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Spherical wave diffraction by a rational wedge

by

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Abstract

In this paper we derive a new expression for the point source Green's function for the reduced wave equation, valid in an angular sector, whose angle is equal to a rational multiple of π. This Green's function is used to find new expressions for the field produced by the diffraction of a spherical wave source by a wedge, whose angle can be expressed as a rational multiple of π. The expressions obtained are in the form of source terms and real integrals which represent the diffracted field. The general result obtained includes as special cases Macdonald's solution for diffraction by a half plane; a solution for the problem of diffraction by a wedge of open angle $3\pi/2$, i.e. a corner; a new representation for the solution of the problem of diffraction by a mixed soft/hard half plane; and a new representation for the point source Green's function for Laplace's equation, valid in an angular sector whose angle is equal to a rational multiple of π.

1. Introduction.

In two previous papers Rawlins (1986a) and (I986b) the solution to the problem of diffraction of a plane and cylindrical wave by a rational wedge is given in terms of geometrical acoustic terms, and real integrals representing the diffracted field. Here we shall give an analogous solution to the problem of diffraction of a spherical acoustic wave by a wedge whose angle can be expressed as a rational multiple of π.

The exact solution of the problem of diffraction by a soft or hard wedge of any angle, in the three dimensional case of spherical acoustic wave incidence has been given by a number of authors, Macdonald (1902) and (1915), Wiegrefe (1912), Bromwich (1915) and Carslaw (1920). The solution in all these works was given in the form of a complex contour integral. For the special case of a wedge which reduces to a half plane, Macdonald (1915) showed how the contour integral could be reduced to an elegant form involving real integrals. Though the form of Macdonald's solution is extremely simple the method used to derive it involved a lot of tedious analysis. The problem of the diffraction of a point source by a half plane had been solved earlier by Carslaw (1899) using a method based on that used by Sommerfeld (1896) in considering diffraction by a plane wave. Carslaw's solution was also expressed in terms of real integrals, however, it was of a different (though equivalent) form to that of Macdonald's. These are the only cases known to the author where the solution to wedge diffraction problems involving a point source can be expressed in terms of real integrals. Here we shall show that such solutions can be given for any wedge whose angle can be expressed as a rational multiple of π.

Our starting point is to use the complex integral representation for the periodic Green's function for an arbitrary angle wedge. We then consider the special case of a wedge whose angle can be expressed as a rational multiple of π. It is then shown, by means of an appropriate integral representation for a Bessel function of order one half that the Green's function for a spherical point source can be derived from the plane wave Green's function for a rational wedge. This enables us to obtain a representation for the Green's function for a spherical source, in the form of source and image terms and real integrals which are convenient for calculation of the diffracted field. We remark that recently there has been much work done on uniform asymptotics for the wedge, see Ciarkowski et al (1984). The results presented here offer a new approach, in that a wedge of any angle can be approximated to any order of accuracy by a rational wedge of angle $p\pi/q$ (p and q integers), and the real integrals obtained in this paper can be asymptotically evaluated without difficulty.

In section 2 we shall give the periodic Green's function for a spherical wave source and a wedge of arbitrary angle. The Green's function is in the form of a complex contour integral. Some of the important properties of the Green[1]s function are stated, and appropriate expressions, in terms of this Green's function, are given for various diffraction problems. In section 3 we shall consider in detail the special case of evaluating the complex contour integral representation of the Green's function for a wedge whose angle can be expressed as a rational multiple of π. In section 4 we shall give expressions, for the Green's function for special cases of wedge angles. In section 5 we shall give solutions to three specific problems in diffraction theory which are special cases of the more general result obtained in section 4. The first problem is that of diffraction by a soft or hard half plane by a spherical source whose solution was given in different forms by Carslaw (1899), and Macdonald (1902), (1915). The second is the diffraction by a soft or hard wedge of open angle $3\pi/2$ by a spherical source; this corresponds to a physical situation often met in applications, viz diffraction around corners of buildings. No solution in the form presented here has hitherto appeared in the scientific literature. The last is a new result for the problem of diffraction by a soft/hard half plane by a spherical source. Finally in section 6 we shall give the appropriate Green's function for Laplace's equation by taking the long wave limit in the Green's function for the reduced wave equation. This Green's function, for a point source and a rational wedge, is required for the solution of boundary value problems involving rational angle wedges in compressible fluid dynamics, electrostatics, etc...

In order not to disrupt flow of the arguments in the main text of the paper, various proofs of results needed have been placed in appendices at the end of the paper.

2. Periodic Green's function for a wedge.

The periodic Green's function $G_\alpha\,(r,\theta,z,r_0,\theta_0,z_0;k)$ for a three dimensional wedge situation in a space $0 < r < \infty,\ 2\pi - a \le \theta \le 2\pi,\ -\infty < z < \infty,$ where (r, θ, z) are circular cylinder coordinates has been shown by Carslaw (1920) to be given by

$$G_a(r,\theta,z,r_0,\theta_0,z_0;k) = \frac{1}{2\alpha i}\int_C \frac{e^{-ikR(\zeta)}}{R(\zeta)}\cdot\frac{\sin(\pi\zeta/a)}{\cos(\pi\zeta/a) - \cos(\pi(\theta-\theta_0)/a)}d\zeta, \qquad (1)$$

where $R(\zeta) = \sqrt{r^2 + r_0^2 + (z-z_0)^2 - 2rr_0\cos\zeta},$ and the square root is specified by $-\pi/2 \le \arg R(\zeta) \le \pi/2.$ The contour of integration C is such that the starting point is given by $i\infty + c_1$ and the termination point is given by $i\infty + c_2$ where $-\pi < c_1 < 0,\ \pi < 2\pi.$ The contour of integration C lies below the branch point $\zeta = \alpha = \cosh^{-2}((r^2 + r_0^2 + (z-z_0)^2)/2rr_0)$ and does not intersect the branch cut: Re $\zeta = 0,\ 0 < \mathrm{Im}\,\zeta < 8$, see fig 2.

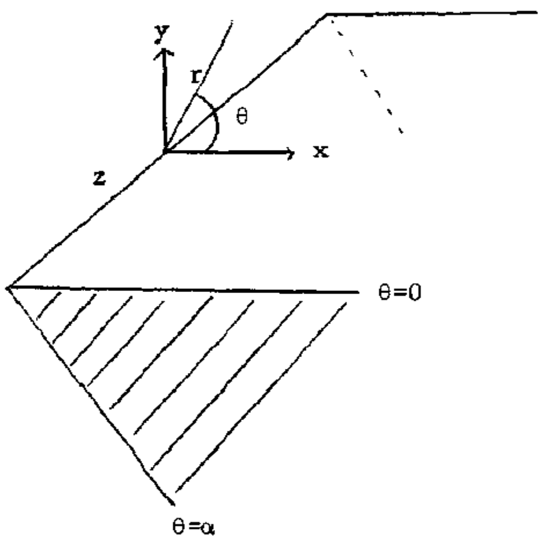

fig 1.

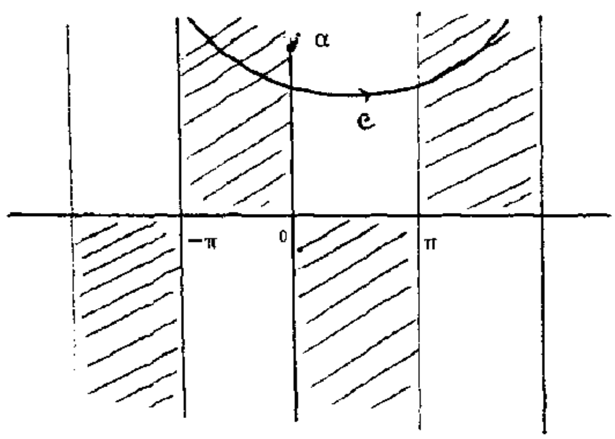

fig 2.

It has been shown by Carslaw that $G_\alpha(r,\theta,z,r_0,\theta_0,z_0;k)$ has the following properties

(i) $(\nabla^2 + k^2)G_\alpha = 0$, where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$,

for all points $P = (r, \theta, z)$, $P_0 = (r_0, \theta_0, z_0)$, such that $P_0 \neq P$.

(ii) $G_\alpha(r, \theta, z, r_0, \theta_0, z_0; k) = G_\alpha(r, \theta + 2a, z, r_0, \theta_0, z_0; k)$.

(iii) $G(r,\theta,z,r_0,\theta_0,z_0;k)$ is finite and continuous for all $P \neq P_0$.

(iv) $G_\alpha(r,\theta,z,r_0,\theta_0,z_0;k) = G_\alpha(r_0,\theta_0,z_0,r,\theta,z;k)$.

(v) $G_\alpha(r,\theta,z,r_0,\theta_0,z_0;k) \sim \frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)}$, as $p \to p_0$,

$$\sim 0, \text{ as } r \to \infty. \tag{2}$$

One can use the Green’s function given above to derive solutions to various diffraction problems in wedge shaped regions. To be specific we shall discuss acoustic waves. The solution u_h or u_s of the problem of a spherical wave*

$$u_0 = \frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)}, \tag{3}$$

diffracted by a rigid wedge $\partial u_h / \partial\theta = 0$ for $\theta = 0$ and $\theta = \alpha$) or a soft wedge ($u_s = 0$ for $\theta = 0$ and $\theta = \alpha$) is given by

$$u_h = G_\alpha(r,\theta,z,r_0,\theta_0,z_0;k) + G_\alpha(r,\theta,z,r_0,-\theta_0,z_0;k), \tag{4}$$

$$u_h = G_\alpha(r,\theta,z,r_0,\theta_0,z_0;k) - G_\alpha(r,\theta,z,r_0,-\theta_0,z_0;k), \tag{5}$$

respectively.

The solution $u_{s/h}$ of the problem of a spherical wave (3) diffracted by a wedge whose face $\theta = 0$ is rigid ($\partial u_{h/s}/\partial\theta = 0$) and whose face $\theta = \alpha$ is soft ($u_{h/s} = 0$) is given by $u_{h/s} = G_{2\alpha}(r,\theta,z,r_0,\theta_0,z_0;k) + G_{2\alpha}(r,\theta,z,r_0,-\theta_0,z_0;k)$

$$- G_{2\alpha}(r,\theta,z,r_0,2\alpha-\theta_0,z_0;k) - G_{2\alpha}(r,\theta,z,r_0,-2\alpha+\theta_0;k). \tag{6}$$

(Footnote: The wave is assumed to have time harmonic variation e^{iwt}, but this factor will not be shown explicitly in the rest of the paper.)

3. Point source Green’s function for a rational wedge

For a wedge, whose angle a can be expressed as a rational multiple of π , i.e. $\alpha = p\pi/q$ where p and q are integers, the point source Green's function becomes

$$G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k) = \frac{1}{2\pi ip}\int_c \frac{e^{-ikR(\zeta)}}{R(\zeta)}.\frac{q\sin(q\zeta/p)}{\cos(q\zeta/p)-\cos((\theta-\theta_0)q/p)}d\zeta. \quad (7)$$

We can use the results of Rawlins (1986b) to give the representation

$$\frac{e^{-ikR(\zeta)}}{R(\zeta)} = \frac{k}{i}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{H^2_{\frac{1}{2}}[kR(\zeta)]}{\sqrt{kR(\zeta)}} ,$$

$$= -\frac{ke^{i\pi/4}}{\sqrt{2\pi}}\int_{\infty+ic}^{0} e^{-\frac{1}{2}(t+k^2R^2(\zeta)/t)}\frac{dt}{t^{3/2}}, \quad (8)$$

where c > 0, (the contour of integration is as shown in fig 5 appendix A in awlins(1986b

By substituting the representation (8) into the expression (7) and inter-changing the order of integration (which is permissible since the integrals are uniformly convergent) gives

$$G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k) = -\frac{ke^{i\pi/4}}{\sqrt{2\pi}}\int_{\infty+ic}^{0} e^{-\frac{i}{2}(t+k^2(r^2+r_0^2+(z-z_0)^2)/t)}G_{\frac{p\pi}{q}}(r,\theta,\theta_0;\frac{k^2r_0}{t})\frac{dt}{t^{3/2}}$$

where (9)

$$G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) = \frac{1}{2\pi ip}\int_c e^{ikr\cos\zeta}\frac{q\sin(q\zeta/p)}{\cos(\zeta q/p)-\cos((\theta-\theta_0)q/p)}d\zeta, \quad (10)$$

i.s the plane wave Green's function for a rational wedge. It has been shown Rawlins (1986a) that the integral (10) can be written in the alternative form:

$$G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1}\sum_N H[\pi-|\theta-\theta_0+2\pi mp/q+2\pi pN|]e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}$$

$$+\frac{1}{2p}\sum_{m=0}^{q-1}e^{ikr\cos(\theta-\theta_0+2\pi mp/q)-i\pi/(2p)}\frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}$$

$$.\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q)}H^{(2)}_{\frac{1}{p}}(x)dx$$

$$+\frac{1}{2ip}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\frac{e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}}{\sin((\theta-\theta_0+2\pi mp/q)}\Big\{$$

$$+\,e^{in\pi/(2p)}\sin((n+1)(\theta-\theta_0+2\pi mp/q)\sin(n\pi/p)\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q)}H^{(2)}_{\frac{(p-n)}{p}}(x)dx$$

$$-\,e^{i(n+1)\pi/(2p)}\sin(n(\theta-\theta_0+2\pi mp/q)\sin((n+1)\pi/p)\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi mp/q)}H^{(2)}_{\frac{(p-1-n)}{p}}(x)dx$$

(11)

where the summation over N is for all integer values of N which can make the argument of the Heaviside step function $\left\{H[x]=\begin{matrix}1 & x>0\\ \frac{1}{2} & x=0\\ 0 & x<0\end{matrix}\right\}$ non negative.

Thus on substituting the expression (11) into (9) and interchanging the order of integrations results in having to evaluate integrals of the form:

$$-\frac{ke^{i\pi/4}}{\sqrt{2\pi}}\int_{\infty+ic}^{0} e^{-\frac{i}{2}(t+k^2(r^2+r_0^2+(z-z_0)^2/t)}e^{ik^2 r r_0\cos\psi/t}\int_{\infty}^{k^2 rr_0/t} e^{-ix\cos\psi}H^{(2)}_{\nu}(x)dx\,\frac{dt}{t^{3/2}}\ \text{nu}$$

which is shown in appendix A to be equal to

$$-2\frac{e^{\frac{i\nu\pi}{2}}}{\pi}\int_0^{\infty}\frac{\cosh(\nu t)}{\cosh t+\cos\psi}\cdot\frac{e^{ikR(\pi-it)}}{R(\pi-it)}dt.$$

Thus

$$G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k)=\sum_{m=0}^{q-1}\sum_{N}H[\pi-|\theta-\theta_0+2\pi mp/q+2\pi pN|]\frac{e^{-ikR(\theta-\theta_0+2\pi mp/q)}}{R(\theta-\theta_0+2\pi mp/q)}$$

$$-\frac{1}{\pi p}\sum_{m=0}^{q-1}\frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}\int_0^{\infty}\frac{\cos(t/p)}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\cdot dt$$

$$-\frac{1}{\pi p}\sum_{m=0}^{q-1}\sum_{n=1}^{p=2}\Bigg\{\frac{\sin((n+1)(\theta-\theta_0+2\pi mp/q)/p)\sin(n\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}$$

$$\cdot\int_0^{\infty}\frac{\cos((p-n)t/p)}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\cdot dt$$

$$-\frac{\sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin((n+1)\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}\cdot$$

$$\cdot\int_0^{\infty}\frac{\cos((p-1-n)t/p)}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\Bigg\}, \qquad (12)$$

where $\sum_{N}$ means that summation is only carried out for those values of N which satisfy the inequality $-\pi < \theta - \theta_0 + 2\pi mp/q + 2\pi pN < \pi$..

Thus the solution $u(r,\theta)$ of the problem of diffraction of the spherical source $u_0 = e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$ by a soft or hard wedge of open angle $\alpha = p\pi/q$ is given by

$$u_s(r,\theta,z) = G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k) - G_{\frac{p\pi}{q}}(r,\theta,z,r_0,-\theta_0,z_0;k) \qquad (13)$$

and

$$u_h(r,\theta,z) = G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k) + G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k) \qquad (14)$$

where $G_{\frac{p\pi}{q}}$ is given by the expression (12). Similarly the solution of the problem of diffraction of the point source $u_0 = e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$ by a wedge whose face $\theta = 0$ is soft, and whose other face $\theta = p\pi/q$ is hard is given by

$$u_{s/h}(r,\theta,z) = G_{\frac{2p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k) + G_{\frac{2p\pi}{q}}(r,\theta,z,r_0,-\theta_0,z_0;k)$$
$$- G_{\frac{2p\pi}{q}}(r,\theta,z,r_0,-\tfrac{2p\pi}{q}+\theta_0,z_0;k) - G_{\frac{2p\pi}{q}}(r,\theta,z,r_0,\tfrac{2p\pi}{q}-\theta_0,z_0;k), \quad (15)$$

where $G_{\frac{2p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k)$ is given by the expression (12) with p replaced by 2p.

An asymptotic expression for $G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k)$ can be obtained from the asymptotic expression for the plane wave Green's function, see Rawlins (1986a), and using the result of appendix B. Thus for $kr \to \infty$ we have

$$G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k) = \sum_{m=0}^{q-1}\sum_{N} H[\pi - |\theta-\theta_0+2\pi mp/q+2\pi pN|]\frac{e^{ikR(\theta-\theta_0+2\pi mp/q)}}{R(\theta-\theta_0+2\pi mp/q)}$$

$$+\frac{k}{2ip}\sum_{m=0}^{q-1}\frac{\sin(\theta-\theta+2\pi mp/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)|\cos((\theta-\theta_0+2\pi mp/q)/2)|}\,.$$

$$.\int_{\infty}^{\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0+2\pi mp/q)\cosh\xi]d\xi$$

$$+\frac{k}{2ip}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\frac{1}{\sin((\theta-\theta_0+2\pi mp/q)/p)|\cos((\theta-\theta_0+2\pi mp/q)/2)|}\Bigg\{$$

$$\sin((n+1)(\theta-\theta_0+2\pi mp/q)/p)\sin(\pi/p)\int_{\infty}^{\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0+2\pi mp/q)\cosh\xi]d\xi$$

$$-\sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin((n+1)\pi/p)\int_{\infty}^{\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0+2\pi mp/q)\cosh\xi]d\xi\Big\}$$

$$+0((kR)^{-3/2}) \qquad (16)$$

Where

$$\xi(\theta_0)=\sinh^{-1}\left\{\frac{2\sqrt{r\,r_0}\left|\cos(\theta-\theta_0+2\pi mp/q)/2\right|}{R(\theta-\theta_0+2\pi mp/q)}\right\}. \qquad (17)$$

The integrals appearing in the above expression (16) can be expressed in terms of the Fresnel integrals, whose properties are well known, for details see Jones' book (1986) p 562.

4. Special cases of wedge angles

$\boxed{P=1}$

$$G_{\frac{\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k)=\sum_{m=0}^{q-1}\sum_N H[\pi-\left|\theta-\theta_0+2\pi m/q+2\pi N\right|]\frac{e^{-ikR(\theta-\theta_0+2\pi m/q)}}{R(\theta-\theta_0+2\pi m/q)}\cdot(18)$$

$\boxed{q=1}$

$$G_{p\pi}(r,\theta,z,r_0,\theta_0,z_0;k)=\sum_N H[\pi-\left|\theta-\theta_0+2\pi pN\right|]\frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)}$$

$$-\frac{1}{\pi p}\frac{\sin(\theta-\theta_0)\sin(\pi/p)}{\sin((\theta-\theta_0)/p)}\int_0^{\infty}\frac{\cos(t/p)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{\pi p}\sum_{n=1}^{p-2}\left\{\frac{\sin((n+1)(\theta-\theta_0)/p\sin(n\pi/p)}{\sin((\theta-\theta_0)/p)}\int_0^{\infty}\frac{\cosh((p-n)t/p)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\right.$$

$$\left.-\frac{\sin(n(\theta-\theta_0)/p)\sin((n+1)\pi/p)}{\sin((\theta-\theta_0)/p)}\int_0^{\infty}\frac{\cos((p-1-n)t/p)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\right\}\cdot \qquad (19)$$

______,, _____.

P=2

$$G_{\frac{2\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k)=\sum_{m=0}^{q-1}\sum_{N} H[\pi-|\theta-\theta_0+4\pi m/q+4\pi N|]\frac{e^{-ikR(\theta-\theta_0+4\pi m/q)}}{R(\theta-\theta_0+4\pi m/q)}$$

$$-\frac{1}{\pi}\sum_{m=0}^{q-1}\cos\left[\frac{(\theta-\theta_0+4\pi m/q)}{2}\right]\int_0^\infty \frac{\cosh(t/2)}{\cosh t+\cos(\theta-\theta_0+4\pi m/q)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\cdot \qquad (20)$$

The last expression (20) can be put in an alternative form by using the results of appendix C. Thus

$$G_{\frac{2\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;k)=\sum_{m=0}^{q-1}\sum_{N} H[\pi-|\theta-\theta_0+4\pi m/q+4\pi N|]\frac{e^{-ikR(\theta-\theta_0+4\pi m/q)}}{R(\theta-\theta_0+4\pi m/q)}$$

$$+\frac{ik}{2}\sum_{m=0}^{q-1}\mathrm{sgn}\left[\cos\left(\frac{(\theta-\theta_0+4\pi m/q)}{2}\right)\right]\int_{\xi(\theta_0)}^{\infty} H_1^{(2)}[kR(\theta-\theta_0+4\pi m/2)\cosh\xi]d\xi\ , \qquad (21)$$

where

$$\xi(\theta_0)=\sinh^{-1}\left\{\frac{2\sqrt{r\,r_0}\,|\cos(\theta-\theta_0+4\pi m/q)/2|}{R(\theta-\theta_0+4\pi m/q)}\right\}.$$

5. <u>Some specific .problems in diffraction theory.</u>

<u>Macdonald's solution for a half plane.</u>

In terms of the Green's function, the solution for the problem of diffraction of a spherical wave $u_0(r,\theta)=e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$ by a soft, or hard half plane is given by

$$\begin{aligned} u_h\ (r,\theta,z)\ &= G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k)\ + G_{2\pi}(r,\theta,z,r_0,-\theta_0,z_0;k) \\ u_s\ (r,\theta,z)\ &= G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k)\ - G_{2\pi}(r,\theta,z,r_0,-\theta_0,z_0;k) \end{aligned} \qquad (22)$$

respectively .

Putting q = 1 in the expression (21) gives

$$G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k)=\sum_N H\,[\pi-|\,\theta-\theta_0+4\pi N|]\ \frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)}$$

$$-\frac{ik}{2}\mathrm{sgn}\left[\cos\left(\frac{(\theta-\theta_0}{2}\right)\right]\int_\infty^{|\xi(\theta_0)|} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi\ , \qquad (23)$$

where $$\xi(\theta_0) = \sinh^{-1}\left\{\frac{2\sqrt{r\,r}\ \cos((\theta-\theta_0)/2)}{R(\theta-\theta_0)}\right\}. \tag{24}$$

Now for $0 < \theta_0 < 2\pi$, and $0 < \theta < 2\pi$, then $|\theta-\theta_0| < 2\pi$, so that the argument of the Heaviside step function in (23) can only be positive if N = 0. Hence

$$G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k) = H[\pi-|\theta-\theta_0|]\ e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$$

$$-\frac{ik}{2}\operatorname{sgn}[\cos((\theta-\theta_0)/2]\int_{\infty}^{|\xi(\theta_0)|} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi\ ,$$

$$= H[\cos((\theta-\theta_0)/2]\frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)} - \frac{ik}{2}\operatorname{sgn}[\cos((\theta-\theta_0)/2]\ .$$

$$\cdot\int_{\infty}^{|\xi(\theta_0)|} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi.$$

If $\cos((\theta-\theta_0)/2) > 0$ then

$$G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k) = \frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)} - \frac{ik}{2}\int_{\infty}^{|\xi(\theta_0)|} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi\ \cdot$$

Now using the fact that

$$\frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)} = -\frac{ik}{2}\int_{-\infty}^{\infty} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi,$$

we can write

$$G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k) = -\frac{ik}{2}\left\{\int_{-\infty}^{\infty} + \int_{\infty}^{\xi(\theta_0)}\right\} H_1^{(2)}\ [kR(\theta-\theta_0)\cosh\xi]d\xi\ ,$$

$$= -\frac{ik}{2}\int_{-\infty}^{\xi(\theta_0} H_1^{(2)}\ [kR(\theta-\theta_0)\cosh\xi]d\xi\ \cdot \tag{25}$$

If $\cos((\theta-\theta_0)/2) < 0$ then

$$G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k) = \frac{ik}{2}\int_{\infty}^{-\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi\ ,$$

$$= -\frac{ik}{2}\int_{-\infty}^{\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi\ \cdot \tag{26}$$

Hence for any sign of $\cos((\theta-\theta_0)/2)$ we have

$$G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k) = \frac{ik}{2}\int_{-\infty}^{\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi\ \cdot \tag{27}$$

The expression for $G_{2\pi}(r,\theta,z,r_0,\theta_0,z_0;k)$ can be found in exactly the same manner for $0 < \theta + \theta_0 < 4\pi$, i.e.

$$G_{2\pi}(r,\theta,z,r_0,-\theta_0,z_0;k) = H\,[\pi-|(\theta+\theta_0)|]\ \frac{e^{-ikR(\theta+\theta_0)}}{R(\theta+\theta_0)}$$

$$+H\,[\pi-|(\theta+\theta_0-4\pi)|]\ \frac{e^{-ikR(\theta+\theta_0)}}{R(\theta+\theta_0)}-\frac{ik}{2}\mathrm{sgn}[\cos((\theta-\theta_0)/2]\cdot$$

$$\int_{\infty}^{|\xi(-\theta_0)|} H_1^{(2)}[kR(\theta+\theta_0)\cosh\xi]d\xi, \qquad (28)$$

where

$$\xi(-\theta_0)=\sinh^{-1}\left\{\frac{2\sqrt{r\,r}\ \cos((\theta+\theta_0)/2)}{R(\theta+\theta_0)}\right\}. \qquad (29)$$

Hence

$$G_{2\pi}\ (r,\theta,z,r_0,-\theta_0,z_0;k)\ =\ H[\cos((\theta+\theta_0)/2)]\ e^{-ikR(\theta+\theta_0)}/R(\theta+\theta_0)$$

$$-\frac{ik}{2}\mathrm{sgn}[\cos((\theta+\theta_0)/2]\int_{\infty}^{|\xi(-\theta_0)|} H_1^{(2)}[kR(\theta+\theta_0)\cosh\xi]d\xi\ ,$$

$$=\ -\frac{ik}{2}\int_{-\infty}^{\xi(-\theta_0)} H^{(2)}[kR(\theta+\theta_0)\cosh\xi]d\xi. \qquad (30)$$

Thus the solution of the problem of diffraction of a spherical wave by a hard or soft half plane is given by substituting the expression (27) and (30) into (22) giving

$$u_h\,(r,\theta,z)=\ -\frac{ik}{2}\int_{-\infty}^{\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi$$

$$-\frac{ik}{2}\ \cdot\ \int_{-\infty}^{\xi(-\theta_0)} H_1^{(2)}[kR(\theta+\theta_0)\cosh\xi]d\xi,$$

(31)

$$u_s\,(r,\theta,z)=\ -\frac{ik}{2}\int_{-\infty}^{\xi(\theta_0)} H_1^{(2)}[kR(\theta-\theta_0)\cosh\xi]d\xi$$

$$+\frac{ik}{2}\ \cdot\ \int_{-\infty}^{\xi(-\theta_0)} H_1^{(2)}[kR(\theta+\theta_0)\cosh\xi]d\xi,$$

where $\xi(\pm\theta_0)$ are given by (24) and (29) respectively. This result agrees with that of Macdonald (1915).

If we use the expression (20) with q = 1 in (22) we obtain directly Carslaw's (1899) solution.

Diffraction by a right angle wedge, (a corner) .

In terms of the Green's function the solution for the problem of diffraction of the spherical wave $u_0(r,\theta) = e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$ by a soft, or hard wedge of open angle $\alpha = 3\pi/2$ is given by

$$u_{\substack{s\\h}}(r,\theta,z) \;\; G_{\frac{3\pi}{2}}(r,\theta,z,r_0,\theta_0,z_0;k) \mp G_{\frac{3\pi}{2}}(r,\theta,z,r_0,-\theta_0,z_0;k) \tag{32}$$

where $G_{\frac{2\pi}{3}}(r,\theta,z,r_0,\theta_0,z_0;k)$ is given by putting p = 3 and q = 2 in the expression (12). After some simplification it is not difficult to show that

$$G_{\frac{3\pi}{2}}(r,\theta,z,r_0,\theta_0,z_0;k) = \sum_N H\,[\pi - |\theta-\theta_0+6\pi N|]e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$$

$$+\sum_N H\,[\pi - |\theta-\theta_0+(3+6N)\pi|\,]e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$$

$$-\frac{1}{2\sqrt{3}\pi}\left(\frac{\sin(\theta-\theta_0)}{\sin((\theta-\theta_0)/3)}-1\right)\int_0^\infty \frac{\cosh(t/3)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{2\sqrt{3}\pi}\left(\frac{\sin(\theta-\theta_0)}{\sin((\theta-\theta_0)/3)}-1\right)\int_0^\infty \frac{\cosh(t/3)}{\cosh t-\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{\sqrt{3}\pi}\cos((\theta-\theta_0)/3)\int_0^\infty \frac{\cosh(2t/3)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{1}{\sqrt{3}\pi}\cos((\theta-\theta_0)/3)\int_0^\infty \frac{\cosh(2t/3)}{\cosh t-\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\;.$$

Now using the identity $\sin\psi/(2\sin\psi/3) - 1 = \cos 2\psi/3$ and the fact that for $\frac{3\pi}{2} < \theta-\theta_0 < \frac{3\pi}{2}$, $-\pi < \theta-\theta_0+6\pi N < \pi$ will only be satisfied by N = 0, and $-\pi < \theta-\theta_0 < (3+6N)\pi < \pi$ is not satisfied by any integer N , then

$$G_{\frac{3\pi}{2}}(r,\theta,z,r_0,\theta_0,z_0;k) = H\,[\pi-|\theta-\theta_0|]e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0)$$

$$-\frac{1}{2\pi\sqrt{3}}\cos(2(\theta-\theta_0)/3)\int_0^\infty \frac{\cosh(t/3)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{2\pi\sqrt{3}}\cos(2(\theta-\theta_0)/3)\int_0^\infty \frac{\cosh(t/3)}{\cosh t-\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{\sqrt{3}\pi}\cos((\theta-\theta_0)/3)\int_0^\infty \frac{\cosh(2t/3)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{1}{\sqrt{3}\pi}\cos((\theta-\theta_0)/3)\int_0^\infty \frac{\cosh(2t/3)}{\cosh t-\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\,\cdot \tag{33}$$

Similarly for $0<\theta+\theta_0<3\pi$ it can be shown that

$$G_{\frac{2\pi}{3}}(r,\theta,z,r_0,-\theta_0,z_0;k)=H\,[\pi-\left|\theta+\theta_0\right|]e^{-ikR(\theta+\theta_0)}/R(\theta+\theta_0)$$

$$+H\,[\pi-\left|\theta+\theta_0+3\pi\right|]e^{-ikR(\theta+\theta_0+3\pi)}/R(\theta+\theta_0+3\pi)$$

$$-\frac{1}{2\pi\sqrt{3}}\cos(2(\theta+\theta_0)/3)\int_0^\infty \frac{\cosh(t/3)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{2\pi\sqrt{3}}\cos(2(\theta+\theta_0)/3)\int_0^\infty \frac{\cosh(t/3)}{\cosh t-\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{\sqrt{3}\pi}\cos((\theta+\theta_0)/3)\int_0^\infty \frac{\cosh(2t/3)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{1}{\sqrt{3}\pi}\cos((\theta+\theta_0)/3)\int_0^\infty \frac{\cosh(2t/3)}{\cosh t-\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\,\cdot \tag{34}$$

If we substitute (33) and (34) into (32) we get the exact solution for the diffraction of a spherical wave by a soft or hard wedge.

<u>Diffraction by a hard/soft half plane.</u>

In terms of the Green's function the solution for the problem of diffraction of the point source $u_0(r,\theta)=e^{-ikR(\theta+\theta_0)}/R(\theta-\theta_0)$ by a hard/soft half plane is given by

$$u_{h/s}\,(r,\theta,z,r_0,\theta_0,z_0)\;=\;G_{4\pi}\,(r,\theta,z,r_0,\theta_0,z_0;k)\;+\;G_{4\pi}\,(r,\theta,z,r_0,-\theta_0,z_0;k)$$
$$-G_{4\pi}\,(r,\theta,z,r_0,4\pi-\theta_0,z_0;k)\;-G_{4\pi}\,(r,\theta,z,r_0,-4\pi+\theta_0,z_0;k)\,. \tag{35}$$

By putting p = 4 in the expression (19) we obtain

$$G_{4\pi}(r,\theta,z,r_0,\theta_0,z_0;k)=\sum_N H\,[\pi-\left|\theta-\theta_0+8\pi N\right|]\,\frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)}$$

$$-\frac{1}{4\pi\sqrt{2}}\cdot\frac{\sin(\theta-\theta_0)}{\sin((\theta-\theta_0)/4)}\int_0^\infty \frac{\cosh(t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{4\pi}\left\{\frac{\sin((\theta-\theta_0)/2)}{\sqrt{2}\sin((\theta-\theta_0)/4)}\int_0^\infty \frac{\cosh(3t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\right.$$

$$\left.-\int_0^\infty \frac{\cosh(t/2)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\right\}-$$

$$-\frac{1}{4\pi}\left\{\frac{\sin(3(\theta-\theta_0)/4)}{\sin((\theta-\theta_0)/4)}\int_0^\infty\frac{\cosh(t/2)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\right.$$

$$\left.-\frac{\sin((\theta-\theta_0)/2)}{\sqrt{2}\sin((\theta-\theta_0)/4)}\int_0^\infty\frac{\cosh(t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt\right\}. \tag{36}$$

For $-2\pi < \theta - \theta_0 < 2\pi$ the only value of N which satisfies $-\pi < \theta - \theta_0 + 8\pi < \pi$ is N = 0. Hence

$$G_{4\pi}(r,\theta,z,r_0,\theta_0,z_0;k) = H[\pi-|\theta-\theta_0|]\cdot\frac{e^{-ikR(\theta-\theta_0)}}{(\theta-\theta_0)}$$

$$+\frac{1}{4\pi}\left(1-\frac{(\sin(3(\theta-\theta_0)/4)}{\sin((\theta-\theta_0)/4)}\right)\int_0^\infty\frac{\cosh(t/2)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\int_0^\infty\frac{\cosh(3t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)(1-2\cos((\theta-\theta_0)/2))\int_0^\infty\frac{\cosh(t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt, \tag{37}$$

$$= H[\pi-|\theta-\theta_0|]\cdot\frac{e^{-ikR(\theta-\theta_0)}}{R(\theta-\theta_0)}$$

$$-\frac{1}{2\pi}\cos((\theta-\theta_0)/2)\int_0^\infty\frac{\cosh(t/2)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\int_0^\infty\frac{\cosh(3t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)(1-2\cos((\theta-\theta_0)/2))\int_0^\infty\frac{\cosh(t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\cdot \tag{38}$$

In a similar manner it is not difficult to show that

$$G_{4\pi}(r,\theta,z,r_0,-\theta_0,z_0;k) = H[\pi-|\theta+\theta_0|]\cdot\frac{e^{-ikR(\theta+\theta_0)}}{R(\theta+\theta_0)}$$

$$-\frac{1}{2\pi}\cos((\theta+\theta_0)/2)\int_0^\infty\frac{\cosh(t/2)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)(1-2\cos((\theta+\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt. \quad (39)$$

$$G_{4\pi}(r,\theta,z,r_0,4\pi-\theta_0,z_0;k) = H\,[\pi-|\theta+\theta_0-4\pi|]\,\frac{e^{-ikR(\theta+\theta_0)}}{R(\theta+\theta_0)}$$

$$-\frac{1}{2\pi}\cos((\theta+\theta_0)/2)\int_0^\infty \frac{\cosh(t/2)}{\cosh t+\cos(\theta+\theta_0)}\,\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)(1-2\cos((\theta+\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt. \quad (40)$$

$$G_{4\pi}(r,\theta,z,r_0,-4\pi+\theta_0,z_0;k) = -\frac{1}{2\pi}\cos((\theta-\theta_0)/2)\int_0^\infty \frac{\cosh(t/2)}{\cosh t+\cos(\theta-\theta_0)}\,\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)(1-2\cos((\theta-\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)}{\cosh t+\cos(\theta-\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt. \quad (41)$$

By substituting the expressions (38) to (41) into (36) gives the solution for diffraction by a hard/soft half plane as:

$$u_{h/s}(r,\theta,z) = H\,[\pi-|\theta-\theta_0|]e^{-ikR(\theta-\theta_0)}/R(\theta-\theta_0) + H\,[\pi-|\theta+\theta_0|]e^{-ikR(\theta+\theta_0)}/R(\theta+\theta_0)$$

$$-H[\pi-|\theta+\theta_0-4\pi|]\frac{e^{-ikR(\theta+\theta_0)}}{R(\theta+\theta_0)}-\frac{1}{\sqrt{2}\pi}\cos((\theta-\theta_0)/4))\int_0^\infty \frac{\cosh(3t/4)}{\cosh t+\cos(\theta-\theta_0)}\,\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$-\frac{1}{\sqrt{2}\pi}\cos((\theta-\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)}{\cosh t+\cos(\theta+\theta_0)}\cdot\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{1}{\sqrt{2\pi}}\cos((\theta-\theta_0)/4)(1-2\cos((\theta-\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)}{\cosh t+\cos(\theta-\theta_0)}\,\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt$$

$$+\frac{1}{\sqrt{2\pi}}\cos((\theta+\theta_0)/4)(1-2\cos((\theta+\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)}{\cosh t+\cos(\theta+\theta_0)}\,\frac{e^{-ikR(\pi-it)}}{R(\pi-it)}\,dt. \quad (42)$$

6. Green's function for the Laplacian for a rational wedge.

In this final section we remark that if we let k = 0 in all the previous results we obtain the appropriate expressions for the solution of Laplace's equation for a rational wedge. For example the general Green's function given by (12) would become

$$G_{\frac{p\pi}{q}}(r,\theta,z,r_0,\theta_0,z_0;0)=\sum_{m=0}^{q-1}\sum_{N} H\,[\pi-|\theta-\theta_0+2\pi mp/q+2\pi pN|]\,\frac{1}{R(\theta-\theta_0+2\pi mp/q)}$$

$$-\frac{1}{p\pi}\sum_{m=0}^{q-1}\frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(\pi/p)}{\sin R((\theta-\theta_0+2\pi mp/q)/p}\int_0^\infty \frac{\cosh(t/p)\qquad\qquad dt}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)R(\pi-it)}$$

$$-\frac{1}{\pi p}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\left\{\frac{\sin((n+1)(\theta-\theta_0+2\pi mp/q)/p)\sin(n\pi/p)}{\sin(((\theta-\theta_0+2\pi mp/q)/p)}\right.$$

$$\cdot\int_0^\infty \frac{\cosh((p-n)t/p)\qquad\qquad dt}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)R(\pi-it)}$$

$$-\frac{\sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin((n+1)\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}$$

$$\left.\cdot\int_0^\infty \frac{\cosh((p-1-n)t/p)\qquad\qquad dt}{(\cosh t+\cos(\theta-\theta_0+2\pi mp/q))R(\pi-it)}\right\} \quad (43)$$

where $\sum_N$ means that summation is only carried out for integer values of N which satisfy the inequality $-\pi < \theta-\theta_0 + 2\pi mp/2 + 2\pi pN < \pi$. The expression (43) satisfies all the conditions (i)-(v) of (2) with k set equal to zero.

Appendix A

Here we derive an alternative representation for the expression

$$I = -\frac{ke^{i\pi/4}}{\sqrt{2\pi}} \int_{\infty+ic}^{0} e^{-\frac{i}{2}(t+k^2(r^2+r_0^2+(z-z_0)^2)/t)} e^{i k^2 r r_0 \cos\psi/t} .$$

$$\int_{\infty}^{k^2 r r_0/t} e^{-ix\cos\psi} H_\nu^{(2)}(x)\, dx \frac{dt}{t^{3/2}}, \quad 0 < \nu < 1. \qquad (A.1)$$

We use the result (see Rawlins (1986b) appendix B)

$$e^{-ik^2 r r_0 \cos\psi/t} \int_{\infty}^{k^2 r r_0/t} e^{-ix\cos\psi} H_\nu^{(2)}(x)\, dx$$

$$= -2\frac{e^{\frac{\nu\pi i}{2}}}{\pi} \int_0^\infty e^{-ik^2 r r_0 \cosh u/t} \frac{\cosh \nu u}{\cosh u + \cos\psi}\, du ,$$

in (A.1) which gives

$$I = -\frac{ke^{i\pi/4}}{\sqrt{2\pi}} \int_{\infty+ic}^{0} e^{-\frac{i}{2}(t+k^2(r^2+r_0^2+(z-z_0)^2)/t)} \left\{ -\frac{2}{\pi} e^{\frac{i\nu\pi}{2}} \cdot \right.$$

$$\left. \cdot \int_0^\infty e^{-ik^2 r r_0 \cosh u/t} \frac{\cosh \nu u}{\cosh u + \cos\psi}\, du \right\} \frac{dt}{t^{3/2}}.$$

$$= -\frac{2e^{i\nu\pi/4}}{\pi} \int_0^\infty \frac{\cosh \nu u}{\cosh u + \cos\psi} \left\{ -\frac{ke^{i\pi/4}}{\sqrt{2\pi}} \int_{\infty+ic}^{0} e^{-\frac{1}{2}(t+k^2(r^2+r_0^2+(z-z_0)^2+2r r_0 \cosh u)/t)} \frac{dt}{t^{3/2}} \right\}$$

$$= -\frac{2e^{i\nu\pi/2}}{\pi} \int_0^\infty \frac{\cosh \nu u}{\cosh u + \cos\psi} \cdot \frac{e^{-ik\mathcal{R}(\pi-iu)}}{\mathcal{R}(\pi-iu)}\, du .$$

Appendix B

Here we evaluate the integral

$$I = -\frac{k e^{i\pi/4}}{\sqrt{2\pi}} \int_{\infty+ic}^{0} e^{-\frac{i}{2}\,(t + k^2 (r^2 + r_0^2 + (z - z_0)^2)/t)} e^{i\,k^2\, r\, r_0 \cos\psi/t}\,.$$

$$\int_{\infty}^{k\sqrt{2 r r_0/t}\,|\cos\psi/2|} e^{-iv^2} dv \frac{dt}{t^{3/2}}$$

We can rewrite this as $(v = k\sqrt{2 r r_0/t}\,|\cos\psi/2|u)$

$$I = -\frac{k e^{i\pi/4}}{\sqrt{2\pi}} \int_{\infty+ic}^{0} e^{-\frac{i}{2}\,(t + k^2 (r^2 + r_0^2 + (z - z_0)^2 - 2 r r_0 \cos\psi)/t)}\,.$$

$$.\,|\cos\psi/2|\; k\sqrt{2 r r_0}\; \int_{\infty}^{1} e^{-i2k^2 r r_0 \cos^2(\psi/2) u^2/t} du\, \frac{dt}{t^2}\,,$$

$$= k\sqrt{2 r r_0}\,|\cos(\psi/2)| \int_{\infty}^{1} \left(-\frac{k e^{i\pi/4}}{\sqrt{2\pi}} \right) \int_{\infty+ic}^{0} e^{-\frac{i}{2}(t + k^2 (R^2(\psi) + 4 r r_0 \cos^2(\psi/2) u^2)/t)} \frac{dt}{t^2} du\,.$$

$$= k\sqrt{2 r r_0}\,|\cos(\psi/2)| \int_{\infty}^{1} \left(-\frac{k e^{i\pi/4}}{\sqrt{2\pi}} \right) \frac{\pi H_1^{(2)}[k\{R^2(\psi) + 4 r r_0 \cos^2(\psi/2) u^2\}^{\frac{1}{2}}]}{k\{R^2(\psi) + 4 r r_0 \cos^2(\psi/2) u^2\}^{\frac{1}{2}}}$$

$$= -k\sqrt{2 r r_0}\,|\cos(\psi/2)| \sqrt{\frac{\pi}{2}} e^{i\pi/4} \int_{\infty}^{1} \frac{H_1^{(2)}[k\sqrt{(R^2(\psi) + 4 r r_0 \cos^2(\psi/2) u^2)}]}{\sqrt{(R^2(\psi) + 4 r r_0 \cos^2(\psi/2) u^2)}} du$$

Now let $2\sqrt{r r_0}\,|\cos\psi/2|\,u = R(\psi) \sinh\xi$, then

$$I = -k \frac{\sqrt{\pi} e^{i\pi/4}}{2} \int_{\infty}^{\xi_0} H_1^{(2)}[\,k R(\psi) \cosh\xi\,]\; d\xi$$

where $\qquad \xi_0 = \sinh^{-1}\left\{ \frac{2\sqrt{r r_o}\,|\cos\psi/2|}{R(\psi)} \right\}$

Appendix C

We shall here give an alternative representation for the integral

$$I = \int_0^\infty \frac{\cosh(t/2)}{\cosh t + \cos\psi} \quad \frac{e^{-kR(\pi - it)}}{R(\pi - it)}\, dt \tag{1.B}$$

$$= 2\int_0^\infty \frac{e^{-ik\sqrt{(r^2+r_0^2+(z-z_0)^2+2r\,r_0\cosh t)}}}{\sqrt{(r^2+r_0^2+(z-z_0)^2+2r\,r_0\cosh t)}} \cdot \frac{\frac{d}{dt}(\sinh t/2)}{\cosh t + \cos\psi} dt.$$

Let $v = 2\sqrt{rr_0}\ \sinh(t/2)$ then since $\cosh t = 1 + 2\sinh^2(t/2)$ we get

$$I = 2\sqrt{rr_0}\int_0^\infty \frac{e^{-ik\ (D^2+V^2)}}{\sqrt{(D^2+V^2)}}\ ,\ \frac{dv}{v^2+4rr_0\cos^2(\psi/2)}, D^2 = (r+r_0)^2 + (z-z_0)^2\ , \tag{2.B}$$

We now use the integral representation, see Rawlins (I986b)

$$H^{(2)}_{\frac{1}{2}}\ (z)\ =\ z^{\frac{1}{2}}\ \frac{1}{\pi i}\ \int_{c-i\infty}^{0}\ e^{\frac{1}{2}(t-z^{2/t})}\ \frac{dt}{t^{3/2}}$$

$c > 0,\ \mathrm{Re}\, v > -1, |\arg z| < \pi/2$,

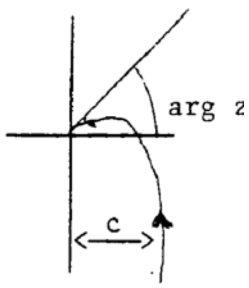

fig 3

and the fact that

$$H^{(2)}_{\frac{1}{2}}(z) = i\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\ e^{-iz}$$

to give

$$\frac{e^{-ikz}}{z}\ =\ -\frac{k}{\sqrt{2\pi}}\ \int_{c-i\infty}^{0}\ e^{\frac{1}{2}(u-k^2z^2/u)}\ \frac{du}{u^{3/2}} \tag{3.B}$$

Let $z = \sqrt{D^2+V^2}$ in (3.B) and then on substituting the resulting expression into (2.B) gives

$$I\ = -\ \frac{k}{\sqrt{2\pi}}\ \int_{c-i\infty}^{0}\ e^{\frac{1}{2}u-\frac{k^2D^2}{2u}}\ 2\sqrt{r\,r_0}\ \int_0^\infty\ \frac{e^{-\frac{k^2v^2}{2u}}\qquad dv}{v^2+4rr_0\cos^2(\psi/2)} \cdot \frac{du}{u^{3/2}}$$

and since $2\sqrt{r\,r_0}\int_0^\infty \frac{e^{-\frac{k^2v^2}{2u}}\qquad dv}{v^2+4rr_0\cos^2(\psi/2)} = \sqrt{\frac{\pi}{2}}\ \frac{k e^{\frac{4k^2 r\,r_0\cos^2(\psi/2)}{2u}}}{u^{\frac{1}{2}}|\cos(\psi/2)|}\ \int_{2\sqrt{r\,r_0}|\cos^2(\psi/2)|}^{\infty}\ e^{-\frac{k^2v^2}{2u}}\ dw$

see Rawlins (1986b) appendix D. we get

$$I = \sqrt{\frac{\pi}{2}}\ \frac{k}{|\cos(\psi/2)|} \int_{2\sqrt{r r_0}|\cos\psi/2|}^{\infty} \left(-\frac{k}{\sqrt{2\pi}}\right) \int_{c-i\infty}^{0} e^{\frac{1}{2}u - \frac{k^2}{2u}(D^2 - 4 r r_0 \cos^2\psi/2 + w^2)} \frac{du}{u^2} dw$$

$$= \sqrt{\frac{\pi}{2}}\ \frac{k}{|\cos(\psi/2)|} \int_{2\sqrt{r r_0}|\cos\psi/2|}^{\infty} \left(-\frac{k}{\sqrt{2\pi}}\right) \int_{c-i\infty}^{0} e^{\frac{1}{2}(u - k^2\{R^2(\psi) + w^2\}/u} \frac{du}{u^2} dw$$

$$= -\frac{\pi}{2}\ \frac{ik}{|\cos(\psi/2)|} \int_{2\sqrt{r r_0}|\cos\psi/2|}^{\infty} \frac{H_1^{(2)}[k\{R^2(\psi) + w^2\}^{\frac{1}{2}}]}{\{R^2(\psi) + w^2\}^{\frac{1}{2}}} dw\ .$$

Let $w = R(\psi)\sinh\xi$ so that

$$I\ =\ -\frac{\pi}{2} \cdot \frac{ik}{|\cos(\psi/2)|} \cdot \int_{\xi_0}^{\infty} H_1^{(2)}\ [\,kR(\psi)\cosh\xi\,]\ d\xi$$

where

$$\xi_0\ = \sinh^{-1}\left\{\frac{2\sqrt{r\, r_0}|\cos\psi/2|}{R(\psi)}\right\}$$

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