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Implicit methods for the simple wave equation

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ABSTRACT

A family of finite difference methods is developed for the numerical solution of the simple wave equation. Local truncation errors are calculated for each member of the family and each is analyzed for stability. The concepts of A_0-stability and L_0-stability, well-used in the literature on other types of partial differential equation, are discussed in relation to second order hyperbolic equations. The numerical methods are extended to cover two-dimensional wave equations and the methods developed in the paper are tested on three problems from the literature.

1. INTRODUCTION

The simple wave equation $\partial^2u/\partial t^2 = \partial^2u/\partial x^2$, with appropriate initial and boundary conditions specified, is solved by approximating the space derivative $\partial^2u/\partial x^2$ by the familiar three point central different replacement and solving the resultant linear system of second order ordinary differential equations for u.

It was shown in [1] that the theoretical solution of this system satisfies a recurrence relation involving the matrix exponential function. The family of numerical methods developed is found by replacing this matrix exponential function with Padé approximants; the family is seen to contain the most widely used explicit and implicit finite difference methods and the explicit method of Twizell [1]. Local truncation errors are calculated for each method.

In the numerical analysis of finite difference methods for second order parabolic equations and first order hyperbolic equations, recent papers have emphasised the concepts of A_0-stability and L_0-stability (see, for instance [2,3,4]). It is shown in §3 of the present paper that, by rewriting the recurrence relation on which the novel methods are based as a system of relations, these stability concepts can be used in the study of numerical methods for second order hyperbolic equations also, and that they are connected to the conventional methods of analysis. Following Lawson and Morris [3] and Gourlay and Morris [4], and using their terminology in relation to second order parabolic equations, amplification symbols are drawn for the numerical methods developed in §2. The graphs of these symbols enable the quick classification of a numerical method as A_0-stable, L_0-stable, conditionally stable, or unstable.

The extension to the wave equation in two space variables is carried out in §4 and in §5 the numerical methods developed in § §2,4 are tested on two problems from the literature on the one-dimensional wave equation and one problem on the two-dimensional wave equation.

2. THE ONE-DIMENSIONAL WAVE EQUATION

2.1 The space discretization and a recurrence relation

Given the simple wave equation in one space variable

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

over a region R = [0<x<1] x [t>0] with boundary conditions

$$u(0,t) = u(1,t) = 0 \quad ; \quad t > 0 \tag{2.2}$$

and initial conditions

$$u(x.0) = f(x), \quad \frac{\partial u}{\partial t}(x, o) = g(x); \quad 0 \leq x \leq 1, \tag{2.3}$$

where f(x) and g(x) are continuous functions of x, one method of solution is to replace the space derivative in (2.1) with a suitable finite difference approximation and then to solve the resulting linear system of second order ordinary differential equations in which t is the independent variable.

The interval $0 \leq x \leq 1$ is divided into N + 1 subintervals each of width h and a uniform grid of width h is superimposed on the space variable so that (N+1)h = 1. The independent variable t is discretized in steps of length ℓ. The region R and its boundary ∂R have thus been discretized at the points $(mh,n\ell)$ where m = 0,1,...,N+1 and n = 0,1,2,... . The solution $u(mh,n\ell)$ of (2.1) at the mesh point $(mh,n\ell)$ will be denoted by u_m^n, the theoretical solution of an approximating finite difference scheme by U_m^n, and the numerical value actually obtained by $\tilde{U}_m^n$.

Following Twizell [1], the space derivative in (2.1) may be replaced by

$$\frac{\partial^2 u}{\partial x^2} = h^{-2}\{u(x-h,t) - 2u(x,t) + u(x+h,t)\} + 0(h^2) . \tag{2.4}$$

Then, (2.1) with (2.4) are applied to all N interior mesh points at time level $t = n\ell$ $(n = 0,1,2,\ldots)$ to produce a system of N second order ordinary equations given by

$$\frac{d^2 \underset{\sim}{U}(t)}{dt^2} = A \underset{\sim}{U}(t) . \tag{2.5}$$

In (2.5), $\underset{\sim}{U}^n = \underset{\sim}{U}(n\ell)$ is the vector of order N having, as elements, the values U_m^n (m - 1 ,2, . . . ,N; n = 0,1,2,...). The matrix A is given by

$$A = h^{-2}\begin{bmatrix} -2 & 1 & & & & 0 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -2 \end{bmatrix} \tag{2.6}$$

which has eigenvalues $\lambda_s = -4h^{-2} \sin^2\{s\pi/2(N+1)\}$, s = 1,2,...,N.

It is known (Twizell [1]) that the solution of (2.5) with (2.3) satisfies the recurrence relation

$$\underset{\sim}{U}(t-\ell) - (\exp(\ell B) + \exp(-\ell B)\} \underset{\sim}{U}(t) + \underset{\sim}{U}(t-\ell) = \underset{\sim}{0} \tag{2.7}$$

with $t = \ell, 2\ell, \ldots$, where $\underset{\sim}{0}$ is the zero—vector of order N and B is a matrix such that $B^2 = A$. It is this recurrence relation which forms the basis for a family of finite difference methods for solving (2.1) with (2.2), (2.3).

2.2 *Solution at the first time level*

Starting values for (2.7) are given by the vector of initial conditions $\underset{\sim}{U}(0) = \underset{\sim}{f}$, obtained from the first equation of (2.3), and the vector $\underset{\sim}{U}(\ell)$. This vector at time $t = \ell$ is not contained explicitly in the initial conditions and must be estimated from(2.3). The estimated vector must be at least as accurate in time as the vectors $\underset{\sim}{U}(t)$ for $t = 2\ell, 3\ell, \ldots$ to be determined from (2.7).

It is easy to verify that

$$\underset{\sim}{U}(\ell) = (I + \tfrac{1}{2}\ell^2 A)\,\underset{\sim}{f} + \ell\,\underset{\sim}{g} + O(\ell^3) \quad , \qquad (2.8)$$

$$\underset{\sim}{U}(\ell) = (I + \frac{1}{2}\ell^2 A + \frac{1}{24}\ell^4 A^2)\,\underset{\sim}{f} + l(I + \frac{1}{6}\ell^2 A)\,\underset{\sim}{g} + 0(\ell^5), \qquad (2.9)$$

$$\underset{\sim}{U}(\ell) = (I + \frac{1}{2}\ell^2 A + \frac{1}{24}\ell^4 A^2 + \frac{1}{720}\ell^6 A^3)\,\underset{\sim}{f} + \ell(I + \frac{1}{6}\ell^2 A + \frac{1}{120}\ell^4 A^2)\,\underset{\sim}{g} + 0(\ell^7) \qquad (2.10)$$

are, respectively, second-, fourth— and sixth-order accurate approximants to $\underset{\sim}{U}(\ell)$ where I is the identity matrix of order N. The choice of (2.8), (2.9) or (2.10) will obviously depend on the accuracy in time of the finite difference method arising from (2.7) with which $\underset{\sim}{U}(\ell)$ will be used.

2.3 Some known difference schemes

Any numerical solution of (2.7) will depend for its accuracy on the approximation to the matrix exponential functions $\exp(\pm\ell B)$. Using the (M,K) Pade approximant to $\exp(\ell B)$ of the form

$$R_{M,K}(\ell B) = [Q_M(\ell B)]^{-1} P_K(\ell, B) + 0(\ell^{M+K+1}) \quad , \qquad (2.11)$$

where P_k and Q_M are matrix polynomials of degrees K and M, respectively, leads to a family of finite difference methods for the solution of (2.1) with (2.2) and (2.3).

The low order (0,1) and (1,0) Pade approximants given, respectively by $\exp(\ell B) \simeq I + \ell B$ and $\exp(\ell B) \simeq (I-\ell B)^{-1}$, lead to inconsistent finite difference replacements. for (2.1). Using the (1,1) Pade approximant, given by $\exp(\ell B) \simeq (I - \frac{1}{2}\ell B)^{-1} (I+ \frac{1}{2}\ell B)$, in (2.7) leads to the well known nine-point implicit scheme given, for example, by Smith [5; p.178], while the use of the (0,2) Padé approximant (the first three terms of the Maclaurin expansion of $\exp(\ell B)$) leads to the familiar five-point explicit scheme given, for instance, in Smith [5; p.177]. Conventional stability analyses show that this implicit scheme is unconditionally stable and that this explicit scheme is stable provided, $r = \ell/h \leq 1$ (for $r = 1$ this explicit scheme is exact). These two methods will be known as M11 and M02, respectively.

Twizell [1] used the (0,4) Pade approximant to $\exp(\ell B)$ in (2.7); this leads to method M04 given by

$$\underset{\sim}{U}(t + \ell) - (2I + \ell^2 A + \frac{1}{12}\ell^4 A^2)\underset{\sim}{U}(t) + \underset{\sim}{U}(t - \ell) = \underset{\sim}{0} \tag{2.12}$$

which, when applied to the mesh point $(mh,n\ell)$, gives a seven point explicit schem which is stable for $r \leq \sqrt{3}$. Due to the presence of A^2 in (2.12), a modified form of the general difference scheme is required for $m = 1$ and $m = N$.

2.4 Looal truncation errors

The principal part of the local truncation error of every finite difference scheme obtained by replacing the matrix exponential terms in (2.7) by Pade approximants, has the form

$$-\frac{1}{12}h^2\frac{\partial^4 u}{\partial x^4} + Cq\,\ell^{q-2}\frac{\partial^q u}{\partial t^q} \tag{2.13}$$

at each interior mesh point where the scheme may be applied. The component $-\frac{1}{12}h^2\partial^4 u/\partial x^4$ arises from the space discretization and the use of (2.4).

The term C_q, where $q = M + K + 1$ for $M + K$ odd and $q = M + K + 2$ for $M + K$ even, is an error constant in time which is related only to the chosen (M,K) Pade approximant. The error constants relating to the Pade" approximants discussed in the present paper are given in Table 1.

Improvements in the time components of the local truncation errors for second order parabolic equations were effected in different ways by Lawson and Morris [3], Gourlay and Morris [4] and Twizell and Khaliq [2], for first order hyperbolic equations by Khaliq and Twizell in [6], and for fourth order parabolic equations by Twizell and Khaliq in [7]. In none of these papers was there any attempt to improve the space components of the principal parts of the local truncation errors. Nevertheless, the numerical results reported in these six papers showed that the improvements in time were justified.

In [6], where the matrices analogous to matrix A defined in (2.6) were also squared, the resulting finite difference schemes were shown, theoretically, to lose accuracy at points adjacent to the boundaries. The numerical results showed, however, that this loss of accuracy did not affect the stability or convergence of the methods. Oliger [8] proved that such loss of accuracy near the boundaries does not affect the overall stability or convergence of numerical methods for solving first order hyperbolic equations. In view of the fact that (2.1) can be written as a system of first order equations, and of the formulation of the numerical methods in §3, Oliger's theory easily carries over to second order hyperbolic equations of the form (2.1).

2.5 The use of higher order Padé approximants

The higher order approximants to be considered are the (2,0), (1,2), (2,1) and (2,2) Padé approximants to the matrix exponential function. The resulting finite difference schemes may be represented in matrix form as follows:

(i) Method M20. Using the (2,0) Padé approximant given by $\exp(\ell B)$ $(I - \ell B + \frac{1}{2}\ell^2 B^2)^{-1}$, the recurrence relation (2.7) becomes

$$(I + \tfrac{1}{4}\ell^4 A^2)\,\underset{\sim}{U}(t+\ell) - (2I + \ell^2 A)\,\underset{\sim}{U}(t) + (I + \tfrac{1}{4}\ell^4 A^2)\,\underset{\sim}{U}(t-\ell) = \underset{\sim}{0}, \qquad (2.16)$$

which, when applied to the general mesh point $(mh,n\ell)$, yields a consistent, thirteen point, implicit, finite difference scheme. The numerical solution vector $\underset{\sim}{U}(t+\ell)$ is found by solving a linear system having a quindiagonal coefficient matrix,

(ii) Method M12. The (1,2) Padé approximant is given by $\exp(\ell B) \simeq (I - \tfrac{1}{3}\ell B)^{-1}(I + \tfrac{2}{3}\ell B + \tfrac{1}{6}\ell^2 B^2)$. Using this replacement in (2.7) the recurrence relation becomes

$$(I - \tfrac{1}{9}\ell^2 A)\,\underset{\sim}{U}(t+\ell) - (2I + \tfrac{7}{9}\ell^2 A)\,\underset{\sim}{U}(t) + (I - \tfrac{1}{9}\ell^2 A)\,\underset{\sim}{U}(t-\ell) = \underset{\sim}{0} \qquad (2.17)$$

which, when applied to any mesh point $(mh,n\ell)$ at time $t = n\ell$, gives a nine point implicit scheme. The vector $\underset{\sim}{U}(t+\ell)$ is found by solving a tri-diagonal linear system.

(iii) Method M21. Using the (2,1) Pade approximant, given by $\exp(\ell B) \simeq (I - \tfrac{2}{3}\ell B + \tfrac{1}{6}\ell^2 B^2)^{-1}(I + \tfrac{1}{3}\ell B)$, in (2.7) gives

$$(I - \tfrac{1}{9}\ell^2 A + \tfrac{1}{36}\ell^4 A^2)\,\underset{\sim}{U}(t+\ell) - (2I + \tfrac{7}{9}\ell^2 A)\,\underset{\sim}{U}(t) = (I - \tfrac{1}{9}\ell^2 A + \tfrac{1}{36}\ell^4 A^2)\,\underset{\sim}{U}(t-\ell). \qquad (2.18)$$

The solution vector $\underset{\sim}{U}(t+\ell)$ is computed by solving a quindiagonal linear system and the implicit finite difference scheme resulting from (2.16) involves thirteen mesh points.

It is seen from Table 1 and the expression given in (2.13) that method M12 has the same principal local truncation error as method M21 and from (ii) and (iii) it is seen that the solution vector $\underset{\sim}{U}(t+\ell)$ is obtained more economically using method M12. Preference for method M12 is, however, dulled by its inferior stability property as will be seen in §3.

(iv) Method M22, The (2,2) Pade approximant is $\exp(\ell B) \simeq (I - \tfrac{1}{2}\ell B + \tfrac{1}{12}\ell^2 B^2)^{-1}(I - \tfrac{1}{2}\ell B + \tfrac{1}{12}\ell^2 B^2)$. A fifteen point, implict finite difference scheme is obtained by using this approximant in (2.7) which becomes

$$(I-\frac{1}{12}\ell^2A+\frac{1}{144}\ell^4A^2)\underset{\sim}{U}(t+\ell)-(2I+\frac{5}{6}\ell^2A+\frac{1}{72}\ell^4A^2)\underset{\sim}{U}(t)=(I-\frac{1}{12}\ell^2A+\frac{1}{144}\ell^4A^2)\underset{\sim}{U}(t-\ell)=\underset{\sim}{C} \tag{2.19}$$

and the solution vector $\underset{\sim}{U}\,(t+\ell)$ is obtained by solving a quindiagonal linear system.

It is seen from Table 1 and (2.13) that method M22 has the same principal.local truncation error as the explicit method M04[1]. However, as will be seen in §3, method M22 has a superior stability property enabling larger time steps to be used.

Deleting the terms in A from (2.19) gives the method

$$(I-\frac{1}{12}\ell^2A)\,\underset{\sim}{U}(t+\ell)-(2I+\frac{5}{6}\ell^2A+)\,\underset{\sim}{U}(t) = (I-\frac{1}{12}\ell^2A)\,\underset{\sim}{U}(t-\ell)=0 \tag{2.20}$$

for which the error constant in time is $C_6 = -\frac{1}{240}$ the principle part of the local truncation error of the nine-point mplicit finite difference scheme resulting from (2.18) also contains the component $-\frac{1}{12}h^2\partial^4u/\partial x^4$. Strictly speaking, this method is not a member of the family which evolves from (2.7) and is, in fact, based on the well known Numerov method for the numerical solution of second order ordinary differential equations; it will be known as method MN.

The order of its error constant in time (though larger in modulus) is the same as the method M22 and, having obviated the need to square the matrix A, formula (2.18) is clearly an attractive alternative to (2.17), the solution at each time step being obtained by solving a tridiagonal linear system instead of a quindiagonal linear system. Unfortunately, the sacrifice to be made is in stability, as will be seen in §3.

3. STABILITY PROPERTIES OF THE METHODS

The recurrence relation (2.7) can be written as

$$\underset{\sim}{U}^{n+1} = C\,\underset{\sim}{U}^{n} - \underset{\sim}{U}^{n-1}\;,\quad C = \exp(\ell B) + \exp(-\ell B)\;. \tag{3.1}$$

Defining $\underset{\sim}{V}^n = \underset{\sim}{U}^{n-1}$, so that $\underset{\sim}{V}^{n+1} = \underset{\sim}{U}^n$, (3.1) can be written as

$$\begin{bmatrix} \underset{\sim}{U}^{n+1} \\ \underset{\sim}{V}^{n+1} \end{bmatrix} = \begin{bmatrix} c & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \underset{\sim}{U}^{n} \\ \underset{\sim}{V}^{n} \end{bmatrix}$$

which is a two-time level scheme of the form

$$\underset{\sim}{W}^{n+1} = Q\,\underset{\sim}{W}^{n} \tag{3.2}$$

Clearly, the constant square matrix Q is of order 2N and the vector $\underset{\sim}{W} = (\underset{\sim}{U}, \underset{\sim}{V})^T$, T denoting transpose, has 2N elements. The known necessary condition for stability is $r(Q) \leq 1$.

The eigenvalues β_r ($r = 1,2,\ldots,2N$) of the matrix Q are given by

$$\det \begin{bmatrix} \mu_s - \beta & -1 \\ 1 & -\beta \end{bmatrix} = 0, \qquad s = 1,2,\ldots, \quad N$$

where μ_s ($s = 1,2,\ldots,N$) are the eigenvalues of the matrix C. Thus

$$\beta^2 - \mu_s \beta + 1 = 0 \qquad (s = 1,\ldots,N)$$

and

$$\beta_{1,s} = \tfrac{1}{2}[\mu_s + (\mu_s^2 - 4)^{\frac{1}{2}}], \beta_{2,s} = \tfrac{1}{2}[\mu_s + (\mu_s^2 - 4)^{\frac{1}{2}}] \tag{3.3}$$

for $s = 1,\ldots,N$. These are the eigenvalues of Q and are also know as amplification factors (Smith [5]). For stability, therefore, it is necessary that $|\beta_{j,s}| \leq 1$ ($j = 1,2$; $s = 1,2,\ldots,N$), but the damping or growth of the wave will depend on the real part of the amplification factors.

Using the terminology of Lawson and Morris [3] and Gourlay and Morris [4] relating to second order parabolic equations, the real parts of the amplification factors will be called the *amplification symbols* or *symbols.*

For some $s = 1,2,\ldots,N$ the real part of $\beta_{1,s}$ will be denoted by $S(z_s)$ and the real part of $\beta_{2,s}$ will be denoted by $\bar{S}(z_s)$, where $z_s = -\ell^2\lambda_s$ and λ_s is an eigenvalue of A; clearly $z_s > 0$ and $\beta_{j,s} = \beta_{j,s}(z)$ for $j = 1,2$. More specifically, the symbols relating to the numerical method based on the (M,K) Pade approximant will be denoted by $S_{M,K}(z)$ and $\bar{S}_{M,K}(z)$, the subscript having been dropped from z_s.

In [2,3,4] the terms A_0-*stable and* L_0-*stable* are used in the context of second order parabolic equations, while in most widely used texts (Smith [5] is a notable example) the terms *unstable, conditionally stable* and *unconditionally stable* are used in relation to all time-dependent partial differential equations. In relation to the present treatment of second order hyperbolic equatons, of which the simple wave equation is a test problem, the two sets of terms may be connected by the following definitions :

Definition 3.1. A numerical method is said to be *unstable* if $|\beta_{1,s}| > 1$ and $|\beta_{2,s}| > 1$ for some $s = 1,2,\ldots,N$.

Definition 3.2 . A numerical method is said to be *conditionally stable* if $|\beta_{1,s}| \le 1$ and $|\beta_{2,s}| \le 1$ $(s = 1,\ldots,N)$ only for some interval of values of the ratio r.

Definition 3.3. A numerical method is said to be unconditionally stable if $|\beta_{1,s}| \le 1$ and $|\beta_{2,s}| \le 1$ for all $s = 1,2,\ldots,N$.

Definition 3.4. The method based on the (M,K) Padé approximant is said to be *A -stable* if $|S_{M,K}(z)| \le 1$ and $|\bar{S}_{M,k}(z)| \le 1$ for all $z > 0$. Clearly an unconditionally stable method is A_o-stable.

Definition 3.5. The method based on the (M,K) Padé approximant is said to be L_o-*stable* if it is A_0-stable and in addition

$$\lim_{Z\to\infty} S_{M,K}(z) = 0 \quad \text{and} \quad \lim_{Z\to\infty} \bar{S}_{M,K}(z) = 0.$$

A wave modelled by an A_0 -stable method will not grow in amplitude while a wave modelled by an L_0 -stable method will be damped.

The stability properties of the numerical methods discussed in §2 will now be analyzed in the light of these definitions (the nomenclature relating to the amplification factors will be simplified to β_1 and β_2) :

(i) Method M11. The amplification factors are given by

$$\beta_1 = \frac{1-\frac{1}{4}z}{1+\frac{1}{4}z} + i\frac{z}{1+\frac{1}{4}z}, \quad \beta_2 = \frac{1-\frac{1}{4}z}{1+\frac{1}{4}z} - i\frac{z}{1+\frac{1}{4}z}, \tag{3.4}$$

giving the single symbol

$$S_{1,1}(z) = \bar{S}_{1,1}(z) = (1 - \tfrac{1}{4}z)/(1 + \tfrac{1}{4}z) \tag{3.5}$$

for this method; the symbol is depicted in Figure 1. The curve tends monotonically to -1 and method M 1 1 is A_o -stable. (The method is well known to be unconditionally stable and this is verified by (3.4) where

$|\beta_1| = |\beta_2| = D$.

(ii) Method M02. For this method the amplification factors are

$$\beta_1 = 1 - \frac{1}{2}z + i(z - \frac{1}{4}z^2)^{\frac{1}{2}}, \quad \beta_2 = 1 - \frac{1}{2}z - i(z - \frac{1}{4}z^2)^{\frac{1}{2}} \tag{3.6}$$

giving

$$S_{0,2} = \bar{S}_{0,2}(z) = 1 - \tfrac{1}{2}z, \quad Z \le 4. \tag{3.7}$$

However, if $z > 4$, there are two distinct amplification symbols; they are

$$S_{0,2}(z) = 1 - \frac{1}{2}z - (\frac{1}{4}z^2 - z)^{\frac{1}{2}}, \bar{S}_{0,2}(z) = 1 - \frac{1}{2}z - (\frac{1}{4}z^2 - z)^{\frac{1}{2}}. \tag{3.8}$$

The symbols are depicted in Figure 2 from which it is seen that S (z) exceeds unity in modulus for z>4 and that $-1 \le S_{0,2}(z), \bar{S}_{0,2}(z) < 1$ for $0 < z \le 4$. Since $z < 4r^2$, this verifies the well known result that M02 is stable only for $r \le 1$.

(iii) Method M04. (Twizell [1]). Here, the amplification factors are

$$\beta_1 = 1 - \tfrac{1}{2}z + \tfrac{1}{24}z^2 + i\left(z - \tfrac{1}{3}z^2 + \tfrac{1}{24}z^3 - \tfrac{1}{576}z^4\right)^{\frac{1}{2}}$$

$$\beta_2 = 1 - \tfrac{1}{2}z + \tfrac{1}{24}z^2 - i\left(z - \tfrac{1}{3}z^2 + \tfrac{1}{24}z^3 - \tfrac{1}{576}z^4\right)^{\frac{1}{2}} \tag{3.9}$$

giving

$$S_{0,4}(z) = \bar{S}_{0,4}(z) = 1 - \tfrac{1}{2}z + \tfrac{1}{24}z^2, \quad z \le 12 \tag{3.10}$$

and

$$S_{0,4}(z) = 1 - \tfrac{1}{2}z + \tfrac{1}{24}z^2 - \left(\tfrac{2}{576}z^4 - \tfrac{1}{24}z^3 + \tfrac{1}{3}z^2 - z\right)^{\frac{1}{2}},$$

$$\bar{S}_{0,4}(z) = 1 - \tfrac{1}{2}z + \tfrac{1}{24}z^2 - \left(\tfrac{1}{576}z^4 - \tfrac{1}{24}z^3 + \tfrac{1}{3}z^2 - z\right)^{\frac{1}{2}}, \tag{3.11}$$

for $z > 12$. The graphs of the amplification symbols are shown in Figure 3. It is easy to show that $|S_{0,4}(z)| = |\bar{S}_{0,4}(z)| \le 1$ for $z \le 12$ and that $\bar{S}_{0,4}(z)$ exceeds unity for $z > 12$, Hence $r \le \sqrt{3}$ for stability, since $z < 4r^2$, this result agreeing with that in [1].

(iv) Method M20. The amplification factors for this method are

$$\beta_1, \beta_2 = \frac{1 - \frac{1}{2}z}{1 + \frac{1}{2}z^2} \pm i\,\frac{\left(z + \frac{1}{2}z^2 + \frac{1}{16}z^3\right)^{\frac{1}{2}}}{1 + \frac{1}{4}z} \tag{3.12}$$

giving

$$S_{2,0}(z) = \bar{S}_{2,0}(z) = (1 - \tfrac{1}{2}z)/(1 + \tfrac{1}{2}z^2). \tag{3.13}$$

It is easy to show that $S_{2,0}(z)$ and $\bar{S}_{2,0}(z)$ attain a minimum value of $-\tfrac{1}{2}(\sqrt{2}-1)$ at $z = 2 + \sqrt{2}$ and that each tends to zero as $z \to \infty$. Method M20 is thus L_0- stable; the graph of the symbol is given in Figure 4.

(v) *Method M12*. Here, the amplification factors are

$$\beta_1, \beta_2 = \frac{1 - \frac{7}{18}z}{1 + \frac{1}{9}z} \pm i\,\frac{\left(z - \frac{5}{36}z^2\right)^{\frac{1}{2}}}{1 + \frac{1}{9}z} \tag{3.14}$$

giving the amplification symbols

$$S_{1,2}(z) = \bar{S}_{1,2}(z) = (1 - \tfrac{7}{18}z)/(1 + \tfrac{1}{9}z), \quad z \le 36/5 \tag{3.15}$$

and

$$S_{1,2}(z) = [1 - \frac{7}{18}z - (\frac{5}{36}z^2 - z)^{\frac{1}{2}}]/(1 + \frac{1}{9}z),$$

$$\bar{S}_{1,2}(z) = [1 - \frac{7}{18}z - (\frac{5}{56}z^2 - z)^{\frac{1}{2}}]/(1 + \frac{1}{9}z) \tag{3.16}$$

for $z > 36/5$. The symbols for the method are depicted in Figure 5 from which it is clear that $S_{1,2}$ (z) exceeds unity in modulus for $z>36/5$. This is equivalent in a von-Neumann analysis of instability arising whenever $r > 3\sqrt{5}/5$.

(vi) Method M21. For this method, the amplification factors are

$$\beta_1,\beta_2 = \frac{1-\frac{7}{18}z}{1+\frac{1}{9}z+\frac{1}{36}z^2} \pm i\frac{(z-\frac{1}{12}z^2+\frac{1}{162}z^3+\frac{1}{1296}z^4)^{\frac{1}{2}}}{1+\frac{1}{9}z+\frac{1}{36}z^2} \tag{3.17}$$

giving the single symbol

$$S_{2,1}(z) = \bar{S}_{2,1}(z) = (1 - \frac{7}{18}z)/(1 + \frac{1}{9}z + \frac{1}{36}z^2) \tag{3.18}$$

which is graphed in Figure 6. It may be shown that the symbol attains a minimum value of $(392 - 441\sqrt{2})/392 \simeq -0.59$ at $z = 18(1 + 2\sqrt{2})/7 \simeq 9.84$ and that it tends monotonically to zero as $z \to \infty$. Method M21 is clearly L_0 -stable.

(vii) *Method M22.* The amplification factors for this method are

$$\beta_1,\beta_2 = \frac{1-\frac{5}{12}z+\frac{1}{144}z^2}{1+\frac{1}{12}z+\frac{1}{144}z^2} \pm i\frac{(z-\frac{1}{6}z^2+\frac{1}{144}z^3)^{\frac{1}{2}}}{1+\frac{1}{12}z+\frac{1}{144}z^2} \tag{3.19}$$

and there is just one amplification symbol, given by

$$S_{2,2}(z) = \bar{S}_{2,2}(z) = (1 - \frac{5}{12}z + \frac{1}{144}z)/(1 + \frac{1}{12}z + \frac{1}{144}z^2), \tag{3.20}$$

which is depicted in Figure 7. A simple analysis shows that the symbol attains its minimum value of -1 at z=12 and tends monotonically to +1 as $z \to \infty$; this shows that the method is A -stable.

(viii) Method MN. As noted in §2, this method is derived from M22 and its amplification factors may be written down from (3.19). They are

$$\beta_1 = \frac{1 - \frac{5}{12}z}{1 + \frac{1}{12}z} + i\frac{(z - \frac{1}{6}z^2)^{\frac{1}{2}}}{1 + \frac{1}{12}z}, \beta_2 = \frac{1 - \frac{5}{12}z}{1 + \frac{1}{12}z} - i\frac{(z - \frac{1}{6}z^2)^{\frac{1}{2}}}{1 + \frac{1}{12}z}. \tag{3.21}$$

The symbols for the method are

$$S_{MN}(z) = \bar{S}_{MN}(z) = \frac{1 - \frac{5}{12}z}{1 + \frac{1}{12}z}, z \le 6 \tag{3.22}$$

and

$$\begin{aligned} S_{MN}(z) &= [1 - \tfrac{5}{12}z - (\tfrac{1}{6}z^2 - z)^{\frac{1}{2}}]/(1 + \tfrac{1}{12}z), \\ \bar{S}_{MN}(z) &= [1 - \tfrac{5}{12}z - (\tfrac{1}{6}z^2 - z)^{\frac{1}{2}}]/(1 + \tfrac{1}{12}z), \end{aligned} \tag{3.23}$$

for $z > 6$. The symbols are shown in Figure 8 from which it is seen that $| S_{MN}(Z) > 1$ for $z > 6$. This shows that the associated finite difference scheme is stable provided $r \le 3\sqrt{2}/2 \simeq 2.12$.

4. THE TWO-DIMENSIONAL WAVE EQUATION

Consider the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}; \quad 0 < x, y < 1, \quad t > 0 \tag{4.1}$$

together with the boundary conditions

$$u(x,y,t) = 0 \; ; \quad \forall \; (x,y) \in \partial R \; , \quad t > 0, \tag{4.2}$$

where ∂R is the boundary of the square region $R = \{(x,y) : 0 < x,y < 1\}$, and the initial conditions

$$u(x,y,0) = F(x,y) \quad , \quad \partial u(x,y,0)/\partial t = G(x,y) \quad ; \quad 0 \le x,y \le 1 \quad . \tag{4.3}$$

This is the problem of a vibrating square membrane fixed round its edges. The functions F(x,y), G(x,y) are given continuous functions of x,y.

The intervals $0 \le x \le 1$ and $0 \le y \le 1$ will each be divided into N+ 1 subintervals each of width h, so that $(N + 1)h = 1$. A square mesh of width h is thus superimposed on the unit square R; the discretization has N^2 mesh points within R and N + 2 equally spaced points along each side of ∂R.

The independent variable t will be discretized in steps of length ℓ as in §2, so that $t = n\ell$ with n = 0.1,2,... . The notations $u^n_{k,m}$, $U^n_{k,m}$ at the mesh point $(x,y,t) = (kh,mh,n\ell)$, which are simple extensions of those of §2, will be used and the $U^n_{k,m}$ $(k,m = 1, \ldots, N;\ n= 0,1,2,\ldots)$ will be elements of the vector

$$\underset{\sim}{U}^n = (U^n_{1,1}, U^n_{2,1}, \ldots, U^n_{N,1}; U^n_{1,2}, U^n_{2,2}, \ldots, U^n_{N,2}, \ldots, U^n_{1,N}, U^n_{2,N}, \ldots, U^n_{N,N}). \tag{4.4}$$

The space derivatives in (4.1) are approximated by the finite difference replacements

$$\frac{\partial^2 u}{\partial x^2} = h^{-2}\{u(x - h, y, t) - 2u(x, y, t) + u(x + h, y, t)\} + 0(h^2) \tag{4.5}$$

and

$$\frac{\partial^2 u}{\partial y^2} = h^{-2}\{u(x, y - h, t) - 2u(x, y, t) + u(x, y + h, t)\} + 0(h^2). \tag{4.6}$$

The differential equation (4.1) is now applied to all N^2 interior mesh points at time level $t = n\ell$, in the order indicated by (4.4), with the space derivatives replaced by (4.5), (4.6). These applications result in a system of N^2 second order ordinary differential equations of the form (2.5).

Recalling that U =0 everywhere on ∂R, the matrix A in (2.5) is now of order N^2 and may be split into the constituent matrices C, D such that A = C + D. The matrix C arises from the use of (4.5) in (4.1); it is block diagonal with tridiagonal blocks, is of order N^2 and has the form

$$C = h^{-2}\begin{bmatrix} c_1 & & & & \\ & c_1 & & \bigcirc & \\ & & \cdot & & \\ & & & \cdot & \\ \bigcirc & & & & \cdot \\ & & & & & c_1 \end{bmatrix} \tag{4.7}$$

where C is the tridiagonal matrix of order N given by

$$C_1 = \begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & & & 0 & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & 1 & & & \\ & & & \cdot & & & \\ & & & 1 & \cdot & & \\ & & & & \cdot & \cdot & \\ & 0 & & & -2 & \cdot & 1 \\ & & & & 1 & & -2 \end{bmatrix}. \tag{4.8}$$

The matrix D arises from the use of (4.6) in (4.1); it is block tri-diagonal with diagonal blocks, if of order N^2 and has the form

$$D = h^{-2}\begin{bmatrix} -2I & I & & & & \\ I & -2I & I & & \bigcirc & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & \bigcirc & & I & -2I & I \\ & & & & I & -2I \end{bmatrix} \tag{4.9}$$

where I is the identity matrix of order N. The eigenvalues of A are now given by

$$\lambda_{i,j} = -4h^{-2}\left(\sin^2\frac{i\pi}{2(N+1)} + \sin^2\frac{i\pi}{2(N+1)}\right)$$

for $i, j = 1, 2, \ldots, N$ so that each $\lambda_{i,j} < 0$.

Solving (4.1) subject to (4.2), (4.3), the analytical solution may, again, be shown to satisfy a recurrence relation of the form (2.7) for $t = \ell, 2\ell, \ldots$. Estimates for the solution of the membrane problem at the first time step may be obtained from (2.8), (2.9) or (2.10).

As in §2, Padé approximants to the matrix exponential functions can be made in (2.7). Using the (0,2) and (1,1) Padé approximants leads to the well known explicit and implicit methods discussed in most relevant texts. The use of higher order Padé approximants leads to a new family of methods which have the same stability classifications as the novel methods for the one-dimensional wave equation discussed in §§2,3. Having used the same space step in the x and y directions it is easy to verify that the method based on the (1,2) Padé approximants is stable for the two-dimensional problem whenever $r \le 3\sqrt{5}/10$.

The principal part of the local truncation error of the numerical method for solving {(4.1),(4.2), (4.3)} based on the (M,K) Pade approximant is given by

$$-\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)+Cq\ell^{q-2}\frac{\partial^q u}{\partial t^q}, \tag{4.10}$$

where $q = M + K+1$ for $M + K$ odd and $q = M + K + 2$ for $M + K$ even. The time error constants C_q are the same as those of §2 and are given in Table 1. To avoid confusion in §5 the methods M11, M02, etc. of §2 will be named T11, T02, etc., respectively, in two space variables.

5. NUMERICAL EXPERIMENTS

To test the behaviour of the methods developed in §§2,4 they were tested on three problems from the literature.

Problem 1 [1], This problem consists of { (2.1), (2.2),(2.3)} with $f(x) = \sin\pi x$ and $g(x) = 0$. The theoretical solution of the problem is

$$u(x,t) = \sin \pi x \cos \pi t \ .$$

The space step h was given the value 0.1, so that N =9, and the time step ℓ was given the values 0,2,0.1,0.05,0.01 giving r = 2,1,0.5,0.1 respectively. It is noted that for r = 2 methods M02, M04, M12 are unstable and, for r = 1 method M02 is exact.

The numerical results obtained are largely in keeping with Table 1 and the analyses of §2.4 and §3, and are presented in Table 2. Ranking the errors in the four columns of Table 2 in increasing magnitude shows that, for larger values of the time step ℓ, the higher order new methods and the existing method of Twizell [1] give the best results. As the time step decreases, it is seen that, to two significant figures, all eight methods produce very similar errors. This indicates that, for small values of ℓ, the space component of the principal part of each local truncation error is dominant. Methods M02, M04, M12 which are unstable for r=2 have, in fact, given very acceptable results for this problem with this value of r. Every error recorded in Table 2 arose at x = 0.5, the midpoint of the space interval, and the computations were carried out using single precision arithmetic on a Honeywell 68 computer. This explains the unexpectedly high error modulus for r = 1 using method M02 which is an exact representation of the simple wave equation for this value of r.

Problem 2 [1]. This problem also consists of {(2.1),(2.2),(2.3)}; here f(x) =sin x and g(x) =0 so that there is a discontinuity between boundary and initial conditions for x= 1. The analytical solution is given by

$$u(x, t) = \sum_{k=1}^{\infty} \frac{2k\,\pi\,(-1)^{k+1}}{k^2\pi^2 - 1} \sin 1 \ \sin k\pi x \cos k\pi t.$$

The same numerical experiments were carried out as for Problem 1 and the maximum relative errors, defined by $|(u-\widetilde{U})/u|$, are given in Table 3.

This time, there is obvious evidence of the instability of the established methods M02, M04 and of method M12 for r = 2; method M02 does, however, give the lowest relative error for r = 1 for which value it is, theoretically, an exact representation of (2.1). Table 3 also shows that, for r = 2 which is close to its stability limit, method MN is beginning to show some evidence of instability. For all other methods, the relative errors are in keeping with the indications of Table 1 and expression (2.14). There is evidence also that the L_0-stable method M21 gives better results than A_0 -stable method M11 which has the same order. This observation was also noted for parabolic problems with discontinuities between initial and boundary conditions in [2,3,4].

Problem 3 [9]. This problem has two space dimensions and is described by {(4.1) , (4.2) , (4.3)} with $F(x, y) = \sin \pi x \sin \pi y$ and $G(x,y) = 0$; there are no discontinuities between $F(x,y)$ and $G(x,y)$ for $x = 0$ or $y = 0$. The theoretical solution is given by

$$u(x,y,t) = \sin \pi X \sin \text{Try} \cos \sqrt{2}\, \pi t \quad .$$

and the new methods T20, T12, T21, T22 were tested with $h=\frac{1}{11}$ (giving 100 grid points at each time level) and $\ell= 0.06$ giving $r = 0.66$. Results for these values of h and ℓ were reported in [10] enabling comparisons to be made with the methods of Lees; results were also determined for the existing methods T11 and T02 which are contained in the family of methods due to Lees as well as the family arising from (2.7). In keeping with [10] absolute errors were computed and the maximum absolute errors which, at each time level, occur at the four mesh points nearest the centre of the square bounded by the lines $x = 0$, $x=1$, $y = 0$, $y = 1$, are given in Table 4 for $t =0.3(0.3)3.0$.

It is seen from Table 4 that the new methods compare favourably to those of Lees [9]. The CPU times for the new method T12 were less than the methods of Lees while the new methods T20, T21, T22, which involve the solution of a quindiagonal system at each time step, were slightly more expensive; the CPU times for the methods of Lees, are very similar to the CPU time for method T11.The improvements in accuracy achieved by T20 and T222, as well as T12, are worth the small extra cost.

6. SUMMARY

A family of finite difference methods for the solution of the simple wave equation in one- and two-space variables has been developed and analyzed in this paper. The concepts of A_0-stability and L_0 -stability, familiar to readers of the literature on second order parabolic equations, were discussed in relation to second order hyperbolic equations.

The new methods were tested on three problems from the literature.

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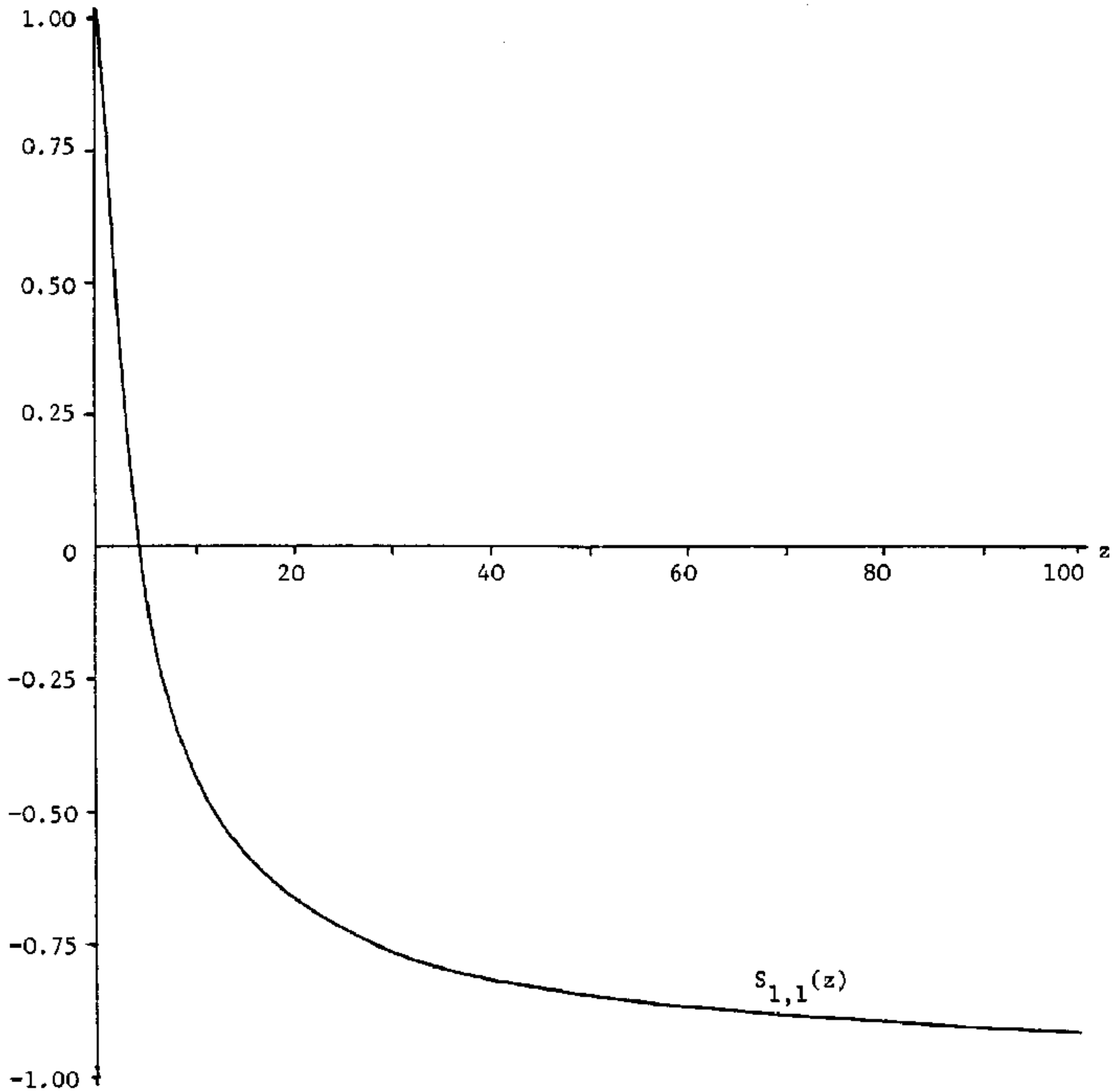

Figure 1: Amplification symbol for method M11.

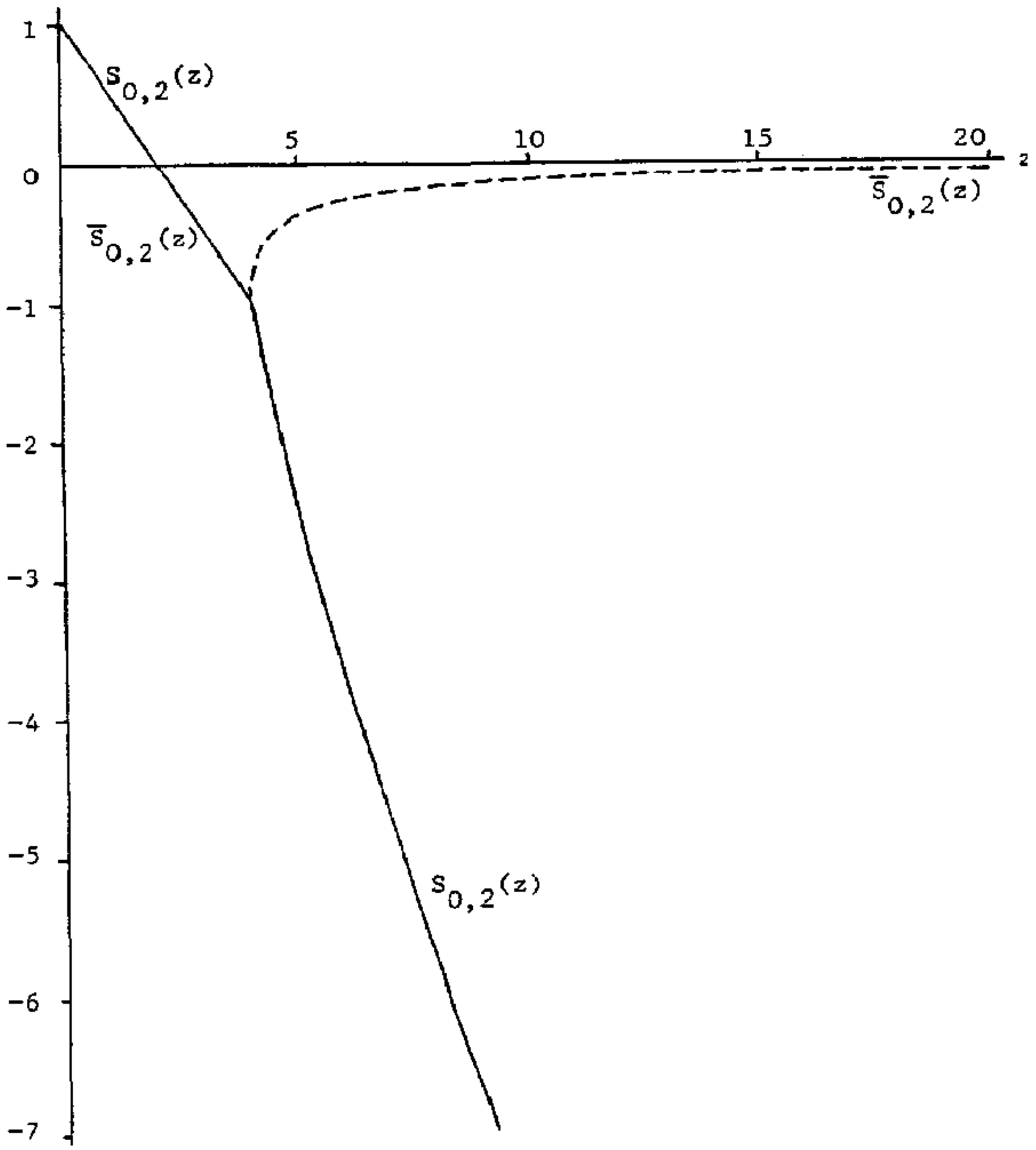

Figure 2: Amplification symbols for method M02.

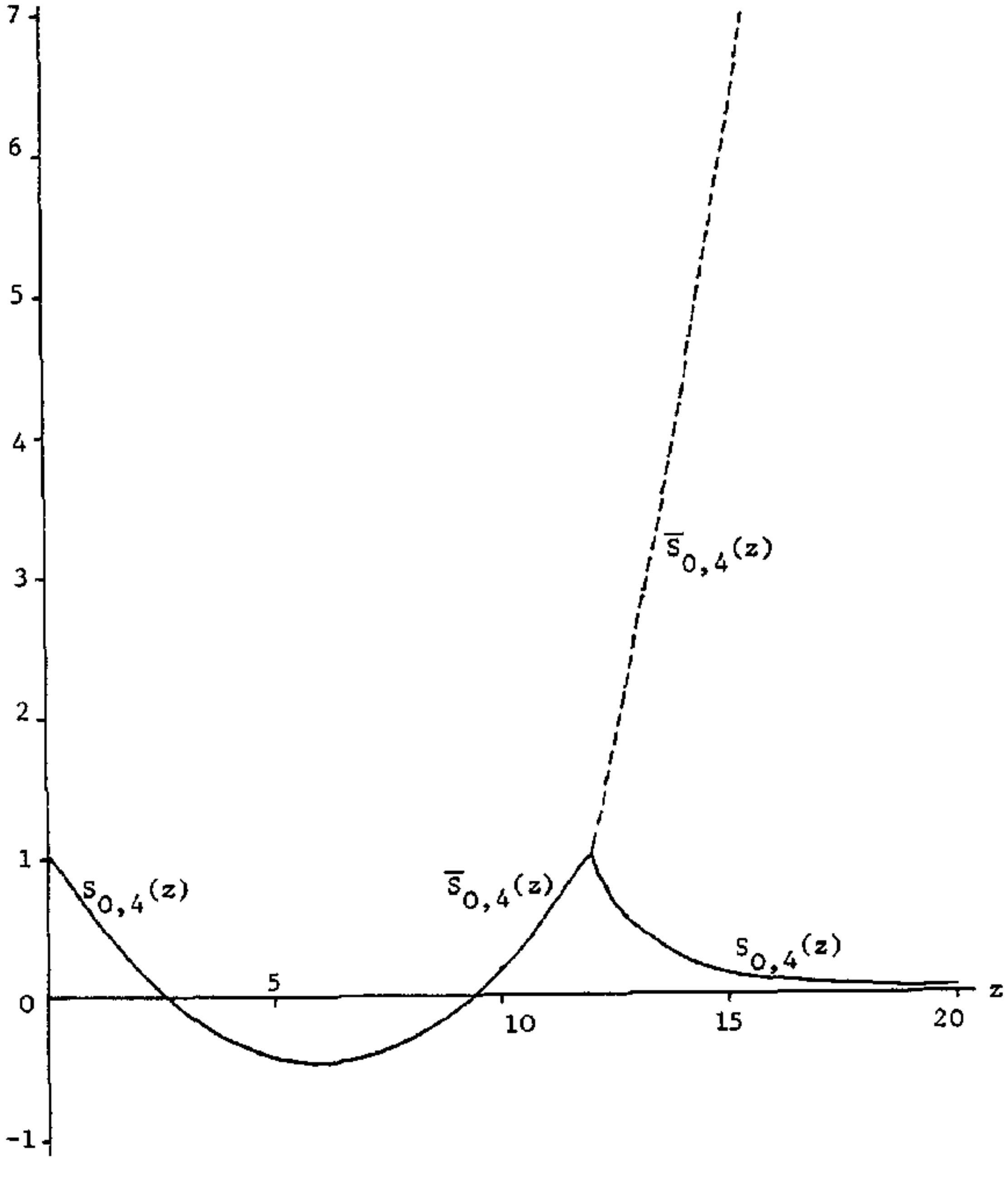

Figure 3: Amplification symbols for method M04.

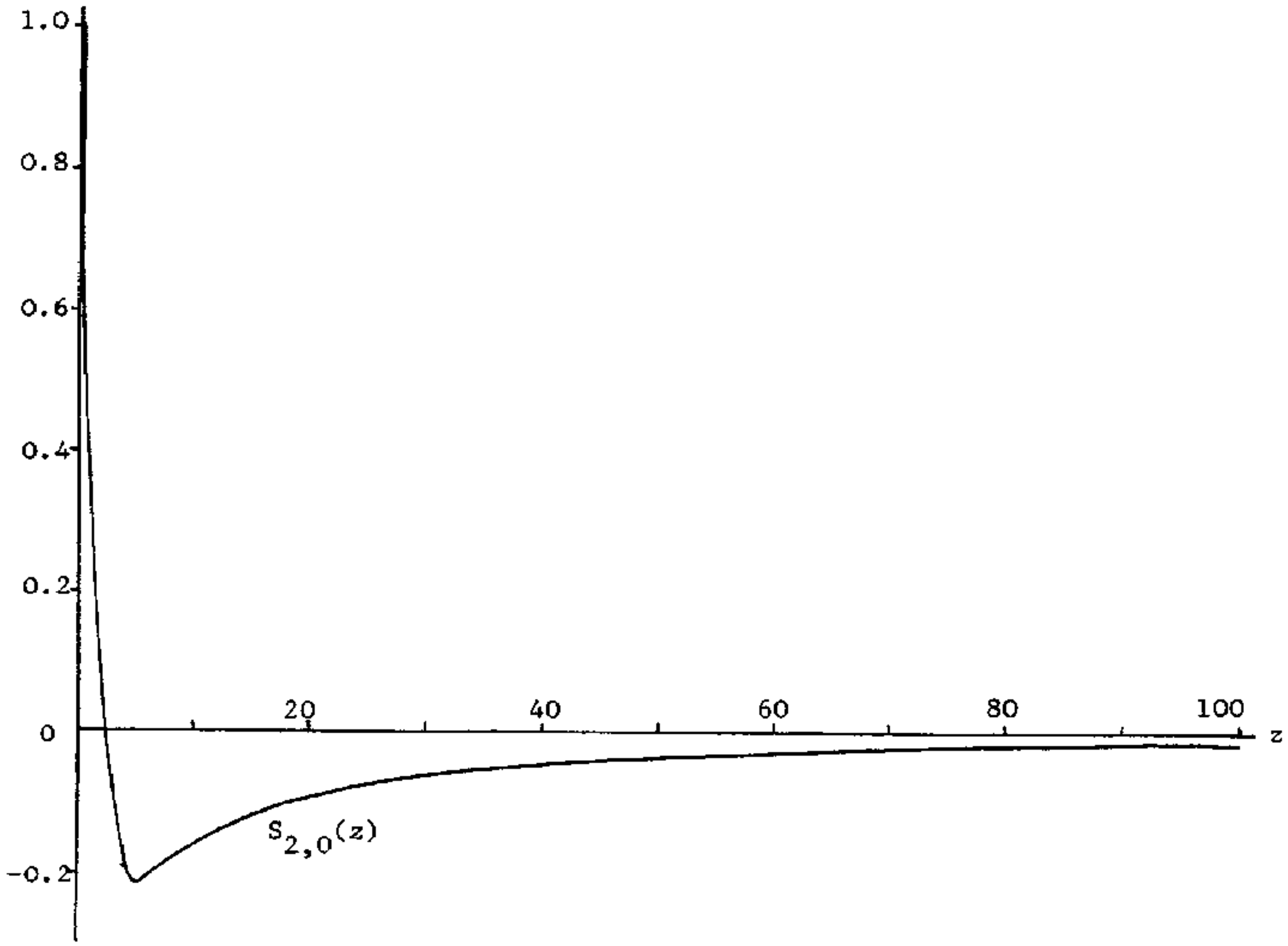

Figure 4: Amplification symbol for method M20.

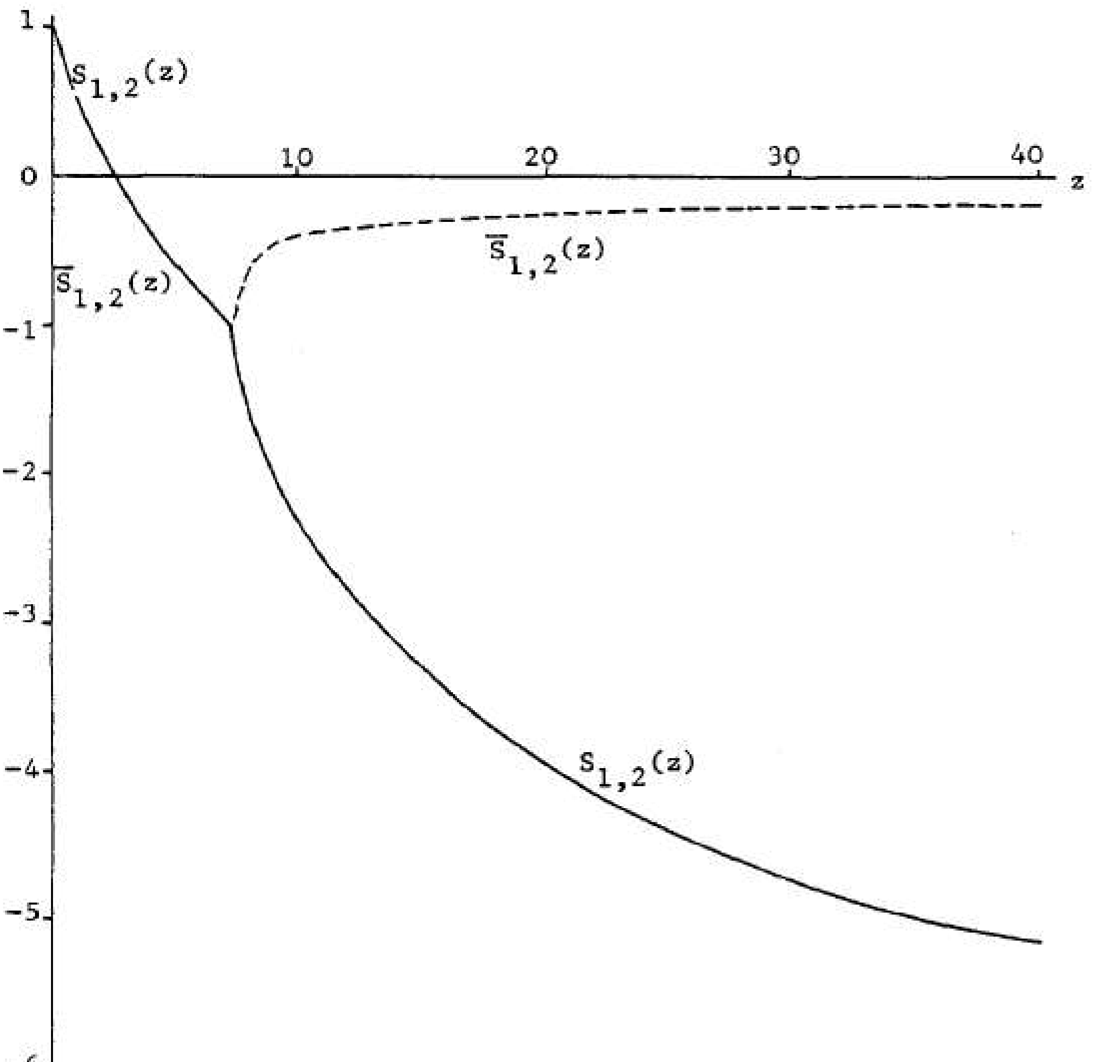

Figure 5: Amplification symbols for method M12.

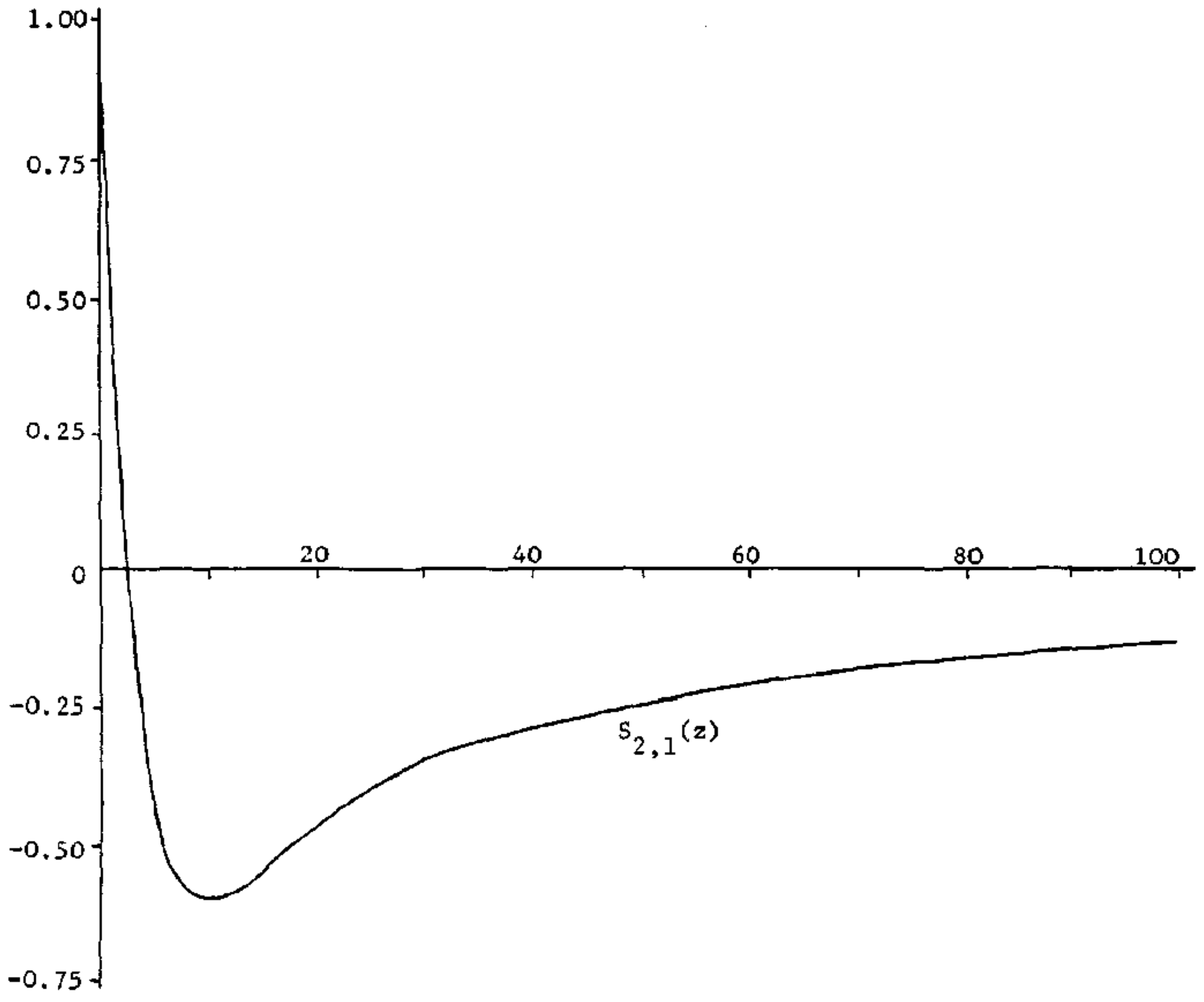

Figure 6: Amplification symbol for method M21.

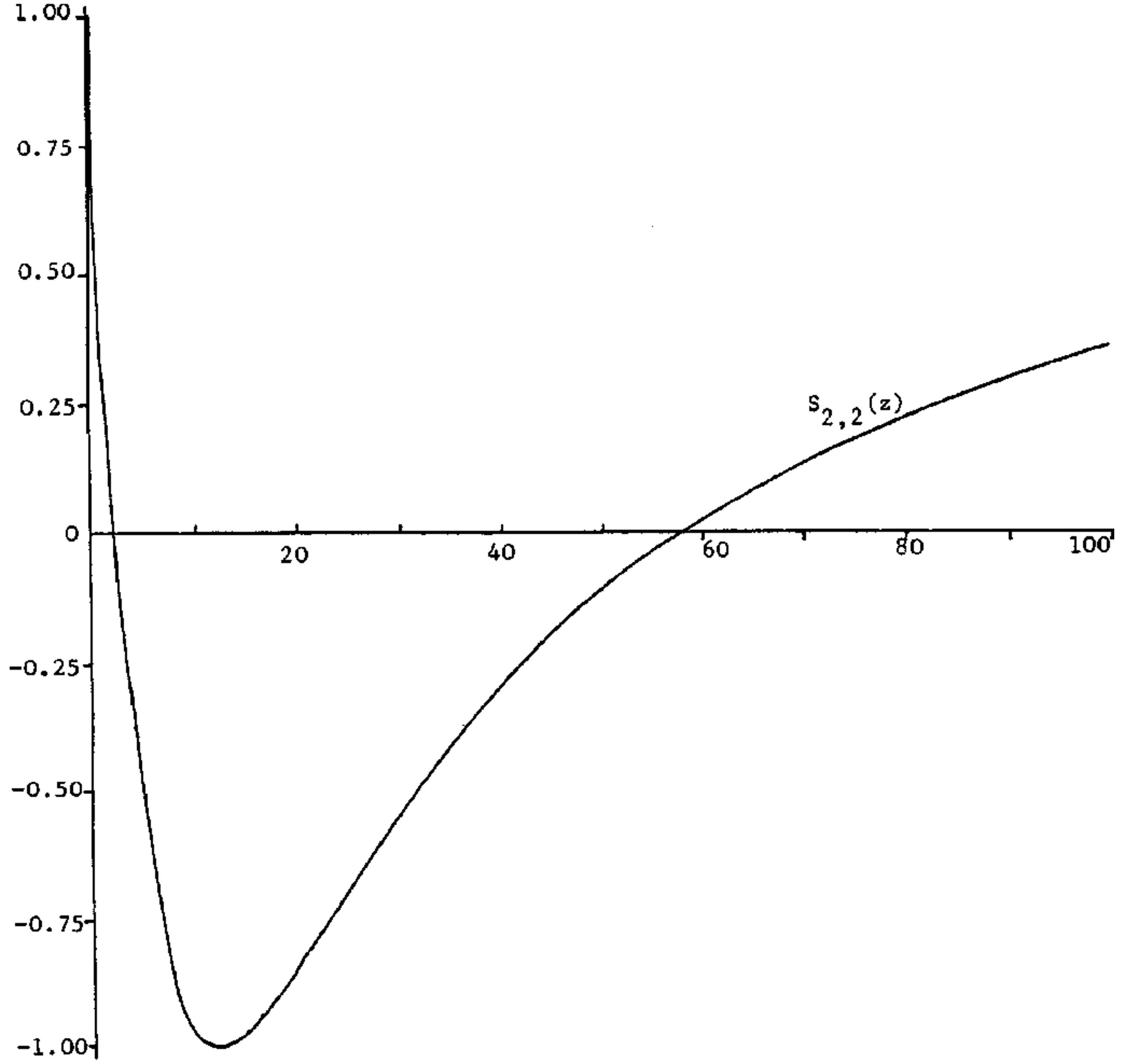

Figure 7: Amplification symbol for method M22 .

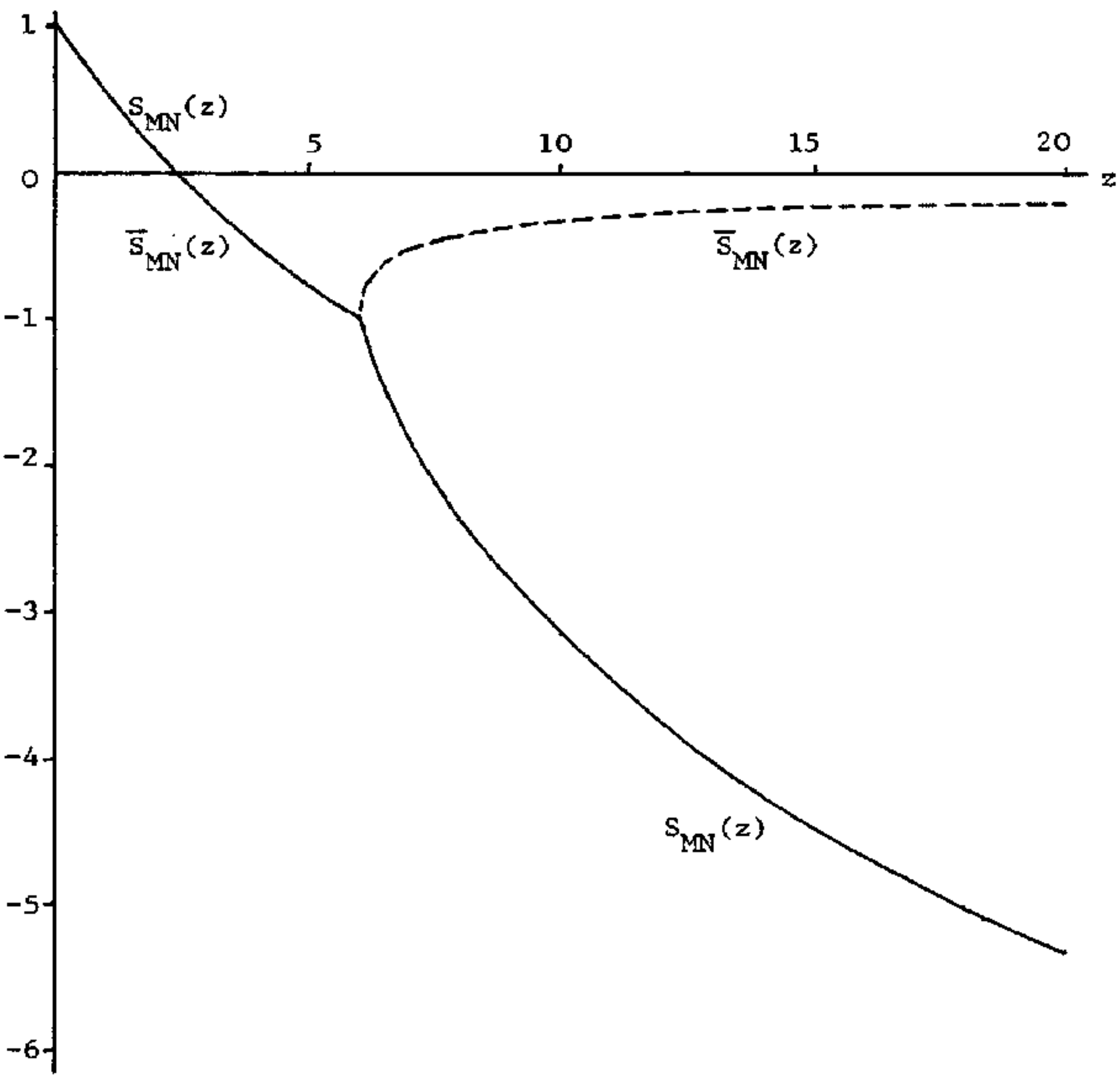

Figure 8: Amplification symbols for method MN.

Table 1: Error constants in time for the consistent finite difference schemes

Method	Error constant
M 11	$C_4 = -1/6$
M0 2	$C_4 = 1/12$
M04	$C_6 = 1/360$
M20	$C_4 = 7/12$
M12	$C_4 = -1/36$
M21	$C_4 = -1/36$
M22	$C_6 = 1/360$
MN	$C_6 = -1/240$

Table 2: Errors in solving Problem 1 for $x = 0.5$, $t = 1.0$ using $h = 0.1$

Method	$\ell = 0.2, r = 2.0$	$\ell = 0.1, r = 1.0$	$\ell = 0.05, r = 0.5$	$\ell = 0.01, r = 0.1$
M11	-0.38E-2	-0.63E-3	-0.18E-3	-0.86E-4
M02	-0.90E-3	+0.22E-7	-0.49E-4	-0.80E-4
M04	-0 .97E-4	-0.84E-4	-0.83E-4	-0.83E-4
M20	-0.24E-1	-0.20E-2	-0.32E- 4	-0.91E-4
M12	-0.32E-3	-0.4 4E-3	-0.9 7E-4	-0.83E-4
M21	-0.15E-5	-0.44E-4	-0.71E-4	-0.83E-4
M22	-0.89E-4	-0.84E-4	-0.83E-4	-0.83E-4
MN	-0.79E-4	-0.82E-4	-0.83E-4	-0.83E-4

Table 3: Maximum relative errors in solving Problem 2 at time t = 1.0using h=0.1

Method	ℓ = 0.2, r =2.0	ℓ = 0.1, r = 1.0	ℓ = 0.05, r = 0.5	ℓ =0-01 , r =0.1
M11	0.12	0.68E-1	0.99E-1	0.38E-1
M02	0.43E+3	0.55E-5	0.20E - 1	0.38E-1
M04	0.43	0.24E-1	0.27E-1	0.31E-1
M20	0.14	0.12	0.44E-1	0.37E-1
M12	0.28E+1	0.14E-1	0.46E-1	0.33E-1
M 21	0.14	0.45E-1	0.21E-1	0.30E-1
M22	0.33	0.14E-1	0.29E-1	0.31E-1
MN	0.40	0.15E-1	0.29E-1	0.33E-1

Methods

Table 4: Maximum absolute errors for Problem 3 at time t = 0.3(0.3)1.0 with $h = \frac{1}{11}$, l = 0.06(r = 0.06)

t	Methods								
							Lees		
	T11	T02	T20	T12	T21	T 22	n = 1	n = ½	n = ½
0.3	0.65E-1	0.12	0.11	0.91E-1	0.13	0.13	0.34E-1	0.18E-1	0.94E-2
0.6	0.31E-1	0.61E-1	0.36E-1	0.46E-1	0.63E-1	0.32E-1	0.40E-1	0.20E-1	0.10E-1
0.9	0.14E-1	0,10	0.55E-1	0.23E-2	0.10	0.75E-1	0.90E-1	0.47E-1	0.25E-1
1.2	0.74E-2	0.10	0.33E-1	0.18E-1	0.12	0.75E-1	0.15	0.74E-1	0.38E-1
1.5	0.14E-1	0.49E-1	0.17E-1	0.13E-1	0.52E-1	0.45E-1	0.57E-1	0.35E-1	0.20E-1
1.8	0.32E-2	0.13	0.10E-2	0.68E-2	0.15	0.85E-1	0.26	0.13	0.69E-1
2.1	0.78E-1	0.13E-1	0.11E-1	0.30E-1	0.18E-1	0.74E-2	0.76E-1	0.27E-1	0.11E-1
2.4	0.13	0.12	0.36E-1	0.31E-1	0.15	0.69E-1	0.31	0.16	0.87E-1
2.7	0.26E-1	0.74E-1	0.40E-1	0.53E-1	0.89E-1	0.29E-1	0.28	0.13	0.62E-1
3.0	0.20	0.94E-1	0.54E-1	0.30E-2	0.11	0.63E-1	0.22	0.13	0.76E-1