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SARD KERNEL THEOREMS ON TRIANGULAR AND RECTANGULAR DOMAINS WITH EXTENSIONS AND APPLICATIONS TO FINITE ELEMENT ERROR BOUNDS

by

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1. Introduction

The purpose of this paper Is to derive error bounds for the finite element analysis of elliptic boundary value problems. As shown in Section 2, the interpolation remainder is an upper bound on the finite element remainder in the appropriate norm. Error bounds are derived for the interpolation remainder by means of extensions of the Sard kernel theorems. The Sard kernel theorems provide a representation of admissible linear functionals on spaces of functions with a prescribed smoothness. If appropriate derivatives of the solution u of the boundary value problem can be found, then these theorems yield computable error bounds. These theorems have been applied to cubatures by Stroud [10] and by Barnhill and Pilcher [1].

The solutions of elliptic boundary value problems are usually assumed to be in a Sobolev space. The Sobolev and Sard spaces are not the same. If (a,b) is the point about which Taylor expansions are taken in the Sard space, then the Sobolev spaces are contained in the Sard spaces of the same order for almost all a and for almost all b. In Section 3, we show that the derivatives occurring in the Sard spaces can be generalized derivatives, so that the derivatives in the two types of spaces are of the same kind.

Some of the functionals of finite element interest are not, in general, admissible for Sard spaces. A precise statement of this is given in Section 2. This problem was avoided by Birkhoff, Schultz and Varga [4] in a way that is appropriate for rectangles, but is inappropriate for triangles because it implies the use of derivatives outside the original region of interest. In Section 4, we extend the kernel theorems and show how to choose the point (a,b) so that the finite element functionals can be applied in an arbitrary triangle. The method can also be used for more general regions.

The finite element functionals do not involve all possible derivatives of a certain order. In Section 5, we prove a Zero Kernel Theorem that states sufficient conditions for certain of the Sard kernels to be identically zero. The Zero Kernel Theorem has various applications, one being that certain mesh restrictions in Birkhoff, Schultz and Varga can be avoided.

We conclude in Section 6 with computed examples of the constants in the error bound for piecewise linear and piecewise quadratic interpolation.

2. The Galerkin Method and Its Relationship to Interpolation.

Finite element analysis means piecewise approximation over a set of geometric "elements". This rather general definition suffices e.g., for computer-aided geometric design, but for elliptic boundary value problems finite element analysis usually means the Galerkin method. If the partial differential equation is the Euler equation for a variational problem, then the Rayleigh-Ritz method is applicable and is the same as the Galerkin method. Thus the Galerkin method is the more general since it does not depend upon the existence of some underlying variational problem. Therefore, we discuss only the Galerkin method in this paper.

Let Ω be a simply connected bounded region that satisfies a restricted cone condition in the xy- plane. For $p > 1$ and ℓ a non-negative integer, $W_0^1(\Omega)$ is the Sobolev space of functions with all ℓ^{th} order generalized derivatives existing and in $L_p(\Omega)$. Usually $p = 2$. We recall that for Ω as in Figure 1, $\frac{\partial u}{\partial x} = u_{1,0}$ is the (1,0) generalized derivative of u means that $u_{1,0}$ is in $L_1(\Omega)$ and

$$\iint_\Omega \frac{\partial u}{\partial x} v \, dx \, dy = \int_c^d \left[u(x, y)\, v(x, y) \Big|_{x=x_1(y)}^{x=x_2(y)} \right] dy - \iint_\Omega u \frac{\partial v}{\partial x} \, dx \, dy \tag{2.1}$$

for all test functions v in $W_2^1(\Omega)$

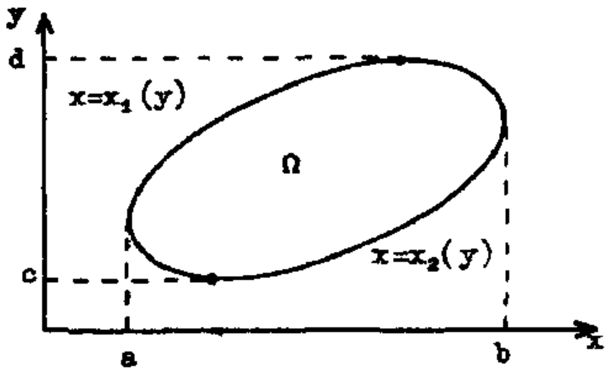

Fig.1

$$V \equiv 0 \text{ on } \partial\Omega \text{ i.e } V \in \overset{0}{W}{}_2^1(\Omega) \text{ then}$$
$$\iint_\Omega u_{1,0}\, v \, dx \, dy = - \iint_\Omega u\, v_{1,0} \, dx \, dy \tag{2.2}$$

A norm for $w_2^\ell(\Omega)$ is the following:

$$\|v\|_{W_2^\partial(\Omega)} = \left\{ \sum_{|\alpha| \le \partial} (\|D^\alpha v\|_{L_2})^2 \right\}^{\frac{1}{2}} \tag{2.3}$$

where $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha = \frac{\partial^\alpha}{\partial x^{\alpha 1} \partial y^{\alpha 2}}$

and the summation in (2.3) is over all α such that $|\alpha| = \alpha_1 + \alpha_2 \le \ell$. The definition of generalized derivative implies that the partials in D^{α} can be taken in any order.

The function space $W_2^{\partial}(\Omega)$ is the completion in the norm (2.3) of $C^m(\Omega)$, m = 0,1 , ... or equivalently of $C^{\infty}(\Omega)$.

Following Varga [11], we consider linear elliptic operators in divergence form:

$$Lu(x, y) = \sum_{|\alpha| \leq \ell} (-1)^{\alpha} D^{\alpha}[P_{\alpha}(x, y) D^{\alpha} u(x, y)] \tag{2.4}$$

where the p_{α} are in $L_{\infty}(\Omega)$. The nonhomogeneous boundary value problem corresponding to L is to find $u \in W_2^{\ell}(\Omega)$ such that :

$$Lu(x,y) = g(x,y) , \qquad (x,y) \in \Omega \tag{2.5}$$

$$D^{\beta}u(x,y) = f_{\beta}(x,y), \qquad (x,y) \in \partial\Omega \text{ for } 0 \leq |\beta| \leq \ell - 1 . \tag{2.6}$$

The homogeneous problem is that all the f_{β} are identically zero, the relevant Sobolev space then being called $\overset{0}{W}{}_2^{\ell}(\Omega)$

A norm in $\overset{0}{W}{}_2^{\ell}(\Omega)$ is $||v||_{\overset{0}{W}{}_2^{\ell}(\Omega)} = \left\{ \sum_{|\alpha| = \ell} ||D^{\alpha} v||^2_{L_2(\Omega)} \right\}^{\frac{1}{2}}$.

Theorem 1 in Section 3 on equivalent norms implies that this is a norm on $\overset{0}{W}{}_2^{\ell}(\Omega)$

$$\text{Let } a(u,v) = \sum_{|\alpha| = \ell} \iint_{\Omega} P_{\alpha}(x, y) D^{\alpha} u(x, y) D^{\alpha} v(x, y) \, dx, \, Dy \tag{2.7}$$

Then the <u>weak problem,</u> corresponding to (2.5) and (2.6) is to find u satisfying (2.6) and such that

$$a(u.v) = (g,v) \tag{2.8}$$

for all v in $\overset{0}{W}{}_2^{\ell}(\Omega)$

The definition of the weak problem can be motivated by the integration of (2.5) by parts with a test function v in $\overset{0}{W}{}_2^{\ell}(\Omega)$

We consider interpolants $\tilde{u}$ to u, where the interpolation conditions are the following:

$$L_i(\tilde{u}) = L_i(u), \quad i = 1, \ldots, m,$$
$$M_j(\tilde{u}) = M_j(u), \quad j = 1, \ldots, n, \qquad (2.9)$$

and the L_i and M_j. are interpolation functionals such that the $L_i(u)$ are unknown and the $M_j.(u)$ are known a priori. Hereafter we assume that the $M_j(u)$ are known from the boundary data (2.6).

$\bar{\Omega}$ is usually discretized and the linear funotionals L_i and M_j based on the discretization, an example being the evaluation of u and its derivatives at certain mesh points. Let v^h be an (m + n)-dimensional subspace of $w_2^\ell(\Omega)$ such that the $L._i$ and M_j are linearly independent over v^h. Then V^h has a basis of functions $\{B_i(x, y)\}_{i=1}^m$ and $\{c_j(x, y)\}_{j=1}^n$ that are biorthonormal with respect to the L_i and M_j [5]. Let S^h be the subset of $w_2^\ell(\Omega)$ which consists of functions v of the form

$$v(x,y) = \sum_{i=1}^m a_i B_i(x,y) + \sum_{j=1}^n M_i(u) \; c_j(x,y)$$

where the a_i are constants. Let s_0^h be the m-dimensional subspace generated by the B_i. The G<u>alerkin method</u> is to find U in S^h such that $a(U,v) = (g,v)$ for all v in s_0^h. (2.10)

The"conforming condition" is that $s^h \subset w_2^\ell(\Omega)$,which is required for the Galerkin method. We also require $s_0^h \subset \overset{0}{w}{}_2^\ell(\Omega)$, which usually follows from the conforming condition.

<u>Lemma.1</u> (Strang [9]). The Galerkin approximation U is the b<u>est approximation</u> from S^h to u in the energy norm induced by

the inner product a(u,v). That is,

$$a(u-U,\ u-U) \leq a(u-\tilde{u},\ u-\tilde{u} \text{ for all } \tilde{u} \text{ in } S^h \tag{2.11}$$

In fact,

$$a(u-U, u-U) + a(\tilde{u}-U, \tilde{u}-U) = a(u-\tilde{u}, u-\tilde{u}). \tag{2.12}$$

Proof: From the definitions of weak problem (2.3) and Galerkin method, (2.10), $a(u-U,v) = 0$ for all v in s_0^h .

Therefore, $a(u-U,\ u-U) = a(u-U, u-\tilde{u})$, and $a(\tilde{u}-U,\ -U) =$ $a(\tilde{u}-u,\ (\tilde{u}-U)$, from which (2.12) follows. Q.E.D.

The normal equations for this beat approximation can be derived as follows:

If $\tilde{u}$ interpolates to u with respect to the functionals L_i and M_j, then

$$\tilde{u}(x,y) = \sum_{i=1}^{m} L_i(u)B_i(x,y) + \sum_{j=1}^{n} M_j(u)\ c_j(x,y) \tag{2.13}$$

Hence $U(x,y) = \sum A_i B_i(x,y) + \sum_{j=1}^{n} M_j(u)\ c_j(x,y)$

and $a(U,B_k) = (g,B_k)$ $k =- 1, \ldots, m.$

Thus

$$\sum_{i=1}^{m} A_i\ a(B_i, B_k) = (g,B_k) - \sum_{j=1}^{n} M_j(u)\, a(c_j,B_k)\ ,$$

$$k = 1, \ldots, m. \tag{2.14}$$

Equations (2.14) yield a method of calculation of the A_i.

Since the B_k are in $\overset{0}{w}{}_2^{\ell}(\Omega)$, the actual normal equations are the following :

$$\sum_{i=1}^{m} A_i a(B_i, B_k) = a(u - \sum_{j=1}^{n} M_j(u) C_j, B_k), \; k = 1,\ldots\ldots, m. \quad (2.15)$$

The norm induced by $a(u,v)$ is equivalent to the $w_2^{\ell}(\Omega)$ norm if a is bounded and $w_2^{\ell}(\Omega)$ - elliptic, i.e.

$$\text{Elliptic : } \rho \, \| v \|^2_{W_2^{\ell}(\Omega)} \leq a(v, v) \text{ for all } v \varepsilon W_2^{\ell}(\Omega) \text{ and some constant } \rho > 0 \quad (2.16)$$

$$\text{Bounded :} | \, a(v, w) \leq \| a \| \, \| v \|_{W_2^{\ell}} \| w \|_{W_2^{\ell}} \text{ for all } v, w \; \varepsilon \; W_2^{\ell}(\Omega) \quad (2.17)$$

$$\text{From (2.4), } \| a \| \leq \max_{|\alpha| \leq \ell} \| P\alpha \| \, L_{\infty}(\Omega) \quad (2.17)$$

Lemma 2. Assumptions (2.16),(2.17) imply that

$$\| u - U \| \, W_2^{\ell} \leq \left\{ \frac{\| a \|}{\rho} \right\}^{\frac{1}{2}} \min_{\tilde{u} \varepsilon S^h} \| u - \tilde{u} \| \, W_2^{\ell} \quad (2.18)$$

Proof : The best approximation property $a(u\text{-}U,u\text{-}U) < a(u - \tilde{u}, u - \tilde{u})$,

ellipticity ,and boundedness imply the conclusion . Q.E.D.

Example. For Poisson's equation, $\ell = 1$, $| |a| | = 1$ and ρ can be taken as one.

Interpolation remainder theory is applicable to the Galerkin method from the best approximation property (2.11) or equivalently, from (2.18), the interpolant being taken as $\tilde{u}$

3. The Sobolev Imbedding Theorems.

The following theorem on equivalent norma [7] was used in Section 2:

Theorem 1. If $F_1, \ldots F_N$ are bounded linear functionals on $w_2^{\ell}(\Omega)$ that are linearly independent over $P_{\ell-1}$, the space of polynomials of degree $\leq \ell - 1$, and $N = \ell(\ell+1)/2$, then the usual

$w_2^\ell(\Omega)$ norm (2.3) can be replaced by the norm

$$+||v|| = \left\{ \sum_{k=1}^{N} |F_k(v)|^2 + \sum_{|\alpha|=\ell} ||D^\alpha v||^2_{L_2(\Omega)} \right\}^{\frac{1}{2}} \tag{3.1}$$

The norm on $\overset{0}{W}{}_2^\ell(\Omega)$ is obtained with the F_k being of the form $\int_{\partial\Omega} D^\beta u \, ds$, $|\beta| < \ell - 1$

The F_k are bounded because lower order derivatives can be bounded in terms of higher order derivatives as follows:

Theorem 2. Let Ω be the union of finitely many star-like regions. If $\ell = 1$, then v in $w_2^1(\Omega)$ implies that

$$\|v\|_{L2}(\Omega_1) \le \|\mathcal{L}\|_{W_2^1(\Omega) \to L_2(\Omega_1)} \|v\|_{W_2^1(\Omega)} \tag{3.2}$$

where Ω_1 is a one-dimensional subset of $\bar{\Omega}$.

If $\ell > 1$, then v in $w_2^\ell(\Omega)$ implies that

$$\|v\|_{C^{\ell-2}(\Omega)} \le \|\mathcal{L}\|_{W_2^\ell(\Omega) \to L_2(\Omega_1)} \|v\|_{W_2^\ell(\Omega)} \tag{3.3}$$

Where $\|\mathcal{L}\|_{x\to y}$ means the norm of the operator imbedding X into Y.

We note from Theorem 2 that point evaluation funotionals are bounded on $W_2^2(\Omega)$. However, these functionals are unbounded on $w_2^1(\Omega)$.

A specific example of Theorem 2 is the following:

Lemma 3. Let Ω be a bounded convex region with B equal to the maximum of B_x and B_y , where B_x is the diameter of Ω along parallels to the x-axis and B_y is dual.

If $u \equiv 0$ on $\partial\Omega$, then

$$\|u\|_{L_2(\Omega)} \le \frac{B}{2} \|u\|_{\overset{0}{W}{}_2^1(\Omega)} \tag{3.4}$$

Proof: Let $\partial\Omega$ be parametrized by the pair of functions $y_1(x) \le y_2(x)$, $a \le x \le b$ or by $x_1(y) \le x_2(y)$, $c \le y \le d$

(see Figure 1). Then

$$u(x, y) - u(x, y_1(x)) = \int_{y_1(x)}^{y} u_{0,1}(x, \tilde{y})d\tilde{y} \tag{3.5}$$

$$u(x, y) - u(x_1(y), y) = \int_{x_1(y)}^{x} u_{1,0}(\tilde{x}, y)d\tilde{x}. \tag{3.6}$$

From (3.5), $|u(x, y)|^2 \le (y - c) \int_{y_1(x)}^{y_2(x)} [u_{0,1}(x, \tilde{y})]^2 d\tilde{y}$,

so that

$$\int_a^b \int_{y_1(x)}^{y_2(x)} |u(x, y)|^2 \, dy \, dx \le \frac{B_y^2}{2} \int_a^b \int_{y_1(x)}^{y_2(x)} [u_{0,1}(x, \tilde{y})]^2 \, d\tilde{y} \, dx. \tag{3.7}$$

A dual result comes from (3.6) and the conclusion follows. Q.E.D.

4. Interpolation Remainder Theory

We review and then extend the Sard kernel theorems in order to obtain interpolation error bounds, including the corresponding constants. Let p and q be positive integers with $n = p + q$. Sard [6] has defined several types of spaces of functions with a prescribed smoothness. The two types of interest for remainder theory are the triangular Spaces $\underline{\underline{B}}_{p,q}$ and the rectangular spaces For $\underline{\underline{B}}_{\lceil p,q \rceil}$ remainders of polynomial precision in two variables, $\underline{\underline{B}}_{p,q}$ is the more useful unless the remainder corresponds to a tensor product rule, in which case $\underline{\underline{B}}_{\lceil p,q \rceil}$ is used. The latter case has been considered much more, building as it does on one-dimensional rules, and many particular results are summarised in Stancu[8]. This paper will be concerned

with interpolation over triangulated polygons Ω ,
so that $\underline{\underline{B}}_{p,q}$ Is the appropriate Sard space.
$\underline{\underline{B}}_{p,q}$ is the space of bivariate functions with Taylor expansions containing derivatives in a certain triangular form.
The Taylor expansions are at the point (x,y) about the point (a , b).
The notation $\underline{\underline{B}}_{p,q}$ means that the derivatives occurring in the Taylor expansions are integrable. In fact, we shall usually consider subspaces of $\underline{\underline{B}}_{p,q}$ in which the derivatives are in L_p . for some $p' \geq 1$•
The space $\underline{\underline{B}}_{p,q}$ depends on the region Ω in which the Taylor expansions take place. Sard let Ω be a rectangle, but this is insufficient for our later purpose of interpolation to functions defined on triangles. However, the boundary value problem assumption that Ω be a bounded region satisfying a restricted cone condition is too general.

<u>Definition 1</u>. Let Ω be a bounded region with the following property: After a rotation (if necessary), there is a point (a,b) in $\bar{\Omega}$ such that for all (x,y) in $\bar{\Omega}$ the rectangle with opposite corners at (a,b) and at (x,y) is contained In $\bar{\Omega}$

<u>Examples.</u> If Ω is a rectangle, then (a,b) can be an arbitrary point in the rectangle. If Ω is a triangle, then (a,b) can be taken as the point on the longest side of the triangle that is at the foot of the perpendicular to this side from eth opposite vertex.

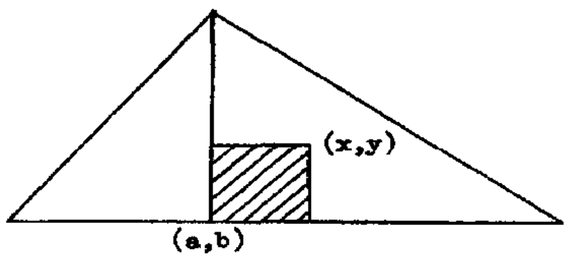

Fig.2.

We assume hereafter that the region Ω of definition of the boundary value problem (2.8) is the union of finitely many regions Ω satisfying the above definition. When Ω is a rectangle, the next theorem is due to Sard.

<u>Theorem 3. Taylor Expansion.</u> Let Ω satisfy Definition 1. Then $u \in \underline{\underline{B}}_{p,q}(\Omega)$ implies that u has the following Taylor expansion at (x,y) about (a,b):

$$u(x,\ y) = \sum_{i+j<n} (x - a)^{(i)} (y - b)^{(j)} u_{i,j}(a,\ b)$$

$$+ \sum_{j<q} (y - b)^{(j)} \int_a^x (x - \tilde{x})^{(n-j-1)} u_{n-j,j}(\tilde{x},\ b)\ d\tilde{x}$$

$$+ \sum_{j<q} (x - b)^{(j)} \int_b^x (y - \tilde{x})^{(n-j-1)} u_{i,n-i}(a,\ \tilde{y})\ d\tilde{y}$$

$$+ \int_b^y (y - \tilde{y})^{(q-1)} \int_a^x (x - \tilde{x})^{(p-1)}\ u_{p,q}(\tilde{x},\ \tilde{y})\ d\tilde{x}\ d\tilde{y} \qquad (4.1)$$

where $(x - a)^{(i)} \equiv (x- a)\ I\ /\ i!$ etc.

Remarks on the proof:

Theorem 3 is proved by several integrations by parts.

$$u(x,\ y) = (x,\ b) + u_{0,1}(x,\ b)(y - b) + \ldots + \int_b^y (y - \tilde{y})^{(q-1)} u_{0,q}(x,\ \tilde{y})\ d\tilde{y} \qquad (4.2)$$

$$u_{0,q}(x,\ \tilde{y}) = u_{0,q}(a,\ \tilde{y}) + u_{1,q}(\varepsilon,\ \tilde{y})(x - a) + \ldots + \int_a^x (x - \tilde{x})^{(p-1)} u_{p,q}(\tilde{x},\ \tilde{y})\ d\tilde{x} \qquad (4.3)$$

This completes the expansion along Sard's "main route" in the Sard index triangle of partial derivatives from (0,0) to (p,q), Figure 3.

Next, univariate expansions are made along the arrows, exactly one expansion for each term of (4.2) after (4.3) has been substituted into (4.2).

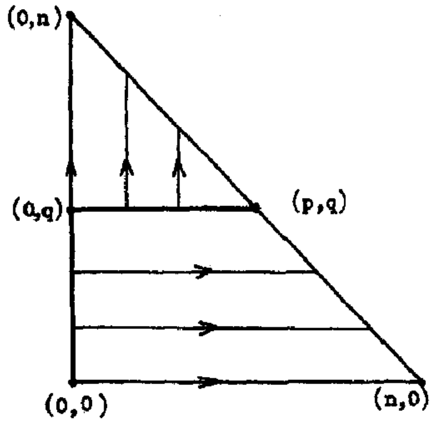

Figure 3.

We have assumed the existence of the generalized derivatives in Table 1. These derivatives need exist only almost everywhere in the variables $\tilde{x}$ and $\tilde{y}$, because these variables are "covered" by integrals in the Taylor expansions. In particular, $u_{p,q}(\tilde{x}, \tilde{y})$ exists a,e. $\tilde{x}$ and a.e. $\tilde{y}$ Our later use of (4.1) only requires that u(x,y) exist a.e. (x,y) and that the derivatives involving x in the first two columns in Table 1 exist a.e. x.

$u_{i,j}(a,b)$ for all $0 \leq i + j < n$.

	$u_{0,n}(a,\tilde{y})$								
	$u_{0,n-1}(a,\tilde{y})$	$u_{1,n-1}(a,\tilde{y})$							
	$\vdots$	$\vdots$	$\ddots$						
	$u_{0,q+1}(a,\tilde{y})$	$u_{1,q+1}(a,\tilde{y})$	$\cdots$	$u_{p-1,q+1}(a,\tilde{y})$					
	$u_{0,q}(a,\tilde{y})$	$u_{1,q}(a,\tilde{y})$	$\cdots$	$u_{p-1,q}(a,\tilde{y})$					Main Route
$u_{0,q}(x,\tilde{y})$		$u_{1,q}(\tilde{x},\tilde{y})$	$\cdots$	$u_{p-1,q}(\tilde{x},\tilde{y})$	$u_{p,q}(\tilde{x},\tilde{y})$				
$u_{0,q-1}(x,\tilde{y})$	$u_{0,q-1}(x,b)$	$u_{1,q-1}(\tilde{x},b)$	$\cdots$	$u_{p-1,q-1}(\tilde{x},b)$	$u_{p,q-1}(\tilde{x},b)$	$u_{p+1,q-1}(\tilde{x},b)$			
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$u_{0,1}(x,\tilde{y})$	$u_{0,1}(x,b)$	$u_{1,1}(\tilde{x},b)$							
$u(x,y)$	$u(x,b)$	$u_{1,0}(\tilde{x},b)$	$\cdots$	$u_{p-1,0}(\tilde{x},b)$	$u_{p,0}(\tilde{x},b)$	$u_{p+1,0}(\tilde{x},b)$	$\cdots$	$u_{n,0}(\tilde{x},b)$	

Table 1.

An importance of these derivatives being generalised rather than ordinary is to make the Sard and Sabolev spaces more compatible. (Sard's statement of this theorem presumes that the derivatives are ordinary.)

The Sard kernel theorems are for admissible functionals defined on functions which have a rectangle as their domain of definition. We extend the definition of admissible functional to regions satisfying Definition 1.

Definition 2. The admissible functionals on $\underline{\underline{B}}_{p,q}(\Omega)$ are of the following form:

$$\begin{aligned} Fu = & \sum_{\substack{i<p \\ j<q}} \int_{\Omega}\int u_{i,j}(\tilde{x},\tilde{y})\, d\mu^{i,j}(\tilde{x},\tilde{y}) \\ & + \sum_{\substack{i+j<n \\ i \geq p}} \int_{a}^{\tilde{a}} u_{i,j}(\tilde{x},b)\, d\mu^{i,j}(\tilde{x}) \\ & + \sum_{\substack{i+j<n \\ j \geq q}} \int_{\beta}^{\tilde{\beta}} u_{i,j}(a,\tilde{y})\, d\mu^{i,j}(\tilde{y}) \end{aligned} \tag{4.4}$$

where the $\mu^{i,j}$ are of bounded variation with respect to their arguments. The line segments $y = b,\ \alpha \leq \overline{x} \leq \overline{\alpha}$ and $x = a, \beta \leq \tilde{y} \leq \tilde{\beta}$ are assumed to be in Ω or, equivalently, the support of the univariate $\mu^{i,j}$ is contained in $\overline{\Omega}$.

Theorem 4. Kernel Theorem, Let Ω satisfy Definition 1 and F be an admissible functional on $\underline{\underline{B}}_{p,q}(\Omega)$. If $u \in \underline{\underline{B}}_{p,q}(\Omega)$, then

$$Fu(x, y) = \sum_{i+j<n} c^{i,j} u_{i,j}(a, b) + \sum_{j<q} \int_{\alpha}^{\tilde{\alpha}} u_{n-j,j}(\tilde{x}, b)\, K^{n-j,j}(\tilde{x})d\tilde{x}$$

$$+ \sum_{i<D} \int_{\beta}^{\tilde{\beta}} u_{i}, n - i(a, \tilde{y})K^{i,n-i}(\tilde{y})d\tilde{y} + \int_{\Omega}\int u_{p,q}(\tilde{x}, \tilde{y})K^{P,q}(\tilde{x}, \tilde{y})d\tilde{x}\, d\tilde{y} \qquad (4.5)$$

$$\text{Where } c^{i,j} = F_{(x,y)}\left[(x, a)^{(i)}(y - b)^{(j)}\right], i + j < n \qquad (4.6)$$

$$K^{n-j,j}(\tilde{x}) = F(x, Y)\left[(x - \tilde{x})^{(n-j-1)}\varphi(a, \tilde{x}, x)(y - b)^{(j)}\right], j < q, \tilde{x} \notin Jx \qquad (4.7)$$

$$K^{i,n-i}(\tilde{y}) = F_{(x,y)}\left[(x - a)^{(i)}(y - \tilde{y})^{(n-i-1)}\psi(b, \tilde{y}, y)\right], i < P.\tilde{y} \notin jy \qquad (4.8)$$

$$K^{p,q}(x, y) = F_{(x,y)}\left[(x - \tilde{x})^{(p-1)}\psi(a, \tilde{x}, x)(y - \tilde{y})^{(q-1)}\psi(b, \tilde{y}, y)\right]\tilde{x} \notin \bar{j}\, x, \tilde{y} \notin \bar{j}\, y \qquad (4.9)$$

The notation F(x,y) means that F is applied to functions in the variables (x,y). The function Ψ is

$$\psi(a,\tilde{x},x) \equiv \begin{cases} 1 & \text{if } a \le \tilde{x} < x \\ -1 & \text{if } x \le \tilde{x} < a \\ 0 & \text{otherwise} \end{cases}$$

Jx is the "jump set" consisting of the points of discontinuity of the total variation functions $|\mu^{n-1-j,j}|(x)$ for $j < q$. Jy is the dual jump set, If $p > 1$, $\bar{J}\,x$ is the jump set consisting of points of discontinuity of $|\mu^{p-1}, j'|(x, \tilde{\beta})$ for $j' < q$, where $|\mu^{p-1}, j'|(x, \tilde{\beta}) \equiv |\mu^{p-1}, j'|(x, y)$ evaluated at $y = \tilde{\beta}$. If $P = 1$, then $\bar{j}x$ is empty. $\bar{J}y$ Is dual.

Remarks on the proof:
The purpose of the function $\psi(a, \tilde{x}, x)$ is to change indefinite Integrals of the form $\int_a^x f(\tilde{x})\, d\tilde{x}$ to definite integrals of the Form $\int_\alpha^{\tilde{\alpha}} \psi(a, \tilde{x}, x) f(\tilde{x}) d\tilde{x}$., The functions $\mu^{i,j}$ are defined in order that Fubini's Theorem can be applied. The jump sets arise because, for example $\frac{\partial^{n-1}}{\partial x^{n-1}} [(x - \tilde{x})^{(n-1)} \psi(a, \tilde{x}, x)]$ is integrated against $\mu^{n-1,0}(\tilde{x})$, , which is undefined at $\tilde{x} = x$ unless $n = 1$.

An advantage of the Sard kernel theorem is that in (4.5) the variables $(\tilde{x}, \tilde{y})$ occurring as arguments of the derivatives are "covered", i.e., they are the variables of integration.
In finite element analysis, the functionals of interest involve derivatives. Since the variables that occur as arguments of the derivatives in the Sard kernel theorem are covered, the order of these derivatives is not increased by applying derivative functionals to them.

The following illustrates what can happen with uncovered variables:
Example of a Taylor expansion with uncovered variables.
The Sard space. $\underline{\underline{B}}_{1,0}$ consists of functions with Taylor expansions of the form

$$u(x, y) = u(a, b) + \int_a^x u_{1,0}(\tilde{x}, y)\, d\tilde{x} + \int_b^y u_{0,1}(a, \tilde{y})\, d\tilde{y} \tag{4.10}$$

The variable y is uncovered in the first integral. If the derivative operator $\frac{\partial}{\partial y}$ is applied to (4.10), then the formal result is

$$u_{0,1}(x, y) = \int_a^x u_{1,1}(\tilde{x}, y)\, d\tilde{x} + u_{0,1}(a, y) \tag{4.11}$$

However, (4.11) assumes the existence of $u_{1,1}$, which is not ensured by the function u being in $\underline{\underline{B}}_{1,0}$

5. Finite Element Remainder Functionals.

If u is an interpolant to u, then the remainder is

$$Ru(x,y) \equiv u(x,y) - \tilde{u}(x,y) \tag{4.12}$$

The finite element remainder functionals of interest for a $2\ell^{th}$ order elliptic boundary value problem are the following:

$$R_{i,j}u(x,y) \equiv \frac{\partial^j}{\partial y^j}\frac{\partial^i}{\partial x^i} Ru(x,y) \text{ for } 0 \leq i + j \leq \ell \tag{4.13}$$

In order to use the Sard kernel theorems, the space $\underline{\underline{B}}_{p,q}$ must be chosen. The interpolant p(x,y) usually has some polynomial precision and the constant n is chosen so that this polynomial precision is at least n- 1. This choice implies that $c^{i,j} = 0$, $0 \leq I + j < n$. p and q are arbitrary positive integers such that $p + q = n$. However, if n is even, then $p = q = n/2$ is a practical choice if Ω and R are symmetric about y = x, because the number of kernels to be calculated is reduced. In general, we let $p + q. \geq \ell + 1$ and if $p+q = \ell + 1$, then $p = \left[\frac{\ell+1}{2}\right]$, the greatest integer in $(\ell + 1)/2$, and

$q = \ell + 1 - \left[\frac{\ell+1}{2}\right]$. In the sequel we consider the result of applying the $R_{\cdot i,j}$ to the Taylor expansion (4.1).

Inadmissible Functionalss an example.

For the Sard space $\underline{\underline{B}}_{1,1}$ the term in $\frac{\partial u(x,y)}{\partial x}$ $R_{1,0}u(x,y)$

is not admissible unless x = a. Dually, $R_{0,1}$, is not admissible on $\underline{\underline{B}}_{1,1}$ unless y = b. Birkhoff, Schultz and Varga [4] considered piecewise Hermite interpolation over a region divided into sub rectangles. They let the point of interpolation (x,y) = (a,b), the point of Taylor expansion. This has the effect of involving derivative values in rectangles containing the region of interest, as we now illustrate. Let T be the right triangle with vertices at (0,0) (1,0), and (0,1). Then in $\underset{=}{B}_{1,1}(T)$ implies that

$$R_{1,0}\, u(a,b) = \int_0^1 K^{2,0}(a,b;\tilde{x})u_{2,0}(\tilde{x},b)\,d\tilde{x} + \int_0^1 K^{0,2}(a,b;\tilde{y})u_{0,2}(a,\tilde{y})d\tilde{y} + \int\int_T K^{1,1}(a,b;\tilde{x},\tilde{y})\,u_{1,1}(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y} \tag{4.14}$$

Hence $||R_{1,0}u(a,b)||L_2\ (T)(a,b)$ involves values of $u_{2,0}$ and $u_{0,2}$. outside T and, in fact, in the whole unit square.

To avoid this difficulty, we apply $R_{1,0}$ and $R_{0,1}$ to the Taylor expansion (4.1) directly. This avoids difficulties of the type $\int_\alpha^{\tilde{\alpha}} \frac{\partial}{\partial x}\psi$ which instead becomes $\frac{\partial}{\partial x}\int_a^x$, for example. It is integrals of the form $\int_\alpha^{\tilde{\alpha}} \frac{\partial}{\partial x}\psi$ that make $R_{1,0}$ inadmissable on $\underset{=}{B}_{1,1}$.

5. Zero Kernels.

It was noted [2] by direct calculation that, for linear interpolation on the triangle T, the kernel $K^{0,2}$ corresponding to $R_{1,0}$ is identically zero. The first clue that such a result held was that in Birkhoff, Schultz and Varga, [p.242] the Kernel "$k_{0,2}(t')$" corresponding to $R_{1,0}$ for bilinear

Hermite interpolation is identically zero instead of what is claimed in that paper.

In general, we let P denote an interpolation functional with remainder R = I - P. We consider the Sard kernels corresponding to the remainder functional $D^{(h,k)}$ R.

Theorem, If f(x,y) $\varepsilon \underline{B}_{p,q}$ is of the form f(x,y) = $p_1(x)$ h(y),

where $P_I(x)$ is a polynomial in x of degree i < h, and if P has the property that

$$P[p_i(x)h(y)] = q(x,y) \tag{5.1}$$

where q(x,y) considered as a function of x alone is a polynomial of degree < h, then the Sard kernels for $D^{(h,k)}$ R have the property that

$$K^{i,P+q-i}(x, y; \tilde{y}) \equiv 0, \quad 0 \leq i <. h \leq p. \tag{5.2}$$

Dually, if f(x,y) = g(x) $q_j(y)$, where $q_j(y)$ is a polynomial in y of degree j < k and

$$P\,[g(x)\ q_j(y)] = s(x,y) \tag{5.3}$$

where s(x,y) considered as a function of y alone is a polynomial of degree < k, then the Sard kernels

$$K^{p+q-j,j}(x,y;\tilde{x}) \equiv 0, \qquad 0 \leq j < k \leq q \tag{5.4}$$

Proof of (5.2): We assume that 0 <h ≤ p. The Sard kernels for the functional $D^{(h,k)}$ R are the (h,k) partial derivatives of the corresponding kernels for R. Let i be an integer such that 0 ≤ i < h. Then the kernel $K^{1,P+q-1}(x,y;\tilde{y})$ corresponding to R is the following:

$$k^{i}_{R}, p + q - i(x, y; \tilde{y}) = R_{(x,y)}\left[(x - a)^{(i)} (y - \tilde{y})^{(p + q - i - 1)} \psi(b, \tilde{y}, y)\right], \tilde{y} \notin J y \quad (5.5)$$

Therefore, the kernel corresponding to $D^{(h,k)} R$ is the following;

$$K^{i,p+q-i}(x, y; \tilde{y}) = \frac{\partial^{h}}{\partial x^{h}} \frac{\partial^{k}}{\partial y^{k}} K^{i}_{R}, p + q - i(x, y; \tilde{y})$$

$$= \frac{\partial^{k}}{\partial y^{k}} \frac{\partial h}{\partial x^{h}} \Big[(x - a)^{(i)} (y - \tilde{y})^{(p+q-i-1)} \psi(b, \tilde{y}, y)$$

$$- P\Big\{(x - a)^{(i)} (y - \tilde{y})^{(p+q-i-1)} \psi(b, \tilde{y}, y)\Big\}\Big]$$

$$= \frac{\partial^{k}}{\partial y^{k}} [0 - 0] = 0, \quad \text{by assumption} \quad (5.1). \quad \text{Q.E.D}$$

Schematic ally, the domain of influence in the ard index space $\underset{=}{B}_{p,q}$ of the functional $D^{(h,k)}$ R is the shaded sub triangle shown in Figure 4.

For given h and k, p should be chosen so that h < p and k < q.

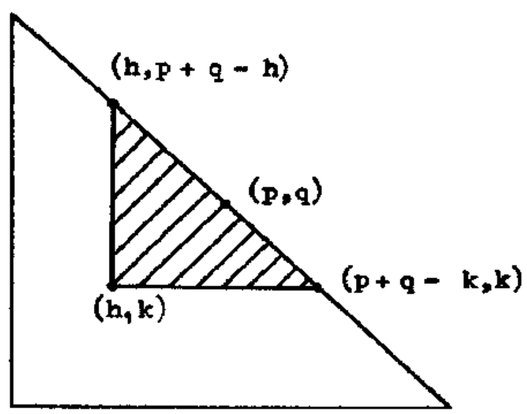

Figure 4.

Many interpolants satisfy hypotheses (5,1) and (5.3) e.g., linear interpolation with $i = j = 0$. (4.13).

We next prove the corollary that (5.1) and (5.3) are <u>always</u> satisfied by tensor product schemes with sufficient polynomial precision. However, we then conclude this Section with an example in which (5.2) does not hold.

<u>Corollary.</u> Tensor product interpolants of polynomial precision at least h-1 in the variable x and at least k - 1 in the variable y satisfy (5.2) and (5.4).

proof: p a tensor product interpolant implies that P is of the form

$$P_x \;\; P_y \quad = \quad P_y \quad P_x \tag{5.6}$$

where p_x is an interpolant in the variable x and p_y is dual in y. Therefore, if $f(x,y) = p_i(x)h(y)$ where $P_i(x)$ is a polynomial in x of degree $i <\;\; h$, then $P[_{Pi}(x)h(y)] = P_y\, p_x\, [p_1(x)h(y)] = P_y[p_1(x)h(y)] = p_i(x)\;\; P_y[h(y)] \equiv q\;\; q(x,y)$. $q(x,y)$ satisfies (5.1) so that (5.2) follows. The argument is dual for (5.4). Q.E.D.

Birkhoff, Schults and Varga considered tensor product piecewise Hermite interpolation on rectangles and they assumed that their meshes were "regular" [4,p. 244]. Their reason for this assumption was the possibility of negative exponents in equation (4.20) in [4]. However, the above Corollary implies that the kernels of the terms corresponding to these negative exponents are identically zero and so no such mesh restriction is needed.

We conclude this section with an example of an interpolant on a triangle such that its $K^{0,2}$ kernel corresponding to $R_{1,0}$ is not identically zero.

Example Let $P\, f(x, y) = f(0,0)\,(1 - x)\,(1 - y)$

$+ f(1,0)\, x\,(1 - y) + f(0,1)(1 - x)y$. Then

$$R_{1,0}\, f(x, y) = f_{1,0}(x, y) - (1 - y)\, f(0,0)$$

$$+ (1 - y)\, f(1,0) - y\, f(0,1) \quad \text{and}$$

$$K^{0,2}(x, y; \tilde{y}) = R_{1,0}(x, y)[(y - \tilde{y})\, \psi\, (b, \tilde{y}, y)]$$

$$= - y(1 - \tilde{y})\, \psi\, (b, \tilde{y}, 1)] \neq 0$$

unless b=1

We note that

P[l.h(y)] = h(0)(l-y) + h(l)(l -x)y, which is not a function of y alone, so that (5.1) is not satisfied.

<u>Error Bounds for Interpolation on a Triangle</u>

In this section, we illustrate how to obtain error bounds for linear and quadratic interpolation on the triangle T with vertices (0,0), (h,0), and (o,h). The linear bivariate polynomial which interpolates the function values of u(x,y) at the vertices of the triangle T is

$$\tilde{u}(x, y) = u(0, 0)\left[1 - \frac{x + y}{h}\right] + u(h, 0)\frac{x}{h} + u(0, h)\frac{y}{h}. \tag{6.1}$$

The quadratic bivariate polynomial which interpolates the function values at the vertices and mid-points of the sides of the triangle T is

$$\tilde{u}(x,y)\;\; u(0,0)\left[1 - \frac{3}{h}(x + y) + \frac{4}{h^2}xy + \frac{2}{h^2}(x^2 + y^2)\right]$$

$$+ u(\frac{h}{2},0)\left[\frac{4x}{h} - \frac{4xy}{h^2} - \frac{4x^2}{h^2}\right] + u\left(0,\frac{h}{2}\right)\left[\frac{4y}{h} - \frac{4xy}{h^2} - \frac{4y^2}{h^2}\right]$$

$$+ u(h,0)\left[\frac{-x}{h} + \frac{2x^2}{h^2}\right] + u(0,h)\left[\frac{-y}{h} + \frac{2y^2}{h^2}\right] + u\left(\frac{h}{2},\frac{h}{2}\right)\frac{4\,xy}{h^2}.$$

(6.2)

The finite element error bounds of interest are those on the $L_2(x,y)$ norm of the following error functions:

$$R\,u(x,y) = u(x,y) - \tilde{u}(x,y), \qquad (6.3)$$

$$R_{1,0}\,u(x,y) = \frac{\partial}{\partial x} R\,u(x,y), \qquad (6.4)$$

$$R_{0,1}\,u(x,y) = \frac{\partial}{\partial y} R\,u(x,y), \qquad (6.5)$$

$L_2(x,y)$ denotes the L_2 norm over the triangle T with respect to (x,y). We also derive bounds on the general L_q (x,y) norm at R u(x,y) for the linear interpolant (6.1). The results obtained are generalisations of those given in Barnhill and Whiteaan [2,3] •

The point (a,b) of the Taylor expansions is taken as (0,0) which satisfies the requirement that for (x,y) □ T the rectangle [0,x] x [0,y] is contained in T. This choice of (a,b) simplifies the Ψ functions of section 4 to the functions of the form

$$(x-\tilde{x})_+^{(i)} = \begin{cases} (x-\tilde{x})^{(i)} & \text{Fot } x > \tilde{x} \\ 0 & \text{otherwise} \end{cases} \qquad (6.6)$$

<u>L_q Bounds on R for Linear Interpolation.</u>

The error functional

$$Ru(x,y) = u(x,y) - \left\{ u(0,0)\left[1-\left(\frac{x+y}{h}\right)\right] + u(h,0)\frac{x}{h} + u(0,h)\frac{y}{h}\right\} \qquad (6.7)$$

is zero for the functions 1, x and y. We thus consider the Sard space $\underline{\underline{B}}_{1,1}$.(T) in which the Taylor expansion is

$$u(x,y) = u(0,0) + x u_{1,0}(0,0) + y\,u_{0,1}(0,0) + \int_0^x (x-\tilde{x})\,u_{2,0}(\tilde{x},0)\,d\tilde{x}$$
$$+ \int_0^y \int_0^x u_{1,1}(\tilde{x},y)\,d\tilde{x}\,d\tilde{y} + \int_0^y (y-\tilde{y})\,u_{0,2}(0,\tilde{y})\,d\tilde{y}. \qquad (6.8)$$

The Sard kernel theorem gives

$$
\begin{aligned}
Ru(x, y) = & \int_o^h u_{2,0}(\tilde{x}, 0)\, k^{2,0}(x, y; \tilde{x})\, d\tilde{x} \\
& + \iint_T u_{1,1}(\tilde{x}, \tilde{y}) k^{1,1}(x, y; \tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y} \\
& + \int_o^h u_{0,2}(0, \tilde{y}) k^{0,2}(x, y; y`) d\tilde{y}
\end{aligned} \tag{6.9}
$$

where, from the symmetry of the kernels $K^{2,0}$ and $K^{0,2}$, the kernel functions are

$$
K^{2,0}(x, y; \tilde{x}) = k^{0,2}(y, x; \tilde{x}) = R(x, y)(x - \tilde{x}) + (x - \tilde{x}) + - \frac{x}{h}(h - \tilde{x}), \tag{6. 10}
$$

$$
K^{1,1}(x, y; \tilde{x}, \tilde{y}) = R_{(x,y)}(x - \tilde{x})_+^0 (y - \tilde{y})_+^0 = (x - \tilde{x})_+^0 (y - \tilde{y})_+^0, \tag{6.11}
$$

and $R_{(x,y)}$ denotes the functional R applied to the functions in the variables x and y. From (6.9) using Holder's inequality and the triangle inequality, we have the bound

$$
\begin{aligned}
||R u(x,y)||L_q(x,y) \le & \\
& || u_{2,0}(\tilde{x},0) ||L_{pi(x`)} \quad || \quad ||k^{2,0}(x,y;\tilde{x})||L_{p1}(\tilde{x}) \quad ||L_q(x,y) \\
+ & || u_{1,1}(\tilde{x},\tilde{y}) ||L_{p2(\tilde{x},\tilde{y})} \quad || \quad ||k^{1,1}(x,y;\tilde{x},\tilde{y})||L_{p2}(\tilde{x},\tilde{y}) \quad ||L_q(x,y) \\
+ & || u_{0,2}(0,\tilde{y}) ||L_{pi(\tilde{y})} \quad || \quad ||k^{0,2}(x,y;\tilde{y})||L_{pi}(\tilde{y}) \quad ||L_q(x,y),
\end{aligned} \tag{6.12}
$$

where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$ and $\frac{1}{p_2} + \frac{1}{p_2'} = 1$. The norms involving

one variable are over [0,h] and those involving two variables are over T, where, for simplicity, we assume the existence of the double integral rather than the more general repeated integral in (6.9). The L_P norms of the kernel functions are

$$||\,k^{1,1}(x,y;\tilde{x},\tilde{y})\,||_{Lp_2}\ (\tilde{x},\tilde{y}) = \begin{cases} (x,y)\ 1/p_2\ , & p_2 < \infty\ , \\ 1\ , & p_2 = \infty\ , \end{cases} \tag{6.13}$$

$$|\,k^{2,0}(x,y;\tilde{x})\,||_{Lp_2}\ (\tilde{x}) = \begin{cases} \frac{(h-x)x}{h} \left(\frac{h}{p_1+1}\right)\frac{1}{p_1}\ , & p_1 < \infty\ , \\ \frac{(h-x)x}{h}\ , & p_2 = \infty\ , \end{cases} \tag{6.14}$$

and $K^{0,2}$ (x,y:y) is dual. The L_q nortas of (6.13) and (6.14) are

$$||\ \ ||k^{1,1}\ (x,y;\tilde{x},\tilde{y})\,||\,L_{p_2}\ (\tilde{x},\tilde{y})\ ||\,L_q\ (x,y)$$

$$= \begin{cases} h^{2\left(\frac{1}{p^2} + \frac{1}{q}\right)} \left\{\beta\left(\frac{q}{p_2} + 1\ ,\ \frac{q}{p_2} + 2\right)\right\}^{1/q} & p_2\ , q < \infty\ , \\ \left(\frac{h^2}{4}\right)^{1/p_2}\ , & P_2 < \infty\ ,\quad q = \infty\ , \\ \left(\frac{h^2}{2}\right)^{1/q}\ , & P_2 = \infty\ ,\quad q \le \infty\ , \end{cases} \tag{6.15}$$

$$|| \quad ||k^{2,0}(x,y;\tilde{x}) \quad ||L_{p_1}(\tilde{x}) \quad ||L_q(x,y)$$

$$= \begin{cases} h^{1} + \frac{1}{p_1} + \frac{2}{q} \left(\frac{1}{p_1+1}\right)^{1/p_1} \{\beta(q+1, q+2)\}^{1/q}, & p_1 \le \infty, q < \infty \\ \frac{h^{1+\frac{1}{p_1}}}{4} \left(\frac{1}{p_1+1}\right)^{1/p_1}, & p_1 \le \infty, q = \infty, \end{cases} \tag{6.16}$$

and $K^{0,2}$ (x,y;y) is dual, with the convention that $\left(\frac{1}{p_1+1}\right)^{1/p_1} = 1$, when $p_1 = \infty$. ß(m,n), m,n > 0, is the Beta function so that if m and n are integers

$$\beta(m,n) = \frac{(m-1)!\ (n-1)}{(m+n-1)!} \quad \bullet \tag{6.17}$$

$$\beta(m+\frac{1}{2,}, n+\frac{1}{2} = w\frac{(m-\frac{1}{2})....(\frac{1}{2})\ (n-\frac{1}{2}).... (\frac{1}{2})}{(m+n)!} \tag{6.18}$$

A sharper bound is obtained by taking the L_q (x,y) norm of the right hand side of (6.9) directly. For example, with $q = p_1' = p_2' = 2$, (6.12) gives the bound

$$||R\ u(x,y)||L_2(x,y) \le ||u_{1,1}(\tilde{x},\tilde{y})||L_2(\tilde{x},\tilde{y}) \frac{h^2}{2\sqrt{3}}$$

$$+\left\{||u_{2,0}(\tilde{x},0)\ ||L_2(\tilde{x}) + ||u_{0,2}(0,\tilde{y})||L_2(\tilde{y})\right\} \frac{h^{5/2}}{6\sqrt{5}}, \tag{6.19}$$

whereas talcing the $L_2(x,y)$ norm directly gives

$$\| Ru(x, y) \Big[\| L_2(x, y) \leq \| u_{1,1}(\tilde{x}, \tilde{y}) \mid \|^2_{L_2(\tilde{x},\tilde{y})} \frac{h^4}{12}$$
$$+ \left\{ \| u_{2,0}(\tilde{x},0) \|^2_{L_2(\tilde{x})} + \| u_{0,2}(0, \tilde{y}) \|^2_{L_2(\tilde{y})} \right\} \frac{h^5}{180}$$
$$+ \| u_{2,0}(\tilde{x},0) \| L_{2(\tilde{x})} \| u_{0,2}(0, \tilde{y}) \| L_2(\tilde{y}) \frac{h^5}{88}$$
$$+ \| u_{1,1}(\tilde{x}, \tilde{y}) \| L_{2(\tilde{x},\tilde{y})} \Big\{ \| u_{2,0}(\tilde{x},0) \| L_2(\tilde{x})$$
$$+ \| u_{0,2}(0, \tilde{y}) \|_{L_2}(\tilde{y}) \Big\} \frac{\pi h^{9/2}}{64/3} \Big]^{\frac{1}{2}} \tag{6.20}$$

The apparent discrepancy In the orders h is implict in the difference between the univariate and bivariate norms.

$\underline{L_2 \text{ Bounds on } R_{1,0} \text{ and } R_{0,1} \text{ for Linear Interpolation.}}$

$R_{1,0}$ and $R_{0,1}$ are symmetric functionals in $\underset{=}{B}_{1,1}$ ..(T).

The functional

$$R_{1,0}\, u(x,y) = u_{1,0}(x,y) + \frac{u(0,0) - u(h, 0)}{h} \tag{6.21}$$

is zero for the functions 1 ,x, and y. The application of this functional to the Taylor expansion (6.9) in the Sard space $B_{1,1}$ gives

$$R_{1,0}\, u(x, y) = R_{1,0}(x, y) \left[\int_o^x (x - \tilde{x}) u_{2,0}(\tilde{x},0) d\tilde{x} \right]$$
$$+ R_{1,0}(x, y) \left[\int_0^y \int_0^x u_{1,1}(\tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y} \right]$$
$$+ R_{1,0}(x, y) \left[\int_0^y (y - \tilde{y}) u_{0,2}(0, \tilde{y})\, d\tilde{y} \right]. \tag{6.22}$$

$R_{1,0}$ is not an admissible functional for the Sard kernel theorem in $\underset{=}{B}_{1,1}$ but the first and last terms in (6.22) can be evaluated in Sard kernel form. Thus

$$R_{1,0}\left[\int_0^x (x - \tilde{x})u_{2,0}(\tilde{x},0)\, d\tilde{x}\right] = \int_0^h u_{2,0}(\tilde{x},0)K^{2,0}(x, y, ; \tilde{x})\, d\tilde{x} \tag{6.23}$$

and

$$R_{1,0}\left[\int_0^y (y - \tilde{y})u_{0,2}(0, \tilde{y})\, d\tilde{y}\right] = \int_0^h u_{0,2}(0, \tilde{y})K^{0,2}(x, y, ; \tilde{x})\, d\tilde{y} \tag{6.24}$$

Where

$$K^{2,0}(x, y; \tilde{x}) = R_{1,0}(x - \tilde{x})+ = (x - \tilde{x})_+^0 - \frac{h - \tilde{x}}{h}, \ \tilde{x} \neq x, \tag{6.25}$$

and

$$k^{0,2}(x, y ; \tilde{y}) = R_{1,0}(y - \tilde{y})+ = 0 \tag{6.26}$$

For the first kernel $\tilde{x} = x$ is a jump set. The second kernel is an example of the Zero Kernel Theorem. The L_p norm of the kernel function (6.25) is

$$|| k_{2,0} (x,y;\tilde{x}) || L_p(\tilde{x}) = \begin{cases} \left(\frac{1}{p+1}\right)^{1/p} \left[\frac{x^{p+1}}{h^p} + h\left(1-\frac{x}{h}\right)^{p+1}\right]^{1/p}, & p<\infty \\ 1 \quad \text{if} \quad p = \infty \end{cases} \tag{6.27}$$

The middle term of (6.22) is evaluated by applying the functional directly to it. Thus

$$R_{1,0}\left[\int_0^y \int_0^x u_{1,1}(\tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y}\right] = \frac{\partial}{\partial x} \int_0^x \int_0^y u_{1,1}(\tilde{x}, \tilde{y})\, d\tilde{y}\, d\tilde{x}$$

$$= \int_0^y u_{1,1}(x, \tilde{y})d\tilde{y} \ , \tag{6.28}$$

where we have assumed the existence of the double integral so that Fubini'a theorem applies. Substitution in (6.22) gives the following!

$$\| R_{1,0}u(x, y) \|_{L_2}(x, y) \leq$$

$$\| u_{2,0}(\tilde{x},0) \| Lp_{1}'(\tilde{x}) \| \; \| K^{2,0}(x, y; \tilde{x}) \| L_{P1}(\tilde{x}) \|_{L_2}(x, y)$$

$$+ \| \int_0^y u_{1,1}(x, \tilde{y})\, d\tilde{y} \|_{L_2}(x, y) \qquad , \qquad (6.29)$$

Where $\frac{1}{P_1} + \frac{1}{p_1'} = 1$ Now

$$\int_0^h \int_0^{h-y} \left| \int_0^y u_{1,1}(x, \tilde{y})\, d\tilde{y} \right|^2 dx \; dy$$

$$\leq \int_0^h \int_0^{h-y} y2/P2 \left\{ \int_0^y |u1,1(x, \tilde{y})|^{p_2'} d\tilde{y} \right\}^{2/p_2'} dx \; dy \qquad (6.30)$$

Where $\frac{1}{P_2} + \frac{1}{P_2} = 1$ and

$$\int_0^{h-y} \left\{ \int_0^y \left| u1,1(x, \tilde{y}) \right|^P d\tilde{y} \right\}^{2/P2} dx\ dy$$

$$\leq (h - y)^{1 - \frac{2}{P_2'}} \left\{ \int_0^{h-y} \int_0^y \left| u_{1,1}(x, \tilde{y}) \right|^{p_2'} d\tilde{y}\ dx \right\}^{2/P_2'}$$

$$\leq (h - y)^{\frac{2}{P_2} - 1} \| u_{1,1}(x, \tilde{y}) \|^2_{L_{p_2'}} (x, \tilde{y}) \qquad (6.31)$$

Provided $P_2' \geq 2$. We thus get

$$\| \int_0^y u_{1,1}(x, \tilde{y})\ (x, \tilde{y})\ d\tilde{y}\ \|\ L_2(x, y)$$

$$\leq \begin{cases} h^{2/P2} \left[\beta\left(2/_{P2}, 2/_{P2} + 1 \right) \right]^{\frac{1}{2}} \|\ u1,1(x, \tilde{y})\ \|\ L_{P_2'}(x, \tilde{y}),\ 2 \leq P_2 < \infty, \\ \dfrac{h^2}{2\sqrt{3}} \|\ u_{1,1}(x, \tilde{y}) \|\ L_\infty(x, \tilde{y}) \end{cases} \qquad (6.32)$$

Where ß(m,n) is the Beta function Lastly

$$\| \quad \|\ K^{2,0}(x,\ y;\ \tilde{x})\ \|\ L_{P_1}(\tilde{x})\ \|\ L_2(x,\ y) = \begin{cases} \dfrac{h^2 \sqrt{11}}{2\sqrt{15}}, & P = 1, \\ \dfrac{h^{3/2}}{2\sqrt{2}}, & P = 2, \\ \dfrac{h}{\sqrt{2}}, & P = \infty \end{cases} \qquad (6.33)$$

The L_2 bound on $R_{0,1}$ u (x,y) is dual

Application to Finite Element Error Bounds

We consider the space of piecewise linear interpolants over a triangulated polygon Ω. This is a suitable sub space v^h of w_2^1 for the Galerkin method described in section 2. For a particular Triangle Te in Ω, a bound on $\| u(x, y) - \tilde{u}(x, y) \| W_2^1(T_e)$ can be obtained from (6.12) and (6.29) with a suitable change of variable from T to T_e. A $W_2^1(\Omega)$ error bound is then given by

$$\| u(x, y) - \tilde{u}(x, y) \| W_2^1(\Omega) = \left\{ \sum_e \| u(x, y) - \tilde{u}(x, y) \|^2_{W_2^1(T_e)} \right\}^{\frac{1}{2}}, \tag{6.34}$$

where $\Omega = \sum_e T_e$.

L_2 Bound on R for Quadratic Interpolation

The error functional R u(x,y) = u(x,y) - u(x,y),
where u(x,y) is the quadratic interpolant (6.2), is zero for the functions 1, x, y, x^2, xy and y^2 • We thus consider the Sard space $B_{2,1}.^{(T)}$. The kernel theorem is

$$\begin{aligned} Ru(x, y) = {} & \int_0^h u_{3,0}(\tilde{x},0)k^{3,0}(x, y; \tilde{x})\, d\tilde{x} \\ & + \int_T\int u_{2,1}(\tilde{x}, \tilde{y})k^{2,1}(x, y; \tilde{x}, \tilde{y})\, d\tilde{x}\, d\tilde{y} \\ & + \int_0^h u_{1,2}(0, \tilde{y})k^{1,2}(x, y; \tilde{y})d\tilde{y} \\ & + \int_0^h u_{0,3}(0, \tilde{y})k^{0,3}(x, y; \tilde{y})d\tilde{y}, \end{aligned} \tag{6.35}$$

where the kernel functions are the following

$$k^{3,0}(x, y; \tilde{x}) = k^{0,3}(y, x; \tilde{x}) = R_{(x,y)}(x - \tilde{x})_+^{(2)}$$

$$= (x - \tilde{x})_+^{(2)} - \left(\frac{h}{2} - \tilde{x}\right)_+^{(2)}\left[\frac{4x}{h} - \frac{4x^2}{h^2}\right] - (h - \tilde{x})^{(2)}\left[-\frac{x}{h} + \frac{2x^2}{h^2}\right], \tag{6.36}$$

$$k^{2,1}(x, y; \tilde{x}, \tilde{y}) = R_{(x,y)}(x - \tilde{x})_+(y - \tilde{y})_+^o$$

$$= (x - \tilde{x})_+(y - \tilde{y})_+^o - \left(\frac{h}{2} - \tilde{x}\right)_+\left(\frac{h}{2} - \tilde{y}\right)_+^o \frac{4xy}{h^2}, \tag{6.37}$$

$$k^{1,2}(x, y; \tilde{y}) = R_{(x,y)}x(y - \tilde{y})_+ = x\left[(y - \tilde{y})_+ - \left(\frac{h}{2} - \tilde{y}\right)_+ \frac{2y}{h}\right]. \tag{6.38}$$

The square of the L_2 norms of the kernel functions are

$$\| k^{3,0}(x, y; \tilde{x} \|^2_{L_2(\tilde{x})}$$

$$= \frac{1}{30}\left[-\frac{x^7}{h^2} + \frac{11}{2}\frac{x^6}{h} - 11x^5 + \frac{83}{8}x^4h - \frac{77}{16}x^3h^2 + x^2h^3 - \frac{1}{16}xh^4\right]$$

$$+ \left(\frac{h}{2} - x\right)_+^o \frac{1}{15}\left[\frac{x^7}{h^2} - \frac{7}{2}\frac{x^6}{h} + 5x^5 - \frac{15}{4}x^4h + \frac{25}{16}x^3h^2 - \frac{11}{32}x^2h^3 + \frac{1}{32}xh^4\right], \tag{6.39}$$

$$\| K^{2,1}(x, y; \tilde{x}, \tilde{y}) \|^2_{L_2} (\tilde{x}, \tilde{y}) = \frac{1}{3} x^3 y + \frac{1}{3} x^2 y^2$$

$$+ \left(\frac{h}{2} - x\right)^0_+ \left(\frac{h}{2} - y\right)^0_+ \left[\frac{4}{3} \frac{x^4 y^2}{h^2} - 2 \frac{x^3 y^2}{h}\right]$$

$$+ \left(\frac{h}{2} - x\right)^0_+ \left(y - \frac{h}{2}\right)^0_+ \frac{2}{3} \frac{x^4 y}{h} - x^3 y$$

$$+ \left(x - \frac{h}{2}\right)^0_+ \left(\frac{h}{2} - y\right)^0_+ \left[- x^2 y^2 + \frac{1}{6} x y^2 h\right], \tag{6.40}$$

$$\| K^{1,2}(x, y; \tilde{y} \|^2_{L_2} (\tilde{y}) = \left(\frac{h}{2} - y\right)^0_+ x^2 \left[\frac{2}{3} \frac{y^4}{h} - \frac{2}{3} y^3 + \frac{1}{6} y^2 h\right]$$

$$+ \left(y - \frac{h}{2}\right)^0_+ x^2 \left[\frac{1}{3} y^3 - \frac{1}{3} y^2 h + \frac{1}{12} y h^2\right], \tag{6.41}$$

and $K^{0,3}(x, y; \tilde{y})$ is the dual of $K^{3,0}$ $(x, y; \tilde{x})$.The L_2 (x,y) norms are

$$\| \; \| K^{3,0}(x, y; \tilde{x}) \| L_2(\tilde{x}) \| L_2(x, y) = \frac{\sqrt{179}}{5 \cdot 3.2^6.\sqrt{3}} h^{7/2}, \tag{6.42}$$

$$\| \; \| K^{2,1}(x, y; \tilde{x}, \tilde{y}) \| L_2(\tilde{x}, \tilde{y}) \| L_2(x, y) = \frac{\sqrt{761}}{3 \cdot 2^4 \cdot \sqrt{3.5.7.2}} h^3, \tag{6.43}$$

$$\| \; \| K^{1,2}(x, y; \tilde{y}) \| L_2(\tilde{y}) \| L_2(x, y) \| L_2(x, y) = \frac{\sqrt{119}}{3 \cdot 2^5 \sqrt{3 \cdot 5 \cdot 7}} h^{7/2}. \tag{6.44}$$

We thus have the bound. :

$$\| R\, u(x, y) \|\, L_2(x, y) \le \left\{ \| u_{3,0}(\tilde{x},0) \|\, L_2(\tilde{x}) + \| u_{0,3}(0, \tilde{y}) \|\, L_2(\tilde{y}) \right\} \frac{\sqrt{179}}{5\cdot 3\cdot 2^6\cdot\sqrt{3}}\, h^{7/2}$$

$$+ \| u_{2,1}(\tilde{x}, \tilde{y}) \|\, L_2(\tilde{x}, \tilde{y}) \frac{\sqrt{761}}{3\cdot 2^4\cdot\sqrt{3\cdot 5\cdot 7\cdot 2}}\, h^3$$

$$+ \| u_{1,2}(0, \tilde{y}) \|\, L_2(\tilde{y}) \frac{\sqrt{119}}{3\cdot 2^5\sqrt{3\cdot 5\cdot 7}}\, h^{7/2} \tag{6.45}$$

We summarise Some results for the quadratic interpolant Functionals $R_{1,0}$ and $R_{0,1}$:

<u>The Functional $R_{1,0}$ for Quadratic Interpolation</u>

The functional

$$R_{1,0}u(x, y) = u_{1,0}(x, y) - u(0,0)\left[-\frac{3}{h} + \frac{4}{h^2}y + \frac{4}{h^2}x\right]$$

$$- u\left(\frac{h}{2},0\right)\left[\frac{4}{h} - \frac{4}{h^2}y - \frac{8}{h^2}x\right] + u(0, \frac{h}{2})\frac{4}{h^2}y$$

$$- u(h,0)\left[-\frac{1}{h} + \frac{4}{h^2}x\right] - u(\frac{h}{2}, \frac{h}{2})\frac{4}{h^2}y \tag{6.46}$$

is admissible for the Sard kernel theorem in $\underline{\underline{B}}_{2,1}$ The kernel theorem is

$$R\,1,0\;u(x,y) = \int_o^h u_{3,0}(\tilde{x},0)\; k^{3,0}(x,y;\tilde{x})\; d\tilde{x}$$

$$+ \int_T\int u_{2,1}(\tilde{x},\tilde{y})\; k^{2,1}(x,y;\tilde{x},\tilde{y})\; d\tilde{x}\, d\tilde{y}$$

$$+ \int_o^h u_{1,2}(o,\tilde{y})\; k^{1,2}(x,y;\tilde{y})\; d\tilde{y}$$

$$+ \int u_{0,3}(0,\tilde{y})\, K_{0,3}(x,y;\tilde{y})\; d\tilde{y} \qquad (6.47)$$

where the kernel functions are the following:

$$K^{3,0}(x,\, y;\, \tilde{x}) = (x - \tilde{x}) + -\left(\frac{h}{2} - \tilde{x}\right)_+^{(2)} \left[\frac{4}{h} - \frac{8x}{h^2}\right] - (h - \tilde{x})^{(2)}\left[-\frac{1}{h} + \frac{4x}{h^2}\right] \qquad (6.48)$$

$$K^{2,1}(x,\, y;\, \tilde{x},\, \tilde{y}) = (x - \tilde{x})_+^0 (y - \tilde{y})_+^0 - \left(\frac{h}{2} - \tilde{x}\right)_+ \left(\frac{h}{2} - \tilde{y}\right)_+^0 \frac{4y}{h^2},\; \tilde{x} \neq x, \qquad (6.49)$$

$$K^{1,2}(x,\, y;\, \tilde{y}) = (y - \tilde{y})_+ - \left(\frac{h}{2} - \tilde{y}\right)_+ \frac{2y}{h}, \qquad (6.50)$$

$$K^{0,3}(x,\, y;\, \tilde{y}) = 0\,. \qquad (6.51)$$

The square of the L_2 norms of the kernel functions are

$$||k^{3,0}(x,y;\tilde{x})\;||^2_{L_2(\tilde{x})} = -\frac{x^5}{h^2} + \frac{17}{12}\;\frac{x^4}{h}\; - 2x^3 - \frac{12}{5}\, x^2 h + \frac{57}{10}\, xh^2 - \frac{23}{24}\, h^3$$

$$+ \left(\frac{h}{2} - x\right)_+^0 \left[-2\frac{x^4}{h} + x^3\right] + \left(x - \frac{h}{2}\right)_+^0 \left[-\frac{2}{3}x^2 h + \frac{3}{8}xh^2 - \frac{1}{48}\, h^3\right], \qquad (6.52)$$

$$\| K^{2,1}(x, y; \tilde{x}, \tilde{y}) \|^2_{L_2}(\tilde{x}, \tilde{y}) = xy + \frac{1}{3} y^2$$

$$+ \left(\frac{h}{2} - x\right)^0_+ \left(\frac{h}{2} - y\right)^0_+ \left[4 \frac{xy^2}{h} - 4 \frac{x^2 y^2}{h^2}\right]$$

$$+ \left(\frac{h}{2} - x\right)^0_+ \left(y - \frac{h}{2}\right)^0_+ \left[2xy - 2 \frac{x^2 y}{h}\right]$$

$$- \left(x - \frac{h}{2}\right)^0_+ \left(\frac{h}{2} - y\right)^0_+ y^2 , \tag{6.53}$$

$$\| K^{1,2}(x, y; \tilde{y}) \|^2_{L_2(\tilde{y})} = \frac{1}{3} y^3 + \frac{1}{6} y^2 h^2 + \left(\frac{h}{2} - y\right)^0_+ \frac{2}{3} \frac{y^4}{h} - y^3$$

$$+ \left(y - \frac{h}{2}\right)^0_+ \left[- \frac{1}{4} y^2 h + \frac{1}{12} y h^2\right]. \tag{6.54}$$

The Functional $R_{0,1}$. for Quadratic Interpolation

The functional

$$R_{0,1} u(x, y) = u_{0,1}(x, y) - u(0,0) \left[- \frac{3}{h} + \frac{4}{h^2} x + \frac{4}{h^2} y\right]$$

$$+ u\left(\frac{h}{2}, 0\right) \frac{4}{h^2} x - u\left(0, \frac{h}{2}\right) \left[\frac{4}{h} - \frac{4}{h^2} x - \frac{8}{h^2} y\right]$$

$$- u(0, h) \left[- \frac{1}{h} + \frac{4}{h^2} y\right] - u\left(\frac{h}{2}, \frac{h}{2}\right) \frac{4}{h^2} x \tag{6.55}$$

is not admissible for the Sard kernel theorem in $\underline{\underline{B}}_{2,1}$. The application of this functional to the Taylor expansion in $\underline{\underline{B}}_{2,1}$ gives

$$
\begin{aligned}
R_{0,1}\, u(x, y) = {} & R_{1,0}(x, y)\left[\int_0^x (x - \tilde{x})^{(2)}\; u_{3,0}(\tilde{x},0)\; d\tilde{x} \right] \\
& R_{1,0}(x, y)\left[\int_0^y \int_0^x (x - \tilde{x}) u_{2,1}(\tilde{x}, \tilde{y}) u_{2,1}(\tilde{x}, \tilde{y})\; d\tilde{x}\; d\tilde{y} \right] \\
& R_{1,0}(x, y)\left[x \int_0^y (y - \tilde{y})\; u_{1,2}(0, \tilde{y})\; d\tilde{y} \right] \\
& R_{1,0}(x, y)\left[\int_0^y (y - \tilde{y})^{(2)}\; u_{0,3}(0, \tilde{y})\; d\tilde{y} \right]
\end{aligned}
\qquad (6.56)
$$

The second term of (6.54) is evaluated by direct application of the functional to it. Thus

$$
R_{0,1}(x, y)\left[\int_0^y \int_0^x (x - \tilde{x})\; u_{2,1}(\tilde{x}, \tilde{y})\; d\tilde{x}\; d\tilde{y} \right]
$$

$$
= \frac{\partial}{\partial y}\left[\int_0^y \int_0^x (x - \tilde{x})\; u_{2,1}(\tilde{x}, \tilde{y})\; d\tilde{x}\; d\tilde{y} \right] - \frac{4}{h^2}\, x \int_0^{h/2} \int_0^{h/2} \left(\frac{h}{2} - \tilde{x} \right) u_{2,1}(\tilde{x}, \tilde{y})\; d\tilde{x}\; d\tilde{y}
$$

$$
= \int_0^x (x - \tilde{x}) u_{2,1}(x, \tilde{y})\; d\tilde{x} - \frac{4}{h^2}\, x \int_0^{h/2} \int_0^{h/2} \left(\frac{h}{2} - \tilde{x} \right) u_{2,1}(\tilde{x}, \tilde{y})\; d\tilde{x}\; d\tilde{y}. \qquad (6.57)
$$

The remaining terms of (6.56) can be evaluated in Sard kernel form. Thus

$$R_{0,1}(x,\ y)\left[\int_0^x (x - \tilde{x})^{(2)} u_{3,0}(\tilde{x},0)\ d\tilde{x}\right] = \int_0^h u_{3,0}(\tilde{x},0)\ K^{3,0}(x,\ y;\ \tilde{x})\ d\tilde{x}, \qquad (6.58)$$

$$R_{0,1}(x,\ y)\left[x\int_0^y (y - \tilde{y})\ u_{1,2}(0,\ \tilde{y})\ d\tilde{y}\right] = \int_0^h u_{1,2}(0,\ \tilde{y})\ K^{1,2}(x,\ y;\ \tilde{y})\ d\tilde{y}, \qquad (6.59)$$

$$R_{0,1}(x,\ y)\left[\int_0^y (y - \tilde{y})^{(2)} u_{0,3}(0,\ \tilde{y})\ d\tilde{y}\right] = \int_0^h u_{0,3}(0,\ \tilde{y}) K^{0,3}(x,\ y;\ \tilde{y}) d\tilde{y} \qquad (6.60)$$

where the kernel functions are

$$K^{3,0}(x,y;\tilde{x}) = 0 \quad , \qquad (6.61)$$

$$K^{1,2}(x,\ y\ \tilde{y}) = x\left[(y - \tilde{y})_+^0 - \left(\frac{h}{2} - \tilde{y}\right)_+ \frac{2}{h}\right] \ ,\ \tilde{y} \neq y\ , \qquad (6.62)$$

and $K^{0,3}(x,y;\tilde{y})$ is the dual of the $R_{1,0}$ kernel $K^{3,0}(x,y;\tilde{y})$, equation (6.48).

The square of the L_2 norm of (6.62)is

$$\|\ K^{1,2}(x,\ y;\ \tilde{y}\ \|_{L_2}^2\ (\tilde{y}) = x^2 y + \frac{1}{6} x^2 h$$

$$+ \left(\frac{h}{2} - y\right)_+^0 \left[2\,\frac{x^2 y^2}{h} - 2x^2 y\right] - \left(y - \frac{h}{2}\right)_+^0 x^2 h \qquad (6.63)$$

A L_2 (x,y) bound on $R_{1,0}$ u(x,y) and $R_{0,1}$ u(x,y) can be obtained as was done above for the functional R.

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