TR/09/94 **June 1994**

A TRIFURCATED WAVEGUIDE PROBLEM II

A.D. Rawlins

w9253398

A TRIFURCATED WAVEGUIDE PROBLEM II

by

A.D. Rawlins

Department of Mathematics and Statistics

Brunel University

Summary

We consider the diffraction of the dominant plane wave mode which propagates out of the end of a semi-infinite waveguide. This waveguide is made up of a half plane with a Dirichlet boundary condition and a half plane with a Neumann boundary condition. This semi-infinite waveguide is symmetrically located inside another infinite waveguide one of whose infinite plates has a Dirichlet and the other a Neumann boundary condition. A closed form solution of the resulting matrix Weiner-Hopf equation is obtained.

1. Introduction

In a previous publication, Rawlins [5], a mathematical model to predict noise in an exhaust system was proposed. This model resulted in a matrix Wiener-Hopf problem. Fortunately the matrix could be explicitly factorized. In the present work we consider another possible mathematical model of an exhaust system. This again results in a matrix Wiener-Hopf problem which is solved exactly. For the justification of the mathematical model see [5].

The trifurcated waveguide problem under consideration is shown in fig. 1. The plates which make up the waveguide have Neumann or Dirichlet boundary conditions on them. The plates are symmetrically positioned relative to the centre line of the system. A fundamental mode is assumed to propagate out of the month of the semi-infinite waveguide. Although some related trifucated waveguide problems can be solved as a result of their symmetry by a direct application of the normal scalar Wiener-Hopf technique, see Rawlins [5], the present problem results in a non-trivial matrix Wiener-Hopf problem. Fortunately we are able to solve this matrix Wiener-Hopf problem explicitly.

To be specific we shall consider an acoustic problem and in section 2 we shall formulate the mathematical problem that we intend to solve. In section 3 we shall solve the problem formulated in section 2. The solution will be expressed as complex contour integrals. In section 4 we shall analytically convert these integrals to infinite series of modes which propagate in the waveguide region. In order not to disrupt the flow of the solution in the main text an appendix has been included at the end of the paper. This appendix includes analytical details required in the main text.

Formulation of the boundary value problem

We shall consider the acoustic diffraction of a plane wave mode propagating out of the open end of a semi-infinite duct; this semi-infinite duct consists of one plate which is rigid and the other which is soft. The semi-infinite duct is situated symmetrically between two infinite plates one of which is soft and the other rigid. The geometry of the problem is shown in fig. 1.

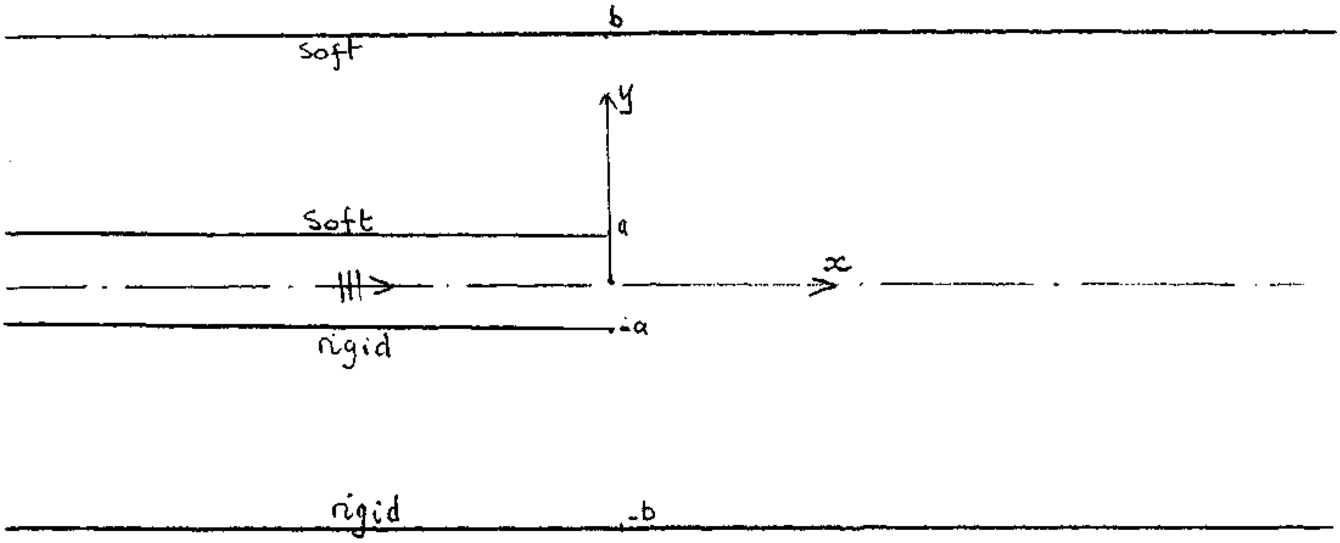

fig. 1

The sound source field, which is located at $y = y_0$, $x = x_0$ ($x_0 < 0$, $-a < y_0 < a$) and propagates modes along the semi-infinite duct. We shall introduce a scalar potential function ϕ (x,y,t) which defines the acoustic pressure and velocity by $p = -\rho_0\, \partial\phi / \partial t$, and $u = \text{grad}\, \phi$ respectively, where ρ_0 is the density of the undisturbed medium.

The incident sound field is assumed to have time harmonic variation $e^{-i\omega t}$, where the wave number $k = \omega/c$, and c is the speed of sound. In the remainder of this paper we shall drop the time dependence, it being tacitly understood that $\phi(x,y,t) = e^{-i\omega t}\phi(x,y)$. To this end we require a representation for the solution ϕ (x,y) of the two-dimensional Helmholtz equation

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0 \tag{1}$$

in the trifurcated duct system which satisfies the following boundary conditions (see fig.l)

$$\phi = 0, \quad y = b, \quad -\infty < x < \infty;$$
$$\phi = 0, \quad y = b, \quad -\infty < x < 0 \;;$$
$$\phi_y = 0, \quad y = -a, \quad -\infty < x < 0; \qquad \text{(2a,b,c,d)}$$
$$\phi_y = 0, \quad y = -b, \quad -\infty < x < \infty;$$

where it is assumed that b > a.

To these boundary conditions we add those conditions at infinity which are relevant to the nature of the propagating modes which various duct regions can sustain. From Rawlins [5] it is not difficult to show that

For $x \to -\infty$, $-a \le y \le a$

$$\phi(x,y) = y^{i\alpha_1 x} \cos\left((y+a)\frac{\pi}{4a}\right) + R_2\, e^{-i\alpha_1 x} \sin\left((y+a)\frac{\pi}{4a}\right) + 0(e^{-i\alpha_2}) \qquad (3)$$

where

$$\alpha_1 = \sqrt{k^2 - (\pi^2/16a^2)},\ \alpha_2 = \sqrt{k^2 - (9\pi^2/16a^2)},\ \alpha_{2n-1} = \sqrt{k^2 - ((2n-1)\pi/4a^2},\ n = 1,2,....$$

If we restrict $\pi/4 < ka < 3\pi/4$ then $\alpha_1 > 0$ and $\alpha_2 = i\sqrt{(3\pi/4a)^2 - k^2}$, so that $\mathrm{Im}\alpha_n > 0$, $\mathrm{Re}\alpha_n = 0$, $n > 1$. Thus in the semi-infinite duct region $-\infty < x < 0$, $-a \le y \le a$ can only sustain the lowest incident and reflected mode.

For $x \to +\infty$, $-b \le y \le b$

$$\phi(x,y) = T\, e^{i\alpha_1 x} \cos\left((y+b)\,\frac{\pi}{4b}\right) + 0(e^{i\alpha_2 x}), \qquad (4)$$

where

$$\hat{\alpha}_1 = \sqrt{k^2 - (\pi/4b)^2)},\ \hat{\alpha}_2 = \sqrt{k^2 - (3\pi^2/4b)^2)},\ \hat{\alpha}_{2n-1} = \sqrt{k^2 - (2n-1)\pi/4b)^2)},\ n = 1,2,....$$

For $x \to -\infty$, $a \le y \le b$

$$\phi(x,y) = T_3\, e^{i\hat{\alpha}_1 x} \sin\left((y-a)\frac{\pi}{(b-a)}\right) + 0(e^{i\hat{\alpha}_2 x}), \qquad (5)$$

where

$$\tilde{\alpha}_1 = \sqrt{(k^2 - (\pi/(b-a))^2)},\ \tilde{\alpha}_2 = \sqrt{(k^2 - (2\pi/(b-a))^2)}, \tilde{\alpha}_{2n-1} = \sqrt{(k^2 - (n\pi/(b-a))^2)},$$

n = 1,2,... .

For $x \to -\infty$, $-b \le y \le -a$

$$\phi(x,y) = T_3\, e^{-ikx} + T_3\, e^{i\tilde{\alpha}_1 x} \cos\left((y+a)\frac{\pi}{(b-a)}\right) + 0(e^{-\tilde{\alpha}_2 x}), \tag{6}$$

Where

$$\tilde{\alpha}_0 = k,\ \tilde{\alpha}_1 = \sqrt{(k^2 - (\pi/(b-a))^2)},\ \tilde{\alpha}_2 = \sqrt{(k^2 - (2\pi/(b-a))^2)},\ \tilde{\alpha}_n = \sqrt{(k^2 - (n\pi/(b-a))^2)},$$

n = 0,1,2,... .

Finally, in order to ensure uniqueness of the solution to the problem we need to specify the "edge condition" at the end of the semi-infinite planes, that is

$$\phi(x,\pm a) = 0(x^{1/2}) \quad \text{and} \quad \phi_y(x,\pm a) = 0(x^{-1/2}) \quad \text{as} \quad x \to 0. \tag{7}$$

Solution of the boundary value problem

For analytic convenience we shall assume that k = Rek + iImk(Rek > Imk ≥ 0). A suitable representation for the total field $\phi(x,y)$ in all space $-\infty < x < \infty$, $|y| \le b$ which satisfies (1) and (2) is given by:

$$\phi(x,y) = -\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x} \frac{\cos\kappa(b+y)}{\kappa \sin\kappa(b-a)} \phi_1^-(\alpha)d\alpha,\ (-b \le y \le -a,\ -\infty < x < \infty); \tag{8}$$

$$\phi(x,y) = -\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{e^{i\alpha x}}{\kappa\cos 2\kappa a}\{\sin\kappa(y-a) + \kappa\cos\kappa(y+a)\ \phi_2^-(\alpha)\}d\alpha$$
$$+ e^{i\alpha_1 x}\cos\left[(y+a)\frac{\pi}{4a}\right],\ (a \le y \le a,\ -\infty < x < \infty); \tag{9}$$

$$\phi(x,y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{\sin\kappa(b-y)}{\sin\kappa(b-a)}\ \phi_2(\alpha)d\alpha,\ (a \le y \le b,\ -\infty < \infty). \tag{10}$$

In the expressions (8) to (10), $\kappa = (k^2 - \alpha^2)^{1/2}$ and the branch cuts are taken to be from k to $i\infty$ and from –k to $-i\infty$. The cut sheet on which we shall work is defined by

$0 \leq$ argk $\leq \pi$, see fig. 2. The real parameter $\mathcal{T}$ is restricted by requiring that the asymptotic behaviour (3), (4) and (5) be achieved. This necessitates that the contour of integration in (8) to (10) lies in the strip:

$$\max\{-\mathrm{Im}k, -\mathrm{Im}\,\alpha_1, -\mathrm{Im}\,\tilde{\alpha}_1,\} < \tau < \min\{\mathrm{Im}k, \mathrm{Im}\,\tilde{\alpha}_1,\}$$

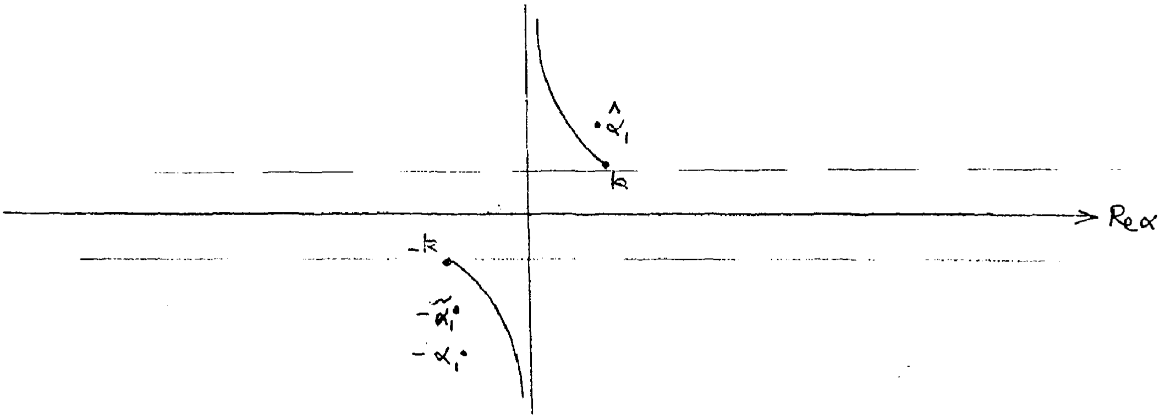

fig.2

It can be shown, see Rawlins [5], that the inequality (11) is certainly satisfied by -Imk < $\mathcal{T}$ < Imk. No singularities of the integrands of (8) to (10) lie within the strip -Imk < $\mathcal{T}$ < Imk. The unknown functions $\phi_1^-(\alpha)$ and $\phi_2^-(\alpha)$ are functions which are analytic and regular in the region $\mathrm{Im}\,\alpha$ < Imk. We must now ensure that two remaining boundary conditions are satisfied; namely that the field and its normal derivative are continuous across y = -a and y = a, (x>0) respectively, that is

$$\phi(x,-a^-) = \phi(x,-a^+), \frac{\partial\phi}{\partial y}(x,a^-) = \frac{\partial\phi}{\partial y}(x,a^+), x > 0\,. \tag{11}$$

By substituting (8) to (10) into (11) we get

$$\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\left\{\frac{\cos\kappa(b+a)\phi_1(\alpha)}{\kappa\sin\kappa(b-a)\cos 2ka} + \frac{\phi_2^-(\alpha)}{\cos 2ka} + \frac{1}{\alpha-\alpha_1}\right\}d\alpha = 0, (x>0) \tag{12}$$

$$\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\left\{\frac{\phi_1^-(\alpha)}{\cos 2\kappa a} + \frac{\kappa\cos\kappa(b+a)\phi_2^-(\alpha)}{\sin\kappa(b-a)\cos 2\kappa a} - \frac{k_1}{\alpha-\alpha_1}\right\}d\alpha = 0, (x>0), \tag{13}$$

where $\kappa_1 = \pi/(4a)$.

A solution of (12) and (13) is given by

$$\frac{\phi_1^-(\alpha)}{\cos 2\kappa a} + \frac{\kappa \cos\kappa(b+a)\phi_2^-(\alpha)}{\sin\kappa(b-a)\cos 2\kappa a} - \frac{\kappa_1}{\alpha-\alpha_1} = \phi_1^+(\alpha) \tag{14}$$

$$\frac{\cos\kappa(b+a)\phi_1^-(\alpha)}{\kappa \sin\kappa(b-a)\cos 2\kappa a} + \frac{\phi_2^-(\alpha)}{\cos 2\kappa a} + \frac{1}{\alpha-\alpha_1} = \phi_2^+(\alpha), \tag{15}$$

where $\phi_1^+(\alpha), \phi_2^+(\alpha)$ are functions that are analytic and regular in the region $\mathrm{Im}\,\alpha$ > -lmk.

The equations (14) and (15) constitute a coupled system of Wiener-Hopf equations which we can write in matrix form as:

$$\psi^+(\alpha) = K(\alpha)\psi^-(\alpha) + D/(\alpha-\alpha_1) \tag{16}$$

where

$$\psi^{\pm}(\alpha) = \begin{pmatrix} \phi_1^{\pm}(\alpha) \\ \phi_2^{\pm}(\alpha) \end{pmatrix}, D = \begin{pmatrix} -\kappa_1 \\ 1 \end{pmatrix}; \tag{17}$$

$$K(\alpha) = \frac{1}{\cos 2\kappa a} \begin{pmatrix} 1 & \dfrac{\kappa\cos\kappa(b+a)}{\sin\kappa(b-a)} \\ \dfrac{\cos\kappa(b+a)}{\kappa \sin\kappa(b-a)} & 1 \end{pmatrix}. \tag{18}$$

The usual Wiener-Hopf technique can be applied in a straightforward manner if the system (16) can be uncoupled into separate Wiener-Hopf equations. However the present problem requires that the matrix function $K(\alpha)$ be factorized as a product of two matrices, one analytic in $\mathrm{Im}\,\alpha$ > -Imk and the other analytic in Ima < Imk. This is a nontrivial procedure and it is not always obvious how to achieve it explicitly, at least in the classical sense, Noble [4] (where the elements of the matrix factors have algebraic behaviour at infinity). The matrix $K(\alpha)$ given by (18) is of a special form that can be factorized by a method of Daniele [1], and see Rawlins [6]. Without going into the details we obtain

$$K(\alpha) = K_+(\alpha)K_-(\alpha) \tag{19}$$

with

$$K_\pm(\alpha) = \sqrt{K_\pm(\alpha)}\begin{pmatrix} \cosh[1/2\kappa t_\pm(\alpha)] & \kappa\sinh[1/2\kappa t_\pm(\alpha)] \\ \frac{1}{\kappa}\sinh[1/2\kappa t_\pm(\alpha)] & \cosh[1/2\kappa t_\pm(\alpha)] \end{pmatrix} \tag{20}$$

where

$$K(\alpha) = \det K = K_+(\alpha)K_-(\alpha) = -\frac{\cos 2\kappa d}{\cos 2\kappa a \sin^2 \kappa(b-a)} \tag{21}$$

$$t(\alpha) \ = \ \frac{1}{k}\ell n\left(\frac{e^{i\lambda}\cos(\kappa a + \pi/4)\cos(\kappa b - \pi/4}{\cos(\kappa a - \pi/4)\cos(\kappa b + \pi/4}\right) \ = \ t_+(\alpha) + t_-(\alpha). \tag{22}$$

Explicit expressions for $K_\pm(\alpha)$ and $t_\pm(\alpha)$ are derived in the appendix. They are:

$$K_-(\alpha) = \frac{i}{(b-a)l^2}\prod_{n=1}^{\infty}\left(\frac{2kbn\pi}{a(b-a)}\right)^2 \frac{\left[(\hat{u}_{4n-}1)^2 - (1^2{}_{4n-1}u)^2\right]\left[(\hat{u}_{4n-}1_3)^2 - (\hat{1}_{4n-3}u)^2\right]}{\left[(u_{4n-1}1)^2 - (1^2{}_{4n-1}u)^2\right]\left[(u_{4n-3}1)^2 - (1\ {}_{4n-3}u)^2\right]\left[(\tilde{u}_n 1)^2 - (\tilde{1}_n u)^2\right]^2}$$

$$K_+(\alpha) = K_-(-\alpha) \tag{23}$$

$$t_-(\alpha) = -\frac{1}{\kappa}\ell n\left[\prod_{n=1}^{\infty} \frac{(\hat{u}_{4n-1}1 + \hat{1}_{4n-1}u)(\hat{u}_{4n-3}1 - \hat{1}_{4n-3}u)((u_{4n-1}1 - 1_{4n-1}u)(u_{4n-3}1 + 1_{4n-3}u)}{(\hat{u}_{4n-1}1 - \hat{1}_{4n-1}u)(\hat{u}_{4n-3}1 + \hat{1}_{4n-3}u)(u_{4n-1}1 + 1_{4n-1}u)(u_{4n-3}1 - 1_{4n-3}u)}\right] - \frac{1}{\kappa}\ell n(-\frac{\alpha - i\kappa}{\kappa})$$

$$t_+(\alpha) = t_-(-\alpha) \tag{24}$$

where $u = (k+\alpha)^{1/2}, l = (k-\alpha)^{1/2}, u_n = (k+\alpha_n)^{1/2}, l_n = (k-\alpha_n)^{1/2}, \alpha_n = (k^2(n\pi/(4a))^2)^{1/2}; \hat{u}_n =$ $(k+\hat{\alpha}_n)^{1/2}, \hat{l}_n = (k + \hat{\alpha}_n)^{1/2}, = (k^2 - (n\pi/(4b))^2)^{1/2}$, $\hat{u}_n = (k+\hat{\alpha}_n)^{1/2}$, $\hat{l}_n = (k+\hat{\alpha}_n)^{1/2}$, $= (k^2 - (n\pi/(b-a)\}^2)^{1/2}$.

The Wiener-Hopf equation (16) may now be solved by using (19) to rewrite (16) as

$$K_-\psi^-(\alpha) + K_+^{-1}(\alpha_1)D/(\alpha - \alpha_1) = K_+^{-1}(\alpha)\psi^+(\alpha) - [K_+^{-1}(\alpha) - K_+^{-1}(\alpha_1)]D/(\alpha - \alpha_1)\ . \tag{25}$$

The left-hand side of the above equation is analytic in $\mathrm{Im}\,\alpha$ < Imk, the right-hand side is analytic in $\mathrm{Im}\,\alpha$ > -Imk. Consequently each side is equal to an entire function $E(\alpha)$, that is, a matrix with polynomial entries. Hence letting

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = K_+^{-1}(\alpha_1)D \tag{26}$$

we have

$$\psi^-(\alpha) = K_-^{-1}(\alpha)[-C/(\alpha-\alpha_1) + E(\alpha)]. \tag{27}$$

The determination of $E(\alpha)$ depends on the asymptotic behaviour of various functions. In the appendix it is shown that as $|\alpha| \to \infty$ in $\mathrm{Im}\,\alpha < \mathrm{Im}k$:

$$K_-(\alpha) = 2 + 0(\alpha^{-1}), t_-(\alpha) = -\frac{i}{\alpha}\ell n\alpha + 0(\alpha^{-1}). \tag{28}$$

Hence

$$K_-(\alpha) = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha^{1/2}+0(1) & i\alpha^{1/2}+0(\alpha) \\ -i\alpha^{1/2}+0(\alpha^{-1}) & \alpha^{1/2}+0(1) \end{pmatrix}. \tag{29}$$

The asymptotic behaviour of $\psi^{\cdot}(\alpha)$ may be found from the edge conditions (7); it is not difficult to show that

$$\phi_1^-(\alpha) = 0(\alpha^{-1/2}), \quad \phi_1^-(\alpha) = 0(\alpha^{-2/2}), \tag{30}$$

as $|\alpha| \to \infty$ in $\mathrm{Im}\,\alpha < \mathrm{Im}k$:

It is now not difficult to show that by substituting (29), and (30) into the expression (27) that the entire matrix $E(\alpha)$ is given by

$$E(\alpha) = -ic_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{31}$$

in order that the asymptotic behaviour (30) be obtained from (27).

The solution can be written after a bit of algebra in a very simple form by introducing the following functions

$$F(u,l) = ul_1 + lu, \tag{32}$$

$$P(l,u) = \frac{e^{i\pi/4}(1-iu)e^{1/2\kappa_-(\alpha)}}{2\sqrt{k}\sqrt{K_-(\alpha)}},$$

$$= 1\left(\frac{b-a}{2}\right)^{1/2}\prod_{n=1}^{\infty}\left(\frac{2n\pi kb}{a(b-a)}\right)\frac{(u_{4n-1}l+l_{4n-1}u)(u_{4n-3}l-l_{4n-3}u)((u_n l)^2-(l_n u)^2)}{(\hat{u}_{4n-1}l+\hat{l}_{4n-1}u)(\hat{u}_{4n-3}l-\hat{l}_{4n-3}u)}. \quad (33)$$

Thus one obtains:

$$\Psi^-(\alpha) = \frac{P(u_1,l_1)}{(\alpha-\alpha_1)}\begin{pmatrix} P(l,u)F(u,l) \ + \ P(l,-u)F(-u,l) \\ [P(l,-u)F(-u,l) \ - \ P(l,u)F(u,l)]/\kappa \end{pmatrix}. \quad (34)$$

It is not difficult to show that P(l,u), P(l,-u) have neither zeros nor poles in Imα < imk. They do have branch points at α = -k, but the particular combinations in (34) are invariant under the transformation u → -u which means that $\psi^-(\alpha)$ is indeed analytic in Imα < Imk.

We now substitute the expression (34) into the integral representaitons (8) to (10) and obtain the field representations for the different regions as:

Region A (-b < y < -a, x < 0)

$$\phi(x,y) = -\frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\frac{\cos\kappa(b+y)}{\kappa\sin\kappa(b-a)}\{P(l,u)F(u,l)+P(l,-u)F(-u,l)\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (35)$$

Region B (-a < y < a, x < 0)

$$\phi(x,y) = e^{i\alpha_1 x}\cos[(y+a)\pi/4a] + \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}}{\cos 2ka}\{[(\sin k(y-a)-\cos\kappa(y+a)P(l,u)F(u,l)$$

$$+(\sin\kappa(y-a)+\cos\kappa(y+a))P(l,u)F(-u,l)]/\kappa)\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (36)$$

Region C (a < y < b, x < 0)

$$\phi(x,y) = \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\frac{\sin\kappa(b-y)}{\sin\kappa(b-a)}\{[P(l,-u)F(-u,l)-P(l,u)F(u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (37)$$

Region D (-b < y < b, x > 0)

$$\phi(x,y)=\frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}e^{i\alpha x}\,\frac{\kappa\sin\kappa(b-y)}{\cos 2\kappa b}\{[\cos(\kappa a+\pi/4)\cos(\kappa b-\pi/4)F(u,-1)/P(u,-1)$$

$$-\cos(\kappa a-\pi/4)\cos(\kappa b+\pi/4)F(u,l)/P(u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}\,. \qquad (38)$$

In the expressions (35) to (38) the pole $\alpha = \alpha_1$ lies above the contour of integration, that is $\mathcal{T} < \mathrm{Im}\alpha_1$. We also note that the term in the curley bracket { } of (35) to (37) has no singularities in $\mathrm{Im}\alpha > -\mathrm{Im}k$. Thus the only singularities in $\mathrm{Im}\alpha < \mathrm{Im}k$ of the integrands of (35) and (37) occur at the zeros of $\kappa\sin\kappa$ (b-a) = 0, that is $\alpha = -\tilde{\alpha}_n = \sqrt{(k^2-(n\pi/(b-a))^2)}$, n = 1,2,... . The only singularities of the integrand of (36) occur at the zeros of cos2ka = 0 that is $\alpha = -\alpha_{2n-1} = \sqrt{(k^2-((2n-1)\pi/4a)^2)}$ are the only singularities in $\mathrm{Im}\alpha < \mathrm{Im}k$. The only singularities in $\mathrm{Im}\alpha > -\mathrm{Im}k$ of the integrand (38) occur at the zeros of cos2kb = 0, that is $\alpha = \hat{\alpha}_{2n-1} = \sqrt{(k^2-(2n-1)\pi/(4b))^2)}$, n = 1,2,..., and also the pole $\alpha = \alpha_1$.

4. Mode field representation

An application of Cauchy's residue theorem to the complex integrals (35) to (38) then gives the field in the various regions as a sum of waveguide modes.

Region A (-b < y < -a, x < 0)

$$\phi(x,y)=-\frac{P(u_1,l_1)P(\sqrt{2k},0)\sqrt{2}u_1e^{-ikx}}{\sqrt{k}(b-a)(k+\alpha_1)}$$

$$-\frac{P(u_1,l_1)}{(b-a)}\sum_{n=1}^{\infty}\frac{e^{-i\tilde{\alpha}_n x}(-)^n}{(\tilde{\alpha}_n+\alpha_1)\tilde{\alpha}_n}\cos\tilde{\kappa}_n(b+y)\{P(\tilde{u}_n,\tilde{l}_n)F(\tilde{l}_n,\tilde{u}_n)+P(\tilde{u}_n,\tilde{l}_n)F(-\tilde{l}_n,\tilde{u}_n)\}. \qquad (39)$$

Region B (-a < y < a, x < 0)

$$\phi(x,y) \;=\; e^{i\alpha_1 x}\cos[(y+a)\pi/4a]$$

$$+\frac{P(u_1,l_1)}{2a}\sum_{n=1}^{\infty}\frac{e^{-i\alpha_m x}(-)^n}{\alpha_m(\alpha_m+\alpha_1)}\{[(\sin\kappa_m(y-a)-\cos\kappa_m(y+a))P(u_m,l_m)F(l_m,u_m)$$

$$+(\sin\kappa_m(y-a)+\cos\kappa_m(y+a))P(u_m,-l_m)F(-l_m,u_m)]\},\ (m=2n-1). \quad (40)$$

Region C (a < y < b, x < 0)

$$\phi(x,y)=-\frac{(P(u_1,l_1)}{(b-a)}\sum_{n=1}^{\infty}\frac{e^{-i\tilde{\alpha}_n x}(-)^n\sin\tilde{\kappa}_n(b-y)}{(\tilde{\alpha}_n+\alpha_1)\tilde{\alpha}_n}\{P(\tilde{u}_n,-\tilde{l}_n)F(-\tilde{l}_n,\tilde{u}_n)-P(\tilde{l}_n,\tilde{u}_n)F(\tilde{u}_n,\tilde{l}_n)\}. \quad (41)$$

Region D (-b ≤ y ≤ b, x > 0)

$$\phi(x,y)=\frac{-P(u_1,l_1)}{2b}\sum_{n=1}^{\infty}\frac{(-)^{[\frac{n}{2}]}e^{i\hat{\alpha}_m x}\hat{\kappa}_m\sin\hat{\kappa}_m(b-y)}{\hat{\alpha}_m(\hat{\alpha}_m-\alpha_1)}\frac{F(\hat{u}_m,(-)^{n+1}\hat{l}_m)}{P(\hat{u}_m,(-)^{n+1}\hat{l}_m)}\cos(\hat{\kappa}_m a+(-)^{n+1}\pi/4),(m=2n-1)$$

(42)

where [x] denotes the largest integer ≤ x.

Conclusion

We have solved a new diffraction problem in closed form by using matrix factorization. The present approach would not be substantially changed if we included a flow in the region -a < y < a. This would be a possible model for an exhaust system, where exhaust gases flow out of the system. It is hoped to present this solution in the near future.

We also remark that if we considered the same duct geometry and boundary conditions, but with the incident wave propagating in the region x > 0, | y | < b from z = ∞, see fig. 3, then we would obtain a similar matrix Wiener-Hopf problem which can be solved exactly. As a special case of this latter problem by letting a → 0 one would obtain the solution to the problem considered by Lüneburg and Hurd [3].

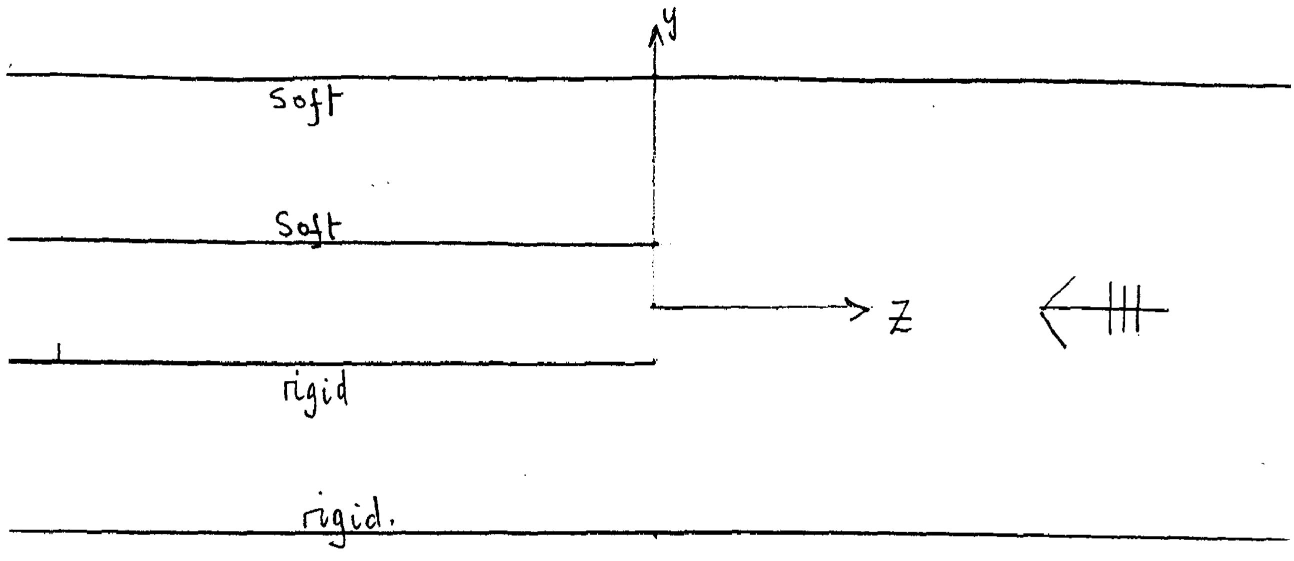

fig. 3

References

[1] Daniele, V.G., On the factorization of Wiener-Hopf matrices in problems solvable with Hurd's method. IEEE Trans. on Antennas and propagation, AP-26, 614-616 (1978).

[2] Kranzer and Radlow, J., An asymptotic method for solving perturbed Wiener-Hopf problems. J. Math. Mech. 14, 41-60 (1965).

[3] Luneburgh, E. and Kurd, R.A., Two waveguide bifurcation problems. Z. Angew. Math. Mech. (ZAMM) 65, 551-559 (1985).

[4] Noble, B. The Wiener-Hopf Technique. Pergamon, London, 1958.

[5] Rawlins, A.D., A trifurcated waveguide problem. Proc. R. Soc. Lond. (submitted).

[6] Rawlins, A.D., A note on the factorization of matrices occurring in Wiener-Hopf problems. IEEE Trans, on Antennas and Propagation, AP-28, 933-934 (1980).

Appendix

Here we shall carry out the explicit factorization of (23) and (24), and give their asymptotic behaviour as $|\alpha| \to \infty$. For $K(\alpha)$ given by (23) we can write it in a more convenient form

$$K(\alpha) = -\frac{\cos(\kappa b + \pi/4)\cos(\kappa b - \pi/4)}{\cos(\kappa a + \pi/4)\cos(\kappa a - \pi/4)\sin^2\kappa(b-a)}$$

and if we let

$$K(\alpha, a) = \cos(\kappa a + \pi/4)\cos(\kappa a - \pi/4) = K_+(\alpha, a)K_-(\alpha, a)$$

$$S(\alpha, (b-a) = \sin^2\kappa(b-a) = S_+(\alpha, (b-a))S_-(\alpha, (b-a),$$

$$K_\pm(\alpha) = i\frac{K_\pm(\alpha, b)}{K_\pm(\alpha, a)S_\pm(\alpha, b-a)}$$

Now from Rawlins [5] it has been shown that

$$K_-(\alpha, a) = K_+(-\alpha, a) = \prod_{n=1}^{\infty}\left(\frac{a}{(2n-1)\pi k}\right)^2 (u_{4n-1}l - l_{4n-1}u)(u_{4n-1}l + l_{4n-1}u)$$

$$\times(u_{4n-3}l - l_{4n-3}u)(u_{4n-3}l - l_{4n-3}u);$$

And that

$$K_\pm(\alpha, a) = 0(e^{a|\alpha|}) \text{ as } |\alpha| \to \infty \text{ in } \mathrm{Im}\alpha > \text{-Im}k \text{ and } \mathrm{Im}\alpha < \mathrm{Im}k \text{ respectively.}$$

From the well known product for the sine function

$$\sin z = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

it is not difficult to show that

$$S_-(\alpha, (b-a)) = S_+(-\alpha, (b-a)) = (b-a)l^2\prod_{n=1}^{\infty}\left(\frac{(b-a)}{2kn\pi}\right)^2 ((\tilde{u}_n l)^2 - (\tilde{l}_n u)^2)^2 .$$

Thus

$$K_{-}(\alpha)=\frac{i}{(b-a)l^{2}}\prod_{n=1}^{\infty}\left(\frac{2kn\pi b^{2}}{a(b-a)}\right)\frac{((\hat{u}_{4n-1}\,l)^{2}-(\hat{l}_{4n-1}\,u)^{2})((\hat{u}_{4n-3}\,l)^{2}(\hat{l}_{4n-3}\,u)^{2})}{((u_{4n-1})^{2}-(l_{4n-1}u)^{2})((u_{4n-3})^{2}-(l_{4n-3}u)^{2})((\tilde{u}_{n}\,l)^{2}-(\tilde{l}_{n}\,u)^{2})^{2}}\,.$$

Now since K(-α) = K(α) then by Radlow and Kranzer [2] we have since K(α) = 4+0(1) that

$$K_{\pm}(\alpha) = 2 + o(l)\,.$$

The factorization of

$$t(\alpha)=\frac{1}{\kappa}\,1n\left[\frac{e^{i\pi}\cos(\pi/4-\kappa b)\cos(\pi/4+\kappa a)}{\cos(\pi/+\kappa b)\cos(\pi/4-\kappa a)}\right]$$

follows directly from Rawlins [5], as

$$t_{-}(\alpha)=\frac{-1}{\kappa}\;in\left(\frac{-\alpha-ik}{k}\right)+\frac{-1}{\kappa}\;1n\left[\prod_{n=1}^{\infty}\frac{(\hat{u}_{4n-1}\,l-\hat{l}_{4n-1}\,u)(\hat{u}_{4n-3}\,l+\hat{l}_{4n-3}\,u)(u_{4n-1}l+l_{4n-1}u)(u_{4n-3}l-l_{4n-3}u)}{(\hat{u}_{4n-1}\,l+\hat{l}_{4n-1}\,u)(\hat{u}_{4n-3}\,l-\hat{l}_{4n-3}\,u)(u_{4n-1}l-l_{4n-1}u)(u_{4n-3}l+l_{4n-3}u)}\right.$$

where

$$t_{+}(\alpha) = t_{-}(-\alpha)$$

and

$$u=(k+\alpha)^{1/2}u_{n}=(k+\alpha)^{1/2},\; l=(k-\alpha)^{1/2},\; l_{n}=(k-\alpha)^{1/2}\;,$$

$$\alpha_{n}=(k^{2}-(n\pi/4a))^{2})^{1/2},\hat{u}_{n}=(k+\hat{\alpha}_{n}),\hat{l}_{n}=(k-\hat{\alpha}_{n}),\;\hat{\alpha}_{n}=(k^{2}-(n\pi/(4b))^{2})^{1/2}.$$

The behaviour of $t_{-}(\alpha)$ as $\alpha \to \infty$ in Imα < Imk can be shown to be

$$t_{-}(\alpha)=-\frac{i}{\alpha}\ell n(\alpha)+0(\alpha^{-1})\,.$$