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# Asymptotic Behaviour of Zeros of Bieberbach Polynomials

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**Abstract** Let  $\Omega$  be a simply-connected domain in the complex plane and let  $\pi_n$  denote the  $n$  th degree Bieberbach polynomial approximation to the conformal map  $f$  of  $\Omega$  onto a disc. In this paper we investigate the asymptotic behaviour (as  $n \rightarrow \infty$ ) of the zeros of  $\pi_n$ ,  $\pi_n'$  and also of the zeros of certain closely related rational approximants to  $f$ . Our results show that, in each case, the distribution of the zeros is governed by the location of the singularities of the mapping function  $f$  in  $\mathbb{C} \setminus \Omega$ , and we present numerical examples illustrating this.

**Keywords** : Bieberbach polynomials, Bergman kernel function, conformal mapping, zeros of polynomials.

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## 1 Introduction

Let  $\Omega$  be a simply-connected domain of the complex plane  $\mathbb{C}$ , whose boundary  $\partial\Omega$  is a closed Jordan curve, and let  $\zeta \in \Omega$ . Then, by the Riemann mapping theorem, there exists a unique conformal mapping  $w = f_\zeta(z)$  of  $\Omega$  onto a disc  $\{w : |w| < r_\zeta\}$ , such that

$$f_\zeta(\zeta) = 0, \quad f_\zeta'(\zeta) = 1.$$

The radius  $r_\zeta$  of this disc is called the conformal radius of  $\Omega$  with respect to  $\zeta$ .

For the inner product

$$(g, h) := \iint_{\Omega} g(z) \overline{h(z)} dm,$$

Where  $dm$  is the 2-dimensional Lebesgue measure, we consider the Hilbert space

$$L^2(\Omega) := \{g : g \text{ analytic in } \Omega, \|g\|^2 = (g, g) < \infty\}.$$

Let  $K(z, \zeta)$  denote the *Bergman kernel function* of  $\Omega$  which has the reproducing property

$$g(\zeta) = (g, k(\cdot, \zeta)), \quad \forall g \in L^2(\Omega). \quad (1.1)$$

(cf. [1], [3], [4], [8]). Then it is known (cf. [4, p.34]) that  $r_\zeta = (\pi K(\zeta, \zeta))^{-1/2}$  and that for  $z \in \Omega$

$$f_\zeta'(z) = \frac{K(z, \zeta)}{K(\zeta, \zeta)}, \quad f_\zeta(z) = \frac{1}{K(\zeta, \zeta)} \int_{t=\zeta}^z K(t, \zeta) dt. \quad (1.2)$$

Next let  $Q_n(z) = \gamma_n z^n + \dots$ ,  $\gamma_n > 0$ , be the sequence of orthonormal polynomials for the inner product  $(\cdot, \cdot)$ , ie.

$$\iint_{\Omega} Q_k(z) \overline{Q_l(z)} dm = \delta_{k,l}.$$

Since  $\Omega$  is a Jordan region, it is known (cf. [4, p. 17]) that  $\{Q_n\}_0^\infty$  forms a complete orthonormal system for  $L^2(\Omega)$  and consequently, from the reproducing property (1.1), that  $K(\cdot, \zeta)$  has the  $L^2(\Omega)$ -convergent Fourier series expansion

$$K(z, \zeta) = \sum_{j=0}^{\infty} \overline{Q_j(\zeta)} Q_j(z). \quad (1.3)$$

The *Bieberbach polynomials*  $\pi_n$  for  $\Omega$  are defined by

$$\pi_n(z) := \frac{1}{K_{n-1}(\zeta, \zeta)} \int_{t=\zeta}^z K_{n-1}(t, \zeta) dt, \quad (1.4a)$$

Where  $K_{n-1}(\cdot, \zeta)$  denotes the partial Fourier sum

$$K_{n-1}(z, \zeta) := \sum_{j=0}^{n-1} \overline{Q_j(\zeta)} Q_j(z). \quad (1.4b)$$

These polynomials satisfy

$$\pi_n(\zeta) = 0, \quad \pi_n'(\zeta) = 1 \quad \|\pi_n'\|^2 = \frac{1}{K_{n-1}(\zeta, \zeta)},$$

and provide approximations to the mapping function  $f_\zeta$  in the sense that  $\pi_n \rightarrow f_\zeta$  locally uniformly in  $\Omega$  (cf. [4, p.34]). The latter is a direct consequence of the fact that convergence in the norm of  $L^2(\Omega)$  implies uniform convergence on each compact subset of  $\Omega$  (cf. [4, p. 26]). More generally, if  $\{s_j\}_1^\infty$  is any complete orthonormal system for  $L^2(\Omega)$  and

$$\hat{\pi}_n(z) := \frac{1}{\hat{K}_{n-1}(\zeta, \zeta)} \int_{t=\zeta}^2 \hat{K}_{n-1}(t, \zeta) dt, \quad (1.5a)$$

Where

$$\hat{K}_{n-1}(z, \zeta) = \sum_{j=1}^{n-1} s_j(\zeta) \overline{s_j(z)}, \quad (1.5b)$$

then  $\hat{\pi}_n \rightarrow f_\zeta$  locally uniformly in  $\Omega$  (cf. [4, p.32]).

In practice the success of the above method for approximating  $f_\zeta$  depends critically on the choice of the denning orthonormal system. In particular, if the mapping function  $f_\zeta$  has singularities either on the boundary  $\partial\Omega$  or close to  $\partial\Omega$  in  $C \setminus \overline{\Omega}$ , then it is essential that the orthonormal system contains functions that reflect the corresponding singularities of  $K(\cdot, \zeta)$  (cf. [7], [9], [10]). For this reason, the problem of determining the location and nature of the singularities of  $f_\zeta$  is of considerable practical interest.

The purpose of this paper is to describe the asymptotic behaviour (as  $n \rightarrow \infty$ ) of the zeros of the Bieberbach polynomials  $\pi_n$ , and also of the zeros of certain rational approximants  $\pi_n$  of the type studied in [7], [9] and [10] (see also [4, p.36]). Our results show that the distributions of these zeros, and also of the zeros of the derivatives  $\pi_n'$  and  $\hat{\pi}_n'$ , are governed by the location of the singularities of the mapping function  $f_\zeta$  in  $C \setminus \Omega$ . From the more practical point of view, the results of several numerical experiments indicate that the distributions of the zeros of  $\pi_n$  and  $\pi_n'$  can help to determine the approximate location of these singularities.

The paper is organized as follows: Section 2 contains the statements of our main result concerning the zeros of Bieberbach polynomials (Theorem 2.2) and of an intermediate result (Theorem 2.1) which is needed for our proofs. In Section 3 we present three examples illustrating the results stated in Section 2, and make several observations regarding the distributions of the zeros of  $\pi_n$  and  $\pi_n'$  in relation to the singularities of  $f_\zeta$ . In Section 4 we give the proofs of Theorems 2.1 and 2.2. Finally, in Section 5 we treat the problem of the distribution of zeros of rational approximants  $\hat{\pi}_n$  of the type studied in [7], [9], and [10].

Our main results will be given in terms of a *normalized counting measure for zeros*, which is defined as follows: If  $P$  is a polynomial of degree  $n$  with zeros  $z_1, z_2, \dots, z_n$  (some of which may be repeated), then the measure  $v(P)$  is defined by

$$v(P)(B) := \frac{1}{n} \cdot [\# \text{ of zeros of } P \text{ in } B], \quad (1.6)$$

for any Borel set  $B \subseteq C$ . Thus,  $v(P)$  is a probability measure on the Borel subsets of  $C$ .

## 2 Statements of Results for Bieberbach Polynomials

Let  $w = \phi(z)$  denote the conformal mapping of  $D := \overline{C} \setminus \overline{\Omega}$  onto  $\{w : |w| > 1\}$ , normalized by  $\phi(\infty) = \infty$  and  $\phi'(\infty) > 0$ , and observe that the Green function of  $D$  with pole at  $\infty$  is given by  $g_D(z, \infty) = \log |\phi(z)|$ . Further, for each  $\sigma > 1$ , let  $\Gamma_\sigma$  denote generically the locus

$$\Gamma_\sigma := \{z : |\sigma(z)| = \sigma\} = \{z : g_D(z, \infty) = \log \sigma\}, \quad (2.1)$$

and set  $\Gamma_1 := \partial\Omega$ . Finally, let  $\overline{\Omega}_\sigma$  denote the collection of points interior to the level curve  $\Gamma_\sigma$ .

For the compact set  $\overline{\Omega}_\sigma$ ,  $\sigma \geq 1$ , there exists a unique probability measure  $\mu_\sigma$  supported on  $\Gamma_\sigma$  that minimizes the energy integral

$$I[\mu] := \iint \log |z - t|^{-1} d\mu(z) d\mu(t) \quad (2.2)$$

over all probability measures supported on  $\overline{\Omega}_\sigma$  (cf. [5, § 16.4], [13]). The measure  $\mu_\sigma$  is called the *equilibrium distribution* for  $\overline{\Omega}_\sigma$  and the *logarithmic capacity* of  $\overline{\Omega}_\sigma$  is defined by

$$\text{cap}(\overline{\Omega}_\sigma) := \exp(-I[\mu_\sigma]). \quad (2.3)$$

In terms of the mapping function  $\Phi$  we have that

$$\text{cap}(\overline{\Omega}_\sigma) = \sigma / \Phi'(\infty). \quad (2.4)$$

Before giving our main result it is convenient to state

**Theorem 2.1** *With the notations and assumptions of Section I,*

$$\limsup_{n \rightarrow \infty} |Q_n(\zeta)|^{1/n} = \frac{1}{\rho} (\leq 1), \quad (2.5)$$

where  $\rho (\geq 1)$  is the largest index such that  $f_\zeta$  has an analytic (single-valued) extension throughout  $\Omega_\rho$ .

Notice that if  $f_\zeta$  has a singularity on  $\Gamma_1 = \partial\Omega$ , then  $\rho = 1$ . Moreover, if  $\rho < \infty$ , then  $f_\zeta$  has at least one singularity on  $\Gamma_\rho$ .

We can now state our main result concerning the zeros of Bieberbach polynomials.

**Theorem 2.2** *Suppose that the constant  $\rho$  of Theorem 2.1 is finite and let  $A \subseteq \mathbb{N}$  be a sequence for which*

$$\lim_{\substack{n \rightarrow \infty \\ n \in A}} |Q_n(\zeta)|^{1/n} = \frac{1}{\rho}. \quad (2.6)$$

*Then in the weak-star topology of measures, the normalizing counting measures for the zeros of  $\pi_n$  and  $\pi'_n$  satisfy*

$$v(\pi_{n-1}) \xrightarrow{*} \mu_\rho \text{ and } v(\pi'_{n-1}) \xrightarrow{*} \mu_\rho, \text{ as } n \rightarrow \infty, n \in A, \quad (2.7)$$

Where  $\mu_p$  is the equilibrium distribution for  $\overline{\Omega_p}$ .

In (2.7), the first convergence means that

$$\lim_{\substack{n \rightarrow \infty \\ n \in A}} \int f d\nu(\pi_{n+1}) = \int f d\mu_p$$

for every function  $f$  continuous on  $C$  having compact support. From this it follows (cf. [6, pp. 8,9]) that if  $B$  is any Borel set, then

$$\mu_p(B) \leq \liminf_{n \rightarrow \infty} \nu(\pi_{n+1})(B) \leq \limsup_{n \rightarrow \infty} \nu(\pi_{n+1})(B) \leq \mu_p(\overline{B}),$$

where  $\overset{\circ}{B}$  denotes the interior of  $B$ .

Since  $\text{supp}(\mu_p) = \Gamma_p$ , we immediately deduce from (2.7) the following.

**Corollary 2.3** *With  $p$  as in Theorem 2.2, every point on  $\Gamma_p$  is an accumulation point of the zeros of the Bieberbach polynomials  $\{\pi_n\}_0^\infty$  and of the derived sequence  $\{\pi'_n\}_1^\infty$ . Consequently,  $\{\pi_n\}_0^\infty$  cannot converge uniformly in any neighbourhood containing a point of  $\Gamma_p$ .*

**Remark 1** Theorem 2.2 (and Corollary 2.3) does not preclude the possibility that there exist accumulation points of zeros of  $\{\pi_n\}$  or  $\{\pi'_n\}$  that lie off of the level curve  $\Gamma_p$ . However, the number of zeros of  $\{\pi_{n+1}\}_{n \in \Lambda}$  and of  $\{\pi'_{n+1}\}_{n \in \Lambda}$  that can lie on a given compact set disjoint from  $\Gamma_p$  is  $o(n)$ .

**Remark 2** For  $p > 1$ , Corollary 2.3 also follows from the maximal geometric convergence of the sequences  $\{\pi_n\}_0^\infty$  and  $\{\pi'_n\}_1^\infty$  and Walsh's extension of the Jentzsch theorem [15]. However, this argument does not apply to the important case  $p = 1$ .

### 3 Examples

**3.1** Consider the case where  $\Omega = \{z : |z| < 1\}$ . Then

$$Q_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \quad n = 0, 1, \dots,$$

and hence

$$k(z, \zeta) = \frac{1}{\pi} \cdot \sum_{j=0}^{\infty} (j+1)(\bar{\zeta}z)^j = \frac{1}{\pi} \cdot \frac{1}{(1-\bar{\zeta}z)^2} \quad \zeta, z \in \Omega,$$

$$k_{n-1}(z, \zeta) = \frac{1}{\pi} \cdot \sum_{j=0}^{n-1} (j+1)(\bar{\zeta}z)^j = \frac{1}{\pi} \cdot \frac{n(\bar{\zeta}z)^{n+1} - (n+1)(\bar{\zeta}z)^n + 1}{(1-\bar{\zeta}z)^2}$$

Also,



$$f_\zeta(z) = (1 - |\zeta|^2) \left( \frac{z - \zeta}{1 - \bar{\zeta}z} \right),$$

so that the mapping function  $f_\zeta$  has a simple pole at the point  $z = 1/\bar{\zeta}$  but is otherwise analytic in the extended plane. Therefore:

(i) Since  $\Phi(z) = z$ , the constant  $p$  in Theorems 2.1 and 2.2 is

$$p = \frac{1}{|\zeta|}.$$

This is, indeed, the reciprocal of

$$\lim_{n \rightarrow \infty} |Q_n(\zeta)|^{1/n} = \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n+1}{\pi}} \zeta^n \right|^{1/n} = |\zeta|.$$

(ii) If  $\zeta \neq 0$ , then according to Theorem 2.2 ( which holds with  $\Lambda = N$  )

$$v(\pi_n) \xrightarrow{*} \mu_{1/|\zeta|} \quad \text{and} \quad v(\pi'_n) \xrightarrow{*} \mu_{1/|\zeta|},$$

where  $\mu_{1/|\zeta|}$  is the equilibrium distribution

$$d\mu_{1/|\zeta|} = \frac{|\zeta|}{2\pi} ds, \quad \text{for the disc } |z| \leq 1/|\zeta|. \quad \text{That is,}$$

where  $s$  denotes arc length on the circle  $|z| = 1/|\zeta|$ . In other words,  $d\mu_{1/|\zeta|}$  is the uniform distribution on the circle  $\zeta$ . This limit behaviour can be verified directly from the explicit formulae for  $\pi'_n(z) = \{k_{n-1}(z, \zeta)/k_{n-1}(\zeta, \zeta)\}$  and  $\pi_n(z)$ .

3.2 Let  $\Omega$  be the rectangle  $\Omega = \{z = x + iy; |x| < 2, |y| < 1\}$  and set  $\zeta = 0$ . Then, the mapping function  $f_0$  is analytic on  $\partial\Omega$ , but its analytic extension has a simple pole at each of the points

$$z = 2(2k + il), \quad k, l = 0, \pm 1, \pm 2, \dots, \quad k + l = \text{odd}.$$

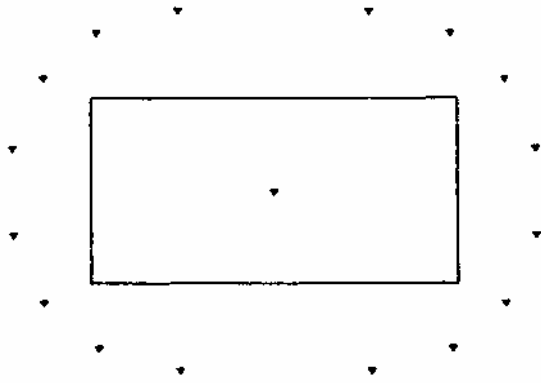
Thus, the singularities of  $f_0$  nearest to  $\partial\Omega$  occur at the points  $z = \pm 2i$  and, consequently, the value of  $\rho$  in Theorems 2.1 and 2.2 is (to 5 significant digits)

$$\rho = \Phi(2i) = 1.4095$$

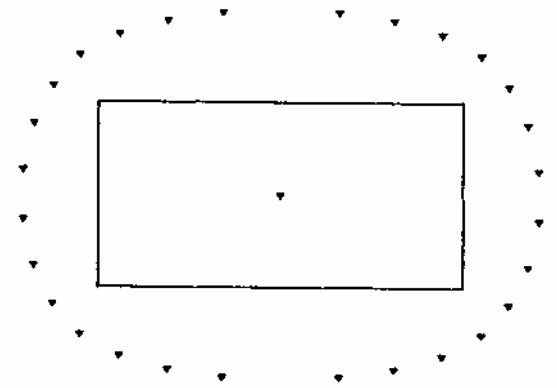
(cf. [10, p. 662]).

In Figure 1 we have plotted the zeros of the Bieberbach polynomials  $\pi_{17}$  and  $\pi_{29}$  and in Figure 2 those of their derivatives  $\pi'_{17}$  and  $\pi'_{29}$ . These zeros were obtained by using the Fortran conformal mapping package BKMPACK of Warby [16] ( for computing the Bieberbach polynomials ) and the NAG zero finding subroutine C02AEF.

Figures 3 and 4 contain, respectively, plots of the images of the zeros of  $\pi_{17}, \pi_{29}$  (with the exception of  $z = 0$ ) and of  $\pi'_{17}, \pi'_{29}$ , under the conformal map  $\phi: C \setminus \bar{\Omega} \rightarrow \{w: |w| > 1\}$ . These images were obtained from an accurate approximation to  $\phi$ , which was again computed by using BKMPACK

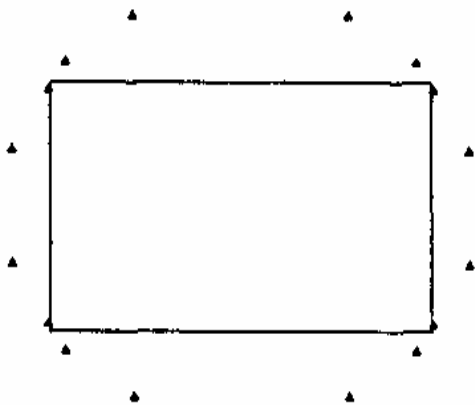


(a)  $n = 17$

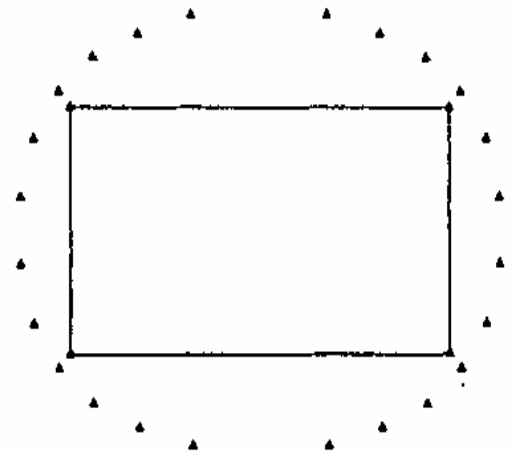


(b)  $n = 29$

**Figure 1:** Zeros of  $\pi_n$

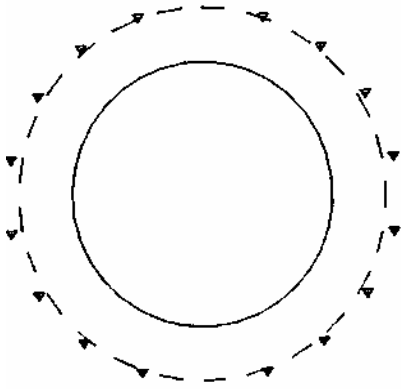


(a)  $n = 17$

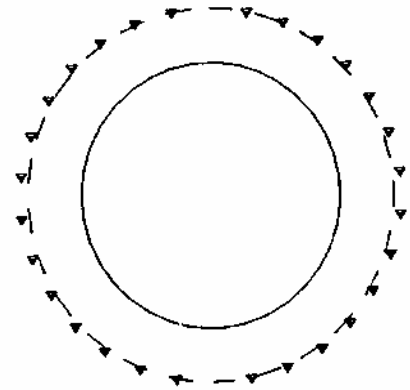


(b)  $n = 29$

**Figure 2 :** Zeros of  $\pi'_n$

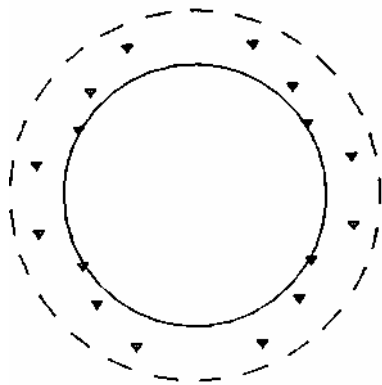


(a)  $n=17$

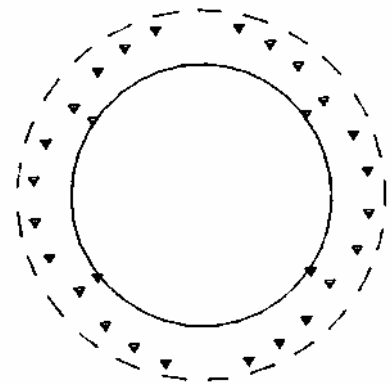


(b)  $n=29$

**Figure 3** : Images of zeros of  $\pi_n$



(a)  $n=17$



(b)  $n=29$

**Figure 4** : Images of zeros of  $\pi'_n$

We observe the following in connection with the plots in Figures 1-4 :

(i) The plots in Figures 3 and 4 illustrate the position of the images of the zeros relative to the circles  $|w| = 1$  and  $|w| = P = 1.4095$ , and hence the closeness of the zeros to the level curve  $\Gamma_\rho$ .

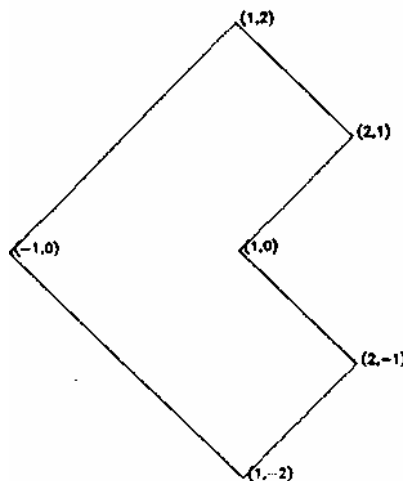
(ii) As predicted by Theorem 2.2, the zeros of the Bieberbach polynomials appear to be approaching the level curve  $\Gamma_\rho$  that corresponds to the nearest singularities of  $f_0$ . Although the zeros appear to thin out near the two singular points  $\pm 2i$ , where  $f_0$  becomes unbounded, Theorem 2.2 assures us that they do approach these points as  $n$  increases.

(iii) The behaviour of the zeros of the derivatives is similar to that described above, except that now there is always a zero close to each of the four corners of  $\Omega$ . This reflects the fact that  $f_0'$  is zero at each of these points.

3.3 Let  $\Omega$  be the L-shaped domain illustrated in Figure 5 and take  $\zeta = 0$ . In this case, the mapping function  $f_0$  has a branch point singularity at the re-entrant corner  $z_c = 1$ , in the sense that

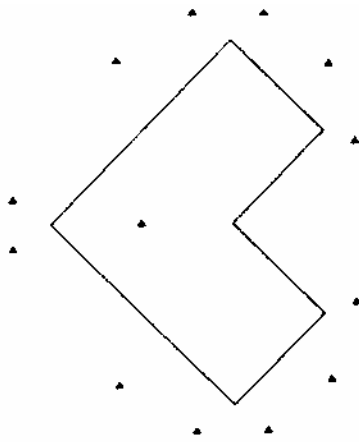
$$f_0(z) - f_0(z_c) - (z - z_c)^{2/3}, \quad \text{as } z \rightarrow z_c.$$

In addition,  $f_0$  has simple pole singularities in  $C \setminus \overline{\Omega}$ , of which the closest to  $\partial\Omega$  occur at the points  $-1 \pm i$  (cf. [10, p.663]). Since  $f_0$  has a singularity on  $\Gamma_1 = \partial\Omega$ , it follows that  $\rho = 1$  in Theorems 2.1 and 2.2.

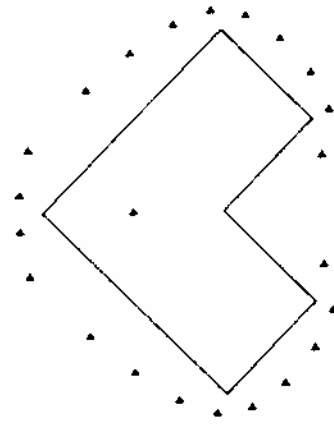


**Figure 5** : L-shaped region  $\Omega$

In Figure 6 we have plotted the zeros of the Bieberbach polynomials  $\pi_{13}$  and  $\pi_{23}$  and in Figure 7 those of their derivatives  $\pi'_{13}$  and  $\pi'_{23}$ . These zeros were again computed by using the conformal mapping package **BKMPACK** and the **NAG** zero finding routine **C02AEF**.

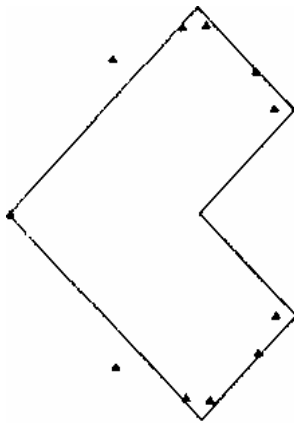


(a)  $n = 13$

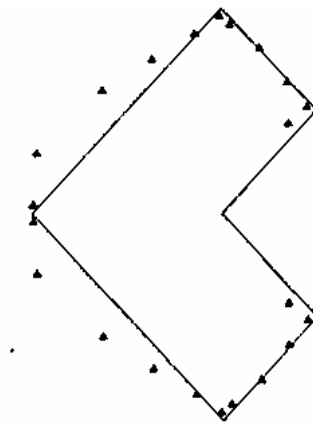


(b)  $n = 23$

**Figure 6 :** Zeros of  $\pi_n$



(a)  $n = 13$



(b)  $n = 23$

**Figure 7:** Zeros of  $\pi'_n$

As predicted by Theorem 2.2, the zeros of both the Bieberbach polynomials and their derivatives appear to be approaching the boundary  $\partial\Omega$ . In both cases, the zeros appear to thin out near the re-entrant corner  $z_c = 1$ , where  $f_0'$  becomes unbounded. They do, however, approach  $z_c$  as  $n$  increases. A similar, but less pronounced, thinning out occurs near the parts of  $\partial\Omega$  which are close to the two points  $-1 \pm i$ , where  $f_0$  becomes unbounded. Finally, in the case of the derivatives, there are always zeros close to each of the right-angled corners, reflecting the fact that  $f_0'$  is zero there.

#### 4 Proofs of Theorems 2.1 and 2.2

To establish Theorems 2.1 and 2.2 we shall make use of several lemmas. The first is due to S.N. Bernstein and J.L. Walsh.

**Lemma 4.1** ([14, p.77]) *Let  $E \subset \mathbb{C}$  be a compact set whose complement  $U := \overline{\mathbb{C}} \setminus E$  is connected and regular with respect to the Dirichlet problem. Let  $g_U(z, \infty)$  denote the Green function for  $U$  with pole at  $\infty$ .  $P_n$  is a polynomial of degree at most  $n$  and*

$$\|P_n\|_{L^\infty(E)} := \max_{Z \in E} |P_n(z)| \leq M,$$

then

$$|P_n(z)| \leq M \exp\{n g_U(z, \infty)\}, \quad z \in U. \quad (4.1)$$

**Lemma 4.2** ([14, p.28]) *If  $G$  is a bounded simply-connected domain and  $R > 1$  is given, then there exists a closed Jordan region  $E \subset G$  such that the closed region  $\overline{G}$  lies interior to the level curve*

$$l_R := \{z : g_{\overline{\mathbb{C}} \setminus E}(z, \infty) = \log R\}. \quad (4.2)$$

Combining the above lemmas we shall establish

**Lemma 4.3** *The orthonormal polynomials  $Q_n$  of Section 1 satisfy*

$$\lim_{n \rightarrow \infty} \|Q_n\|_{L^\infty(\overline{\Omega})}^{1/n} = 1. \quad (4.3)$$

**Proof** The argument is similar to that in [14, p.96]. Let  $R > 1$  be given. Then, by Lemma 4.2, there exists a closed Jordan region  $E \subset \Omega$  such that  $\overline{\Omega}$  lies interior to the level curve  $l_R$  of (4.2). Let  $r := \text{dist}(E, \partial\Omega)$ . From the well-known estimate

$$|g(z)|^2 \leq \frac{1}{\pi r^2} \iint_{\Omega} |g|^2 dm, \quad z \in E, \quad (4.4)$$

which holds for every  $g \in L^2(\Omega)$  (cf. [4, p.4]), we get

$$\| Q_n \|_{L_\infty(E)}^2 \leq \frac{1}{\pi r^2} \iint_{\Omega} | Q_n |^2 dm = \frac{1}{\pi r^2},$$

and so, by Lemma 4.1,

$$\| Q_n \|_{L_\infty(I_R)} \leq \frac{R^n}{r\sqrt{\pi}}. \quad (4.5)$$

Since  $\bar{\Omega}$  lies interior to  $I_R$ , we have by the maximum principle that

$$\| Q_n \|_{L_\infty(\bar{\Omega})} \leq \| Q_n \|_{L_\infty(I_R)}.$$

Thus, from (4.5),

$$\limsup_{n \rightarrow \infty} \| Q_n \|_{L_\infty(\bar{\Omega})}^{1/n} \leq R.$$

But as  $R$  is arbitrary, letting  $R \downarrow 1$  yields

$$\limsup_{n \rightarrow \infty} \| Q_n \|_{L_\infty(\bar{\Omega})}^{1/n} \leq 1.$$

Moreover, the inequality

$$\liminf_{n \rightarrow \infty} \| Q_n \|_{L_\infty(\bar{\Omega})}^{1/n} \geq 1$$

is an easy consequence of the fact that the  $Q_n$ 's are orthonormal. Thus, (4.3) holds.  $\square$

In the terminology of [11, §3 ], Lemma 4.3 shows that the measure  $dm$  on  $\bar{\Omega}$  is *completely regular*. Consequently ( cf. [11, Proposition 3.2 ] ), the leading coefficients  $\gamma_n$  of the polynomials  $Q_n$  satisfy

$$\lim_{n \rightarrow \infty} \gamma_n^{1/n} = \frac{1}{\text{cap}(\bar{\Omega})}. \quad (4.6)$$

Weremarkthatinthespecial case when  $\Omega$  is bounded by an analytic Jordan curve, then estimates finer than that in (4.6) can be obtained for the  $\gamma_n$ 's (cf. [4, p.12 ] ).

We can now give the :

**Proof of Theorem 2.1** Again the proof is essentially the same as that given by Walsh [14, p.130]. From (1.2) and (1.3), we have

$$K(\zeta, \zeta) f'_\zeta(z) - \sum_{n=0}^{\infty} \overline{Q_n(\zeta)} Q_n(z). \quad (4.7)$$

ie. the constants  $\overline{Q_n(\zeta)}$  are the Fourier coefficients of the function  $K(\zeta, \zeta) f'_\zeta(z)$ :

$$\overline{Q_n(\zeta)} = K(\zeta, \zeta) \iint_{\Omega} f'_\zeta \overline{Q_n} dm. \quad (4.8)$$

Now suppose that  $p > 1$  so that  $f_\zeta$  (and hence  $f'_\zeta$ ) is analytic on  $\Omega_\rho \supset \bar{\Omega}$ , and let  $p_n$  denote the polynomials of respective degrees at most  $n$  of best uniform approximation to  $f'_\zeta$  on  $\bar{\Omega}$ . From a result of Walsh [14, p.90 ], we have

$$\limsup_{n \rightarrow \infty} \| f'_\zeta - p_{n-1} \|_{L_\infty(\bar{\Omega})}^{1/n} \leq \frac{1}{\rho}. \quad (4.9)$$

Furthermore, by the orthogonality property of the  $Q_n$ 's, equation (4.8) can be written as

$$\overline{Q_n(\zeta)} = K(\zeta, \zeta) \iint_{\Omega} (f_{\zeta}' - p_{n-1}) \overline{Q_n} dm \quad (4.10)$$

Thus, from (4.10) and the Cauchy-Schwarz inequality, we get

$$\limsup |Q_n(\zeta)|^{1/n} \leq \frac{1}{\rho}. \quad (4.11)$$

Note that in the case when  $f_{\zeta}'$  is not analytic on  $\overline{\Omega}$ , that is  $\rho = 1$ , inequality (4.11) remains valid.

Next, we suppose that

$$\limsup_{n \rightarrow \infty} |Q_n(\zeta)|^{1/n} = \frac{1}{\sigma} \leq \frac{1}{\rho}. \quad (4.12)$$

and show that this leads to a contradiction. Indeed, from (4.3) and Lemma 4.1, we have for  $\sigma > \tau > \rho$

$$\limsup_{n \rightarrow \infty} \|Q_n\|_{L_{\infty}(\Omega_{\tau})}^{1/n} \leq \tau,$$

and so, by (4.12),

$$\limsup_{n \rightarrow \infty} \|Q_n(\zeta) Q_n(\cdot)\|_{L_{\infty}(\Omega_{\tau})}^{1/n} \leq \frac{\tau}{\sigma} < 1.$$

Thus, the series in (4.7) converges uniformly on  $\Omega_{\tau}$  to an analytic extension of  $K(\zeta, \zeta) f_{\zeta}'$ . But this contradicts the definition of  $\rho$  as being the largest index for which  $f_{\zeta}'$  (and, equivalently,) is analytic throughout  $\Omega_{\rho}$ .  $\square$

In the proof of Theorem 2.2 we shall make use of the following result due to Blatt, Saff and Simkani, which generalizes an earlier theorem of G. Szegő [12].

**Lemma 4.4** ([2]) *Let  $S$  be a compact set with positive capacity and suppose that the monic polynomials  $P_n(z) = z^n + \dots$ , which are given for a subsequence of indices  $n$ , say  $n \in \Lambda \subseteq \mathbb{N}$ , satisfy*

$$(a) \quad \limsup_{n \rightarrow \infty} \|P_n\|_{L_{\infty}(S)}^{1/n} \leq \text{cap}(S), n \in \Lambda,$$

and

$$(b) \quad \lim_{n \rightarrow \infty} v(P_n)(A) = 0, n \in \Lambda, \text{ for all closed sets } A \text{ contained in the (2-dimensional) interior of the polynomial convex hull of the set } S.$$

Then, in the weak-star sense,

$$v(P_n) \xrightarrow{*} \mu_s, \text{ as } n \rightarrow \infty, n \in \Lambda,$$

where  $\mu_s$  is the equilibrium distribution for  $S$ .

By the *polynomial convex hull* of  $S$  we mean the complement of the unbounded component of  $\overline{C} \setminus S$ .



**Proof of Theorem 2.2** With the subsequence  $\wedge$  as in (2.6) and with the assumption that  $\rho < \infty$ , we shall first establish that

$$v(\pi'_{n+1}) \xrightarrow{*} \mu_\rho, \text{ as } n \rightarrow \infty, n \in \wedge \quad (4.13)$$

Since

$$\pi'_{n+1}(z) = \frac{1}{K_n(\zeta, \zeta)} \sum_{j=0}^n \overline{Q_j(\zeta)} Q_j(z) \quad (4.14)$$

we see that the leading coefficient of  $\pi'_{n+1}$  is given by

$$\lambda_n := \frac{Q_n(\zeta) \gamma_n}{K_n(\zeta, \zeta)}, \quad (4.15)$$

that is,  $\pi'_{n+1} = \lambda_n z^n + \dots$ . Notice that since  $\rho < \infty$ , (2.6) gives  $\lambda_n \neq 0$  for  $n \in \wedge$  sufficiently large. Setting  $P_n := \pi'_{n+1} / \lambda_n, n \in \wedge$ , we shall show that the hypotheses (a) and (b) of Lemma 4.4 are valid with  $S = \overline{\Omega_\rho}$ .

Since  $K_n(\zeta, \zeta) \rightarrow K(\zeta, \zeta) > 0$  as  $n \rightarrow \infty$ , we deduce from (4.15), (2.6) and (4.6) that

$$\lim_{\substack{n \rightarrow \infty \\ n \in \wedge}} |\lambda_n|^{1/n} = \frac{1}{\rho \text{cap}(\overline{\Omega})} = \frac{1}{\text{cap}(\overline{\Omega_\rho})}. \quad (4.16)$$

Furthermore, it can be seen from the proof of Theorem 2.1 that  $\pi'_{n+1}$  converges locally uniformly in  $\Omega_\rho$  to  $f'_\zeta$  and that

$$\limsup_{n \rightarrow \infty} \|\pi'_{n+1}\|_{L^\infty(\overline{\Omega_\rho})}^{1/n} \leq 1. \quad (4.17)$$

Thus, from (4.16) and (4.17) we have

$$\limsup_{\substack{n \rightarrow \infty \\ n \in \wedge}} \|P_n\|_{L^\infty(\overline{\Omega_\rho})}^{1/n} \leq \text{cap}(\overline{\Omega_\rho}), \quad (4.18)$$

which establishes property (a) of Lemma 4.4. Also, since  $f'_\zeta$  can have at most a finite number of zeros in any compact subset  $A \subset \Omega_\rho$ , property (b) of Lemma 4.4 follows from the theorem of Hurwitz. Hence

$$v(P_n) = v(\pi'_{n+1}) \xrightarrow{*} \mu_\rho, \text{ as } n \rightarrow \infty, n \in \wedge.$$

Finally, we note that the leading coefficient of  $\pi'_{n+1}$  differs from that of  $\pi_{n+1}$  only by a factor  $1 / (n + 1)$  which does not affect the  $n$ th root estimates needed for applying Lemma 4.4. Thus, by the same reasoning as above, we get that

$$v(\pi_{n+1}) \xrightarrow{*} \mu_\rho, \text{ as } n \rightarrow \infty, n \in \wedge. \quad \square$$

## 5 Rational Approximants

Suppose that the mapping function  $f_\zeta$  is analytic on  $\overline{\Omega}$  (so that the constant  $\rho$  of Theorem 2.1 is greater than one), and assume that  $f_\zeta$  is analytic on  $\Gamma_\rho$  except for simple poles at the  $l$  points  $\alpha_j \in \Gamma_\rho$ ,  $j = 1, 2, \dots, l$ . Let  $\hat{\rho} (> \rho)$  denote the largest index such that

$$\hat{f}_\zeta(z) := f_\zeta(z) \prod_{j=1}^l (z - \alpha_j) \quad (5.1)$$

has an analytic (single-valued) extension throughout  $\Omega_\rho$ . Then, as discussed in [10], improved rates of convergence can be obtained in the Bergman kernel method when the defining orthonormal system is constructed by orthonormalizing the set consisting of the monomials  $1, z, z^2, \dots$  and the  $l$  rational functions

$$\eta_j(z) := \frac{1}{(z - \alpha_j)^2}, \quad j = 1, 2, \dots, l, \quad (5.2)$$

that reflect the singularities of  $K(\cdot, \zeta)$  at the points  $\alpha_j$ ,  $j = 1, 2, \dots, l$ . Our goal in this section is to study the zeros of the resulting approximations to  $f_\zeta$ .

Consider the nested spaces

$$S_k := \text{span}\{\eta_1, \dots, \eta_k\}, \quad 1 \leq k \leq l,$$

$$S_{1+j} := \text{span}\{\eta_1, \dots, \eta_l, 1, z, \dots, z^{j-1}\}, \quad j = 1, 2, \dots.$$

Let  $\{s_1, s_2, \dots, s_n\}$  denote an orthonormal basis for  $S_n$ ,  $n = 1, 2, \dots$  and, for each  $n$ , write

$$s_n(z) = \frac{P_n(z)}{q(z)^2}, \quad (5.3)$$

where

$$q(z) := \prod_{j=1}^l (z - \alpha_j) \quad (5.4)$$

and  $P_n$  is a polynomial. We note the following regarding the polynomials  $p_n$ :

(i) For  $n > l$ ,  $P_n$  has the form

$$P_n(z) = \hat{\gamma}_n q(z)^2 z^{n-l-1} + \dots = \hat{\gamma}_n z^{n+l-1} + \dots, \quad (5.5)$$

where we can assume  $\hat{\gamma}_n > 0$ .

(ii) For  $n > l$ ,  $P_n$  satisfies the  $l$  linear homogeneous constraints

$$\text{Res} \left( \frac{P_n}{q_2}, \alpha_k \right) = \frac{d}{d_z} \left\{ \frac{P_n(z)}{\prod_{\substack{j=1 \\ j \neq k}}^l (z - \alpha_j)^2} \right\} \Big|_{z=\alpha_k} = 0, \quad k = 1, \dots, l, \quad (5.6a)$$

which, for brevity, we denote by

$$L(P_n) = 0, \quad n > l. \quad (5.6b)$$

(For  $l = 1$  the empty product in (5.6a) is taken to equal 1 .)

(iii) Additional constraints hold for  $n = 1, \dots, l$ , and  $P_n$  has degree at most  $2l - 2$  for such  $n$ .

(iv) Since

$$(s_n, s_k) = \iint_{\Omega} \frac{P_n}{q^2} \overline{\left(\frac{P_k}{q^2}\right)} dm = \iint_{\Omega} P_n \overline{P_k} \frac{dm}{|q|^4}, \quad (5.7)$$

the polynomials  $P_n$  are orthonormal with respect to the weighted measure  $dm / |q|^4$  on  $\Omega$ . Hence, for  $n > l$ ,

$$\hat{P}_n(z) := \frac{P_n(z)}{\hat{\gamma}_n} = z^{n+1-1} + \dots \quad (5.8)$$

solves the extremal problem

$$\min_p \iint_{\Omega} |p|^2 \frac{dm}{|q|^4}, \quad p(z) = z^{n+1-1} + \dots, \quad L(p) = 0. \quad (5.9)$$

We next observe that the rational functions  $s_n = P_n / q^2$ ,  $n = 1, 2, \dots$ , form a complete orthonormal system for  $L^2(\Omega)$ . Therefore, the function  $K(\zeta, \zeta) f'_\zeta = K(\cdot, \zeta)$  has the Fourier series expansion

$$K(\zeta, \zeta) f'_\zeta(z) \sim \sum_{j=1}^{\infty} \overline{\left(\frac{P_j(\zeta)}{q(\zeta)^2}\right)} \frac{P_j(z)}{q(z)^2} \quad (5.10)$$

and, as in [10], this leads us to consider rational function approximations to  $f'_\zeta$  given by

$$\hat{\pi}_n(z) := \frac{1}{\hat{K}_{n-1}(\zeta, \zeta)} \int_{t=\zeta}^z \hat{K}_{n-1}(t, \zeta) dt, \quad (5.11)$$

where

$$\hat{K}_{n-1}(z, \zeta) := \sum_{j=1}^{n-1} \overline{\left(\frac{P_j(\zeta)}{q(\zeta)^2}\right)} \frac{P_j(z)}{q(z)^2}. \quad (5.12)$$

We observe that  $\hat{\pi}_n$  is rational, since

$$\hat{\pi}'_n(z) = \frac{\hat{K}_{n-1}(z, \zeta)}{\hat{K}_{n-1}(\zeta, \zeta)} \quad (5.13)$$

is clearly rational and has zero residue at each of its poles. Indeed,  $\hat{\pi}'_n$  and  $\hat{\pi}_n$  have, respectively, the forms

$$\hat{\pi}'_n(z) = \frac{g_n(z)}{q(z)^2}, \quad \text{and} \quad \hat{\pi}_n(z) = \frac{h_n(z)}{q(z)}, \quad (5.14)$$

where  $g_n$  and  $h_n$  are polynomials. Thus, for the study of the zeros of  $\hat{\pi}_n$  and  $\hat{\pi}'_n$  we define the normalized counting measures  $\nu(\hat{\pi}_n)$  and  $\nu(\hat{\pi}'_n)$  by

$$\nu(\hat{\pi}'_n) := \nu(g_n), \quad \nu(\hat{\pi}_n) := \nu(h_n), \quad (5.15)$$

where  $g_n$  and  $h_n$  are the polynomials in (5.14).

We can now establish the following analogues of Theorems 2.1 and 2.2.

**Theorem 5.1** *With the assumptions of this section,*

$$\limsup_{n \rightarrow \infty} \left| \frac{P_n(\zeta)}{q(\zeta)^2} \right|^{1/n} = \frac{1}{\hat{\rho}} < \frac{1}{\rho}. \quad (5.16)$$

**Theorem 5.2** *Suppose that the constant  $\hat{\rho}$  is finite and let  $\wedge \subseteq \mathbb{N}$  be a sequence for which*

$$\lim_{\substack{n \rightarrow \infty \\ n \in \wedge}} \left| \frac{P_n(\zeta)}{q(\zeta)^2} \right|^{1/n} = \frac{1}{\hat{\rho}} \quad (5.17)$$

Then

$$v(\mu_{n+1}) \xrightarrow{*} \mu_{\rho} \quad \text{and} \quad v(\mu_{n+1}) \xrightarrow{*} \mu_{\rho}, \text{ as } n \rightarrow \infty, \quad n \in \wedge, \quad (5.18)$$

where  $\mu_{\rho}$  is the equilibrium distribution for  $\overline{\Omega_{\rho}}$ .

Since the proofs of the above results are similar to those of Theorems 2.1 and 2.2, we shall provide only a sketch of the details.

We first observe from (5.7) and (5.10) that the constants  $\overline{p_n(\zeta)} / \overline{q(\zeta)^2}$ , which are the Fourier coefficients of  $K(\zeta, \zeta) f_{\zeta}'$  in the basis  $\{s_n\}_1^{\infty}$ , are the same as the Fourier coefficients in the expansion of the function

$$F(z) := K(\zeta, \zeta) q(z) f_{\zeta}'(z) \quad (5.19)$$

in terms of the orthonormal basis of polynomials  $\{P_n\}_1^{\infty}$  (with respect to the measure  $dm / |q|^4$  on  $\overline{\Omega}$ ). The latter expansion can easily be shown to converge maximally to  $F$  on  $\overline{\Omega}$  in the sense of Walsh [14, p. 79]. (We note that the weight  $1 / |q|^4$  does not present any difficulties, since it is bounded from above and below by positive constants on  $\Omega$ .) Thus,

$$\limsup_{n \rightarrow \infty} \|F - g_n\|_{L_{\infty}(\Omega)}^{1/n} = \frac{1}{\hat{\rho}}, \quad (5.20)$$

from which it follows (by the same reasoning as that used in the proof of Theorem 2.1) that (5.16) holds.

Next, to prove Theorem 5.2, we can imitate the proof of Theorem 2.2 provided that we first establish the analogue of (4.6), ie.

$$\lim_{n \rightarrow \infty} \hat{\gamma}_n^{1/n} = \frac{1}{\text{cap}(\overline{\Omega})}. \quad (5.21)$$

This can be seen as follows. Let  $T_n(z) = z^n + \dots$  be the monic polynomial of degree  $n$  that satisfies

$$\|T_n\|_{L_{\infty}(\overline{\Omega})} = \min \|P\|_{L_{\infty}(\overline{\Omega})},$$

where the minimum is taken over all monic polynomials  $p = z^n + \dots$  of degree  $n$ . As is well-known (cf. [13, § III.5])

$$\lim_{n \rightarrow \infty} \|T_n\|_{L_\infty(\bar{\Omega})}^{1/n} = \text{cap}(\bar{\Omega}). \quad (5.22)$$

since  $L(q^2 T_{n-l-1}) = 0$  for  $n > l$ , we see from the extremal property (5.8)-(5.9) of  $\hat{P}_n$  that

$$\frac{1}{\hat{\gamma}_n^2} = \iint_{\Omega} \frac{|\hat{P}_n|^2}{|q|^4} dm \leq \iint_{\Omega} \frac{|q^2 T_{n-l-1}|^2}{|q|^4} dm,$$

and so from (5.22) we get that

$$\liminf_{n \rightarrow \infty} \hat{\gamma}_n^{1/n} \geq \frac{1}{\text{cap}(\bar{\Omega})}. \quad (5.23)$$

On the other hand, it is known (cf, [13, § III.5]) that for any monic polynomial  $p(z) = z^n + \dots$  of degree  $n$ ,

$$[\text{cap}(\bar{\Omega})]^n \leq \|P\|_{L_\infty(\bar{\Omega})}.$$

Thus,

$$[\text{cap}(\bar{\Omega})]^{n+l-1} \leq \|\hat{P}_n\|_{L_\infty(\bar{\Omega})} = \frac{1}{\hat{\gamma}_n} \|P_n\|_{L_\infty(\bar{\Omega})},$$

and so

$$\limsup_{n \rightarrow \infty} \hat{\gamma}_n^{1/n} \leq \frac{1}{\text{cap}(\bar{\Omega})} \cdot \limsup_{n \rightarrow \infty} \|P_n\|_{L_\infty(\bar{\Omega})}^{1/n}. \quad (5.24)$$

Finally, it is easily verified that the orthonormal polynomials  $P_n$  satisfy

$$\lim_{n \rightarrow \infty} \|P_n\|_{L_\infty(\bar{\Omega})}^{1/n} = 1.$$

Hence, (5.24) yields

$$\limsup_{n \rightarrow \infty} \hat{\gamma}_n^{1/n} \leq \frac{1}{\text{cap}(\bar{\Omega})},$$

and this, together with (5.23), establishes (5.21).

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