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SCATTERING BY ONE-DIMENSIONAL ROUGH SURFACES

S.N. Chandler-Wilde & C.R. Ross

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S. N. Chandler-Wilde* C. R. Ross[†]

Abstract

We consider the two-dimensional Dirichlet boundary value problem for the Helmholtz equation in a non-locally perturbed half-plane. We look for a solution in the form of a double-layer potential using, as fundamental solution, the Green's function for the impedance half-plane. This leads to a boundary integral equation which can be solved for any bounded and continuous boundary data provided the boundary itself does not differ too much from the flat boundary $\{(x_1,h) \in R^2 : x_1 \in R\}$ $(h > 0)$. We show this by calculating the symbol of the integral operator in the integral equation in the flat boundary case, and then using standard operator perturbation results. Continuous dependence of the solution on the shape of the boundary is shown.

1 Introduction

We consider the two-dimensional Dirichlet boundary value problem for the Helmholtz equation in a non-locally perturbed half-plane. This Dirichlet boundary value problem arises from a study of time harmonic wave scattering by one-dimensional surfaces, in particular acoustic scattering of an incident field u^i by a sound soft boundary ∂D where the total field $u^t = u^i + u^s$ vanishes, so that the scattered field u^s satisfies the Dirichlet condition $u^s = -u^i$ on ∂D. The identical boundary condition arises in the P polarization case over a perfectly conducting grating in electromagnetic scattering [5].

A rigorous study of the locally perturbed half-plane case using integral equation methods has been undertaken in Gartmeier [6] and Willers [11], and for the diffraction grating (periodic boundary) case in Kirsch [9]. The results we present here appear to be the first attempt to give a rigorous existence proof using integral equation methods for the more general rough surface scattering case. In contrast to the above cited papers and the well researched case of scattering by bounded obstacles [4], the boundary value problem we discuss does not lead to an integral equation with a compact operator. Thus the compact operator theory of Riesz and Fredholm cannot be applied.

In Section 2 we formulate the boundary value problem. In our boundary value problem we insist that the solution be bounded in the horizontal direction (parallel to the boundary) and tempered in the vertical direction (away from the boundary). Also, for real wave number k, we demand that the solution satisfies a limiting absorption condition. In Section 3 we discuss the Green's function G_1 for the Helmholtz equation in a half-plane with an impedance boundary condition. In Section 4 we state properties of a modified double-layer potential on ∂D, in which we replace the usual fundamental solution by the Green's function G_1. We show that this double-layer potential satisfies the boundary value problem in the case Im $k > 0$ provided that the density satisfies a second kind integral equation on the boundary ∂D. In Section 5 we consider the flat boundary case where the integral operator

*Department of Mathematics and Statistics, Brunei University, Uxbridge UB8 3PH, U.K.
[†]Dept. of Civil Engineering, University of Bradford, Bradford BD7 1DP, U.K.

K is a convolution operator. By computing the Fourier-transform of the kernel of K we prove uniqueness and existence of the solution of the integral equation in this case (for k > 0 as well as Imk > 0) using standard results in convolution operator theory [8]. A simple function analytical argument extends this result to boundaries which do not differ too much from the fiat boundary. Finally, we prove existence of a solution to the boundary value problem for this case (including the limiting absorption principle for the case k > 0). Throughout the paper, we will write $x, y \in R^2$ with $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $y' = (y_1, y_2)$. By U we denote the upper half-plane $U = \{x \in R^2 : x_2 > 0\}$. $x . y$ stands for the euclidian scalar product in $\mathbf{R^2}$.

2 Formulation of the Boundary Value Problem

Throughout, suppose that $K \in C$ (the wave-number) satisfies Im k ≥ 0 and Re k > 0. Let $f \in C^{1,\alpha}(R)$ with $f(x) > \in > 0$, $\forall x \in R$, and define the region D by

$$D := \{(x_1, x_2) \in R^2 : x_2 > f\{x_1)\}$$

so that the boundary of D is

$$(1) \qquad \partial D = \{ (x_1 , f(x_1)) : x_1 \in R \} .$$

Let BC(∂D) denote the set of bounded and continuous functions denned on ∂D.
The boundary value problem is as follows: given g $g \in BC(\partial D)$, determine $u \in C^2(D) \cap C(\overline{D})$ such that

1. u is a solution of the Helmholtz equation, i.e. $\Delta u + k^2 u = 0$ in D
2. $u = g$ on ∂D
3. For some $n \in N$ and C > 0

$$| x_2^{-n} u(x)| \leq C, \quad \text{var } \forall x \in D ,$$

so that u is tempered in the X2-direction and bounded in the x_1 -direction.
For real k we insist that u also satisfies a limiting absorption principle. Let $u^{(\lambda)}$ denote a solution to the above problem for wave number $K = \lambda$. Then the additional condition is

4. I f k > 0 then there exists a solution $u^{(K+i\in)}$ for all sufficiently small $\in > 0$ such that $u^{(k+i\in)} \to u^{(K)}$ as $\in \to 0$, uniformly on compact subsets of $\overline{D}$.

3 The Green's Function for the Impedance Half-Plane

In order to establish existence results for the above boundary value problem using integral equation methods, we introduce the following fundamental solution of the Helmholtz equation.

Let $\beta \in C$, Re β > 0. Define

$$(2) \qquad G\beta (x, y) := \varphi (x, y) + \varphi' (x, y) + P \quad x, y) , \qquad x, y \in \overline{U} , x \neq y ,$$

where

$$\varphi(x, y) := -\frac{i}{4} H_0^{(1)} (k \mid x - y \mid),$$

$$\varphi' (x, y) := -\frac{i}{4} H_0^{(1)} (k \mid x - y' \mid)$$

and

(3) $$P_\beta(x,y) := \frac{i\kappa\beta}{\pi}\int_{-\infty}^{\infty} \frac{e^{i((x-y').(-s,\sqrt{\kappa^2-s^2}))}}{\sqrt{\kappa^2-s^2}\,(\sqrt{\kappa^2-s^2}+\kappa\beta)}ds\ ,$$

with Im $\sqrt{\kappa^2-s^2} \geq 0$. Note that Pβ depends on the components of x and y only in the combinations $x_2 - y_2$ and $|x_1 - y_1|$. We will therefore write

(4) $$P\beta(x,y) = \hat{P}\beta\,(x - y'),$$

With the function $\hat{P}\beta$ denned accordingly. It is easy to see from (3) that $\hat{P}\beta \in C^\infty(U)$ and that all the derivatives of $\hat{P}\beta$ depend continuously on K for Imk ≥ 0, Rek > 0. The function Gβ is the velocity potential in U (sum of incident and scattered fields) resulting from an incident field $u^i = \Phi(x,y)$ scattered on ∂U with admittance β. In particular, (where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$)it is straightforward to see that

$$\Delta G_\beta(x,y) + \kappa^2 G_\beta(x,y) = 0,\ \ x,y \in U, x \neq y$$

and

$$\frac{\partial G_\beta}{\partial x_2}(x,y) + i\kappa\beta G_\beta(x,y) = 0\ ,\ \ x \in \partial U, y \in U.$$

The reasons why we use Gβ rather than the standard fundamental solution Φ will become clear in due course. An important reason is the faster asymptotic decay rate of Gβ(x, y) as |x — y| → ∞ with x and y close to the boundary ∂U. This decay rate is quantified in the first lemma, in which we consider, as in most of the sequel, just the specific case β = 1. Here $grad_y G_1$ denotes the gradient of G_1 taken with respect to y.

LEMMA 3.1. Let $C_1 > 0$. Then, for some constant C depending only on C_1 and Re k, there holds

(5) $$|G_1(x,y)| \leq C\frac{x_2}{|x-y|^{3/2}}$$

and

(6) $$|grad_y G_1(x,y)| \leq C\frac{x_2}{|x-y|^{3/2}}$$

for all K ∈ C with Re k > 0 and 0 ≤ Imk ≤ C_1 and for all x , y ∈ U with \x — y\ ≥ 1 and $0 \leq y_2 \leq C_1$.

A proof of this lemma, using a Laplace-transform type integral representation for G_1 and Watson's Lemma, is given in [3]. For further properties and integral representations for Gβ see [2].

4 The Double-Layer Potential

With our Green's function G_1 we are able to construct the following velocity potential. Given a function φ ∈ BC(∂D) we call the integral

(7) $$u(x) = \int_{\partial D} \frac{\partial G_1(x,y)}{\partial v(y)}\varphi(y)ds(y),\qquad x \in D$$

the double-layer potential with density φ. Here, as in the following, v{y) stands for the normal vector at y ∈ ∂D pointing into D.

First of all we note that u is well denned, that is the integral over ∂D exists for ∈ D: the integrand is integrable at infinity since *f* ∈ BC^1(R) and $\frac{\partial G_1(x,y)}{\partial v(y)} = 0\left(\frac{x_2}{|x-y|^{3/2}}\right)$ as |x — y| →∞, from Lemma 3.1.

We will look for a solution to the boundary value problem in the form of such a double-layer potential; the following lemmata show that u, given by (7), satisfies conditions 1 and 3 of our boundary value problem. Proofs of these results and of Theorem 4.1, which use Lemma 3.1 and standard results on double-layer potentials (see [7] and [4]), are given in [3].

LEMMA 4.1. T*he double-layer potential u satisfies* $u \in C^2(D)$ *and*

$$\Delta u + \kappa^2 u = 0 \text{ in } D.$$

LEMMA 4.2. *For the double-layer potential u there exists C > 0 such that*

$$|x_2^{-1/2} u(x)| \leq C \| \varphi \|_\infty$$

for all $x \in D$ *and* $\varphi \in BC(\partial D)$.

To establish conditions under which the double-layer potential satisfies the boundary condition of our boundary value problem we give the following jump relations for our double-layer potential u.

THEOREM 4.1. *The double-layer potential u with density can be continuously extended from D to* $\overline{D}$ *with limiting values*

(8) $$u_+(x) = \int_{\partial D} \frac{\partial G_1(x,y)}{\partial \nu(y)} \varphi(y) ds(y) - \frac{1}{2}\varphi(x), \qquad x \in \partial D,$$

where $$u_+(x) := \lim_{h \to 0, h > 0} u(x + h\nu(x)).$$

The integral exists as an improper integral.

The double-layer potential u satisfies the boundary condition of the boundary value problem provided that φ satisfies

(9) $$g(x) = \int_{\partial D} \frac{\partial G_1(x,y)}{\partial \nu(y)} \varphi(y) ds(y) - \frac{1}{2}\varphi(x), \qquad x \in \partial D.$$

Defining $\overline{\varphi}, \overline{g}, \in BC(R)$ by

(10) $$\overline{\varphi}(s) := (s, f(s)), \ \overline{g}(s) := g(s, f(s)), \ s \in R,$$

and parametrizing the integral in the obvious way we obtain the following integral equation problem: determine $\overline{\varphi} \in BC(R)$ such that

(11) $$-2\overline{g}(s) = -2\int_{\infty}^{+\infty} \frac{\partial G_1(x,y)}{\partial \nu(y)} \overline{\varphi}(t)\sqrt{1+f'(t)^2}\, dt + \overline{\varphi}(s), \qquad s \in R,$$

where x = (s,f(s)),y = (t,f(t)). We can summarize our results in the following theorem.

THEOREM 4.2. If $\overline{\varphi} \in BC(R)$ satisfies the integral equation (11), φ is defined in terms of by (10) and Imx, > 0, then u, defined by (7), satisfies the boundary value problem of Section 2. □

In the lossless medium case k > 0 it is still the case that, if $\overline{\varphi}$ satisfies (11), then u satisfies conditions 1 - 3 of the boundary value problem. But in this case we additionally have to show that u satisfies the limiting absorption condition. As we will see later on, where we can show the existence of a solution $\overline{\varphi}$ to our integral equation (9), we can easily show that the corresponding double-layer potential satisfies the limiting absorption principle and thus the boundary value problem for k > 0, too.

5 The Integral Operator and Existence Results

With our integral equation (11) in mind, we define the kernel k by

(12) $$k(s,t) := -2\frac{\partial G_1(x,y)}{\partial v(y)}\sqrt{1+f'(t)^2}, \qquad s,t \in R, \qquad s \neq t,$$

where x = (s,f(s)),y = (t,f(t)). Using this kernel we define the integral operator K for $\Psi \in BC(R)$ by

(13) $$(K\psi)(t) := \int_{-\infty}^{+\infty} k(s,t)\,\psi(t)dt, \qquad t \in R.$$

Whenever we wish to denote explicitly the dependence of the kernel and operator on the boundary function f we will write kf and Kf for k and K, respectively.

Our integral equation (11) can be written in terms of the operator K and I, the identity operator, as

(14) $$-2\overline{g} = (I+K) \quad .$$

It is not difficult to see, using Lemma 3.1, that K : BC(R) BC(R) and that the mapping is bounded but not compact.

We now turn our attention to the existence of a solution to our boundary value problem. Here we will be able to prove an existence result for arbitrary boundary data on a surface ∂D which does not differ too much from the flat boundary $\partial D_h := \{(x_1,h),\ x_1 \in R\}$, where h > 0. Before we elaborate on that, we work out the kernel for the case of the flat boundary ∂D_h- Note that then k depends only on s — t so the corresponding integral operator, which we will denote by k , is then a convolution operator. With x,y $\in \partial D_h$ we find that the first term of G_1 does not contribute to k. We have

(15) $$k(s,t) = -2\frac{\partial}{\partial y_2}\left(-\frac{i}{4}H_0^{(1)}(k\,|\,x-y'|) + P_1(x,y)\right),$$

where x = (s, h), y — (t,h). We now write k(s,t) = K_h(s — t) with

$$\kappa_h(p) = -2\frac{\partial}{\partial y_2}\left(-\frac{i}{4}H_0^{(1)}(\kappa\sqrt{P^2+(h+y_2)^2} + \hat{p}_1(\kappa p, \kappa(y_2+h))\right)\Big|_{y2=h} \cdot$$

We know (see [8], pp 200 - 201) that for the convolution operator K, the inverse of I+ K, exists as an operator from *BC*(**R**) into BC(**R**) if and only if

$$(1+F\kappa_h)(w) \neq 0, \qquad \forall\, w \in R,$$

where F is the Fourier-Transform operator defined, for Ψ $L_1(R)$, by

$$(F\psi)(t) := \int_{-\infty}^{\infty} e^{ist}\psi(s)ds,\ t \in R.$$

Taking the Fourier-transform of the Hankel-function from [2] and that of $\hat{P}_1$ from [2] (or equation (3)) we calculate that

(16) $$(F\kappa_h)(w) = \frac{\sqrt{\kappa^2-w^2}-\kappa}{\sqrt{\kappa^2-w^2}+\kappa}\,e^{\,i2h\sqrt{\kappa^2-w^2}}\ .$$

From (16) we can deduce that $(Fk_h)(w) \neq -1$ for $w \in R$. This is because

(17) $$\left|(F\kappa_h)(w)\right| \leq \left|\frac{\sqrt{\kappa^2-w^2}-\kappa}{\sqrt{\kappa^2-w^2}+\kappa}\right| \leq 1$$

since $\sqrt{\kappa^2 - w^2}$ and κ lie in the first quadrant of the complex plane; and we have equality in the second inequality in (17) if and only if $k^2 = w^2$ when $(F\kappa\ )(W) = 1$. Thus the inverse of $I + K$ exists and is bounded as an operator from BC(R) into BC(R).

At this point we would like to justify further our choice of G_I as our fundamental solution in the double-layer potential. If we use the simpler Dirichlet Green's function $G_D = \Phi - \Phi'$ (which also satisfies the properties in Lemma 3.1) instead of G_I, the Fourier-transform of the corresponding convolution kernel k_h is

$$(F\kappa_h)(w) = -e^{i2h\sqrt{\kappa^2 - w^2}}.$$

Then $(Fk_h)(w) = -1$ for some $w \in \mathbf{R}$ if $kh \geq \pi$ so that $I + K$ is not invertible for $k_h \geq \pi$. This illustrates the well posedness problems we would have with the integral equation (11) if we replaced G_I with the Dirichlet Green's function G_D- We also have to rule out the Green's function-for the Neumann boundary half-plane $G_N := \Phi + \Phi'$ and the free field Green's function Φ as both of these do not secure an integrable double-layer potential with arbitrary bounded and continuous boundary data on a truly non-locally perturbed half-plane.

The following continuity properties of K will be useful. We introduce the norm $\|f\|_{BC^1(R)} := \|f\|_{\infty,R} + \|f\|_{\infty,R}$. Where necessary, we will explicitly indicate that the operator K_f depends on the wave number k by writing K_f as $K_f^{(k)}$. The norm $\|K\|_\infty$ is the induced operator norm of $K : BC(\mathbf{R}) \to BC(\mathbf{R})$.

THEOREM 5.1. [3] For some $C_1, C_2 > 0$ let $B := \{f \in C^{1,\alpha}(\mathbf{R}) : f(s) \geq C_1,\ s \in \mathbf{R}$, *and* $\|f\|_{C^{1,\alpha}(R)} \leq C_2\}$. There holds

$$\sup_{\|f-g\|_{BC^1}(R) \leq \epsilon,\, f,g \in B} \|K_f - K_g\|_\infty \to 0 \text{ as } \epsilon \to 0^+$$

and

$$\sup_{f \in B} \|K_f^{(\kappa + i\epsilon)} - K_f^{(\kappa)}\|_\infty \to 0 \text{ as } \epsilon \to 0^+ .$$

This result again relies heavily on Lemma 3.1.

Now we can easily prove the invertibility result we mentioned earlier on. Let $f_h(s) = h$, $s \in R$, denote the function corresponding to the flat boundary ∂D_h. Note that $K f_h = K$.

THEOREM 5.2. For all $C > 0$ there exists $e > 0$ such that if $\|f\|_{C^{1,\alpha}(R)} \leq C$ and $\|f\|_{C^{1,\alpha}}(R) \leq \epsilon$ *then the inverse of* $I + K_f$ *exists and is bounded as an operator from BC(R) into BC(R).*

Proof. Due to Theorem 5.1 there exists $\epsilon > 0$ such that

$$\sup_{\substack{\|f - f_h\|_{BC^1(R)} \leq \epsilon \\ \|f\|_{C^{1,\beta}}(R) \leq C}} \|K - K_f\|_\infty < \frac{1}{\|(I+K)^{-1}\|_\infty}$$

The proof is completed by a standard Neumann series argument in functional analysis, see for example [1], Theorem 5.

The following theorem summarizes the main results of the paper so far and proves existence of a solution to our boundary value problem (including the limiting absorption principle for the case κ > 0) provided ∂D does not differ too much from a flat boundary.

THEOREM 5.3. *For all* $C > 0$ *there exists* $\epsilon > 0$ *such that if* $\|f\|_{C^{1,\alpha}} \leq C$ *and* $\|f - f_h\|_{BC^1(R)} \leq \epsilon$ *then the integral equation (11) has exactly one solution* $\bar{\varphi}$ *and (9)* *exactly one solution φ (φ and $\bar{\varphi}$ related by (10)) for every $g \in BC(\partial D)$. Furthermore, φ depends continuously on g, and the double-layer potential u, given by (7), satisfies the boundary value problem stated in Section 2 and, for some $C > 0$ independant of g,*

(18) $$|x_2^{-1/2} u(x)| \leq C \|g\|_\infty$$

for all $x \in D$.

Proof. The unique solvability of the integral equation for every $g \in BC(\partial D)$ follows from Theorem 5.2. Also from Theorem 5.2 we have that the inverse of $I + K_f$ is bounded, which yields the continuous dependence of the solution φ on g and furthermore, from Lemma 4,2, the bound (18). By Theorem 4.2 and the remark following it, the double-layer potential u satisfies conditions 1-3 of the boundary value problem. In the case $k > 0$ we have further, from the second part of Theorem 5.1 and the standard functional analytic arguments referred to in the proof of Theorem 5.2, that $\bar{\varphi}^{(K+te)}$ exists for all sufficiently small $\in > 0$ and $\|\bar{\varphi}^{(K+tc)} — \bar{\varphi}^{(k)}\|_\infty \to 0$ as $\in — 0$, where $\bar{\varphi}^{(\lambda)}$ denotes the solution of equation (11) for wave-number $k = \lambda$. The limiting absorption principle then follows from the bound in Lemma 4.2, on noticing (see [3]) that the constant in this bound is independant of Imk provided that Imk is restricted as in Lemma 3.1.□

We now come to the final theorem of the paper, in which we show the continuous dependence of the solution to the boundary value problem given by the double-layer potential (4) on the boundary of the region itself. This result is important for inverse scattering problems corresponding to our boundary value problem. In the same way as we use the notation K_f to denote the operator K generated by the boundary function $f \in C^{1,\alpha}(R)$, we will write ∂D_f for the correspo nding boundary, φ_f will denote the corresponding solution to the integral equation (11) for boundary data g_f, and let u_f denote the double-layer potential with density . We say a sequence of functions (gf_n) $n \in N$ frorn $BC(\partial D_{fn})$ is convergent to a function g_f in $BC(\partial D_f)$ if

$$\lim_{n\to\infty} \sup_{t\in R} |gf_n(t, f_n(t)) - gf(t, f(t))| = 0$$

For easy reference, we define the set

$$F := \left\{ f \in C^{1,\alpha}(R) : \|K - K_f\|_\infty < \frac{1}{\|(I+K)^{-1}\|_\infty} \right\}.$$

Arguing as in the proof of Theorem 5.2, we have existence and boundedness of $(I + K_f)^{-1}$ for all $f \in F$.

THEOREM 5.4. *Let B be as defined in Theorem* 5.1. *Let $f_n \in F \cap B$, $n \in N$, $f \in F \cap B$* and $\|f_n - f\|BC^1(R) —> 0$ as $n —\infty$. Furthermore, let $gj_n \in BC(\partial Df_n)$, $n \in N$, $g_f \in BC(\partial D_f)$ and $gf_n \to gf$ as $n \to \infty$. Then $uf_n — u_{fn}$ uniformly on compact subsets of $D^s := \{x \in R^2 : \sup_{n \in N} f_n(x_2) < x_2\}$.

This theorem is proved in two steps, first we show

(19) $$\|\varphi f_n - \varphi_f\| \to 0 \text{ as } \|f_n - f\|_{BC^1(R)} \to 0,$$

where φ_{fn} is the corresponding density to the double-layer potential u_{fn}, with the help of Theorem 5,1. The second step, showing

$$\|u_{f_n} - u_f\|_{\infty, G} \to 0,$$

for all compact $G \subset D^s$, is then easy to show using (19) and Lemma 3.1.

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