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Surface Representation in Computer-Aided Geometric Design

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I. INTRODUCTION

A precursor to a finite element analysis of a curved surface such as an aircraft or car body, is that of the mathematical description of the surface itself. This topic is a subject of the theory of computer- aided *geom tric design* or *computational geometry* and is related to finite element methods through its use of piecewise defined interpolation and approximation theory. In particular, a curved surface in c*,a.g.d.* is usually represented as a piecewise defined vector valued function [x(u,v),y(u,v), z(u,v)] , where the parametric variables (u, v) are defined over rectangular elements.

The use of vector valued,, rather than scalar valued, functions gives the designer much greater flexibility in the representation of the curved surface. Also, the almost exclusive use of rectangular elements enables continuity conditions between those elements to be handled relat1vely easily, whilst simplifying the data structure required for the complete surface and simplifying routines such as those required for plots and cross-sections. Indeed, the special problems associated with vector valued surface representation seems to have precluded the use of triangular elements favoured by the finite element analyst and more research is needed in this area. There are, however, situations peculiar to vector valued representations, where a surface patch requiring a non- rectangular domain of definition, such as a triangle or pentagon, can occur within a rectangular patch framework.. This paper, after reviewing the subject of rectangular patch representations, will consider a recent development in the representation of such surfaces.

A starting point for the development of curved surface representations is to consider a wire frame model as illustrated in the simple example of Fig. 1. Each portion of the surface bounded by four curved sides is to be defined by a rectangular patch, that is a vector valued function with a rectangular domain of definition. The example also shows how a triangular patch can occur within a rectangular patch framework. The next section gives a brief review of local rectangular patch representations.

For more details the interested reader should consult references [2],[6], and [7],

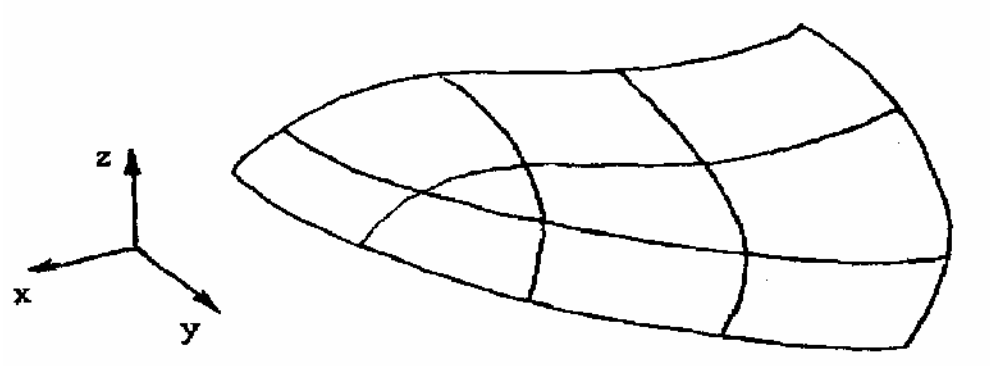


FIG. 1. Example of a wire frame model.

2. SURFACE REPRESENTATION OVER RECTANGLES

In briefly reviewing the subject of rectangular surface patch representations, the following structure can be discerned: Let univariate curves be defined by

$$\underline{r}(u) = \sum_{i=0}^{m} \alpha_i(u)\ \underline{a}_i \ , \qquad u \in [a,\ b] \ , \tag{2.1}$$

$$\underline{r}(v) = \sum_{j=0}^{n} \beta_i(v)\ \underline{b}_j \ , \qquad v \in [c,\ d] \ , \tag{2.2}$$

where $\underline{r} \equiv [x,y,z]$, $\underline{a}_i , \underline{b}_j \in R^3$, and the $\{\alpha_i\}$ and $\{\beta_j\}$ are polynomial basis functions of degree m and n respectively. Then, for $(u,v) \in [a,b]x[c,d]$, three types of bivariate surfaces can be defined:

2.1. The Lofted Surfaces

$$\underline{r}_1(u,\ v) = \sum_{i=0}^{m} \alpha_i(u)\ \underline{a}_i(v) \ ,\ \underline{r}_2(u,\ v) = \sum_{j=0}^{n} \beta_i(v)\ \underline{b}_j(u) \ . \tag{2.3}$$

2.2. The Tencor Product Surface.

$$\underline{t}(u,\ v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_i(u)\ \beta_j(v)\ \underline{c}_{i,j} \tag{2.4}$$

2.3. *The Boolean Sum Blended Surface*

$$\underline{p}(u,\ v) = \underline{r}_1(u,\ v) + \underline{r}_2(u,\ v) - \underline{t}(u,\ v) \tag{2.5}$$

The tensor produce surface will be familiar to the finite, element analyst, where the $\{\alpha_i\}$ and $\{(\beta_j\}$ are defined by Hermite interpolation bases. The Boolean sum blended surface is a generalization introduced by Coons [4], its application being restricted to interpolatory bases. In this case the lofted surfaces can be written as

$$\underline{r}_1 = p_1[\underline{f}] \quad \text{and} \quad \underline{r}_2 = p_2[\underline{f}] \tag{2.6}$$

P_1 denoting a linear operator which acts on a bivariate vector valued function $\underline{f}(u,v)$ considered as a function of u, and P_2 denoting an operator which acts on $\underline{f}(u,v)$ considered as a function of v . The tensor product and Boolean sum blended surfaces are then defined by

$$\underline{t} = P_1 P_2[\underline{f}] \;,\; \underline{p} = (P_1 \oplus P_2)[\underline{f}] \equiv (p_1 + p_2 - p_1 p_2)[\underline{f}]. \tag{2.7}$$

Such operators have interesting approximation theoretic properties which have been studied by Gordon [8], The Boolean sum blended form has been used in finite element theory, for example in the derivation of the mapping techniques of Gordon and Hall [9] and in the systematic derivation of serendipty type elements [5] . For the purposes of c. a. g. d. , the Boolean sum blended surface has the useful property of combining the interpolation properties of the two lofted surfaces, where it is assume d that $\underline{f}$ is such that P_1 and P_2 commute .

2. 4, Pati nal Forms

Extra degrees of freedom can be introduced by replacing (2,1) and (2,2) with the rational forms

$$\underline{r}(u) = \frac{1}{\upsilon(u)} \sum_{i=0}^{m} \alpha_i(u)\, \upsilon_i \underline{a}_i \;, \qquad \upsilon(u) = \sum_{i=0}^{m} \alpha_i(u)\, \upsilon_i \;, \tag{2.8}$$

$$\underline{r}(v) = \frac{1}{\omega(v)} \sum_{j=0}^{n} \beta_j(v)\, \omega_j \underline{b}_j \;, \qquad \omega(v) = \sum_{j=0}^{n} \beta_j(v)\, \omega_j \;, \tag{2.9}$$

Appropriate modificat ions can then be made to the bivariate surface descript ions. Quadratic rational curves and surfaces are useful in that they can be used to represent conies and quadrics.

2.5, Continuity Between Surface Patches

Suppose $\underline{p}(u,v)$ and $\underline{q}(u,v)$ define two regular parametric representations on $[0,1]x[0,1]$. Then the patches join with c^0 continuity of position if, for example,

$$\underline{p}(1,v) = \underline{q}(0,v) \tag{2.10}$$

Conditions sufficient for C^1 (tangent plane) and C^2 (curvature) continuity across this common boundary are then given by

$$c_1 \underline{p}_{1,0} (1,v) = c_2 \underline{q}_{1,0} (0,v) , c_1 ,c_2 >0, \tag{2,11}$$

$$c_1^2 \underline{p}_{2,0} (1,v) + k_1 \underline{p}_{1,0} (1.v) = c_2^2 \underline{q}_{2,0} (0,v) + k_2 \underline{q}_{1,0} (0,v) . \tag{2.12}$$

The above conditions match tangent and curvature along the v = constant direction and clearly more general continuity conditions could be formulated. However, the above conditions, and their duals across other boundaries, suffice for most practical cases.

2. 6. Examples

2.6.1 Linear interpetation

The lofted surfaces are defined in operator form on the square [0,1]x[0,1] by

$$p_1 [\underline{f}] (u,v) = (1-u) \underline{f}(0,v) + u \underline{f}(1,v) ,$$
$$P_2 [\underline{f}] (u,v) = (1-v) \underline{f}(u,0) + v \underline{f}(u-1) , \tag{2.13}$$

The tensor product and Boolean sum blended forms are then defined by (2.7). The tensor product surface interpolates $\underline{f}$ at the corners of the square, whereas the Coons' Boolean sum blended surface matches $\underline{f}$ along the entire boundary .

2. 6. 2. *Cubic Hermite interpolation*

Let the cubic Hermite basis functions be defined on [0,1] by

$$\varphi_0(t) = (1-t)^2 (2t+1) , \quad \Phi_1(t) = (1-t)^2 t$$
$$\psi_0(t) = t^2(-2t+3) , \quad \square_j (t) = t^2 (t-1) . \tag{2,14}$$

Then the lofted surfaces are defined on [0,1]x[0,1] by

$$p_1 [\underline{f}](u, v) = \sum_{i=0}^{1} [\varphi_i (u) \underline{f}_{i,0} (0, v) + \psi_i (u) \underline{f}_{i,0} (1, v)] ,$$

$$p_2[\underline{f}](u,v) \; \sum_{j=0}^{1} [\varphi_j(v) \; \underline{f}_{0,j}(u,0) + \psi_j(v) \; \underline{f}_{0,j}(u,1)]. \qquad (2.15)$$

The tensor product and Boolean sura blended forms are then defined by (2.7), where, for commutivity of the operators, we assume that $\partial^2 \underline{f}/\partial u \partial v = \partial^2 \underline{f}/\partial v \partial u$ at the corners of the square.

2.6.3. The cubic as a convex combination

For a given constant k, $0 < k \leq -3$, let cubic basis functions be defined on $[0, 1]$ by

$$\gamma_0(t) = (1-t)^2(2t-kt+1) \quad , \quad \gamma_1(t) = k(1-t)^2 t \quad , \qquad (2.16)$$

$$\gamma_2(t) = kt^2(1-t) \quad , \quad \gamma_3(t) = t^2(-2t+k+3-k)$$

Then with $\alpha_i = \gamma_i$ $\beta_j = \gamma_j$, and $m = n = 3$, univariate curves (2.1) and (2.2) can be defined. The lofted surfaces and tensor product surface are then defined by (2.3) and (2.4) respectively. The restriction that $0 < k \leq 3$ ensures that $\gamma_i(t) \geq 0$ on $[0,1]$ and since $\Sigma\gamma_i = 1$ it follows that the curves and surfaces are defined by convex combinations. The basis (2.16) is related to the cubic Hermite basis (2.14) in that, in (2.1),

$$\underline{r}(0) = \underline{a}_0 \quad , \quad \underline{r}'(0) = k(\underline{a}_1 - \underline{a}_0) \; ,$$
$$\underline{r}(1) = \underline{a}_3 \quad , \quad \underline{r}'(1) = k(\underline{a}_3 - \underline{a}_2) \; . \qquad (2.17)$$

The examples *2.6.2* and *2.6.3* can be used to construct piecewise defined C^1 surfaces. The Coons' Boolean sum blended cubic Hermite surface has the property that it matches a function $\underline{f}$, and cross boundary tangent vector $\underline{f}_n$ say, along the entire boundary of the square $[0,1]x[0,1]$. The tensor product cubic Hermite surface is a special case of the Boolean sum blended form which is used in many c.*a. g. d.* systems. The tensor product form of the cubic convex combination with $k = 3$ is an example of a surface used by Bezier [3], where the basis functions are the Bernstien polynomials. The representation as a convex combination gives the designer some control over the behaviour of the curve or surface, for example, the representation will lie in the convex hull of the coefficients. The case $k = 2$ corresponds to that used by Ball [1] in a rational lofted form, where the variable coefficients of the rational lofted form are also defined by rational forms.

3. SURFACE REPRESENTATION OVER TRIANGLES AND PENTAGONS

We now consider the problem of constructing surface patches with non-rectangular domains of definition but which occur within a rectangular patch framework. More specifically, we consider the need for triangular and pentagonal patches as illustrated by the wire frame model problems of Fig. 2. Such

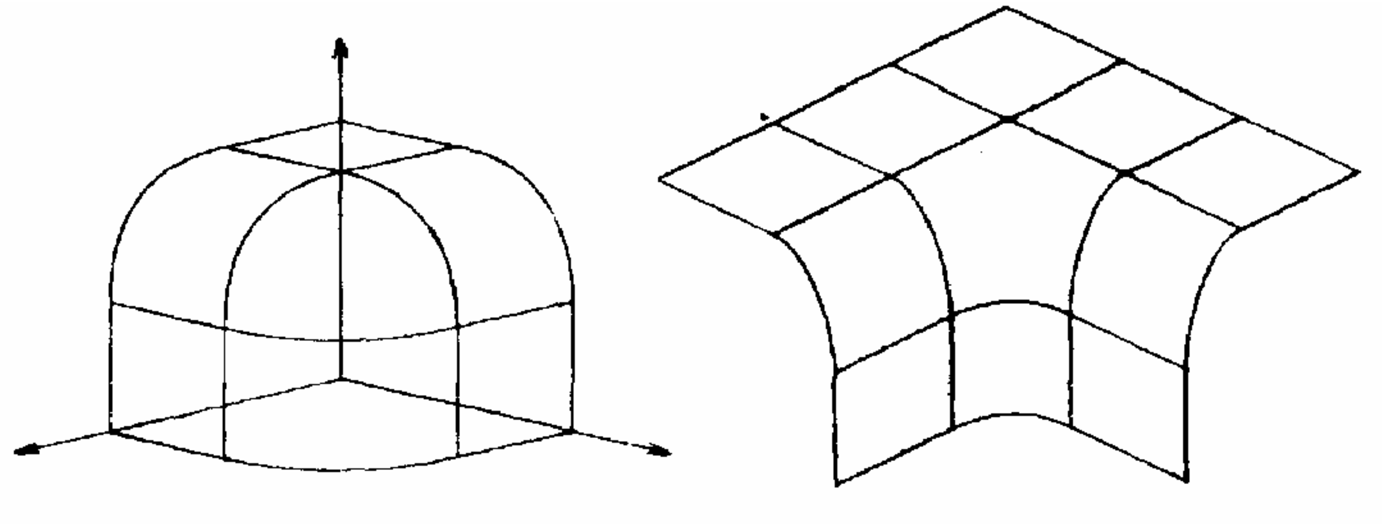


FIG. 2. Triangular and pentagonal patch model problems.

problems are intrinsically vector valued, there being no equivalent scalar valued problems, since the domains of definition of the surface patches cannot lie in a common domain in R^2. One approach to this problem is to divide the non-rectangular patch in to a subsystem of rectangular patches and this has been considered by Bezier [3] and Handscomb [12], Here, however, we briefly describe a method due to the author and P. Charrot which specifically uses non-rectangular domains. Further details can be found in references [10] and [11].

We restrict the discussion to the C^1 case, where continuity of function and tangent plane is required across the boundary curves of the wire frame model. The boundary curve and cross boundary tangent vector are assumed to be defined on each side by an appropriate rectangular patch representation. The method is then to construct Boolean sum blended interpolants which match function and tangent plane conditions on two adjacent sides of the non—rectangular domain. An appropriate convex combination of these interpolants is then formed which is designed to give interpolation to the function and tangent plane along the entire boundary.

3.1. Boolean Sum Blended Taylor Interpolant

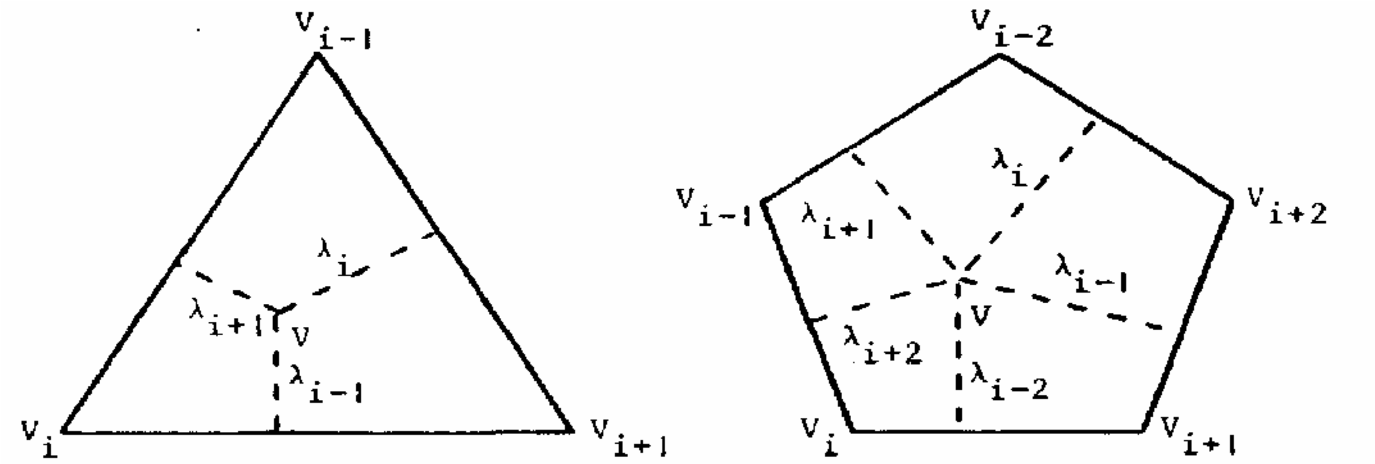
Let lofted Taylor interpolants be defined by

$$T_1[\underline{f}](u,v) = \underline{f}(0,v) + u\,\underline{f}_{1,0}(0,v) \;,$$
$$T_2[\underline{f}](u,v) = \underline{f}(u,0) + v\,\underline{f}_{0,1}(u,0) \;. \qquad (3.1)$$

Then the Boolean sum blended surface

$$\underline{p}(u,v) = (T_1 + T_2 - T_1T_2)\,[\underline{f}]\,(u,v) \qquad (3.2)$$

has the property that it interpolates $\underline{f}$ and the tangent plane of $\underline{f}$ on u=0 and v=0.



FIG, 3. The triangular and pentagonal domains.

3.2. The Triangular Surface Patch

It is convenient to choose the domain as an equilateral triangle of height unity and to define λ_i as the perpendicular distance of a general point V to the side opposite the vertex V_i, i=1,2,3. The boundary curve $\underline{f}$ and the cross boundary tangent vector $\underline{f}_{n\,i}$ are assumed to be given along each side $\lambda_i = 0$. Choosing $u = \lambda_{i+1}$ and $v = \lambda_{i-1}$ as the independent variables, and interpreting $\underline{f}_{n\,i+1}$. along λ_{i-1} = constant, $\underline{f}_{n\,i-1}$ along λ_{i+1} = constant, allows us to define the Boolean sum blended interpolant, $\underline{P}i(v)$ say, which interpolates the function and tangent plane on $\lambda_{i+1} = 0$ and $\lambda_{i-1} = 0$. The triangular surface patch is then defined by

$$\underline{P}(v) = \sum_{i=1}^{3} \alpha_i(v)\;\underline{P}_i(v) \;,\quad \alpha_i(v) = \lambda_i^{\,2}(3 - 2\lambda_i + 6\lambda_{i+1}\lambda_{i-1}) \;. \qquad (3.3)$$

The leading term λ_i^2 in the definition of $\alpha_i(V)$, together with the property that $\Sigma\alpha_i(V) = 1$, ensures that the function and tangent plane of $\underline{P}(V)$ on $\lambda_i = 0$ is an average of those of $\underline{P}_{i+1}(v)$ and $\underline{P}_{i-1}(V)$ Thus $\underline{P}(V)$ interpolates the function and tangent plane along the entire boundary of the triangle.

3.3, *The. Pentagonal Surface Patch*

The domain is now chosen to be a regular pentagon of height unity and, as before, λ_i is defined as the perpendicular distance of a general point V to the side opposite the vertex V_i, i=1, . . , 5. A Boolean sum blended Taylor interpolant $\underline{P}i(V)$ could now be defined with respect to the variables λ_{i+2} and λ_{i-2}. We Prefer, however, to choose the variables

$$u = \lambda_{i+2}/(\lambda_{i-1} + \lambda_{i+2}), \quad v = \lambda_{i-2}/(\lambda_{i+1} + \lambda_{i-2}), \tag{3.4}$$

in the definition of the Boolean sum blended Taylor interpolant. The tangent vector $\underline{f}_{n\,i+2}$ is then interpreted along v=constant which is along the radial line joining the point of intersection of $\lambda_{i+1} = 0$ and $\lambda_{i-2} = 0$ to the point V. Similarly, the tangent vector $f_{n\,i-2}$. is interpreted along the radial direction u=constant. The resulting surface $\underline{P}_i(V)$ interpolates the function and tangent plane on $\lambda_{i+2} = 0$ and $\lambda_{i-2} = 0$. *The* pentagonal surface patch is now defined by

$$\underline{p}(v) = \sum_{i=1}^{5} \alpha_i(v)\, \underline{p}_i(v), \tag{3.5}$$

$$\alpha_i(v) = \lambda_{i+1}^2 \lambda_i^2 \lambda_{i-1}^2 \Big/ \sum_{k=1}^{5} \lambda_{k+1}^2 \lambda_k^2 \lambda_{k-1}^2 . \tag{3.6}$$

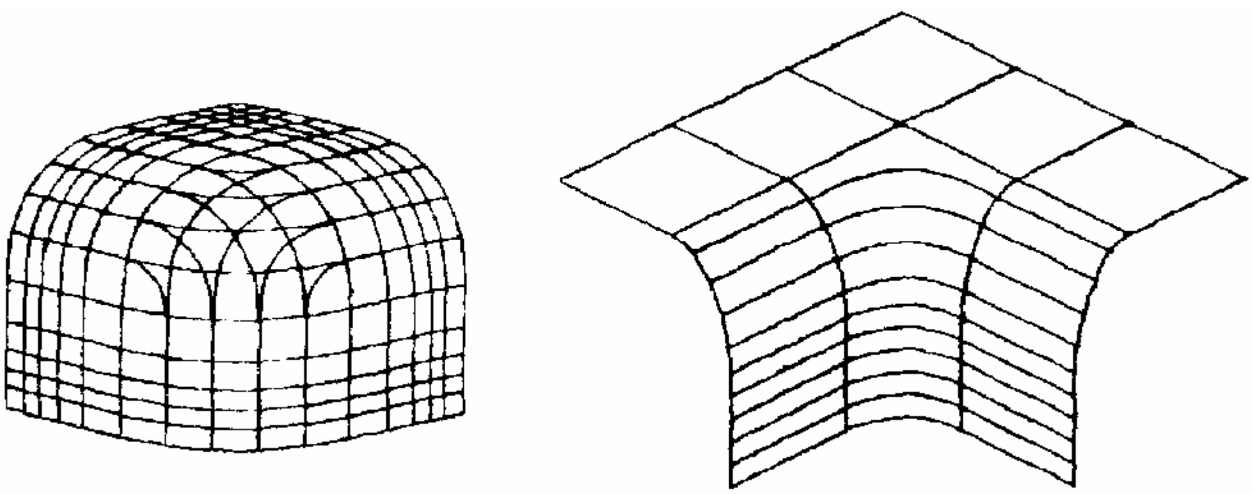


FIG. 4. Cross—sections through surface patches ofmodel problems.

By app lying argriments similar to those used for the triangular surface patch, the properties of the rational function (3.6) ensure that p(V) interpolates the function and tangent plane along the entire boundary of the pentagon, (An alternative rational weight function for the triangle, corresponding to that used for the pentagon, is $\alpha_i(V) = \lambda_i^2/\Sigma\lambda_k^2$. A polynomial weight function for the pentagon cannot be found.)

The implementation of the triangular and pentagonal surface patches for the model problems of Fig. 2 is shown in Fig.4, where the plotting lines are cross sections. In these examples, the rectangular patches are tensor product cubic Hermite surfaces. Further examples can be found in references [10] and [11]-

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