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by

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1. Introduction.

In this paper we are concerned with establishing the existence, both locally and globally in time of solutions for a model equation describing the longitudinal vibration of the material in a straight, thin cylindrical rod. We refer to Green [3], Love [6] and Showalter [10]for a discussion of the linear problem and to Jannzemis [4] for the more general nonlinear setting.

The equation of motion is obtained most simply by setting $\delta L=0$ where L is the following Lagrangian

$$L \equiv \int_{t_1}^{t_2} \int_{\Omega} [\tfrac{1}{2} u_t^2 + \tfrac{1}{2} \beta_t^2 (u_x) - W(u_x, \beta(u_x))] \, dx \, dt \qquad (1.1)$$

$$(t_1, t_2) \times \Omega \subseteq (0,\infty) \times (-\infty, \infty) \ .$$

In particular, we choose here $\Omega = [0,1]$ and refer elsewhere, [7] , for the case when $\Omega = R$.

The term $W(u_x,\beta)$ denotes a generally nonlinear strain-energy function which depends both on u_x (x,t), the longitudinal displacement gradient at time t of a material point at x measured from some chosen point along the rod, and on $\beta(u_x (x,t))$, a measure of the lateral deformation undergone during the motion. Throughout, we make only very mild assumptions concerning the constitutive terms $\beta(.)$, $W(., .)$ and their first derivatives, although some extra restrictions must be imposed if smoothness of solutions is to be obtained.

We adopt the following notation:

$$\frac{d\beta}{d\varphi}(\phi) = \beta'(\phi) \quad , \quad \frac{d^n\beta}{d\phi^n}(\phi) = \beta^{(n)}(\phi) \quad , \quad n = 2, 3, \ldots , \qquad (1.2)$$

$$\frac{\partial W}{\partial \varphi}(\phi,\beta)+\frac{\partial W}{\partial \beta}(\phi,\beta)\beta' \equiv DW(\phi,\beta), \text{ and } DW(\phi,\beta)+\beta'\beta^{(2)}X^2 \equiv \sigma(\phi,X). \qquad (1.3)$$

This implies the Euler equation of (1.1) nay be written

$$u_{tt} - (\beta'^2(u_x)u_{xtt} + \sigma(u_x , u_{xt}))_x = 0 \quad . \qquad (1.4)$$

As an example, the special case when $\beta'(.) = 1$ which is also discussed in [8] reduces the equation (1.4) to the form

$$u_{tt} - u_{xxtt} - \sigma (u_x)_x = 0 \quad . \qquad (1.5)$$

Here we consider the Cauchy problem for the rather general situation in which (1.4) occurs, with the following list of hypotheses on the constitutive terms.

(HI) (a) Let $\beta(.) \in C^2(R)$, $W(., .) \in C^1(R \times R)$ and assume each of $\beta (.)$, $\beta' (.)$, $\beta^{(2)} (.)$, $\frac{\partial W}{\partial \phi}(\phi,.)$ and $\frac{\partial W}{\partial \beta}(. ,\beta)$ to be locally Lipschitz continuous. Therefore, in particular for all ϕ_i , $X_i \in R$, $i = 1,2$ such that for some $R > 0$ $|\phi_i| < R$, $|X_i| < R$, there exists a constant $\Gamma (R) > 0$, such that

$$|\sigma(\phi_1, X_1) - \sigma(\phi, X_2)| < \Gamma(R) \ \{|\phi_1 - \phi_2| + |X_1 - X_2|\} \quad .$$

(b) Let $\beta' (.)$ satisfy, for all $\phi \in \mathbb{R}$,

$$B'^2 (\phi) \geq \alpha$$

where α is a positive constant.

(c) $\sigma (0,0) = 0$.

The above set (H1) will be used later to demonstrate local existence, however only hypothesis (H1)(a) is important for the method of proof used.

(H1)(c) is inserted for physical reasons, whereas (H1)(b) may be relaxed to let $\beta'^2(\phi) > 0, \ \forall\phi \in IR$, [8], (This latter condition is of physical interest when one considers the constraint $u_x > -1$ required to avoid material inversion).

To extend these solutions globally in time, the following additional set will be required.

(H2) There exists $J_1, J_2 \geq 0$ such that for all $\phi \in IR$

(a) $W(\phi, \beta(\phi)) \geq -J_1$,

(b) $|DW(\phi, \beta(\phi))| \leq W(\phi, \beta(\phi)) + J_1 + J_2$,

(c) $\beta'^2(\phi) \leq W(\phi, \beta(\phi)) + J_1 + J_2$,

(d) $\beta^{(2)^2}(\phi) \leq \beta'2(\phi) + J_2$.

Again the first of these conditions is fundamental while those remaining do not impose severe restriction and may be modified quite easily.

It is well known that the Cauchy problem for equations (1.4) with $\beta'(\,.\,) \equiv 0$ in general does not permit C^1 solutions existing globally in time (see, for example, Lax [5]) since it is possible for the deformation gradient $u_x(x, t)$ to become infinite at some $x \in \Omega$, $t < \infty$, however in the situation we consider, globally unique solutions are found whose regularity depends only on the smoothness of the initial data and on $\sigma(\,.\,,\,.)$. §2 and § 3 of this paper deal respectively with the local and global existence problems.

We now provide an appropriate Banach space setting together with some notation.

The space $L^p(0,1)$ denotes the class of measurable real-valued functions on (0,1) for which

$$\|f\|_p \equiv \left(\int_0^1 |f(x)^p dx\right)^{1/p} < \infty \ , 1 \leq p < \infty \ , \qquad (1.6)$$

or $\|f\|_\infty \equiv \text{ess sup}\,|f(x)| < \infty$, when $p = \infty$. (1.7)

The Sobolev space $W^{m,p}(0,1)$, $m \in \mathbb{N}$ consists of those functions in $L^p(0,1)$ all of whose generalised derivatives up to and including order m belong to $L^P(0,1)$. We define a norm on $W^{m,p}(0.1)$ by

$$\|f\|_{m,p} = \left(\sum_{j=0}^{m} \left\| \frac{d^j f}{dx^j} \right\|_p^p \right)^{1/p}, \quad 1 \le p < \infty, \tag{1.8}$$

$$\text{or} \quad \|f\|_{m,\infty} = \sum_{j=0}^{m} \left\| \frac{d^j f}{dx^j} \right\|_\infty, \text{ when } p = \infty. \tag{1.9}$$

$W^{m,p}(0,1)$ denotes the subspace of $W^{m,p}(0,1)$ consisting of those functions in $W^{m,p}(0,1)$ which together with their generalised derivatives of order less than or equal to m - 1 vanish at x = o and x = 1.

Finally let $A \subseteq \mathbb{R}$ and X be a Banach space with norm $\|\ \|_X$. Then for $K \in \mathbb{N} \cup \{0\}$, $C^k(A,X)$ is the class of k-times continuously differentiable mappings $u(t)$, $\frac{d^j u}{dt^j}(t) : A \to X$, $0 \le j \le k$.

$C^k(A;X)$ has the norm

$$|u|_{k,A,X} \equiv \sum_{j=0}^{k} \sup_{t \in A} \left\| \frac{d^j u}{dt^j}(t) \right\|_X. \tag{1.10}$$

When A = [0,T] and X = $L^\infty(0,1)$, we simplify this notation to read

$$|u|_T \equiv |u|_{0,[0,T],L^\infty(0,1)}, \tag{1.11}$$

$$\text{and} \quad |||u|||_T \equiv |u|_{1,[0,T],L^\infty(0,1)}. \tag{1.12}$$

For information concerning the above spaces, which are all Banach, we refer to Adams [1] and Yosida [12] There the following simple facts and definitions may also be obtained.

<u>Definition</u> A sequence $\{u_n\}$ of functions in $L^\infty(0,1)$ converges weak-* in $L^\infty(0,1)$ to u if and only if

$$\int_0^1 u(x)v(x)dx \rightarrow \int_0^1 u(x)v(x) \quad dx \quad , \forall v(x) \in L^1(0,1) \quad ,$$

which we write as $u_n(x) \overset{*}{\rightharpoonup} u(x)$ in $L^\infty(0,1)$.

Similarly, the sequence $\{u_n\}$ in $W^{1,\infty}(0,1)$ converges weak-* in $W^{1,\infty}(0,1)$ to u if and only if

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,1) \text{ and } \frac{du_n}{dx} \overset{*}{\rightharpoonup} \frac{du}{dx} \quad \text{in } L^\infty(0,1) \quad ,$$

written $u_n \overset{*}{\rightharpoonup} u$ in $w^{1,\infty}(0,1)$.

It may be shown that every bounded sequence $\{u_n\}$ in $W^{1,\infty}(0,l)$ contains a subsequence $\{u_n\}$ which converges weak-* in $W^{1,\infty}(0,l)$ to a member x of $W^{1,\infty}(0,1)$, where $\| x \|_{1,\infty} \leq \underset{\mu \to \infty}{\lim} \| u_\mu \|_{1,\infty}$

Furtnermore it is easy to demonstrate that, in particular if $u \in W_0^{m,p}(0,1)$, then identifying classes of measure zero $u \varepsilon C^{m-1}([0,l])$, and for $0 \leq j \leq m-1$

$$\left\| \frac{d^j u}{dx^j} \right\|_p \leq \sup_{x \in [0,1]} \left| \frac{d^j u}{dx^j} \right| \leq \left\| \frac{d^{j+1}}{dx^{j+1}} \right\|_p , 1 \leq p \leq \infty \qquad (1.13)$$

it is immediate that when $u \varepsilon w_0^{m,p}(0,1)$, $\left\| \frac{d^m u}{dx^m} \right\|_p$ forms an equivalent norm for $\| u \|_{m,p}$

2. Local Existence.

We now consider the initial boundary-value problem, equation (1.4) together with Cauchy data given by

$$u(x,0) \equiv u_0(x) \quad , \quad u_t(x,0) \equiv u_1(X) \quad , \tag{2.1}$$

and boundary conditions

$$u(0,t) = 0 \quad , \quad u(1,t) = 0 \quad , \quad t > 0 \quad . \tag{2.2}$$

The following Theorem concerns solutions of a nonlinear integro-differential equation related to (1.4) through variation of constants. The Green's function possesses standard properties described in an appropriate setting, for instance, in Stone [1.1], We demonstrate that there exist unique solutions locally to this related equation and subsequently we verify in Lemma 1 that these solutions are weak solutions of (1.4).

To set up the problem, we note that for an arbitrary function $V(x,.) \varepsilon W^{1,\infty}(0,1)$, by hypothesis (H1)(b) the operator $I - \frac{\partial}{\partial x}\beta^{2}(v_x)\frac{\partial}{\partial x}$ defined on $W_0^{1,\infty}(0,1)$ is uniquely invertible. Thus integrating equation (1.4) twice with respect to time, substituting $\beta'^2(u_x)$ for $\beta'^2(v_x)$ and using (2.1), (2.2) we obtain the equation

$$u(x,t) = U_0(x) + tu_1(x) - \int_0^t\int_0^1 (t-\eta)\, G_\xi(x,\xi;v)\sigma(u_\xi, u_{\xi\eta})d\xi d\eta \tag{2.3}$$

where we have integrated by parts with respect to ξ in the last term. Here $G(x, \xi ; v)$ is the Green's function found on inverting the above operator together with (2.2). $G(x, \xi ; v)$ may be explicitly represented in the usual way. In order to solve the partial differential equation (1.4) we first show that (2.3) has a solution $u(x,t)$ such that $v(x,t) \equiv u(x,t)$; more succinctly, defining the right side of (2.3) to be $A_v u$ we prove there exists a solution $u(x,t)$ to the equation

$$u = A_u u \quad . \tag{2.4}$$

Theorem 1.

Let hypotheses (HI) (a) , (b) and (c) be given and let u_0 (x) , u_1 (x) belong to $w_0^{1,\infty}$ (0,1). Then there exists a unique solution

$u(x, t) \in C^2([o, \tau[; W^{1,\infty}(0,1))$ which satisfies equation (2.4) and (2.2)

for some maximal interval $[0, \tau[$, $\tau > 0$. If $\tau < \infty$, $\| u(. , t)\|_{1,\infty} + \| u_t(. , t)\|_{1,\infty} \to \infty$ as $t \to \tau -$.

Proof We use the contraction mapping principle (see [2]). Let T > 0. We start by demonstrating (2.3) has a fixed point in the space $C^1([0, T]; W_0^{\infty}(0,1))$ for some T sufficiently small. To do this it is sufficient to verify that (see (1.12))

$$B_1(R) \equiv \{u \in C^1([0, T]; W_0^{1,\infty}(0,1) : |||u|||_T + |||u_x|||_T \leq R\} \qquad (2.5)$$

is mapped into itself under the action of A_v for R large enough; then, on checking A_v indeed to be contractive under conditions on R and T which non-trivially intersect those found earlier, the hypotheses of the contraction mapping principle will be met implying the existence locally in time of a unique solution for (2.3).

The equivalence of the norms $\|\cdot\|_{1,\infty}$ and $\|\partial_x \cdot\|_\infty$ on $w_0^{1,\infty}(0,1)$ (see 1.13) permits the use of

$$B(R) \equiv \{u \in C^1([0,T] ; w_0^{1,\infty}(0,1)) : |||u_x|||_T \leq R\} \qquad (2.6)$$

in place of $B_1(R)$. Taking the derivative of (2.3) gives

$$\begin{aligned}(A_v u)_x(x,t) &= u_0'(x) + tu_1'(x) \\ &- \int_0^t \int_0^1 (t-\eta) G_{\xi x}(x,\xi;v)\sigma(u_\xi, u_{\xi\eta}) d\xi\, d\eta \\ &- \int_0^t (t-\eta)[\beta'(v_x)]^{-2}\sigma(u_x, u_{x\eta}) d\eta \end{aligned} \qquad (2.7)$$

from which it is readily found by hypothesis (HI) (b) that

$$|||\,(A_V u)_x\,|||_T \le \|\,u_0'\,\|_\infty + (1+T)\,\|\,u_1'\,\|_\infty$$

$$+\,(c+\alpha^{-1})T(1+T)\;|\,\sigma(u_x, u_{x\eta})\,|_T \ , \qquad (2.8)$$

(see (1.11)) where c is a finite constant. Thus by hypotheses (HI) (a) , (c), (2.8) implies that

$$|||\,(A_V u)_x\,|||_T \le \|\,u_0'\,\|_\infty + (1+T)\,\|\,u_1'\,\|_\infty$$

$$+(c+\alpha^{-1})T\ (1+T)\ RT\ (R) \qquad (2.9)$$

Hence choosing R, T > 0 to satisfy

$$\|\,u_0'\,\| + (1+T)\,\|\,u_1'\,\|_\infty + (c+\alpha^{-1})T(1+T)R\,\Gamma\ (R) < R \qquad (2.10)$$

and noting $A_V\,(0\ ,\ t) = A_V\ u\ (1\ ,\ t) = 0,\ t\ \ge\ 0$, we have that $A_V B\ (R) \subset B(R)$. This result holds on replacing u(x , t) by v(x , t) ε B(R) in (2.7) and it remains to be seen that A_V v is a contraction, i.e. for θ ε [0 , l[we show that there exist R, T > 0 such that for all v, V ∈ B(R)

$$|||\,(A_V v)_x - (A_V v)_x\,|||_T \le \theta\,|||\,v_x - v_x\,|||_T \,. \qquad (2.11)$$

By (2.7) and hypotheses (H1) (a) , (b) , (c) ,

$$|||\,(A_V v)_x - (A_V v)_x\,|||_T \ \le T(1+T)\,|\int_0^1 (G_{\xi x}(x,\xi;v) - G_{\xi x}(x,\xi;v))\sigma(v_\xi, v_{\xi t})d\xi$$

$$+\int_0^1 (G_{\xi x}(x,\xi;v)(\sigma(v_\xi, v_{\xi t}) - \sigma(v_\xi, v_{\xi t}))d\xi\,|\,T$$

$$+\,T(1+T)\,|\,[\beta\beta(v_x)\beta'(v_x)]^{-2}\beta'^{\,2}(v_x)\sigma(v_x, v_{xt})$$

$$\beta'^{\,2}(v_x)\sigma(v_x, v_{xt}))\,|\,T$$

$$\le T(1+T)\ R\,\Gamma(R)\,|\int_0^1 |G_{\xi x}(x,\xi;v) - G_{\xi x}(x,\xi;v)\,|\,d\xi\,|\,T$$

$$+\,T(1+T)\ \Gamma(R)\,|||\,v_x - v_x\,||_T\,|\int_0^1 (G_{\xi x}(x,\xi;v)\,|\,d\xi\,|\,T$$

$$+T(1+T)\alpha^{-2}\,|\beta'^2(v_x)(\sigma(\sigma_x, v_{xt})-\sigma(v_x, v_{xt}))$$
$$+\sigma(v_x, v_{xt})(\beta'^2(v_x)-\beta'^2(v_x))|_T$$
$$\leq T(1+T)\Gamma(R)\,(Rc'\alpha^{-1}\,|\beta'^2(v_x)-\beta'^2(v)|_T$$
$$+c'|||\,v_x - v_x\,|||_T$$
$$+\alpha^{-2}\,|||\,v_x - v_x\,|||_T|\beta'^2(v_x)|_T$$
$$+\alpha^{-2}R\,|\beta'^2(v_x)-\beta'^2(v_x)|_T)$$

Here the last inequality obtains on applying Cauchy-Schwarz to the terms under the integrals. c' > 0 is a finite constant. Letting $\gamma(R)$ be the Lipschitz constant for B'^2, by hypotheses (Hl) (a) , (b) we obtain finally

$$|||\,(A_v\, v)_x - (Avv)x\,|||_T \leq T\,(1+T)\,\Gamma\,(R)\,(c'[\alpha^{-1}R\gamma(R)+1]$$

$$+\alpha^2[2R\gamma 2R\gamma+\beta'^2(0)])\,|||\,v_x - v_x\,||| \tag{2.12}$$

$$\equiv T(1+T)\;f\;(R)|||\,v_x\text{-}v_x||| \tag{2.13}$$

Thus $A_v\,v$ is contractive whenever

$$T(1+T)\,f\,(R) \leq \theta < 1. \tag{2.14}$$

It is easy to see that taking R sufficiently large to dominate the terms $\|u_0'\|_\infty$ and $\|u_1'\|_\infty$ in (2.1O) makes it possible, by choosing T to be small, for (2.10) and (2.14) both to be satisfied - which means the conditions for a unique solution u(x , t) of (2.4) to exist have been met.

We finish by showing u(x , t) e $C^2([0,\tau[;\ W_0^{1,\infty}\ (0,1))$ over a maximal interval of existence $[0, \tau [,\tau > 0$. On differentiating (2.3) twice with respect to time (we now consider $u \equiv v$) the right side of the equation obtained belongs to the class $C\,([\,0,T]\,;W_0^{1,\infty}\,(0,1)\,)$ for every interval [0,T] in which

$u(x, t) \in -C^1([0, T]; W_0^{1,\infty}(0, l))$ and so $u_{tt} \in C([0, T]; W_0^{1,\infty}(0,l))$ also. Let τ be the supremum of the T defined above and let $0 < \varepsilon < \tau$. We apply a standard continuation argument to prove $\| u(.,t)\|_{1,\infty} + \| u_t(.,t)\|_{1,\infty}$ cannot remain bounded as $t \to \tau$ in the case $\tau < \infty$. For suppose the converse to be true. Then, since the length of the interval of existence was shown to depend only on $\| u_0 \|_{1,\infty}$, $\| u_1 \|_{1,\infty}$ and on a σ, we may take new initial data $u(x, \tau - \varepsilon)$, $u_t(x, \tau - \varepsilon)$ and ε sufficiently small for the method of local existence to extend the solution to the interval $[\tau - \varepsilon, \tau - \varepsilon]$, thus violating the maximality of τ. This completes the proof of Theorem 1.

Remark Regularity of the above solutions may be demonstrated very simply on making further assumptions. Specifically, if we allow σ to be m times continuously differentiable it follows immediately on differentiating the right side of (2.3) m + 2 times with respect to t, that $u \in C^{m+2}([0, \tau[; W_0^{1,\infty}(0,1))$ with τ as before. In addition, if we let $\sigma^{(m)}$ be locally Lipschits continuous and consider u_0 and u_1 to belong to the subspace $W^{m+1,2}(0,l) \cap W_0^{1,2}(0,1)$, $m \geq 1$, of $W_0^{1,\infty}(0,l)$ then the preceding proof may be extended by replacing $B_1(R)$ in (2.5) with

$$B_2(R) \equiv \{u \in C^1([0,1]; W^{m+1,2}(0,1) \cap W_0^{1,2}(0,1) : \left|\partial^{m+1} u\right|_{1[0.T], L^2} \leq R\}$$

and, repeating the essential steps earlier, this shows $u \in C^{m+2}([0, \tau[; W^{m+1}(0,l) \cap W_0^{1,2}(0,1))$. The interval $[0,\tau[$ is again unchanged.

The foregoing remark makes it evident that under suitable conditions there occurs no 'loss of derivatives' with time for solutions to the integral equation. Let us now define, for a piecewise continuous function $\phi(x,.)$,

$$[\varphi\varphi(,.)](x) = \lim_{\in \to 0+} \begin{cases} \varphi(\in,.) \ , \quad x = 0 \ , \\ \varphi(x+\in,.) - \varphi(x-\in,.) \ , \ x \in]0\ ,1[\ , \\ -\varphi(1-\in,.) \ , \quad x = 1 \ . \end{cases} \qquad (2,15)$$

The lemma which follows illustrates to some extent the way in which initial discontinuities in the data evolve. More exactly, we can elaborate on the above assertion to show that no new discontinuities can travel into a region of initially smooth data from a region where there exists a 'jump' in the value of a derivative.

<u>Lemma 1</u> Let u(x , t) be the solution to (2.4), and suppose $u_0(x)$, $u_1(x) \in C^1([0,1] \setminus Y_n)$ where $Y_n \subset]0,1[$, n =0,1,2,..., is a set of n arbitrary points at which $u_0'(x)$ or $u_1'(x)$ has a jump discontinuity. Then $u(x,t) \in C^2([0,\tau[; C^1([0,1] \setminus Y_n))$, ie. for $t > 0$, $u_x(x,t)$, $u_{xt}(x,t)$ and $u_{xtt}(x,t)$ possess jump discontinuities at most for $x \in Y_n$.

<u>Proof</u> By Theorem 1 we have that u(x,t) satisfies the following:-

$$u_x(x,t) = u(x) + tu_1'(x) - \int_0^t \int_0^1 (t-\eta) G_{\xi x}(x,\xi;u)\sigma)\sigma_\xi, u_{\xi\eta})d\xi d\xi$$

$$-\int_0^t (t-\eta)[\beta'(u_x)]^{-2} \sigma(u_x, u_{x\eta})d\eta \ , \qquad (2.16)$$

$$u_{xt}(x,t) = u_1'(x) - \int_0^t \int_0^1 G_{\xi x}(x,\xi;u)\sigma)\sigma_\xi, u_{\xi\eta})d\xi d\xi$$

$$-\int_0^t [\beta'(u_x)]^{-2} \sigma(u_x, u_{x\eta})d\eta \qquad (2.17)$$

and

$$u_{xtt}(x,t) = -\int_0^1 G_{\xi x}(x,\xi;u)\sigma(u_\xi, u_{\xi t})d\xi - [\beta'(u_x)]^{-2} \sigma(u_x, u_{xt}). \quad (2.18)$$

It is easily verified that the map

$$\varphi(x) \to \int_0^1 G_{\xi x}\varphi(\xi)\,d\xi$$

takes $L^P(0,1)$ into $C([0,1])$ for $1 \leq p \leq \infty$, implying that the third, second and first terms respectively on the right sides of (2.16), (2.17) and (2.18) are continuous. Therefore, by (HI) (a), (2.16), (2.17), and for $[0,T] \subset [0,\tau[$

$$
\begin{aligned}
&|[u_x(\cdot,t)](x)| + |[u_{xt}(\cdot,t)](x)| \\
&\leq |[u_0'(\cdot)](x)| + (1+T)|[u_1'(\cdot)](x)| \\
&\quad + (1+T)\alpha^{-1}\Gamma(R)\int_0^t (|[u_x(\cdot,\eta)](x) + |[u_{x\eta}(\cdot,\eta)](x)|)d\eta \\
&\leq \{|u_0'(\cdot)](x)| + (1+T)|[u_1'(\cdot)](x)|\} \exp\{t(1+T)\alpha^{-1}c\Gamma\Gamma(R), \qquad (2.19)
\end{aligned}
$$

where $c < \infty$. The last line comes from Gronwall's lemma.

Hence, for $x \notin Y_n$, $t \in [0,\tau.[$

$$[u_x(\cdot,t)](x) = [u_{xt}(\cdot,t)](x) = 0 \quad . \qquad (2.20)$$

Finally from (2.18)

$$|[u_{xtt}(\cdot,t)](x)| \leq \alpha^{-1}\Gamma(R)(|[u_x(\cdot,t)(x)| + |[u_{xt}(\cdot,t)](x)|)$$

implies that when (2.20) holds

$$[u_{xtt}(\cdot,t)](x) = 0 \quad . \qquad (2.21)$$

<u>Remark</u> As in the Remark after Theorem 1, the result of the Lemma can inductively be extended to higher derivatives in both x and t given suitable conditions of smoothness on u_0, u_1 and on σ.

The solution to (2.4) will next be used to demonstrate there exist solutions to the partial differential equation (1.4) when interpreted in a weak sense. The existence of more regular solutions for which (2.4) holds directly is then an immediate consequence of our earlier remark.

Let (. , .) and < . , . >denote inner products over $L^2(0, 1)$ and $L^2(\,]t_1, t_2[\, x\,]0, 1[\,)$

respectively, where $0 \le t_1 \le t_2 < \tau$

We have the following result.

Lemma 2

Let hypotheses (H1) (a), (b) and (c) be given. Then for every

$\varphi(x, t) \in C^1([0,\tau[\,; W_0^{1,1}(0,1))$, the solution u(x,t) to (2.4) satisfies

$$< u_t, \varphi_t > + < \beta'^2 u_{xt}, \varphi_{xt} > + \beta^t \beta^{(2)} u_{xt}, \varphi_x >$$

$$= < DW, \varphi_x > + (u_t, \varphi)|_{t_1}^{t_2} + (\beta'^2 u_{xt}, \varphi_x)|_{t_1}^{t_2} \;, \tag{2.22}$$

where $$DW(u_x, \beta(u_x)) = \frac{\partial W}{\partial u_x} + \frac{\partial W}{\partial \beta}\beta'(u_x) \;, \tag{2.23}$$

(cp. (1.3)).

Proof In the following, integration by parts is justified by Theorem 1. We define

$$A = < \beta'\beta^{(2)} u_{xt}^2, \varphi_x > - < \beta'^2 u_{xtt}, \varphi_x >, \tag{2.24}$$

and

$$B = < \beta'^2 u_{xt}, \varphi_{xt} > + < \beta'\beta^2 u_{xt}, \varphi_x > . \tag{2.25}$$

(2.22) is written in the form

$$< u_t, \varphi_t > + A - < DW, \varphi_x > = (u_t, \varphi)|_{t_1}^{t_2} + (\beta\beta^2 u_{xt}, \varphi_x)|_{t_1}^{t_2} + A - B \;. \tag{2.26}$$

Noting that $A - B = -(\beta\beta^2 u_{xt}, \varphi_x)|_{t_1}^{t_2}$, it remains to be seen that the integral equation solution u(x , t) satisfies

$$< u_t, \varphi_t > + A - < DW, \varphi_x > = (u_t, \varphi)|_{t_1}^{t_2} \tag{2.27}$$

By (2.18)

$$\beta'^2 u_{xtt} + \beta'\beta^{(2)} u_{xt}^2 + DW + \beta'^2 \int_0^1 G_{\xi x}\, \sigma\, d\xi = 0\,,. \qquad (2.28)$$

Thus by (2.28) and the t-derivative of (2.3)(v=u), the left side of (2.27) becomes

$$-< u_t, \varphi_t > + < \beta'^2 \int_0^1 G_{\xi} x\, \sigma\, d\xi\,,\ \varphi_x >$$

$$= < u_1, \varphi_t > - < \int_0^t \int_0^1 G_{\xi} \sigma\, d\xi\, d\eta, \varphi_t > + < \beta'^2 \int_0^1 G_{\xi x}\, \sigma\, d\xi, \varphi_x > . \qquad (2.29)$$

But

$$< u_1, \varphi_t > = (u_1, \varphi)\Big|_{t_1}^{t_2}$$

$$= (u_t, \varphi)\Big|_{t_1}^{t_2} - (u_t - u_1, \varphi)\Big|_{t_1}^{t_2}$$

$$= (u_t, \varphi)\Big|_{t_1}^{t_2} + \int_0^t \Big(\int_0^1 G_{\xi}\, \sigma\, d\xi, \varphi(.,t_2)\Big) d\eta$$

$$- \int_0^{t_1} \Big(\int_0^1 G_{\xi} \sigma\, d\xi, \varphi(.\,, t_1)\Big) d\eta \qquad (2.30)$$

and so from (2.27) it remains to show

$$\int_0^{t_2} \Big(\int_0^1 G_{\xi}\, \sigma\, d\xi, \varphi(.\,, t_2)\Big) d\eta \ \int_0^{t_1} \Big(\int_0^1 G_{\xi}\, \sigma\, d\xi, \varphi(.\,, t_1)\Big) d\eta$$

$$= < \int_0^t \int_0^1 G_{\xi}\, \sigma\, d\xi\, d\eta, \varphi_t > - < \beta'^2 \int_0^1 G_{\xi x}\, \sigma\, d\xi, \varphi_x > \qquad (2.31)$$

which follows immediately on integrating by parts the first term on the right side with respect to η and the second with respect to x . This concludes the proof.

Finally in this Chapter we prove a result on the dependence of the solution of (1.4) on the data u_0 , u_1 . This will imply, of course, that the solution solving (2.4) is also the unique solution for (1.4) .

<u>Lemma 3</u> Let hypotheses (HI) (a) - (c) be given and suppose u(x , t), u_{mn} (x,t) are solutions of (2.4) corresponding to initial data $u_0(x), u_1(x)$ and $u_{0m}(x)$, $u_{1n}(x)$ respectively, where $\{u_{0m}(x)\}$, $\{u_{1n}(x)\}$ are bounded sequences in $W^{1,\infty}(0,1)$ such that

$$u_{0m}(.) \to u_0(.) \text{ , in } W_0^{1,2}(0,1) \text{ ,} \tag{2.33}$$

$$u_{1n}(.) \to u_1(.) \text{ , in } W_0^{1,2}(0,1) \text{ ,} \tag{2.34}$$

as $m , n \to \infty$.

Then for all t ε. [0 , T] , T < , as $m , n \to\infty$,

$$u_{mn}(.,.) \to u(.,.), \text{in } C^1([0,T]; W_0^{1,2}(0,1)) \tag{2.35}$$

$$u_{mn}(.,t) \overset{*}{\rightharpoonup} u(.,t) \text{ weak}^{-*} \text{, in } W^{1,\infty}(0,1) \tag{2.36}$$

and

$$u_{mn_t}(.,t) \overset{*}{\rightharpoonup} u_t(.,t) \text{ weak}^{-*} \text{, in } W^{1,\infty}(0,1) . \tag{2.37}$$

<u>Proof</u> (We preface this proof with the usual remark that in particular higher order t-derivatives converge in the above sense also, on assuming extra smoothness of the constitutive term σ .)

From the boundedness of the sequences $\{u_{0m}\}$, $\{u_{1n}\}$ in $W^{1,\infty}(0,1)$, (2.37) and (2.34) we have (see Chapter 1)

$$u_{0m}(.) \overset{*}{\rightharpoonup} u_0(.) \text{ weak}^{-*} \text{, in } W^{1,\infty}(0,1) \text{ ,} \tag{2.38}$$

and

$$u_{1n}(.) \overset{*}{\rightharpoonup} u_1(.) \text{ weak}^{-*} \text{, in } W^{1,\infty}(0,1) . \tag{2.39}$$

Also (2.36) and (2.37) follow from (2.35) provided $\{u_{mn}(.,t)\}$, $\{u_{mn_t}(.,t)\}$ stay bounded for $t \in [0,T]$ in $W^{1,\infty}(0,1)$. This can be ensured as a result of Theorem 1 by suitably bounding the $W^{1,\infty}(0,1)$

norms of $\{u_{0m}(\,.\,)\}$ and $(u_{1n}(\,.\,)\}$, and means the choice of τ before (2.35), strictly the infimum of the existence intervals over all the data given, converges towards τ of Theorem 1 as $m\,,\,n \to\infty$.
To verify (2.35), let

$$\delta_{0m} \equiv \|u_{0m} - u_0\|^2_{1,2} \quad , \qquad (2.40)$$

$$\delta_{1n} \equiv \|u_{1n} - u_1\|^2_{1,2} \quad , \qquad (2.41)$$

and

$$w_{mn}(x\,,t) \equiv u_{mn}(x\,,t) - u(x\,,t) \qquad (2.42)$$

Then by Lemma 2, $<.\,,.>$ representing the inner product over $L^2(\,]\,0\,,t\,[x]o\,,\,1\,[\,)$,

w_{mn} satisfies the relation

$$< w_{mn_t}, \varphi_t > + < \beta'^2_{mn} u_{mn_{xt}} - \beta'^2 u_{xt}, \varphi_{xt} >$$

$$+ < \beta'_{mn}\beta^{(2)}_{mn} u^2_{mn_{xt}} - \beta'\beta^{(2)} u^2_{xt}, \varphi_x >$$

$$= < DW_{mn} - DW, \varphi_x > + (w_{mn_t}, \varphi)\big|_0^T + (\beta\beta^2 u_{mn_{xt}} - \beta'^2 u_{xt}, \varphi_x)\big|_0^T \quad ,$$

$$(2.43)$$

where $W_{mn} \equiv W(u_{mn_x}\,,\,\beta(u_{mn_x})\,)\,,\ \beta_{mn} \equiv \beta(u_{mn_x})$. (2.44)

Since, in particular (2.43) holds for $\varphi = w_{mn_t}$, integration by parts and rearrangement implies

$$\tfrac{1}{2}\| w_{mn_t} \|^2_2 \Big|_0^t + < \beta'^2_{mn}(u_{mn_{xtt}} - u_{mn_{xt}} >$$

$$= < \beta'^2 - \beta'^2_{mn} u_{xtt}\,,\, w_{mn_{xt}} >$$

$$+ < DW - DW_{mn}\,,\, W_{mn_{xt}} > \quad . \qquad (2.45)$$

So, by (Hl) (a) , (b) and the proof of Theorem 1 ,

$$\tfrac{1}{2}\| w_{mn_t} \|_2^2 \Big|_0^t + \tfrac{1}{2}\alpha \| w_{mn_{xt}}(.\,,t)\|_2^2 - \tfrac{1}{2}(R\gamma(R) + \beta^2(0))\| w_{mn_{xt}}(.\,,0)\|_2^2$$
$$\leq [\alpha^{-1} + c]\, R\,\gamma(R)\,\Gamma(R) < |w_{mn_x}|\, , |w_{mn_{xt}}| >$$
$$+ c\,\Gamma(R) < |w_{mn_x}| + |w_{mn_{xt}}|, |w_{mn_{xt}}| > \ , \qquad (2.46)$$

where $c < \infty$ and the term u_{xtt} has been majorised using (2.18). Letting

$a = \inf\{1, \alpha\}$, $b = R\,\gamma(R) + \beta'^2(0)$ and $d = \tfrac{1}{2}[(\alpha^{-1} + c)R\gamma(R)\Gamma(R) + 3c\Gamma(R)$,

(2.46) and the inequality $2AB \leq A^2 + B^2$ yield

$$\| w_{mn_t}(.\,,t)\|_{1,2}^2 \leq a^{-1}(b+1)\delta_{1n} + a^{-1}d \int_0^t (\| w_{mn}(.\,,\eta \|_{1,2}^2 + \| w_{mn\eta}(.\,,\eta)\|_{1,2}^2)\, d\eta. \qquad (2.47)$$

Also, from the identity

$$w_{mn}^2(x,t) = w_{mn}^2(x\,,0) + 2\int_0^t w_{mn}(x,\eta)\, w_{mn_\eta}(x,\eta)\, d\eta$$

there follows trivially

$$\| w_{mn}(.\,,t)\|_{1,2}^2 \leq \delta_{0m} + \int_0^t (\| w_{mn}(.\,,\eta)\|_{1,2}^2 + \| w_{mn_\eta}(.\,,\eta)\|_{1,2}^2)\, d\eta\ . \qquad (2.48)$$

(2.47), (2 48) and Gronwall's lemma now combine to form the inequality

$$\| w_{mn}(.,t) \|_{1,2}^2 + \| w_{mn_t}(.\,,t)\|_{1,2}^2 \leq (\delta_{0m} + a^{-1}(b+1)\delta_{1n})\exp[(1+a^{-1}d)T] \qquad (2.49)$$

from which we obtain (2.35) on letting $.\delta_{0m}$, $\delta_{1n} \to 0$.

3. Global Existence.

The conditions we have so far considered (1.4) under are by themselves insufficient to guarantee the global existence of solutions. It is necessary to strengthen hypotheses (HI) (a) - (c) by making some additional demands of the constitutive terms, which we do by applying (H2) (a) - (d) of Chapter 1. This poses a mild restriction but is important in making the anticipated energy estimate relevant for obtaining later $W^{1,\infty}$ bounds, which in turn permit a continuation of the local existence procedure of Theorem 1. (H2) (b) - (d) could also be generalised, however, as they stand only the two types of estimate just mentioned will be needed.

We do not consider existence under uniform Lipschitx hypotheses (Γ no longer dependent on R - see (HI) (a)) due to the obviously severe restriction this makes on the response functions at large values of the strain, although it is not difficult to find that then $u \in C^2([0, \infty[; W_0^{1,2}(0,1))$ (see [9]).

The first part of the continuation argument rests on the following Lemma.

<u>Lemma 4</u> Let $u(x, t) \varepsilon C^2([0, \quad [; W_0^{1,\infty}(0,1))$ be the solution of (2.22) and let $T < \tau$. Then for every $t \in [0,T]$,

$$\int_0^1 \{\tfrac{1}{2} u_t^2 + \tfrac{1}{2}\beta_t^2(u_x) + W(u_x, \beta(u_x))\} dx(t)$$

$$= \int_0^1 \{\tfrac{1}{2} u_1^2 + \frac{1}{2}\beta'^2(u_0')u_1'^2 + W(u_0', \beta(u_0'))\} dx \equiv E. \tag{3.1}$$

<u>Proof</u> In (2.22) we let $t_1 \to 0$ and take $t_2 = t$. It is evident from the integral representation that u(x,t) takes up initial data, and since $u_t \in C^1([0,T]; W_0^{1,\infty}(0, 1)) \subset c([0,T]; W_0^{1,1}(0,1))$ the result follows on letting $\phi = u_t$.

We now state and prove the main Theorem of the Chapter.

Theorem 2

Let hypotheses (H1) (a) , (b) , (c) , (d) and the conditions of Theorem 1 be given. Then the solution u(x , t) of (2.22) is a priori bounded in the space $C^1([0,T]; W_0^{1,\infty}(0,l))$, for every $T < \tau$.

Proof For this proof we take $c > 0$ to be a generic constant. By (1.3) , (2.28), for all $t \in [0,T]$ and almost every $x \in (0,l)$

$$|\beta'^2 u_{xtt} + \beta'\beta^{(2)} u_{xt}^2 + DW| \le c\beta'^2 \int_0^1 |DW + \beta'\beta^{(2)} u_{\xi t}^2| d\xi \tag{3.2}$$

and from Lemma 4, (H1) (b) and (H2) (a) ,

$$\tfrac{1}{2}\int_0^1 u_{\xi t} d\xi \le \alpha^{-1}[E + J_1] \ . \tag{3.3}$$

Consider the integral in (3.2) -Lemma 4, (H2) (b) and (d) , and (3.3) imply

$$\begin{aligned} &\int_0^1 | DW(u_\xi, \beta(u_\xi)) + \beta'(u_\xi)\beta^{(2)}(u_\xi) u_{\xi t}^2 | d\xi \\ &\le \int_0^1 \{W(u_\xi, \beta(u_\xi)) + (\beta'^2(u_\xi) + \tfrac{1}{2} J_2) + J_1 + J_2\} d\xi \\ &\le 2E + \alpha^{-1} J_2 [E + J_2] + J_1 + J_2 \ . \end{aligned} \tag{3.4}$$

We therefore have. from (3.2), that

$$|(\beta'^2 u_{xtt} + \beta'\beta^{(2)} u_{xt}^2 + DW u_{xt}| \le c\ |\beta'^2 u_{xt}|$$

and so, on integrating with respect to t and applying (H2) (c),

$$-J \le \tfrac{1}{2}\beta'^2 u_{xt}^2 + W \le E_1 + c\int_0^t |\beta'^2 u_{x\eta}| d\eta$$

$$\leq E_1 + \tfrac{c}{2}\int_0^t \{\beta'^2 u_{x\eta}^2 + \beta'^2\} d\eta$$

$$\leq E_1 + \tfrac{c}{2}\int_0^t \{\beta'^2 u_{x\eta}^2 + W\} d\eta + \tfrac{ct}{2}(J_1 + J_2) \ , \tag{3.5}$$

where $E_1 \equiv \tfrac{1}{2}\beta'^2 (u_0^1(x)) u_1'^2(x) + W(u'(x))$. (3.6)

Hence by Gronwall's lemma and because $e^a - 1 \leq ae^a$,

$$-J \leq \tfrac{1}{2}\beta'^2 u_{xt}^2 + W \leq [E_1 + \frac{ct}{2}(J_1 + J_2)]e^{\frac{ct}{2}} \ . \tag{3.7}$$

Since (3.7) is valid for almost every $x \in (0 , 1)$, (1.13) and (H1) (b) provide the following estimate for all $t \in [0,T]$:-

$$\| u_t(. , t)\|_{1,\infty} \leq 4\alpha^{-1}\{J_1 + [\bar{E}1 + \frac{ct}{2}(J_1 + J_2)]e^{\frac{ct}{2}} \ . \tag{3.8}$$

From this there follows immediately that

$$\| u(. , t)\|_{1,\infty} \leq \|u_0(.)\|_{1,\infty} + \int_0^t \| u_\eta(. , \eta)\|_{1,\infty} d\eta$$

$$\leq \|u_0(.)\|_{1,\infty} + 4\alpha^{-1} t\{J_1 + [E_1 + \frac{ct}{2}(J_1 + J_2)]e^{\frac{ct}{2}} . \tag{3.9}$$

(3.8) and (3.9) therefore prove the Theorem.

<u>Corollary</u> Under the -conditions of Theorem 1 with Theorem 2, the solution u(x,t) of (2.22) belongs to $C^2([0 , T] ; W_0^{1,\infty}(0,1))$ for every $T > 0$.

<u>Proof</u> The regularity argument given in Chapter 2 leads to u(x , t) being a priori bounded in $C([0 , T] ; W_0^{1,\infty}(0,1))$ - one obtains the estimates simply using (2.18) and (HI) (a) together with (3.8) and (3.9). Thus the final part of the statement of Theorem 1 shows that since $\| u(.,t) \|_{1,\infty} + \| u_t(.,t) \|_{1,\infty}$ remains bounded for every finite $t > 0$, then the maximal interval of existence $[0 , \tau[$ must be unbounded, which proves the result.

<u>Remark</u> The earlier Remark after Theorem 1 can quite easily be seen to remain true when $[0, \tau[$ is $[0, \infty[$ We omit the proof. (See [8])

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