TR/08/88 September 1988

An orthonormalization method for the approximate conformal mapping of multiply-connected domains

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Z1628775

ABSTRACT

We consider the use of an orthonormalization method for constructing approximations to one of the standard conformal maps for multiply-connected domains. The method has been used successfully in [12], but only for the mapping of doubly-connected domains. Our purpose here is to consider its application to the mapping of domains whose connectivity is greater than two.

1. Introduction

Let Ω be a finite N-connected domain, $N \geq 2$, with boundary

$$\partial\Omega, = \bigcup_{j=1}^{N} \Gamma_j$$

where Γ_j; $j = 1,2,...,N$, are closed Jordan curves and assume that:

(a) Γ_1 is the outer component of $\partial\Omega$,, i.e. Γ_1 encloses all the other Γ_j' s.

(b) The origin 0 lies in the interior of Γ_N, i.e. $0 \in \mathrm{Int}(\Gamma_N)$. Also, let $S_{1,N}$ denote a domain consisting of a circular ring

$$R := \{w: r_N < |w| < r_1 \} , \tag{1.1}$$

slit along N-2 arcs of circles

$$|w| = r_j; \quad j = 2,3,...,N-1, \tag{1.2a}$$

where

$$r_N<r_j<r_1; j=2,3,\ldots,N-1. \tag{1.2b}$$

Finally, let f denote a conformal map of Ω onto a slit circular ring of the form $S_{1,N}$ so that Γ_1 , Γ_N and Γ_j; $j = 2,3,...,N-1$, correspond respectively to the outer and inner circles bounding the ring (1.1), and to the arcs of the circles (1.2). This is one of the standard conformal maps considered, for example, in [1: Chap.VI], [3: Kap.V, 5] and [9: Chap.VII].

The choice of the mapping function f can be fixed by requiring that Γ_1 and Γ_N are mapped respectively onto the outer and inner circles of R, and imposing the normalizing condition

$$f(\zeta) = \zeta \tag{1.3}$$

where $\zeta \in \Gamma_1$ is some fixed point on the outer component of $\partial\Omega$. These conditions fix f uniquely, and also determine the value of the outer radius r_1 of R, i.e.

$$r_1 =|\zeta|. \tag{1.4}$$

The conditions also fix uniquely the values r_j; $j = 2,3,...,N$, but these are not known a priori. (That is, the radii r_j; j = 2,3,. ..,N, are unknowns of the problem of determining f: $\Omega \to S_{1,N}$.)

In the present paper we consider the use of an orthonormalization method (ONM) for computing approximations to the mapping function f of the form

$$f_n(z) := z \exp\left\{\sum_{j=1}^{n} a_j u_j(z)\right\}, \tag{1.5}$$

where $\{U_J\}$ is an appropriate set of basis functions. This method has been used successfully in [12], but only in connection with the mapping of doubly-connected domains, i.e. in connection with the case N=2 where $\Omega = Int(\Gamma_1) \cap Ext(\Gamma_2)$ and the associated canonical domain $S_{1,2}$ reduces to a circular ring of the form (1.1). The purpose of the present paper is to show,by means of numerical examples,that the **ONM** can also be used for the mapping of domains whose connectivity N is greater than two. In particular, we show that the method is capable of producing accurate approximations to f, even when the boundary $\partial\Omega$ contains sharp corners, where branch point singularities occur. This is done in the manner described for the case N=2 in [12], by introducing into the basis set $\{U_J\}$ "singular" functions that reflect the asymptotic behaviour of f in the neighbourhood of every corner where a singularity occurs.

We end this introductory section by observing that the problem of determining approximations to f: $\Omega \to S_{1,N}$ has received considerable attention only in the doubly-connected case, i.e. the case $\Omega = Int(\Gamma_1) \cap Ext(\Gamma_2)$ and f: $\Omega \to S_{1,2}$ ∂R. contrast, the case N>2 has received very little attention from the numerical point of view. For example, we are aware of only three recent articles which are concerned with computational aspects of the mappingof N-connected domains with N>2. These are as follows:

(i) The paper by Ellacott [2], where the approximation to f: $\Omega \to S_{1,N}$, N>2 is obtained by using an expansion method based on computing the minimax solution of a real and linear approximation problem.

(ii) The paper by Reichel [16], where a "fast" iterative method, based on the integral equation formulations of Symm [4,5], is used for approximating $f: \Omega \to S_{1,N}$, $N \geq 2$.

(iii) The paper by Mayo [6], where a numerical method, based on an integral equation formulation of Mikhlin [7], is used for the solution of a different but closely related conformal mapping problem. (The problem considered by Mayo differs somewhat from ours,because the canonical domain used in [6] is a slit disc rather than a slit circular ring.)

The above three references also contain numerical examples, but these involve only the mapping of domains with smooth boundaries. One important objective of the present paper is to provide further "test" examples, particularly for the mapping of domains with non-smooth boundaries, and thus to stimulate further work in the area.

2. The orthononnalxzation method (ONM)

The details of the **ONM** given in this section are essentially the same as those given in [12], in connection with the mapping of doubly-connected domains. As was observed in [12], the method follows easily from the theory contained in [1: Chap.VI],[3:Kap.V, 5]and [9: Chap.VII].

Let $f : \Omega \to S_{1,N}$, where Ω and $S_{1,N}$ are respectively the N-connected domain, $N \geq 2$, and the slit circular ring introduced in Section 1, and observe that f can be expressed as

$$f(z) = z \exp\{A(z)\} , \tag{2.1}$$

where

$$A(z) := \log f(z) - \log z , \tag{2.2}$$

is analytic and single-valued in Ω. Also, let $L_2(\Omega)$ be the Hilbert space of allsquare integrable functions which are analytic and possess a single-valued indefinite integral in Ω, and denote the inner product of $L_2(\Omega)$ by $(.,.)$, i.e.

$$(u, v) = \iint_{\Omega} u(z)\, \overline{v(z)}\, dxdy. \tag{2.3}$$

Finally, let

$$\begin{aligned} H(z) &:= A'(z) \\ &= f'(z)/f(z) - 1/Z , \end{aligned} \tag{2.4}$$

and assume that all the components Γ_j ; $j = 1,2,...,N$, of $\partial\Omega$ are piece-wise analytic. Then, the following results are either proved in [3: pp.245-251], or else can be deduced easily from results which are given there; see also [12: pp.685-689].

R2.1 *(Complex form of Green's formula)*

Let $u,v \in L_2(\Omega)$ be continuous on $\partial\Omega$ apart from a finite number of branch point singularities of the form

$$(z - z_0)^{\alpha} \ , Z_0 \in \partial\Omega, \quad -1/2 < \alpha < 0 . \tag{2.5}$$

Then,

$$(u, v) = \frac{1}{2i}\int_{\partial\Omega} u(z)\overline{\mu(z)}dz , \qquad \mu'(z) = v(z). \tag{2.6}$$

R2.2. For each function $u \in L2\ (\Omega)$, which is also continuous on $\partial\Omega$ apart from a finite number of singularities of the form (2.5),

$$(u, H) = i\int_{\partial\Omega} u(z) \log |z| dz. \tag{2.7}$$

where H is the function defined by (2.4).□

R2.3. Recall that $0 \in \mathrm{Int}(\Gamma_N)$ and let $\alpha_j \in \mathrm{Int}\ (\Gamma j) = 2,3,...,N-1$, be N-2 points lying respectively in the interiors of the simply-connected domains bounded by the curves Γ_j; $j = 2,3,...,N-1$. Also, let

$$M_j := r_1/r_j;\ j= 2,3....,N \tag{2.8}$$

Then:

$$2\pi \log M_j = \frac{1}{2i}\int_{\partial\Omega} \frac{1}{z - \alpha_j}\left(\overline{A(z)} + 2\log |z|\right) dz \ ;\ j = 2,3,..., \quad N - 1, \tag{2.9a}$$

and

$$2\pi \log M_N = \frac{1}{i}\int_{\partial\Omega} \frac{1}{z}\log|z|dz - \|H\|^2, \tag{2.9b}$$

where A and H are the functions defined by (2.2) and (2.4), and $||.||^2 = (.,.)$

Let $\{\eta_j\}$ be a complete set of the space $L_2(\Omega)$, and assume that the basis functions η_j satisfy the additional continuity requirement stated in R2.1 and R2.2. (As will be seen later, it is always possible to construct such basis sets.) Then, the results R2.2 and R2.3 suggest the following procedure for computing approximations to the mapping function f and to the quantities M_j; $j = 2,3,...,N$:

(i) Orthonormalize the set $\left\{\eta_j\right\}_{j=1}^{n}$ bymeans of the Gram-Schmidt process to obtain the orthonormal set $\left\{\eta_j^*\right\}_{j=1}^{n}$.

(ii) Approximate the function H by the Fourier series sum

$$H_n(z) = \sum_{j=1}^{n}(H, \eta_j^*)\,\eta_j^*(z), \tag{2.10}$$

where the Fourier coefficients (H, η_j^*) are know by means of (2.7)

(iii) Approximate the mapping function f by

$$f_n(z) = z \exp\{A_n(z)\}, \tag{2.11a}$$

where

$$A_n(z) = \int_\zeta^z H_n(t)\,dt; \tag{2.11b}$$

see Eqs (2.1) and (2.4), and recall the normalizing condition (1.3).

(iv) Approximate the quantities M_j; $j = 2,3,...,N$, by:

$$M_j^{\{n\}} = \exp\left\{\left[\frac{1}{2i}\int_{\partial\Omega}\frac{1}{z-\alpha_j}\left(\overline{A_n(z)} + 2\log|z|\right)dz\right]/2\pi\right\};\ j = 2,3,...,\quad N-1 \tag{2.12a}$$

and

$$M_N^{\{n\}} = \exp\left\{\left[\frac{1}{i}\int_{\partial\Omega}\frac{1}{z}\log|z|dz - \|H_n\|^2\right]/2\pi\right\}; \tag{2.12b}$$

see Eqs (2.9).

3. Computational details

The details of the **ONM** procedure of the present paper can be deduced easily from those given in [12], in connection with the mapping of doubly-connected domains. However, for the sake of completeness, we also give a summary of these details here.

3.1 Choice of the basis set $\{\eta_j\}$

Let $\alpha_j \in \mathrm{Int}(\Gamma_j)$;j=2,3,. . . ,N, be N-l points in the interiors of the curves Γ_J ; j = 2,3,...,N. (In particular, since 0 $\in$ Int (Γ_j) We may take $\alpha_N := 0$.) Then, it is well-known that the functions

$$\left.\begin{array}{ll} & z^m;\ m=0,1,2,\ldots, \\ \text{and} & (z-\alpha_j)^{-m};\ j=2,3,\ldots,N,\ m=2,3,\ldots, \end{array}\right\} \qquad (3.1)$$

form a complete set with respect to the space $L_2(\Omega)$; see e.g[3: p244]. That is, the set (3.1) provides a basis for the **ONM**. However, for the reasons explained in [12], the use of this basis is not recommended in cases where the domain under consideration involves "singular" corners. (By singular corners we mean corners where the function H has branch point singularities.) In such cases the basis set should be constructed as describedin [12], by introducing appropriate "singular" functions into the set (3.1). These singular functions are determined as follows:

Let part of a boundarycurve Γj (j = 1,2,...,N) consist of two analytic arcswhich meet at a point z_C and form therea corner of interior angle $W\pi$, wherew := p/q > 0 is a fraction reduced to its lowest terms. (By interior angle we mean interior to the domain Ω.) Then, it can be shown that the asymptotic expansion of H in the neighbourhood of z_C involves terms that can be written in the following form:

$$\eta_C(z) := \begin{cases} \dfrac{d}{dz}\left\{(z-z_C)^r\right\} & \text{if } Z_c \quad \Gamma_1 \quad , & (3.2a) \\ \dfrac{d}{dz}\left\{\left[1/(z-z_C)-1/(z_C-\alpha_j)\right]^r\right\}, & \text{if } Z_c \quad \Gamma_j\ ,\ j \neq 1,N\ , & (3.2b) \\ \dfrac{d}{dz}\left\{(1/z-1/z)^r\right\}, & \text{if } Z_c \quad \Gamma_N & (3.2C) \end{cases}$$

where, in general,

$$r = k + \ell/w\ ;\quad k = 0,1,2,...,\quad 1 \le 1 \le p\ . \qquad (3.2d)$$

If however $z_c \in \Gamma_1$ and both arms of the corner are straight lines, then

$$r = \ell/w\ ;\quad \ell = 1,2,...\ . \qquad (3.2e)$$

This shows that if $p\neq l$ then a branch point singularity occurs at the point z_c. In such a case, the basis set is formed by introducing into the set (3.1) the first few singular functions of the sequence (3.2), corresponding to the first few fractional values of r.

Further details concerning the construction of appropriate basis sets for dealing with corner and other singularities in expansion methods for numerical conformal mapping can be found in [8,10,12,15]. Here, we only note the following in connection with the use of the **ONM**.

(a) If the domain Ω has M-fold,$M\geq 2$, rotational symmetry about the origin, then the number of basis functions used in the **ONM** can be reduced considerably, in the manner indicated in [12].

(b) The function H may also have "pole-type" singularities, i.e. singularities that occur off the boundary curves in compl$(\Omega \cup \partial\Omega)$.The problem of dealing with such singularities is studied in [13,15], but only in connection with the doubly-connected case N=2.

3.2 Evaluation of inner products

The main computation involved in the construction of the ONM approximations (2.11) and (2.12) occurs in the orthonormalization of the basis set $\{\eta_j\}_{j=1}^{n}$ and, in particular, in the evaluation of the inner products (η_r, η_s)

Since the basis functions (3.1) and (3.2) satisfy the boundary continuity requirement needed for the results R2.1 and R2.2, these inner products are expressed as

$$(\eta_r,\eta_s)=\frac{1}{2i}\int_{\partial\Omega}\eta_r(z)\overline{\mu_s(z)}dz, \quad \mu'_s(z)=\eta_s(z), \tag{3.3}$$

and are then computed by Gauss-Legendre quadrature. Similarly, each of the inner products (H,η^*_j)nvolved in the approximation (2.10) is computed by applying to the contour integral

$$(H, \eta^*_j) = i\int_{\partial\Omega} \eta^*_j \log |z| dz, \tag{3.4}$$

the Gaussian rule used for the evaluation of (3.3). Finally, the same Gaussian rule is used for evaluating the contour integrals involved in the approximations (2.12).

When performing the above quadratures care must be taken to deal with the integrand singularities that occur when, due to the presence of a corner at z_c, the basis set contains singular functions of the form (3.2). The effects of such integrand singularities can either be reduced sufficiently or, in many cases, removed completely, by using suitable parameterizations of the boundary curves Γj. The details concerning the choice of appropriate parameterizations can be deduced from the examples given in [10,11,12].

3.3 Convergence - Error estimates

It follows at once from (2.10) that

$$\lim_{n\to\infty}\|H_n - H\| = 0,$$

and, in the space $L_2(\Omega)$, this norm convergence implies that $H_n(z) \to H(z)$ uniformly in every compact subset of Ω. Hence, we have that $f_n(z)\to f(z)$ uniformly in every compact subset of Ω. In addition, for the doubly-connected case, $\partial\Omega = \Gamma_1 U \Gamma_2$, we have a result due to Gaier [3] which establishes the uniform convergence in $\Omega = \Omega U\partial\Omega$, of the **ONM** approximations $\{f_n\}$ corresponding to the use of the set(3.1), i.e. of the set

$$Z^j; \quad j=0,1,2,\ldots, \qquad z^{-j}; j=2,3\ldots\ldots,$$

This result is proved in [3: p. 250] under the assumption that the two components Γ_1 and Γ_2 of $\partial\Omega$ are analytic Jordan curves, and also in [14] under the somewhat less restrictive assumption that the mapping function f is analytic on $\partial\Omega = \Gamma_1 \cup \Gamma_2$ (we point out that the poor of [3] is not for the ONM, but for an equivalent variational method.) Unfortunately, even for the doubly-connected case, we do not know of any theoretical results which establish the uniform convergence of $\{f_n\}$ in $\overline{\Omega}$, in cases where the components Γ_j of $\partial\Omega$ are general piecewise analytic Jordan curves. We do have, however, a reliable method for computing estimates of the maximum error in the modules of the approximate conformal maps. This is done as follows:

As was previously remarked, the normalizing condition (1.3) implies that the outer radius of the ring R is

$$r_1 = |\zeta| \quad . \tag{3.5}$$

Therefore, the radii r_j; j = 2,3,...,N-1, of the circular arcs corresponding to the boundary curves Γ_j; j = 2,3,...,N-1, and also the inner radius r_N of R can be approximated respectively by

$$r_n^{\{j\}} = |\zeta|/M_j^{\{n\}} \quad J=2,3,\ldots\ldots N \tag{3.6}$$

where $M_j^{\{n\}}$ are the appproximations given by (2.12)

Let

$$\varepsilon_1^{\{n\}} := \max_{z \in \sigma_1} \left| r_1 - |f_n(z)| \right|, \tag{3.7a}$$

and

$$\varepsilon_j^{\{n\}} := \max_{z \in \sigma_j} \left| r_j^{\{n\}} - |f_n(z)| \right|, \quad j = 2,3,\ldots, N, \tag{3.7b}$$

where σ_j; j = 1,2,,..,N, are N sets of "test points" on the boundary curves Γ_j; j = 1,2,...,N, respectively. Then the quantity

$$E_n := \max_{1\le j\le N} \varepsilon_j^{\{n\}} \tag{3.8}$$

gives an estimate of the maximum error in $|f_n(z)|$. We expect this to be a reasonable estimate because, in general, the approximations $M_j^{\{n\}}; j = 2,3, \ldots, N$, are much more accurate than $|f_n(z)|$, $z \quad \partial\Omega$.

3.4 Optimum number of basis functions

The ONM is programmed so that it computes recursively a sequence of approximations $\{f_n\}$, where at each stage the number n of basis functions used is increased by N. (As before, N denotes the connectivity of Ω.) The algorithm also includes a criterion for terminating the process at some "optimum" value n = nopt which gives a "best" approximation f_{nopt} in some pre-defined sense. This optimum number is determined in a manner similar to that used in [8,10,12], by means of the following process:

A minimum number $n_{min} := kN$, $k \geq 1$, of basis functions to be used is defined and, for each $\ell.N$, $\ell \geq k$, the error estimate E_n is computed in the manner described in Section 3.3. If when $n = (\ell+1)N$ the inequality

$$E_{(\ell+1)N} < E_{\ell N}, \tag{3.9}$$

is satisfied, then the approximation $f_{(\ell+2)N}$ is computed. When for a certain ℓ, due to numerical instability, (3.9) no longer holds then the process is terminated and ℓN is taken to be the optimum number nopt of basis functions. To safeguard against "slow" convergence, the process also includes a maximum number n_{max} of basis functions to be used. If the inequality (3.9) is still satisfied when $(\ell+1)N = n_{max}$, then we terminate the process and write $n\overset{*}{o}pt = n_{max}$.

4. Numerical examples

Many numerical examples illustrating the application of the ONM to the mapping of doubly-connected domains, including domains with sharp corners, can be found in references [12,13]. Here, we present six examples whose purpose is to confirm our claim that the ONM can also be used for the mapping of N-connected domains with N>2.

In each example, we identify the domain under consideration by listing the equations of the boundary curves Γ_j ; j = 1,2,...,N, where as before: (a) Γ_1 denotes the outer boundary component which is mapped onto the outer circle of the ring R. (b) Γ_N denotes the inner boundary component which contains the origin in its interior, and which is mapped on to the inner circle of R. In each case, we also list the following:

Special points: These are the point $\zeta \in \Gamma_1$, used in the normalizing condition (1.3), and the points $\alpha_j \in Int(\Gamma j)$; j = 2,3,...,N-1, which occur in the basis set (3.1), the singular functions (3.2b) and the approximations (2.12a).

Basis sets: Here, we list the singular functions used for augmenting the set (3.1), and also indicate the simplifications that occur when the domain under consideration has rotational symmetry.

Boundary test points: These are the N sets of boundary points σj; j = 1,2,...,N, used for computing the error estimates (3.7) and (3.8); see Section 3.3.

In each example, the inner products (3.3), (3.4) and the contour integrals in (2.12) are computed, as indicated in [12: 5], by using a composite Gaussian rule based on the 48-point Gauss-Legendre formula. As was remarked in Section 3.2, if the basis set contains singular functions of the form (3.2) then the resulting integrand singularities are treated by using special parametric representations for the curves Γ_j. These representations can be deduced easily from those given in the examples of [10,11,12].

In each case, the numerical results presented are the values nopt, E_{nopt} and $r_j^{\{nopt\}}$; j =2,3,,...,N, giving respectively the optimum number of basis functions, the estimate of the maximum error in modulus of the ONM approximation f_{nopt}, corresponding ONM approximations to the radii

r_j ; $j = 2,3,...,N$; see Sections 3.3 and 3.4. In presenting the results
we use the following abbreviations:

ONM/MB: Results obtained by using the **ONM** with "monomial basis" {3.1).

ONM/AB: Results obtained by using the **ONM** with "augmented basis",formed by introducing into the set (3.1) singular functions of the form (3.2).

All computations were carried out on a **CYBER 170-720** computer, using programs written in single precision Fortran. Single length working on the **CYBER 170-720**, is between 14 and 15 significant figures.

Example 1 Ellipse/3 circles; see Fig.1(a).

Boundary curves:

$$\Gamma_1 := \{(x,y): x = 2\cos x,\ y = \sin\tau,\ 0 \le \tau < 2\pi\}\ , \tag{4.1a}$$

$$\Gamma_{2,3} := [z: z = \pm 1 + 0.25e^{i\tau},\ 0 \le \tau \le 2\pi\}\ , \tag{4.1b}$$

$$\Gamma_4 := \{z: z = 0.25e^{i\tau}\ 0 \le \tau \le 2\pi\}\ . \tag{4.1c}$$

Special points: $\zeta = 2$, $\alpha_{2,3} = \pm 1$.

Basis sets: Because the domain has twofold rotational symmetry about the origin the monomial set is taken to be:

$$\left.\begin{array}{l} z^{2m-1};\ m = \pm 1, \pm 2, \ldots\ , \\ (z-\alpha_2)^{-m-1} + (-1)^m (z-\alpha_3)^{-m-1};\ m = 1,2,3,\ldots\ . \end{array}\right\} \tag{4.2}$$

and

Since the domain does not involve corners, we do not need to use an augmented basis in this example.

Boundary test points: These are defined by(4.1)with τ=0(π/16)2π.

Numerical results:

$$r_1 = |\zeta| = 2\ .$$

ONM/MB: nopt = 48, $E_{48} = 9.0 \times 10^{-8}$

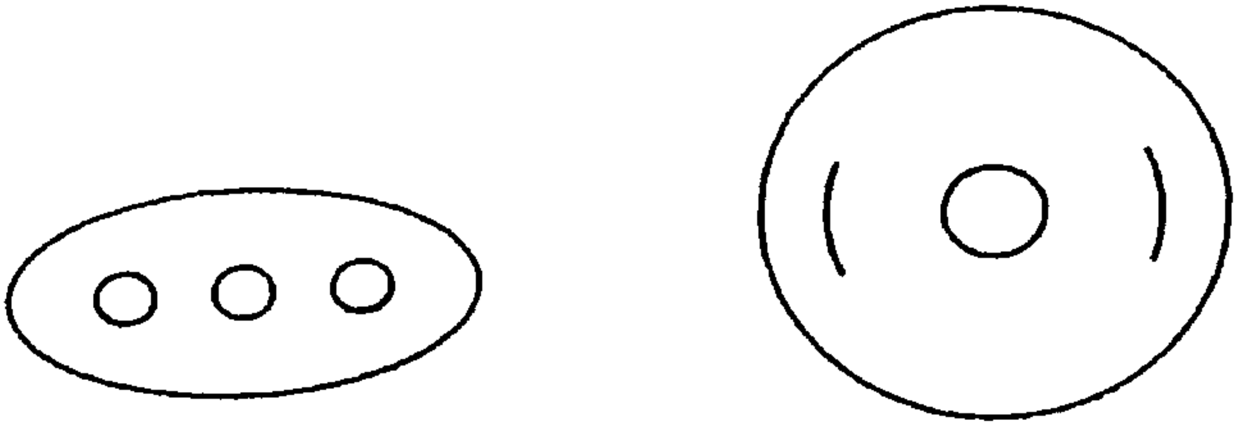
$$r_2^{\{48\}} = r_3^{\{48\}} = 1.435\ 887\ 996, \qquad r_4^{\{48\}} = 0.443\ 083\ 306 \tag{4.3}$$

The plot of the approximate image domain is given in Fig..1(b).

The values (4.3) should be compared with the approximations

$$\tilde{r}_2 = \tilde{r}_3 = 1.4362, \quad \tilde{r}_4 = 0.4430,$$

obtained by Ellacott [2].



(a) (b)

FIGURE 1

Example 2 Ellipse/2 circles; see Fig.2(a)

$$\Gamma_1 := \{(x,y):\ x = 2\cos\tau,\ y = \sin\tau,\ 0 \le \tau \le 2\pi\}, \tag{4.4a}$$

$$\Gamma_2 := \{z:\ z = 1.2 + 0.3e^{i\tau},\ 0 \le \tau \le 2\pi\}\ , \tag{4.4b}$$

$$\Gamma_3 := \{z:\ z = 0.5e^{i\tau},\ 0 \le \tau \le 2\pi\}. \tag{4.4c}$$

Special points: $\zeta = 2$, $\alpha_2 = 1.2$.

Basis sets: Since in this case the domain does not have rotational symmetry, the monomial basis is given by (3.1) with N = 3, i.e.

and $$\left.\begin{array}{l} z^m;\ m = 0,1,2,\ldots\ . \\ z^{-m};\ (z-1.2)^{-m};\ m = 2,3,\ldots\ . \end{array}\right\}$$

The domain does not contain corners and, as in Example 1, we do not need to use an augmented basis.

Boundary test points: These are defined by (4.4)with $\tau=0\ (\pi/8)2\pi$.

Numerical results;

$$r_1, = |\zeta| = 2.$$

OHM/MB: nopt = 48, $E_{48} = 4.1 \times 10^{-5}$

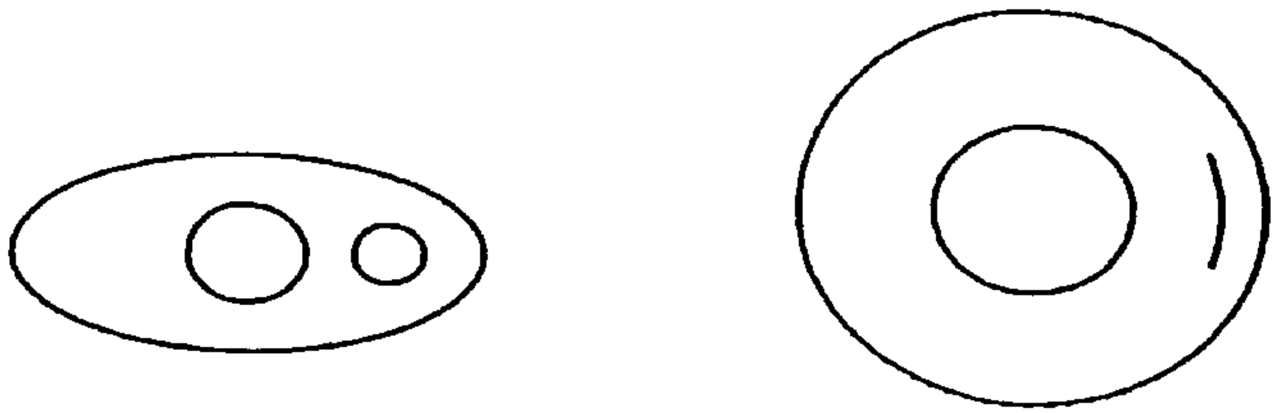
$$r_2^{\{48\}} = 1.621\ 941\ 59, \quad r_3^{\{48\}} = 0.851\ 773\ 08\ . \tag{4.5}$$

The plot of the approximate image domain is given in Fig.2(b).

The values (4.5) should be compared with the approximations

$$\tilde{r}_2 = 1.621\ 941\ 60, \quad \tilde{r}_3 = 0.851\ 773\ 07\ , \tag{4.6}$$

obtained by Reichel in [16}. (Because of the different normalizing condition used in [16], the approximations (4.6) were obtained by multiplying those given in [16] by 4/3.)



(a) (b)

FIGURE 2

Example 3 Square/3 circles; see Fig.3 (a)

Boundary curves:

$$\Gamma_1 := \{(x,y): 1x1 = 2, |y| < 2\} \cup \{(x,y): 1x1 < 2, |y| = 2\}$$

$\Gamma_{2,3} := \{z: z = \pm 1 + 0.2e^{i\tau},\ 0 \leq \tau \leq 2\pi\}$,

$\Gamma_4 := \{z: z = 0.4e^{i\tau}\ \ 0 \leq \tau \leq 2\pi\}$.

Special points: $\zeta, = 2, \quad \alpha_{2,3} = \pm 1$.

Basis sets: Because of the twofold symmetry, the monomial basis is taken to the set (4.2).

The domain does not involve "singular" corners; see Section 3.1. Therefore,as in Examples 1 and 2, we do not need to use an augmented basis.

Boundary test points: Thirty-two equally spaced points on each boundary curve.

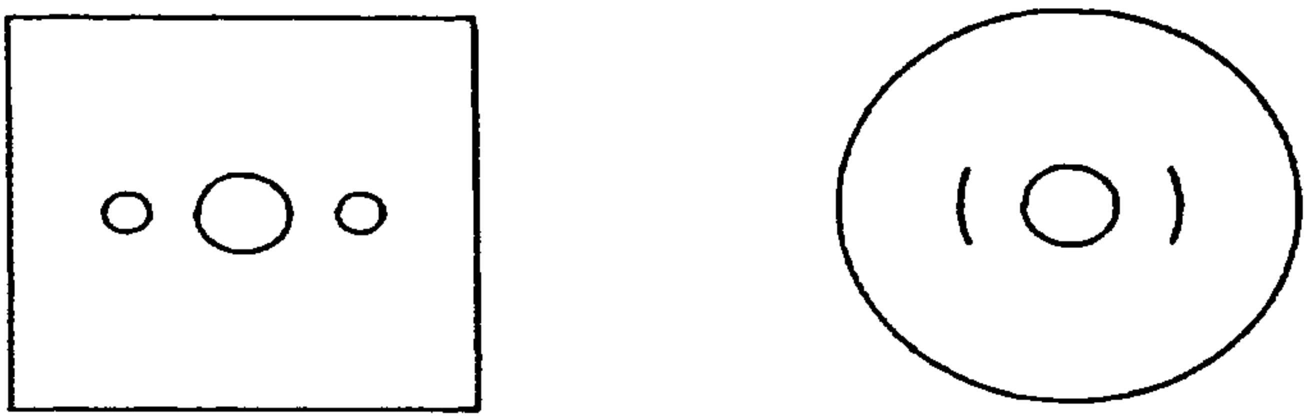
Numerical results:

$$r_1 = |\zeta| = 2 \ .$$

ONH/MB: $\overset{*}{\text{nopt}} = 52, \quad E_{52} = 4.3 \times 10^{-5},$

$$r_2^{\{52\}} = r_3^{\{52\}} = 0.954\ 497\ 52, \quad r_4^{\{52\}} = 0.403\ 650\ 33 \ .$$

The plot of the approximate image domain is given in Fig.3(b).



(a) (b)

FIGURE 3

Remark 1. The numerical results of Examples 1-3, illustrate the high accuracy that can be achieved by the **ONM/MB**, when the domain under consideration does not involve "singular" corners.

Example 4 Circle/2 circles/Square; see Fig.4(a).

Boundary curves;

$$\Gamma_1 := \{z: z = 2e^{i\tau}, 0 \leq \tau \leq 2\pi\} ,$$

$$\Gamma_{2,3} := \{z: z = \pm 1 + 0.4e^{i\tau}, 0 \leq \tau \leq 2\pi\}$$

$$\Gamma_4 := \{(x,y): |x| = 0.2, |y| < 0.2\} \cup \{(x,y): |x| < 0.2, |y| = 0.2\}$$

Special points: $\zeta_1 = 2$, $\alpha_{2,3} = \pm 1$.

Basis sets: Because of the twofold symmetry, the monomial basis is taken to be the set (4.2).

The singular functions corresponding to the branch point singularities at each of the four corners of the square Γ_4 are given by (3.2c) with

$$r = k + 2\ell/3, \quad k = 0,1,2,\ldots, \quad 1 \leq \ell \leq 3 .$$

Because of the symmetry, for each value of r, the singular functions associated with the four corners of Γ_4 can be combined into two "symmetric" singular functions; see [12: Ex.5.3]. In this example, the augmented basis is formed by introducing into the set (4.2) the eight symmetric singular functions corresponding to the values r = 2/3, 4/3, 5/3, 7/3.

Boundary test points: Thirty-two equally spaced points on each of the boundary curves.

Numerical results:

$$r_1 = |\zeta| = 2 .$$

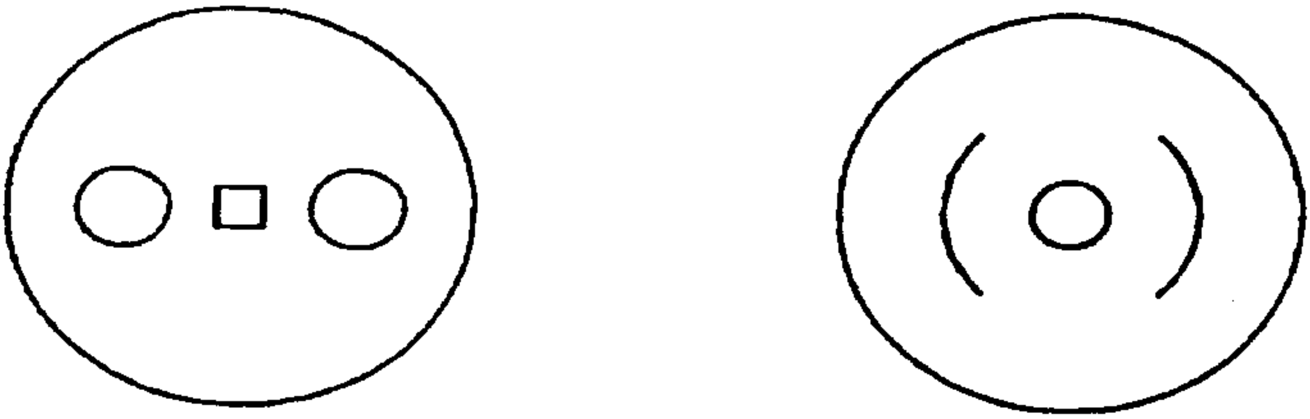
ONM/MB: $\overset{*}{n}opt = 52$, $E_{52} = 3.6 \times 10^{-2}$,

$$r_2^{\{52\}} = r_3^{\{52\}} = 1.107\ 54\ldots.. \quad r_4^{\{52\}} = 0.329\ldots..$$

ONM/AB: nopt = 44, $E_{44} = 8.3 \times 10^{-7}$,

$$r_2^{\{44\}} = r_3^{\{44\}} = 1.107\ 585\ 361, \quad r_4^{\{44\}} = 0.333\ 126\ 300 .$$

The plot of the approximate image domain is given in Fig.4(b).



(a) (b)

FIGURE 4

Example 5 Square/2 squares/Circle; see Fig.5(a)

Boundary curves:

Γ_1: = {(x,y): |x| = 2, |y| < 2} U {(x,y): 1x1 < 2, |y| = 2} ,

$\Gamma_{2,3}$:= {(x,y): lx±l, = 0.4, |y| < 0.4} U {(x,y): lx±l < 0.4, |y| = 0.4},

Γ_4:= {z: z = $0.2e^{i\pi}$, 0 $\leq$ τ $\leq$ 2π} .

Special points: ζ , = 2, $\alpha_{2,3}$ = +1 .

Basis sets: Because of the symmetry the monomial basis is taken to be the set (4.2).

The singular functions corresponding to the branch point singularities at each of the eight corners of the squares Γ_2 and T_3 are given by (3.2b) with

$$r = k + 2\ell/3, \quad k = 0,1,2,\ldots, \quad 1 \leq \ell \leq 3 .$$

Because of the symmetry, for each value of r, the singular functions associated with the eight corners can be combined into four "symmetric" singular functions. In this example the augmented basis is formed by introducing into the set (4.2) the sixteen symmetric singular functions corresponding to the values r = 2/3, 4/3, 5/3, 7/3.

Boundary test points: Thirty-two equally spaced points on each of the boundary curves.

Numerical results:

$$r_1 = |\zeta| = 2 \ .$$

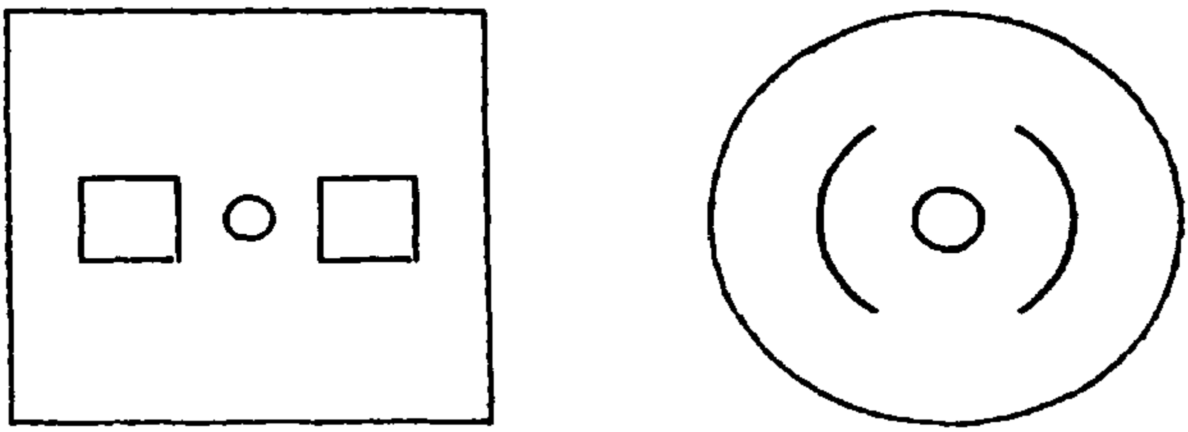
ONM/MB: $\overset{*}{\text{nopt}} = 52, \quad E_{52} = 4.7 \times 10^{-2},$

$$r_2^{\{52\}} = r_3^{\{52\}} = 1.075\ldots\ldots \quad r_4^{\{52\}} = 0.290\ldots\ldots$$

OHH/AB: $\text{nopt} = 40, \quad E_{40} = 5.5 \times 10^{-4},$

$$r_2^{\{40\}} = r_3^{\{40\}} = 1.078\ 579\ 57 \qquad r_4^{\{40\}} = 0.291\ 814\ 83.$$

The plot of the approximate image domain is given in Fig.5(b).



(a) (b)

FIGURE 5

Example 6 Cross/3 circles; see Fig.6(a)

Boundary curves:

$$\Gamma_1 := \bigcup_{j=1}^{4} \gamma_j$$

where

γ_1 := {(x,y): x∈[-3,-1] ∪ [1,3], |y| = 1} ,

γ_2 := {(x,y): 1x1 = 1, y∈ [-3,-l] ∪ [1,3]},

γ_3 := {(x.y) :1x1 < 1, |y| = 3} ,

γ_4 := {(x,y): 1x1 = 3, |y| < 1} ,

$\Gamma_{2,3}$:= {z: z = ±2 + 0.25$e^{1\tau}$ 0 < τ < 2π} ,

Γ_4:= {z: z = 0.4e^{1T}, 0 ≤ τ ≤ 2π>.

Special points: ζ, = 3, $\alpha_{2,3}$ = ±2.

Basis sets: Because of the symmetry the monomial set is taken to be the set (4.2) .

The singular functions corresponding to the branch point singularities at each of the four re-entrant corners of the cross are given by (3.2a) with

$$r = 2\ell/3; \qquad \ell = 1,2.........,$$

Because of the symmetry, for each value of r, the singular functions associated with the four re-entrant corners can be combined into two "symmetric" singular functions. In this example the augmented basis is formed by introducing into the set (4.2) the eight symmetric singular functions corresponding to the values r = 2/3, 4/3, 8/3, 10/3.

Boundary test points: Fifty-six equally spaced points on each boundary curve.

Numerical results:

$$r1 = |\zeta| = 3 .$$

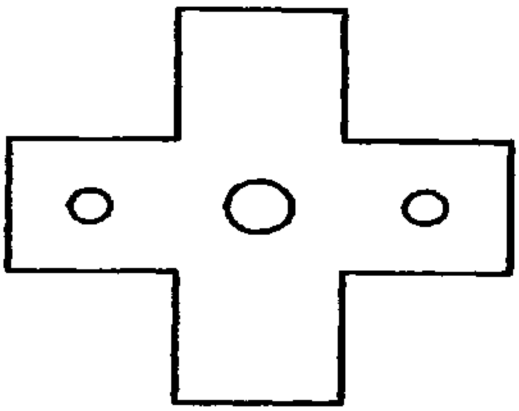
ONM/MB: nopt = 32, $E_{32} = 9.5 \times 10^{-1}$

$$r_2^{\{32\}} = r_3^{\{32\}} = 2.576 \ldots, \quad r_4^{\{32\}} = 0.607$$

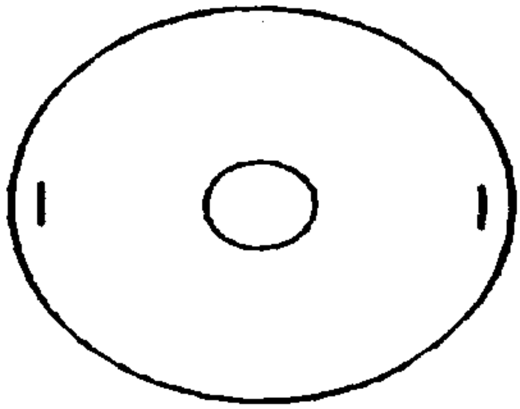
ONM/AB: $\overset{*}{\text{nopt}}$ = 52, $E_{52} = 2.0 \times 10^{-4}$

$r_2^{\{52\}} = r_3^{\{52\}} = 2.646\ 750\ 76, \quad r_4^{\{52\}} = 0.671\ 826\ 15.$

The plot of the approximate image domain is given in Fig.6(b).



(a)



(b)

FIGURE 6

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