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NON-UNIFORM CORNER CUTTING

by

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Abstract The convergence of a non-uniform corner cutting process is investigated. It is shown that the limit curve will be differentiable provided the proportions of the corner cuts are kept within appropriate constraints.

Keywords Subdivision, corner cutting

1. Introduction

In a recent paper [de Boor '87] it was shown that "cutting corners" of a control polygon "always works", in the sense that the limit curve will be Lipschitz continuous. In this paper we wish to show that the limit curve will be differentiable under some appropriate conditions on the corner cutting process. The key to the analysis is the choice of a parameterization which itself satisfies the corner cutting process (rather than using a uniform diadic point parameterization as in [Micchelli and Prautzsch '87] or [Dyn, Gregory, Levin '88] for uniform subdivision schemes).

2. The corner cutting process

Let $f_i^0 \in \mathbb{R}^N$, $i = 0, \dots, n+1$, denote a given sequence of initial of initial control

points in \mathbb{R}^N , ≥ 1 , which are defined at the parameter values

$t_0^0 < t_1^0 < \dots < t_{n+1}^0$. The corner cutting process is then defined by: For

$k = 0, 1, 2, \dots$; for $i = 0, \dots, 2^k n$;

$$(2.1) \quad \begin{cases} f_{2i}^{k+1} = (1-\alpha_i^k) f_i^k + \alpha_i^k f_{i+1}^k & \text{at } t_{2i}^{k+1} = (1-\alpha_i^k) t_i^k + \alpha_i^k t_{i+1}^k, \\ f_{2i+1}^{k+1} = \beta_i^k f_i^k + (1-\beta_i^k) f_{i+1}^k & \text{at } t_{2i+1}^{k+1} = \beta_i^k t_i^k + (1-\beta_i^k) t_{i+1}^k, \end{cases}$$

where

$$(2.2) \quad \alpha_i^k > 0, \quad \beta_i^k > 0 \quad \text{and} \quad j - \alpha_i^k - \beta_i^k > 0.$$

Denote by f^k the control polygon with vertices f_i^k , $i = 0, \dots, 2^k n+1$.

Then (2.1) is a process whereby f^{k+1} is created by corner cutting of the polygon f^k . In general, this process is non-uniform since the proportions

α_i^k, β_i^k of the corner cuts can depend both on i and k .

For the purposes of the analysis, the control points $\left\{ f_i^k \right\}$ are

associated with parameter points $\left\{ t_i^k \right\}$ which also satisfy the corner cutting

process, see (2.1). These parametric points always form a strictly mono-

tonic increasing set $t_0^k < t_1^k < \dots < t_{2^k n+1}^k$ since

$$(2.3) \quad t_i^k < t_{2i}^{k+1} < t_{2i+1}^{k+1} < t_{i+1}^k, \quad i = 0, \dots, 2^k n,$$

for α_i^k, β_i^k satisfying (2.2). The control polygon f^k can thus be identified

unambiguously as the piecewise linear interpolant

$$(2.4) \quad f^k(t) := \left\{ \frac{t_{i+1}^k - t}{t_{i+1}^k - t_i^k} \right\} f_i^k + \left[\frac{t - t_i^k}{t_{i+1}^k - t_i^k} \right] f_{i+1}^k, \quad t \in \left\{ t_i^k, t_{i+1}^k \right\}, \quad i = 0, \dots, 2^k n.$$

We propose to analyse the convergence properties of the component functions

of f^k and hence it suffices from now on to consider the scalar case $N=1$,

see Figure 1.

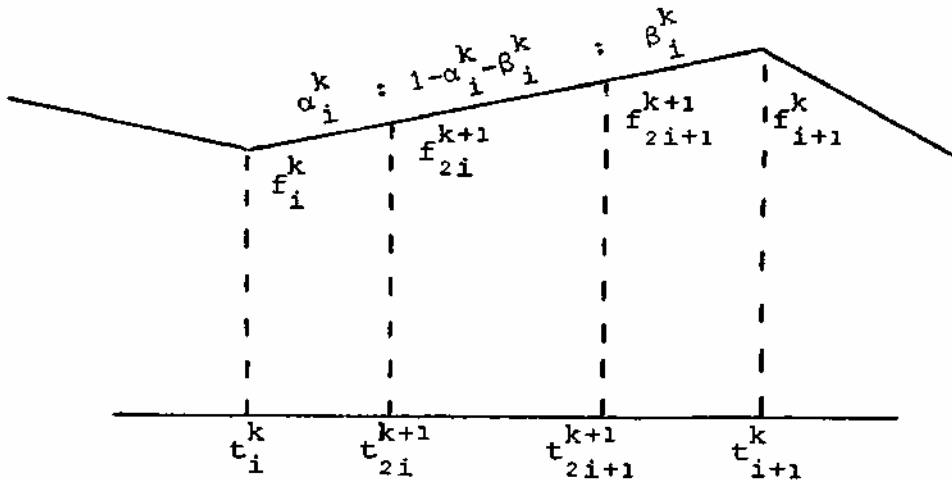


Figure 1

It follows from (2.3) that $\{t_0^k\}_{k=0}^\infty$ and $\{t_{2k=0}^k\}_{k=0}^\infty$ from monotonic increasing and decreasing sequences bounded above and below by t_1^0 and t_n^0 respectively.

Hence there exist

$$(2.5) \quad a := \lim_k t_0^k \leq t_1^0 \quad \text{and} \quad b := \lim_k t_{2k=0}^k \geq t_n^0.$$

We then make use of the uniform norm

$$(2.7) \quad \|f\| = \max_{a \leq t \leq b} |f(t)|, \quad f \in C[a, b],$$

on the interval $[a, b]$.

3. Cutting corners is C^0

Although our main purpose is to find conditions under which the corner cutting process has a C^1 limit, we begin by considering a C^0 analysis. We will show that $\{f^k\}_{k=0}^\infty$ defines a Cauchy sequence in $C[a, b]$ and for this we require the following Lemma:

Lemma 3.1

$$(3.1) \quad |f^{k+p} - f^k| \leq 2 \max_i |\Delta_{f_i}^k| \quad \forall k, p \geq 0,$$

where

$$(3.2) \quad \Delta f_i^k := f_{i+1}^{k+p} - f_i^k .$$

Proof Consider $f^{k+p}(t)$ and $f^k(t)$ on $\left[\begin{matrix} t^{k+p}, t^{k+p} \\ 2^{P_i}, 2^P(i+1) \end{matrix} \right]$. From (2.3) we have

$$(3.3) \quad t_i^k < t_{2^{P_i}}^{k+p} < t_{2^{P(i+1)}}^{k+p} < t_{i+2}^k$$

and since the process (2.1) defines a convex combination we can obtain

$$(3.4) \quad m_i \leq f_j^{k+p} \leq M_i, \quad \forall j = 2^{P_i}, \dots, 2^P(i+1),$$

where

$$(3.5) \quad m_i = \min \left\{ f_i^k, f_{i+1}^k, f_{i+2}^k \right\}, \quad M_i = \max \left\{ f_i^k, f_{i+1}^k, f_{i+2}^k \right\} .$$

Hence

$$m_i \leq f^{k+p}(t) \leq M_i - m_i \leq f^k(t) \leq M_i$$

which gives

$$(3.6) \quad |f^{k+p}(t) - f^k(t)| \leq M_i - m_i \leq |f_{i+1}^k - f_i^k| + |f_{i+2}^k - f_{i+1}^k|$$

$\forall t \in \left[\begin{matrix} t^{k+p}, t^{k+p} \\ 2^{P_i}, 2^P(i+1) \end{matrix} \right]$ and the Lemma follows.

Lemma 3.1 suggests an analysis of the difference process which is obtained from (2.1) as

$$(3.7) \quad \begin{cases} \Delta f_{2i}^{k+1} = (1 - \alpha_i^k - \beta_i^k) \Delta f_i^k, \\ \Delta f_{2i+1}^{k+1} = \beta_i^k \Delta f_i^k + \alpha_{i+1}^k \Delta f_{i+1}^k. \end{cases}$$

Let

$$(3.8) \quad \begin{cases} \bar{\alpha} = \lim_k \max_i \alpha_i^k, & \underline{\alpha} = \lim_k \min_i \alpha_i^k, \\ \bar{\beta} = \lim_k \max_i \beta_i^k, & \underline{\beta} = \lim_k \max_i \beta_i^k. \end{cases}$$

Then we have the following:

Theorem 3.2 (C° convergence) The corner cutting process defined by (2.1) and (2.2) converges to a C° limit if

$$(3.9) \quad \underline{\alpha} > 0, \underline{\beta} > 0 \quad \text{and} \quad 1 - \bar{\alpha} - \bar{\beta} > 0.$$

Proof It follows from the definition of the difference process (3.7) that

$$(3.10) \quad \max_i |\Delta f_i^{k+1}| \leq B_k \max_i |\Delta f_i^k|,$$

where

$$(3.11) \quad B_k = \max_i \left\{ 1 - \alpha_i^k - \beta_i^k, \beta_i^k + \alpha_{i+1}^k \right\}.$$

Moreover, it can be shown that

$$(3.12) \quad B_k \leq B < 1$$

for some constant B , independent of k , if (3.9) holds. Hence the differences are contracting and from Lemma 3.1 it follows that $\{f^k\}_{k=0}^\infty$ defines a Cauchy sequence on $C[a,b]$ which completes the proof.

Conditions (3.9) require $(\underline{\alpha}, \underline{\beta})$ and $(\bar{\alpha}, \bar{\beta})$ to lie strictly within the region Ω_0 depicted in Figure 3.1. In particular $(0,0) < (\underline{\alpha}, \underline{\beta}) \leq (\bar{\alpha}, \bar{\beta}) < (\frac{1}{2}, \frac{1}{2})$

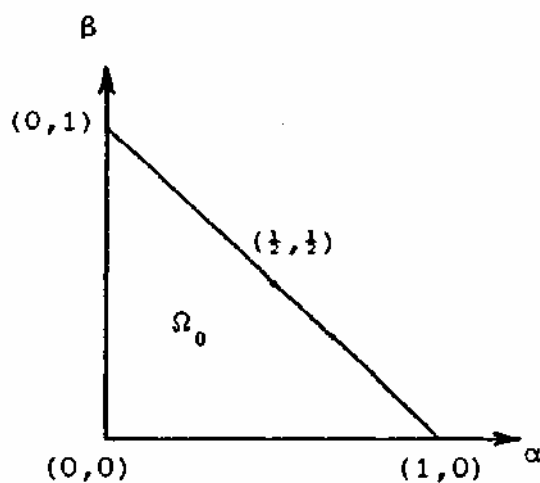


Figure 3.1

is a sufficient condition for a C^0 limit- In [de Boor '87] a different argument is used to prove convergence for a more general corner cutting process. However, our purpose is to find conditions under which the process (2.1) has a C^1 limit and hence we have found it appropriate to develop a separate C^0 analysis here. The C^1 analysis makes use of the following observation:

Remark 3.3 The parameteric points $\left\{t_i^k\right\}_{i=0}^{2^k}$ become dense in $[a,b]$.

Proof Since the parameteric points satisfy the corner cutting process, it follows that

$$(3.13) \quad \begin{cases} \Delta_{t_{2i}^{k+1}} = (1 - \alpha_i^k - \beta_i^k) \Delta_{t_i^k} , \\ \Delta_{t_{2i+1}^{k+1}} = \beta_i^k \Delta_{t_i^k} + \alpha_{i+1}^k \Delta_{t_i^k} , \end{cases}$$

of (3.7), and that

$$(3.14) \quad \max_i |\Delta_{t_i^{k+1}}| \leq B_k \max_i |\Delta_{t_i^k}|$$

of (3.10). Since (3.12) holds (under the conditions (3.9)) we have

$$\lim_k \max_i |\Delta_{t_i^k}| = 0 .$$

4. Cutting corners is C^1

To analyse C^1 convergence, consider the divided difference process defined from (3.7) and (3.13) by

$$(4.1) \quad \begin{cases} d_{2i}^{k+1} = d_i^k \quad \text{at} \quad t_{2i}^{k+1} , \\ d_{2i+1}^{k+1} = (1 - \theta_i^k) d_i^k + \theta_i^k d_{i+1}^k \quad \text{at} \quad t_{2i+1}^{k+1} , \end{cases}$$

where

$$(4.2) \quad d_i^k := \Delta_{f_i^k} / \Delta_{t_i^k} ,$$

$$(4.3) \quad \theta_i^k := \frac{\alpha_{i+1}^k \Delta_{ti+1}^k}{\alpha_{i+1}^k \Delta_{ti+1}^k + (\beta_i^k \Delta_{ti}^k + \alpha_{i+1}^k \Delta_{ti+1}^k)} .$$

We then have:

Theorem 4.1 If the divided difference scheme converges uniformly to $d \in C[a,b]$ (with respect to the parametric points $\left\{ \begin{smallmatrix} k \\ t_i \end{smallmatrix} \right\}$), then the corner cutting scheme converges uniformly to $f \in C^1[a,b]$, where $f' = d$.

Proof Let H_k denote the piecewise cubic Hermite interpolant such that

$$(4.4) \quad H_k \left(\begin{smallmatrix} k \\ t_i \end{smallmatrix} \right) = f_i^k \quad \text{and} \quad H'_k \left(\begin{smallmatrix} k \\ t_i \end{smallmatrix} \right) = d_i^k, \quad i = 0, \dots, 2^k n.$$

Then for $t \in \left(\begin{smallmatrix} k \\ t_i \end{smallmatrix}, \begin{smallmatrix} k \\ t_{i+1} \end{smallmatrix} \right)$ with $\theta = (t - \begin{smallmatrix} k \\ t_i \end{smallmatrix}) / \Delta_{ti}^k + \theta^2(-2\theta + 3) \begin{smallmatrix} k \\ f_{i+1} \end{smallmatrix}$

$$(4.5) \quad H_k(t) = (1-\theta)^2(2\theta+1) \begin{smallmatrix} k \\ f_i \end{smallmatrix} + \theta^2(-2\theta+3) \begin{smallmatrix} k \\ f_{i+1} \end{smallmatrix} \\ + (1-\theta)^2 \theta \Delta_{ti}^k \begin{smallmatrix} k \\ d_i \end{smallmatrix} + \theta^2(\theta-1) \Delta_{ti}^k \begin{smallmatrix} k \\ d_{i+1} \end{smallmatrix} .$$

and

$$(4.6) \quad H'_k(t) = (-3\theta^2 + 2\theta + 1) \begin{smallmatrix} k \\ d_i \end{smallmatrix} + (3\theta^2 - 2\theta) \begin{smallmatrix} k \\ d_{i+1} \end{smallmatrix} .$$

Also, let d be the divided difference control polygon (piecewise linear interpolant) defined for $t \in \left[\begin{smallmatrix} k \\ t_i \end{smallmatrix}, \begin{smallmatrix} k \\ t_{i+1} \end{smallmatrix} \right)$ by

$$(4.7) \quad d_k(t) = (1-\theta) \begin{smallmatrix} k \\ d_i \end{smallmatrix} + \theta \begin{smallmatrix} k \\ d_{i+1} \end{smallmatrix} ,$$

where $d^k \rightarrow d$ uniformly on $C[a,b]$ by hypothesis. Then subtracting (4.6) from (4.7) leads to

$$(4.8) \quad \|d^k - H'_k\| \leq \frac{3}{4} \max_i \left| \begin{smallmatrix} k \\ d_{i+1} \end{smallmatrix} - \begin{smallmatrix} k \\ d_i \end{smallmatrix} \right| .$$

Thus

$$(4.9) \quad \lim_k \|d - H'_k\| \leq \lim_k \|d - d^k\| + \lim_k \|d^k - H'_k\| = 0$$

i.e. $H'_k \rightarrow d$ uniformly. (The right hand side of (4.8) converges necessarily to zero if $\lim_k dk = d \in C[a,b]$ and the parametric points become dense in

$[a,b]$.) We now show that $\{H_k\}_{k=0}^\infty$ converges on $C^1[a,b]$. Assume, without loss of generality, that $f_0^0 = f_1^0 = f_2^0 = 0$. Then at the k th step $f_0^k = f_1^k = f_2^k = 0$ and $d_0^k = d_1^k = 0$. (This reflects the "local support" nature of the corner cutting process.) Thus, necessarily, $f(a) = d(a) = 0$ and $H_k(a) = 0$.

Define

$$(4.10) \quad f(t) := \int_a^t d(\bar{t}) d\bar{t}$$

Then

$$(4.11) \quad \|f - H_k\| = \max_{a < t \leq b} \left| \int_a^t \{d(\bar{t}) - H'_k(\bar{t})\} d\bar{t} \right| \leq (b-a) \|d - H'_k\|.$$

Hence, H_k converges uniformly to $f \in C^1[a,b]$, where $f' = d$. Finally, since

$$(4.12) \quad \|f - f^k\| \leq \|f - H_k\| + \|H_k - f^k\| \leq \|f - H_k\| + \frac{1}{2} \max_i (\Delta t_i^k)^2 \|H_k''\|$$

(using the Cauchy remainder for linear interpolation), it follows that the control polygon f_k of the corner cutting process converges uniformly to $f \in C^1[a,b]$, (where we again note that the parametric points become dense in $[a,b]$, see Remark 3.3).

Theorem 4.1 indicates that, in order to prove C^1 convergence of the corner cutting process, we should find conditions for which the divided difference process (4.1) has a C^0 limit. Now the process (4.1) has the property that the image set

$$(4.13) \quad I_k = \{(t, d^k(t)) \in R^2 : t \in [a,b]\}$$

lies on the initial image set $I_0 \cup k$. It is thus tempting to conclude that $d^k = d^0 \in C[a,b] \cup k$. However, this is an incorrect argument since $d^k(t)$ has been defined as the piecewise linear interpolant with respect to the partition $t_0^k < t_1^k < \dots < t_n^k$ of the original corner cutting process.

Thus the analysis of C^0 convergence of the divided difference process must be constructed with more care and, following the approach of section 3, we have:

Lemma 4.2

$$(4.14) \quad \|d^{k+p} - d^k\| \leq 2 \max_i |\Delta_{d_i}^k| \quad \forall k, p \geq 0 .$$

Theorem 4.3 (C^1 convergence) The divided difference process (4.1) converges uniformly to a C^0 limit (and hence the corner cutting process converges uniformly to a C^1 limit) if

$$(4.15) \quad \bar{\alpha} > 0, \bar{\beta} > 0, 2\bar{\alpha} + \bar{\beta} < 1 \text{ and } \bar{\alpha} + 2\bar{\beta} < 1 .$$

The proof of Lemma 4.2 is identical to that of Lemma 3.1. The proof of Theorem 4.3 requires the following additional lemma:

Lemma 4.4 Let

$$(4.16) \quad r_i^k := \Delta_{t_{i+1}}^k / \Delta_{t_i}^k$$

and assume that (4.15) holds. Then there exist r and R such that

$$(4.17) \quad 0 < r \leq r_i^k \leq R < \infty \quad \forall i, k .$$

Proof From (3.13) we obtain the following non-linear relations:

$$(4.18) \quad \begin{cases} r_{2i}^{k+1} = \left[\beta_i^k + \alpha_{i+1} r_i^k \right] / \left[1 - \alpha_i^k - \beta_i^k \right] \\ r_{2i}^{k+1} = \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] r_i^k / \left[\beta_i^k + \alpha_{i+1} r_i^k \right] . \end{cases}$$

Hence, if $r \leq r_i^k \leq R \quad \forall i, k$, we have

$$\begin{aligned} \left[\beta_i^k + \alpha_{i+1} r \right] / \left[1 - \alpha_i^k - \beta_i^k \right] &\leq r_{2i}^{k+1} \leq \left[\beta_i^k + \alpha_{i+1} R \right] \left[1 - \alpha_i^k - \beta_i^k \right] , , \\ \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] / \left[\beta_i^k / r + \alpha_{i+1} \right] &\leq r_{2i+1}^{k+1} \leq \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] / \left[\beta_i^k / R + \alpha_{i+1} \right] . \end{aligned}$$

Thus, we require r and R such that

$$\left[\beta_i^k + \alpha_{i+1} R \right] / \left[1 - \alpha_i^k - \beta_i^k \right] \leq R, \quad \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] / \left[\beta_i^k / R + \alpha_{i+1} \right] \leq R,$$

and

$$r \leq \left[\beta_i^k + \alpha_{i+1}^k r \right] / \left[1 - \alpha_i^k - \beta_i^k \right], \quad r \leq \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] / \left[\beta_i^k / r + \alpha_{i+1}^k \right].$$

Therefore

$$(4.19) \quad \max_i \left\{ \beta_i^k / \left[1 - \alpha_i^k - \beta_i^k - \alpha_{i+1}^k \right], \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k - \beta_i^k \right] / \alpha_{i+1}^k - \alpha_{i+1}^k \right\} \leq R,$$

and

$$(4.20) \quad r \leq \max_i \left\{ \beta_i^k / \left[1 - \alpha_i^k - \beta_i^k - \alpha_{i+1}^k \right], \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k - \beta_i^k \right] / \alpha_{i+1}^k \right\}$$

provided

$$(4.21) \quad 1 - \alpha_i^k - \beta_i^k - \alpha_{i+1}^k > 0 \quad \text{and} \quad 1 - \alpha_{i+1}^k - \beta_{i+1}^k - \beta_i^k > 0.$$

Condition (4.17) is then obtained under the hypothesis (4.15).

Proof of Theorem 4.3 From (4.1) and (4.3) we obtain

$$(4.22) \quad \begin{cases} \Delta_{d2i}^{k+1} = \theta_i^k \Delta_{d_i}^k, \\ \Delta_{d2i+1}^{k+1} = (1 - \theta_i^k) \Delta_{d_i}^k, \end{cases}$$

where $0 < \theta_i^k < 1$. Thus

$$(4.23) \quad \max_i |\Delta_{d_i}^{k+1}| \leq C_k \max_i |\Delta_{d_i}^k|,$$

where

$$(4.24) \quad C_k = \max_i \left\{ \theta_i^k, 1 - \theta_i^k \right\}$$

and hence

$$(4.25) \quad 0 < C_k < 1.$$

Condition (4.25) is not strong enough for our purposes and we wish to show that

$$(4.26) \quad C_k \leq C < 1$$

for some constant C , independent of k . Now, from (4.3),

$$(4.27) \quad \theta_i^k = \frac{1}{1 + \hat{r}_i^k}, \quad 1 - \theta_i^k = \frac{\hat{r}_i^k}{1 + \hat{r}_i^k},$$

where

$$(4.28) \quad \hat{r}_i^k = \frac{\alpha_{i+1}^k}{\beta_i^k}$$

Furthermore, by Lemma 4.4 (and the hypothesis (4.15)), there exist \hat{r} and \hat{R} such that

$$(4.29) \quad 0 < \hat{r} \leq \hat{r}_i^k \leq \hat{R} < \infty.$$

Thus

$$(4.30) \quad \theta_i^k \leq \frac{1}{1 + \hat{R}} \quad \text{and} \quad 1 - \theta_i^k \leq \frac{1}{1 + \hat{r}} \quad \forall i, k$$

and (4.26) holds. Finally, it now follows from Lemma 4.2 that $\{d^k\}_{k=0}^{\infty}$ defines a Cauchy sequence on $C[a, b]$ and hence has limit $d \in C[a, b]$ say.

Conditions (4.15) for C^1 convergence require $(\underline{\alpha}, \underline{\beta})$ and $(\bar{\alpha}, \bar{\beta})$ to lie strictly within the region Ω_1 depicted in Figure 4.1 (cf. Figure 3.1).

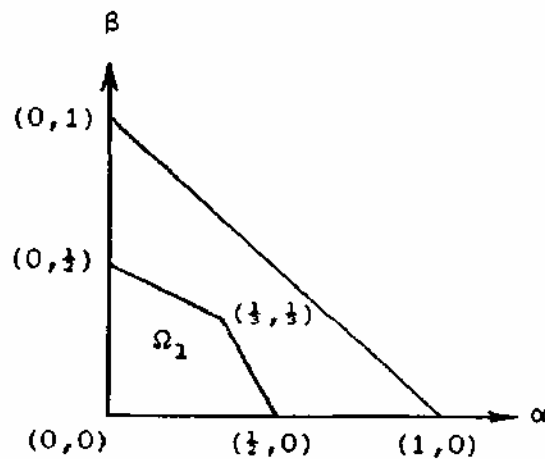


Figure 4.1

In particular, $(0,0) < (\underline{\alpha}, \underline{\beta}) < (\alpha, \beta) < (\frac{1}{3}, \frac{1}{3})$ is a sufficient condition for a C^1 limit (i.e. corner cutting of proportions strictly less than one third ensures a C^1 limit). If $(\bar{\alpha}, \bar{\beta})$ lies strictly outside the region Ω_1 , then convergence to a C^1 limit is no longer guaranteed. For example, with

$\alpha_i^k = \alpha$ and $\beta_i^k = \beta \quad \forall i, k$, it can be shown that (see (4.22))

$$\Delta_{d0}^{k+1} = \theta \Delta_{d0}^k = \theta^k \theta^{k-1} \dots \theta^0 \Delta_{d0}^0$$

will not converge to zero if $2\alpha + \beta > 1$. This violates a necessary C^1 convergence condition. Similarly, by symmetry, $\alpha + 2\beta > 1$ is not allowable.

We have shown that the corner cutting process has a C^1 limit, under the conditions (4.15), with respect to a parameterization which is itself defined by the corner cutting method. We conclude by showing that this parameterization is regular in the case R^N , $N > 1$.

Theorem 4.5 (Regular parameterization) In the case of corner cutting in R^N , $N > 1$, the C^1 limit curve f of Theorem 4.3 is regular, i.e. $f(t) = d(t) \neq 0 \quad \forall t \in [a, b]$, except for the singular cases, where, for some i ,

$$(i) \quad f_i^0 = f_{i+1}^0,$$

or

$$(ii) \quad f_{i+1}^0 = (1-\theta)f_i^0 + \theta f_{i-1}^0 \quad \text{for some } \theta > 0.$$

Proof Let

$$(4.31) \quad J_k = \left\{ d^k(t) \in R^N : t \in [a, b] \right\}$$

be the image set of $d^k(t)$. Then, following an earlier argument, we have $J_k \subset J_0 \quad \forall k$, for a process of the form (4.1). Thus $d(t) = 0$ for some $t \in [a, b]$ implies that $d^0(t) = 0$ for some $t \in [a, b]$ and this can only occur if (i) $d_i^0 = 0$ for some i or (ii) for $d_i^0 = -\theta d_{i-1}^0$ some i and $\theta > 0$.

As a final comment, it should be noted that, because of the local nature of the corner cutting process (2.1), the convergence arguments presented here also apply to the case of closed polygons in R^N , $N > 1$. This case can be treated as periodic data, where $f_n^0 = f_{n+1}^0 = f_1^0$ and in the process $\alpha_{2k_n}^k = \alpha_0^k, \beta_{2k_n}^k = \beta_0^k$.

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