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Abstract

In this paper the most general class of (2x2) - matrices is determined, which permit a Wiener-Hopf factorisation by the procedure of Rawlins and Williams [1]. According to this procedure, the factorisation problem is reduced to a matrix Hilbert problem on a half-line, where the matrix involved in the Hilbert problem is required to have zero diagonal elements.

Introduction

In the work of Rawlins and Williams [1] a Wiener-Hopf factorisation of the matrix

$$\underset{\sim}{A}(\alpha) = \begin{pmatrix} F(K) & G(K)F(K) \\ H(K) & -G(K)H(K) \end{pmatrix} \quad , \tag{1}$$

was carried out. In the expression (1) F, G, and H are analytic functions (except possibly at K = 0) of the variable $K = (k^2 - \alpha^2)^{\frac{1}{2}}$, where α is a complex variable and k a constant with positive real and imaginary parts. The branch of the square root is chosen such that K = k at $\alpha = 0$, with the branch cuts C and C' lying along the half-lines $\alpha = k + \delta$, and $\alpha = -k - \delta$, $\delta \geq 0$, respectively. It was shown in [1] that provided F, G and H do not have any zeros in the cut α-plane and G(K) = - G(-K) then the matrix (1) could be factorised in the form

$$\underset{\sim}{A}(\alpha) = \underset{\sim}{U}(\alpha)\, \underset{\sim}{L}^{-1}(\alpha) \quad ,$$

where $\underset{\sim}{U}(\alpha)$ and $\underset{\sim}{L}(\alpha)$ are non—singular matrices whose elements are analytic for Im(α) > - Im(k), and Im(α) < Im(k), respectively.

The crux of the technique of factorisation depended on being able to assume $\underset{\sim}{U}(\alpha)$ to be analytic everywhere except along the branch cut C through α = -k whilst $\underset{\sim}{L}(\alpha)$ to be analytic everywhere except along the branch cut C' through α= k, and then to show that

$$\underset{\sim+}{A}(\alpha)\underset{\sim-}{A}^{-1}(\alpha) = \begin{pmatrix} 0 & G(\alpha) \\ h(\alpha) & 0 \end{pmatrix} \quad , \tag{2}$$

where g(α), h(α) are specific functions, and where the suffices ± denote values evaluated on the upper side and lower side of the branch cut C : $\alpha = - k - \delta$, $\delta \geq 0$.

Professor J. Boersma in his referee report of [1], asked the question as to whether (1) is the most general matrix, with the same branch cuts, for which the matrix product $\underset{\sim}{A}_{+}(\alpha)\underset{\sim}{A}_{-}^{-1}(\alpha)$ takes the form (2). He conjectured that it would not be. In this note we confirm his conjecture, and give the most general form of the class of (2x2)-matrices which produce zeros in the diagonal for the Hilbert problem.

We shall show that the most general form is:

$$\underset{\sim}{A}(\alpha)=\begin{pmatrix} a_{11}(\alpha) & a_{11}(\alpha)\{F_1(\alpha) \;+\; (k^2-\alpha^2)^{-\frac{1}{2}}F_2(\alpha)\} \\ a_{21}(\alpha) & a_{21}(\alpha)\{F_1(\alpha) \;-\; (k^2-\alpha^2)^{-\frac{1}{2}}F_2(\alpha)\} \end{pmatrix}, \tag{3}$$

with $a_{11}(\alpha)a_{12}(\alpha)F_2(\alpha) \neq 0$ in the cut plane, where $a_{11}(\alpha)$, $a_{21}(\alpha)$ are analytic functions in the cut plane, (with branch cuts C and C'), and $F_1(a)$ and $F_2(\alpha)$ are analytic in the entire α-plane except possibly along the branch cut C'. If further $\underset{\sim}{A}(\alpha)=\underset{\sim}{A}(-\alpha)$ then $F_1(\alpha)=E_1(\alpha)$, $F_2(\alpha)=E_2(\alpha)$ where $E_1(\alpha)$ and $E_2(\alpha)$are analytic in the entire a-plane.

<u>Derivation of the general form (3)</u>

Consider the matrix

$$\underset{\sim}{A}(\alpha)=\begin{pmatrix} a_{11}(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha) \end{pmatrix},$$

where $a_{11}(\alpha)$, $a_{12}(\alpha)$, $a_{21}(\alpha)$,$a_{22}(\alpha)$ are supposed to be analytic functions in the cut α-plane, and det $\underset{\sim}{A}(\alpha) \neq 0$ in the cut a-plane.

Then

$$\underset{\sim}{A}_{+}(\alpha)\,\underset{\sim}{A}_{-}^{-1}(\alpha\alpha = \frac{1}{\det\underset{\sim}{A}(\alpha)}\begin{pmatrix} a_{11}^{+}a_{22}^{-} - a_{12}^{+}a_{21}^{-} & a_{12}^{+}a_{11}^{-} \quad a_{11}^{+}a_{12}^{-} \\ a_{21}^{+}a_{22}^{-} - a_{22}^{+}a_{21}^{-} & a_{22}^{+}a_{11}^{-} \quad a_{21}^{+}a_{12}^{-} \end{pmatrix}, \tag{4}$$

where det $\underset{\sim}{A}\_(\alpha)=(a_{11}^{-}a_{22}^{-} - a_{12}^{-}a_{21}^{-}) \neq 0$.In order that (4) should have the form (2), i,e, zeros on the diagonal, we require

$$a_{11}^{+}a_{22}^{-} = a_{12}^{+}a_{21}^{-}, \quad \text{and} \quad a_{22}^{+}a_{11}^{-} = a_{21}^{+}a_{12}^{-},$$

or, ignoring the trivial situation where $a_{11}^{\pm} \equiv 0$, and/or $a_{21}^{\pm}(\alpha) \equiv 0$,

$$\left(\frac{a_{12}}{a_{11}}\right)^{+} - \left(\frac{a_{22}}{a_{21}}\right)^{-} = 0 \tag{5}$$

$$\left(\frac{a_{22}}{a_{21}}\right)^{+} - \left(\frac{a_{12}}{a_{11}}\right)^{-} = 0 \quad , \tag{6}$$

where $a_{21}(\alpha) \neq 0$, and $a_{11}(\alpha) \neq 0$ on C .

Adding and subtracting (5) and (6) gives

$$\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{+}-\left(\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}\right)^{-} = 0, \quad \alpha\in c \tag{7}$$

$$\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)^{+}+\left(\frac{a_{12}}{a_{11}}\ \frac{a_{22}}{a_{21}}\right)^{-} = 0 . \quad \alpha\in c \tag{8}$$

Using the fact that $\left[\left(k^2-\alpha^2\right)^{\frac{1}{2}}\right]^{\pm} = \pm\left|k^2-\alpha^2\right|^{\frac{1}{2}}$ we can rewrite (8) in the form

$$\left[\left(k^2-\alpha^2\right)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{+}-\left[\left(k^2-\alpha^2\right)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}\right)\right]^{-}=0,\alpha\in C \tag{9}$$

Now provided $a_{11}(\alpha)$ and $a_{21}(\alpha)$ are non-zero in the cut plane and satisfy the conditions

$$\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}=0\left[\left(k^2-\alpha^2\right)^{\mu}\right], \text{ as } \alpha\to\pm k,\ 0\le\mu<1,$$

$$\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}=0\left[\left(k^2-\alpha^2\right)^{v-\frac{1}{2}}\right], \text{ as } \alpha\to\pm k, 0\le\upsilon<1,$$

then the most general solution of (7) and (9) which has no pole singularity at $\alpha=\pm k$ and no other singularities in the cut plane except a branch cut along C' is given by (Muskhelishivili [2])

$$\frac{a_{12}}{a_{11}}+\frac{a_{22}}{a_{21}}=2F_1(\alpha) \tag{10}$$

and

$$\frac{a_{12}}{a_{11}}-\frac{a_{22}}{a_{21}}=2F_2(\alpha)(k^2-a^2)^{-\frac{1}{2}} \tag{11}$$

respectively, where $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire plane except possibly along the branch cut C'. Adding and subtracting (10) and (11) gives

$$a_{12}(\alpha)=a_{11}(\alpha)\{F_1(\alpha)+F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\},$$

$$a_{22}(\alpha)=a_{21}(\alpha)\{F_1(\alpha)-F_2(\alpha)(k^2-\alpha^2)^{-\frac{1}{2}}\}.$$

If $\underset{\sim}{A}(a)=\underset{\sim}{A}(-\alpha)$ then $F_1(\alpha)$ and $F_2(\alpha)$ are analytic in the entire complex plane, as the following analysis will show.

If $\underset{\sim}{A}(\alpha) = \underset{\sim}{A}(-\alpha)$ then $a_{ij}(\alpha) = a_{ij}(-\alpha)$, i,j = 1,2, and in an exactly analogous way one obtains similar equations to (7) and (9) on carrying out evaluations on the branch cut C':

$$\left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^{+} - \left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^{-} = 0, \ \alpha \in C', \tag{7'}$$

$$\left[(k^2 - \alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^{+} - \left[(k^2 - \alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^{-} = 0, \ \alpha \in C', \tag{9'}$$

where now ± corresponds to the lower and upper side of C', respectively. Adding (7) to (7') and (9) to (9') gives

$$\left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^{+} - \left(\frac{a_{12}}{a_{11}} + \frac{a_{22}}{a_{21}}\right)^{-} = 0, \ \alpha \in C \cup C', \tag{7''}$$

$$\left[(k^2 - \alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^{+} - \left[(k^2 - \alpha^2)^{\frac{1}{2}}\left(\frac{a_{12}}{a_{11}} - \frac{a_{22}}{a_{21}}\right)\right]^{-} = 0, \ \alpha \in C \cup C'. \tag{9''}$$

Thus the most general solution of (7") and (9") which has no pole singularity at $\alpha = \pm k$ and no other singularities in the cut $\alpha-$plane is given by:

$$a_{12}(\alpha) = a_{11}(\alpha)\,\{E_1(\alpha) + E_2(\alpha)\,(k^2 - \alpha^2)^{-\frac{1}{2}}\},$$

$$a_{22}(\alpha) = a_{21}(\alpha)\,\{E_1(\alpha) - E_2(\alpha)\,(k^2 - \alpha^2)^{-\frac{1}{2}}\},$$

where $E_1(\alpha)$ and $E_2(\alpha)$ are analytic in the entire $\alpha-$plane.

If in particular we let $a_{11}(\alpha) = F(K)$, $a_{21}(\alpha) = H(K)$, $E_1(\alpha) = 0$, and $E_2(\alpha) = KG(K)$, (the condition $G(K) = -G(-K)$ ensures that KG(K) is an entire function) we obtain the special form considered in [1] .

Following the procedure outlined in Rawlins and Williams [1] a particular factorisation of the matrix (3), which will be useful in applications, is given by $\underset{\sim}{A}(\alpha) = \underset{\sim}{U}^{(0)}(c)\,[\underset{\sim}{L}^{(0)}(\alpha)]^{-1}$ where

$$\underset{\sim}{U}^{(0)}(\alpha) = \begin{bmatrix} [W_1(\alpha)]^{\frac{1}{2}}[W_2(\alpha)]^{\frac{1}{2}} & (k+\alpha)^{\frac{1}{2}}[W_1(\alpha)]^{\frac{1}{2}}[W_2(\alpha)]^{\frac{1}{2}} \\ [W_1(\alpha)]^{\frac{1}{2}}/[W_2(\alpha)]^{\frac{1}{2}} & -(k+\alpha)^{\frac{1}{2}}[W_1(\alpha)]^{\frac{1}{2}}/[W_2(\alpha)]^{\frac{1}{2}} \end{bmatrix},$$

$W_1(\alpha)$ and $W_2(\alpha)$ are solutions of the standard Hilbert problems on the half-line C:

$$[\ell n W_1(\alpha)]^+ - [\ell n W_2(\alpha)]^- = \ell n[g(\alpha)h(\alpha)] ,$$
$$[(k+\alpha)^{\frac{1}{2}}\ell n W_2(\alpha)]^+ - [(k+\alpha)^{\frac{1}{2}}\ell n W_2(\alpha)]^- = i|k+\alpha|^{\frac{1}{2}}\ell n[g(\alpha)/h(\alpha)],$$

Where

$$g(\alpha) = (a_{12}^+(\alpha)a_{11}^-(\alpha) - a_{11}^+(\alpha)\,a_{12}^-(\alpha))/\det \underset{\sim}{A}_-(\alpha) = a_{11}^+(\alpha)/a_{21}^-(\alpha) ,$$

$$h(\alpha) = (a_{21}^+(\alpha)a_{22}^-(\alpha) - a_{22}^+(\alpha)\,a_{22}^-(\alpha))/\det \underset{\sim}{A}_-(\alpha) = a_{21}^+(\alpha)/a_{11}^-(\alpha) .$$

The set of solutions for $W_1(\alpha), W_2(\alpha)$ is further restricted by the requirement that the factor matrix $\underset{\sim}{L}^{(0)}(\alpha)$ is non-singular at $\alpha = -k$ and its elements should be analytic in the region $\mathrm{Im}(\alpha) < \mathrm{Im}(k)$. It is interesting to note that the functions $F_1(\alpha)$, $F_2(\alpha)$ have dropped out completely. This means that for all matrices of the form (3) the factorisation problem reduces to the same Hilbert problem!

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