, \qquad \text{dim span}\{v_2,\ v_3, \cdots, v_{N_0}\} = 2; \\ (iii).\ \ |\lambda_i| < \lambda_2, \qquad i = N_0 + 1,.....,3n+4 \end{cases} \tag{4.16}$$

Remark 4.6. The *Extraordinary Point analysis is still valid for other subdivision algorithms.*

§5. Conclusions

In this paper, an interpolatory subdivision algorithm for surfaces over arbitrary triangulations is introduced and its convergence properties over nonuniform triangulations studied and the local subdivision matrix analysis is presented. It is proved that the algorithm produces smooth surfaces over arbitrary triangular networks if the shape parameters are chosen properly.

The analyses of the scheme here are different from the analyses[18,19] of the scheme over uniform data. In fact, the analysis presented here is a pointwise analysis. The Block-Circulant Matrix theory is used here. This technique is quite suitable for the nonuniform analysis.

§6. Graphic Examples

Here, we present a graphic example of the subdivision algorithm with the shape control parameter $w = -1/12$ to show the smoothing process of the scheme. The initial data comes from the standard unit cube and the initial triangulation of the unit cube is produced by adding six diagonal lines on each face of the cube (the direction of each line is either (1,1,0), (0,1,1), or (0,1,1) therefore there is a symmetry in the triangulation as shown in Figure 5.) Hence, there are 8 vertices, 12 triangles and 18 edges in the initial triangulation. All the vertices are irregular vertices: four of them are 4-poked vertices and the other four are 5-poked vertices. The graphics are plotted on the Postscript Laser Printer at the Computer Centre of Brunei University, U.K.

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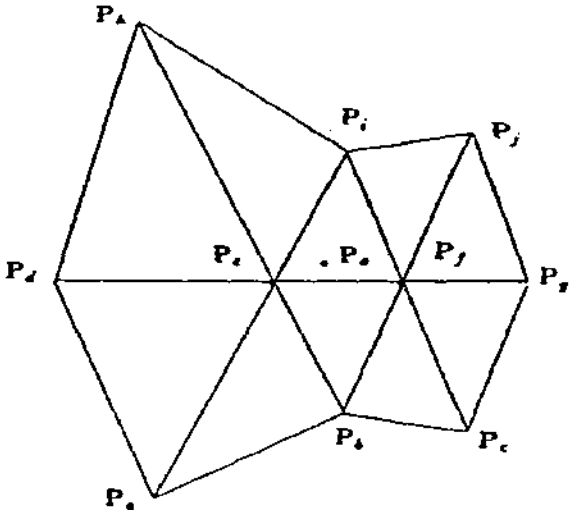

Figure 1. Geometric construction of the scheme.

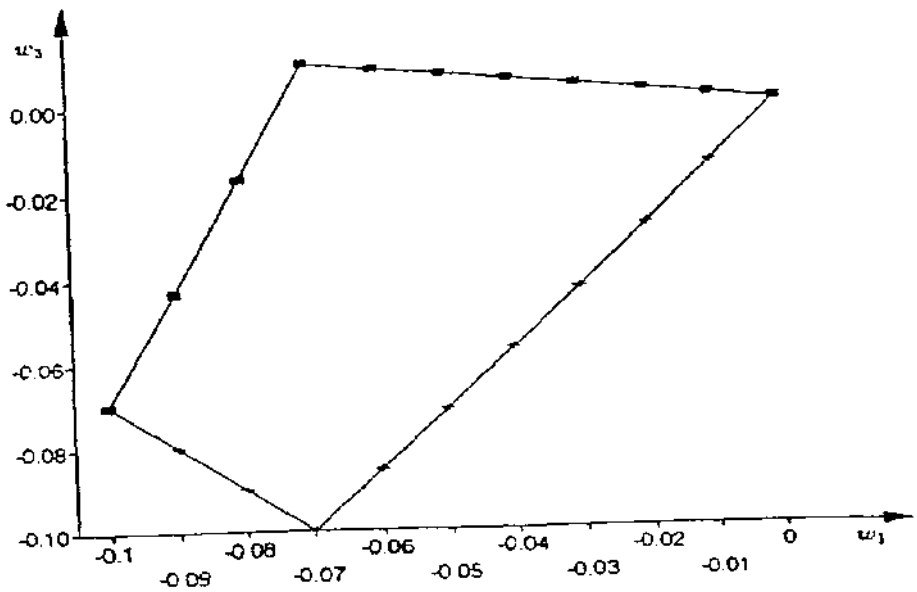

Figure 2. The C^1 convergence region Ω of the scheme.

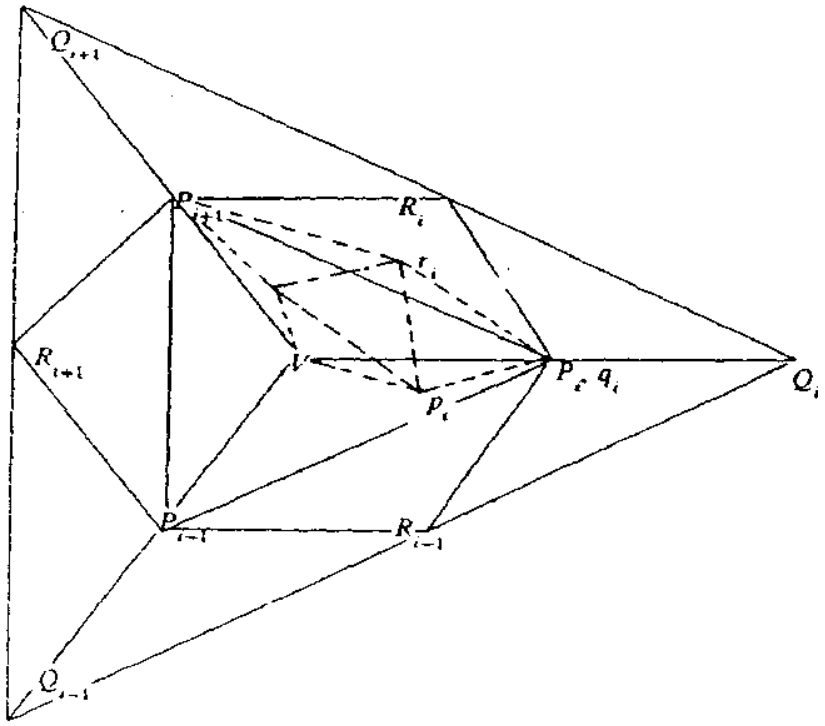

Figure 3. The scheme at a 3-poked vertex.

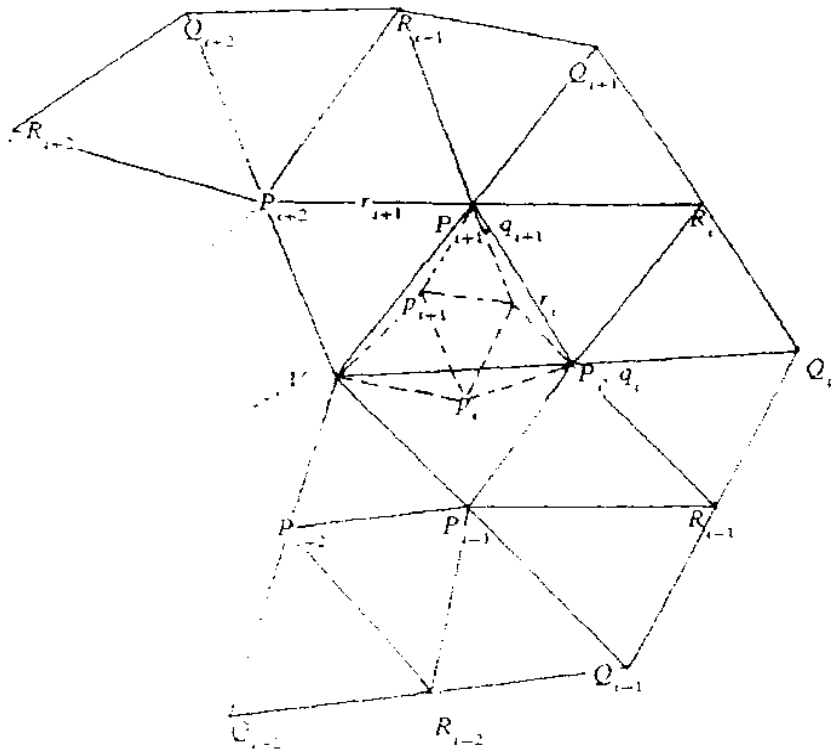

Figure 4. The scheme at a n-poked vertex. ($n > 3$).

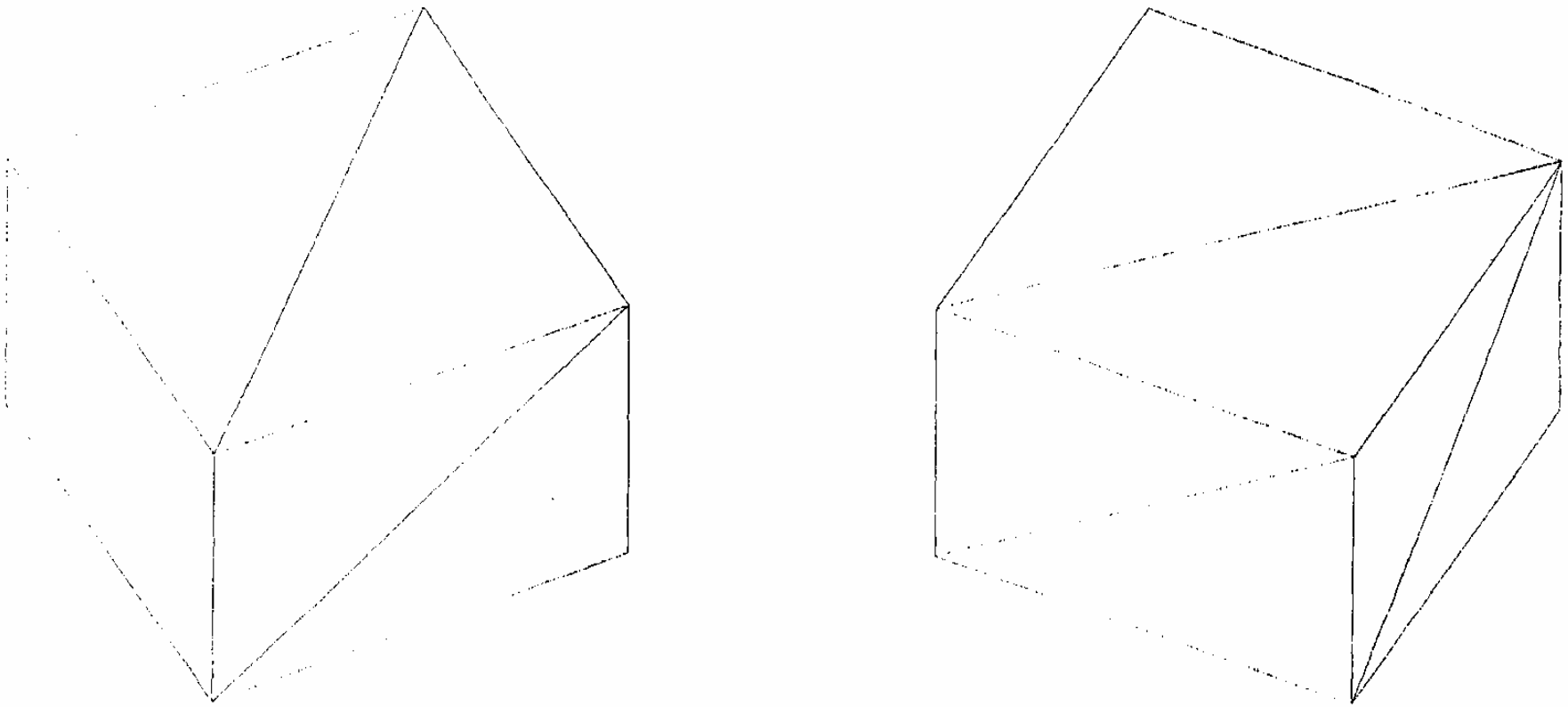

Figure 5. The intial triangulation ($k = 0$).

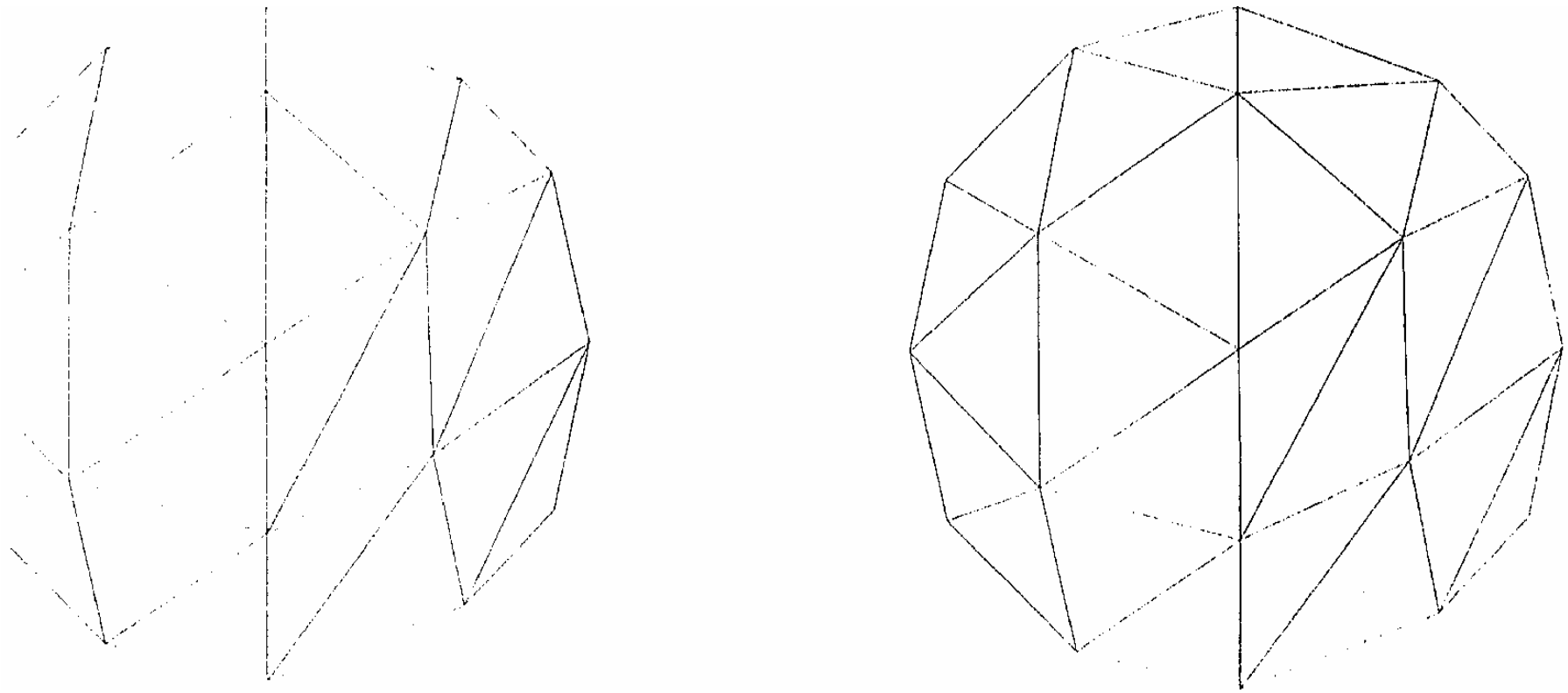

Figure 6. The piecewise linear surface after the first subdivision *(k* = 1)

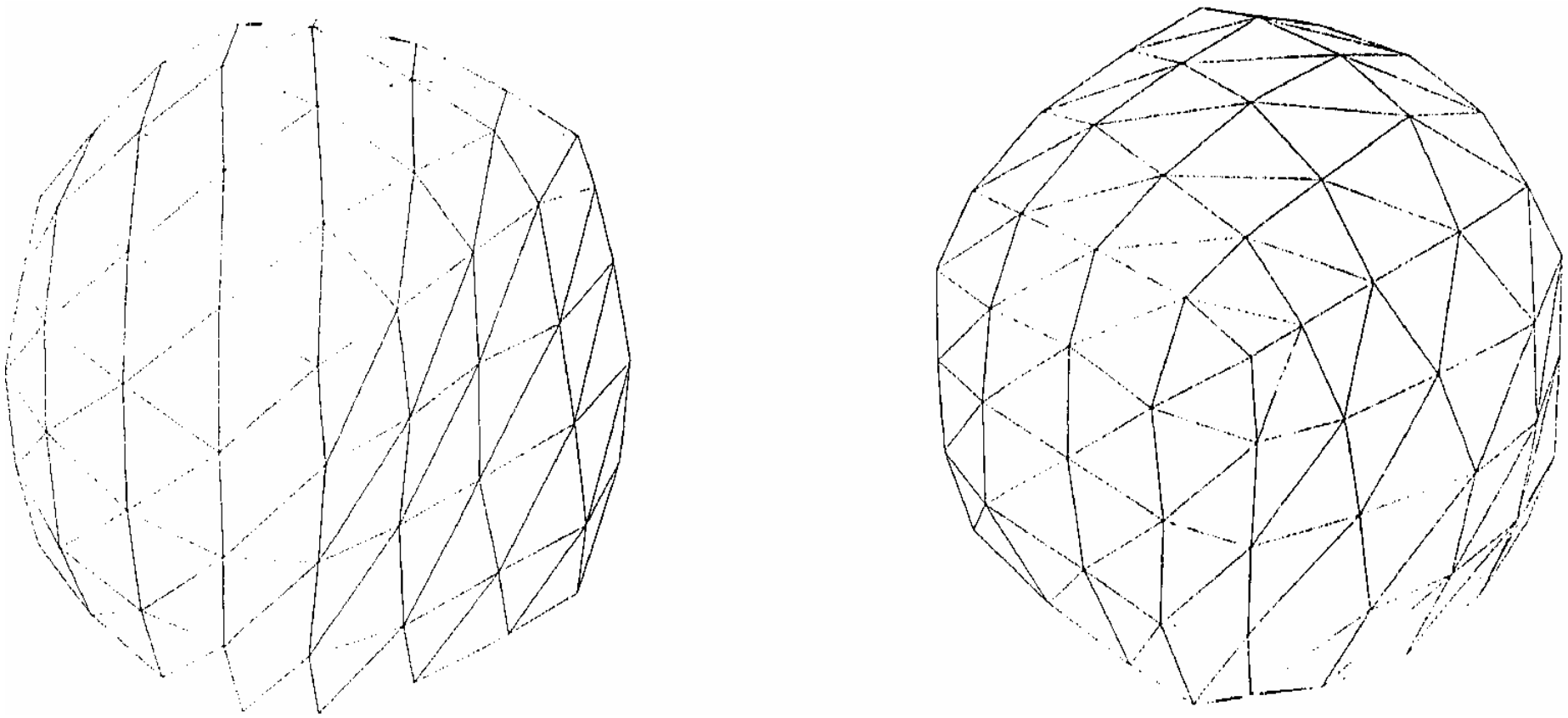

Figure 7. The piecewisc linear surface after the second subdivision *(k* = 2),

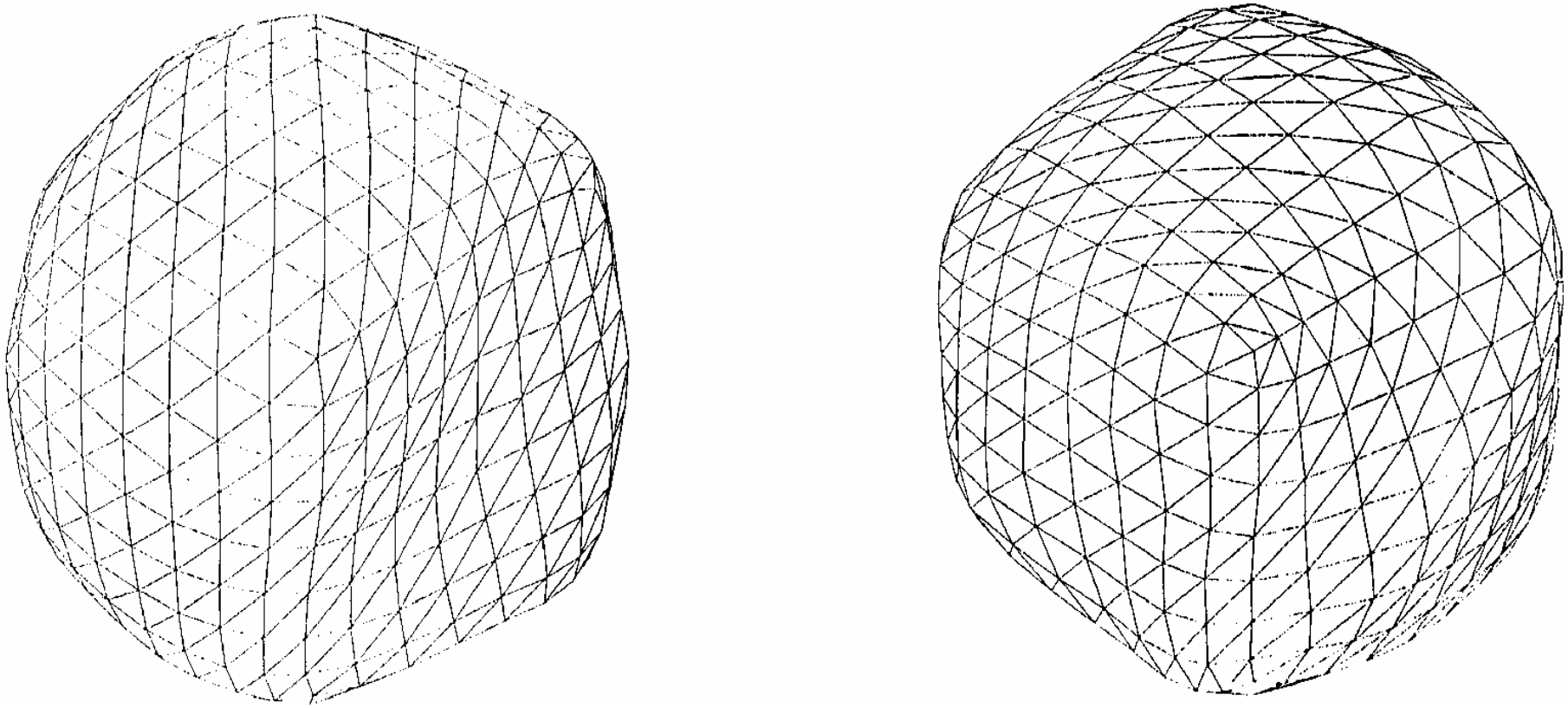

Figure 8. The piecewise liner surface after the third subdivision (k = 3).

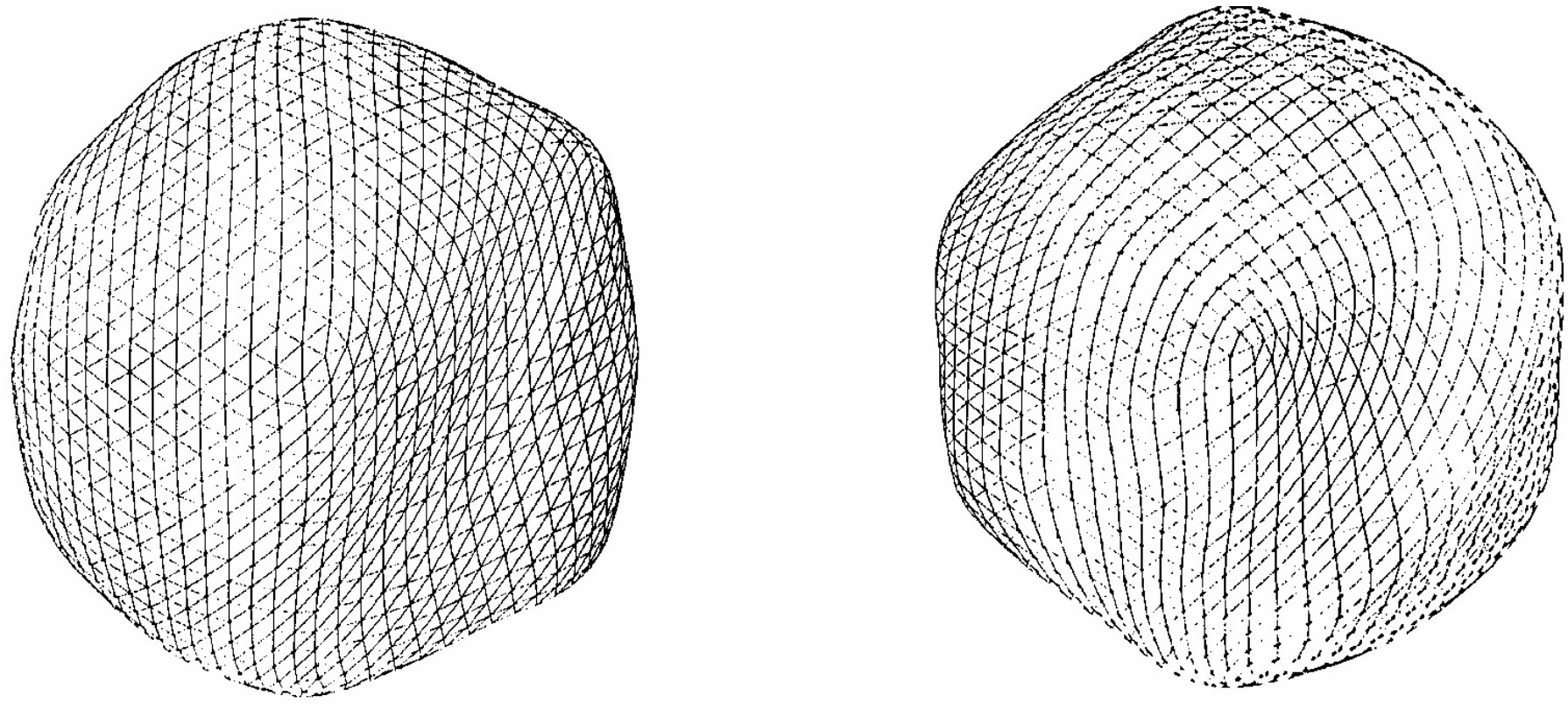

Figure 9. The piecewise liner surface after the forth subdivision (k = 4).

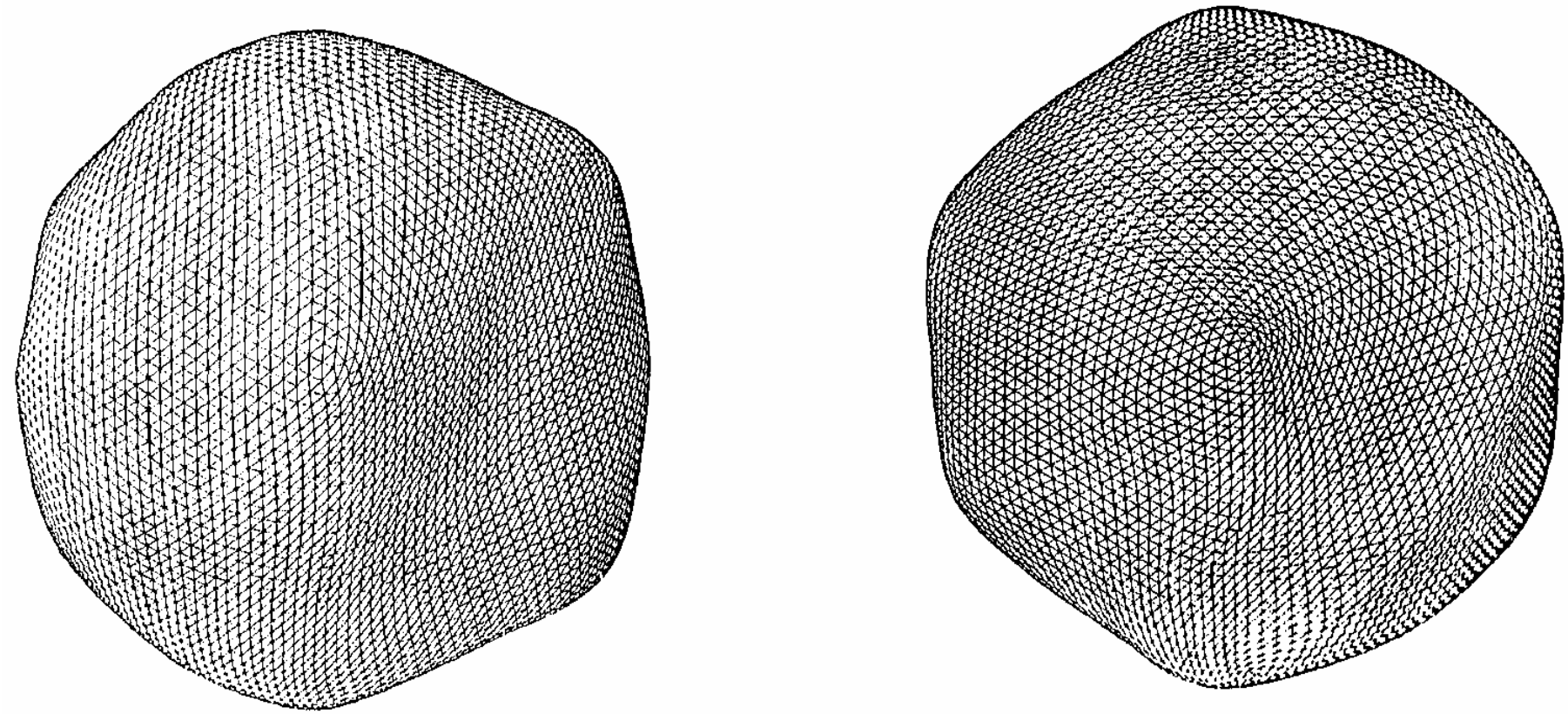

Figure 10. The piecewise liner surface after the fifth subdivision (k = 5).