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Abstract

The Dirichlet and impedance boundary value problems for the Helmholtz equation in a half-plane with bounded continuous boundary data are studied. For the Dirichlet problem the solution can be constructed explicitly. We point out that, for wavenumbers $k > 0$, the solution, although it satisfies a limiting absorption principle, may increase in magnitude with distance from the boundary. Using the explicit solution we propose a novel radiation condition which we utilise in formulating the impedance boundary value problem. By reformulating this problem as a boundary integral equation we prove uniqueness and existence of solution for a certain range of admissable impedance boundary data.

1 Introduction

We give in this paper a rigorous account of the Dirichlet and impedance boundary value problems for the Helmholtz equation in the half-plane U = $\{(x_1,x_2) \in \mathrm{R}^2 : x_2 > 0\}$, with arbitrary bounded and continuous boundary data.

The Dirichlet problem is much the easier in that, using the Dirichlet Green's function for the half-plane, a solution can be written down explicitly as a double-layer potential on the boundary $\Gamma = \{(x_1, 0) : x_1 \in \mathbf{R}\}$, with density the given boundary data. In the case $k > 0$ we point out that, although this solution is the physically correct one, in that it is the unique solution satisfying the limiting absorption principle, the solution radiated from the boundary does not necessarily decay or remain bounded but may grow algebraically at a rate not exceeding $h^1/^2$, where h is the distance from the boundary. We construct a solution achieving this growth rate.

This preliminary study of the Dirichlet problem is of assistance in formulating the impedance boundary value problem, which has been studied previously as a model of outdoor sound propagation [9, 4, 2], Specifically, as a radiation condition for the impedance problem, we suppose that in some half-plane $U_h = \{(x_1, x_2) \in \mathrm{R}^2 :> x_2 > h\}$, with $h > 0$, the solution u can be written as a double-layer potential on the boundary $\Gamma_h = \{(x_1,h) : x_1 \in \mathbf{R}\}$, with some bounded continuous density, so that u satisfies a Dirichlet problem in the half-plane U_h. It is anticipated that this radiation condition, which appears to be novel and is a generalisation of the usual radiation condition for plane wave scattering by a one-dimensional diffraction grating [11, 10], will prove useful in formulating a wider range of diffraction problems, e.g. plane wave scattering by infinite rough surfaces [8, 6].

Here we show that this radiation condition is sufficiently strong to establish a form of Green's representation theorem, enabling the reformulation of the impedance boundary value problem as a second kind boundary integral equation. For admittance boundary

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data β for which $\|\beta - 1\|_\infty$ is sufficiently small, the integral operator in the equation is a contraction mapping, and existence, uniqueness, and stability results for the impedance boundary value problem follow by a standard Neumann's series argument (see the final Theorem 4.4).

2 Notation and Preliminaries

Throughout, $x = (x_1, x_2)$, $y = (y_1, y_2)$ will denote points in R^2. For $h \geq 0$, U_h will denote the half-plane, $U_h = \{x : x_2 > h\}$ and Γ_h its boundary, $\Gamma_h = \{x : x_2 = h\}$. We will abbreviate U_0 and Γ_0 by U and Γ, respectively. For all $x \in U_h$, x'_h will denote the image of x in Γ_h, i.e. $x'_h := (x_1, 2h - x_2)$. We abbreviate x'_0 by x'.

For the most part our function space notation is standard. For $S \subset \mathbf{R}^2$, $C(S)$ will denote the set of functions continuous on S, and $BC(S)$ the set of functions bounded and continuous on S. The set $BC(S)$ with the normal vector space operations and the supremum norm, $\|\psi\|_\infty := \sup_{x\in S} |\psi(x)|$, forms a Banach space.

For $u \in C(U)$ and $h > 0$, define $u_h \in C(\mathbf{R})$ by $u_h(s) := u((s,h))$, $s \in \mathbf{R}$, so that u_h is the restriction of u to Γ_h If $u \in C(\overline{U})$ then we can define u_0 by the same formula with h=0. If $u \in C^1(U)$, define also $\partial u_h / \partial h \in C(\mathrm{R})$ by $\partial u_h / \partial h(s) = \partial u((s,h)) / \partial h$, $s \in \mathrm{R}$, so that $\partial u_h / \partial h$ is the restriction of $\partial u / \partial x_2$ to Γ_h.

Many of the equations presented can be written compactly using a convolution notation. For $\varphi \in L_1(\mathrm{R})$ and $\psi \in L_p(\mathrm{R})$ define $\varphi * \psi$ by

$$(1) \qquad \varphi * \psi(s) := \int_{-\infty}^{+\infty} \varphi(s-t)\psi(t)\,dt.$$

From Young's Theorem, $\varphi * \psi(s)$, defined by (1), exists for almost all $s \in \mathbf{R}$, and $\varphi * \psi \in L_p(\mathbf{R})$ with

$$(2) \qquad \|\varphi * \psi\|_p \leq \|\varphi\|_1 \|\psi\|_p.$$

For $p = \infty$ we have that $\varphi * \psi(s)$ is well-defined for every $s \in \mathbf{R}$ and that $\varphi * \psi \in BC(\mathbf{R})$.

For $\{\psi_n\} \subset BC(\mathbf{R})$, $\psi \in BC(\mathbf{R})$, say that ψ_n converges strictly to ψ and write $\psi_n \xrightarrow{s} \psi$ if $\sup_n \|\psi_n\|_\infty < \infty$ and $\psi_n(s) \to \psi(s)$ uniformly on finite intervals of R. Then [1], if $k \in L_1(\mathbf{R})$ and $\psi_n \xrightarrow{s} \psi$, then $k * \psi_n \xrightarrow{s} k * \psi$. More generally,

$$(3) \qquad \|K_n - K\|1 \to 0, \psi_n \xrightarrow{s} \psi \Rightarrow K_n * \psi_n \xrightarrow{s} K * \psi.$$

We introduce a few further notations. For $x \in \mathrm{R}^2$ and $A > 0$ let $B_A(x)$ denote the open ball $B_A(x) := \{y \in \mathrm{R}^2 : |y - x| < A\}$. Let

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x, y \in R^2, \ x \neq y,$$

so that Φ is the standard fundamental solution of the Helmholtz equation in R^2.

3 The Dirichlet Problem

We first consider the following Dirichlet boundary value problem:

BVP1. Given $f \in BC(\mathbf{R})$ and $k \in \mathbf{C}$ *with* $Im k \geq 0$, $Re k > 0$, *find* $u \in C(\overline{U}) \cap C^2(U)$ *satisfying*

(i) the Helmholtz equation, $\Delta u + k^2 u = 0$ *in* U;

(ii) for some $a \in \mathbf{R}$,

(4) $$\sup_{x\in U} |(1+x_2)^a u(x)| < \infty;$$

(iii) $u = f$ *on* Γ.

REMARK 3.1. Note that if $u \in C^2(U)$ satisfies (i) and (ii) then, by standard regularity arguments, $u \in C^\infty(U)$ and a similar bound to (4) holds for all the derivatives of u, except in a neighbourhood of Γ. In particular, for all $h > 0$,

(5) $$\sup_{x\in U_h} |(1+x_2)^a \operatorname{grad} u(x)| < \infty.$$

The above boundary value statement contains no radiation condition and is not uniquely solvable when $k > 0$: for example $u(x) = \sin(kx)$ satisfies BVPl with $f = 0$ when $k > 0$; though not when $\operatorname{Im} k > 0$ for then (4) is violated.

To write down a particular solution of BVPl we introduce the Dirichlet Green's function, $G_{D,h,}$ for the half-plane $U_{h.}$ For $h > 0$ define

(6) $$G_{D,h}(x,y) := \Phi(x,y) - \Phi(x'_h,y) \quad x,y \in \overline{U}_h \; x \neq y.$$

For $\operatorname{Im} k > 0$ (for which $G_{D,h}(x, y)$ decays exponentially as $|x-y| \to \infty$) we can obtain a form of Green's representation theorem for u, the solution of BVPl (cf. [7]): applying Green's second theorem to u and $G_{D,h}(x,.)$ in the region $U_h \cap B_R(0) \setminus B_\epsilon(x)$, and letting $\in \to 0$ and $R \to \infty$ (and noting, from (4) and (5), that u and grad u have at most algebraic growth at infinity), we obtain that

$$u(x) = \int_{\Gamma h} \frac{\partial G_{D,h}(x,y)}{\partial y_2} u(y)ds(y), x \in U_{h,} ,$$

(7) $$= 2\int_{\Gamma h} \frac{\partial \Phi(x,y)}{\partial y_2} u(y)ds(y), x \in U_{h,} .$$

Defining, for $h > 0$,

$$K_h(s) := 2\frac{\partial \Phi((s,h),y)}{\partial y_2}\bigg|_{y=0}, s \in \mathbf{R},$$

(8) $$= \frac{ihk\, H_1^{(1)}(k\sqrt{s^2+h^2})}{2\sqrt{s^2+h^2}}, s \in \mathbf{R},$$

(7) can be written more compactly as

(9) $$u_H = \kappa_{H-h} * u_h, \; H > h.$$

From standard asymptotic properties of the Hankel function it is easy to establish that, for $0 < h \leq 1$ and some constant $C > 0$,

(10) $$|\kappa_h(s)| \leq \begin{cases} C\dfrac{h}{s^2+h^{2,}} & |s| \leq 1, \\ C|s|^{-3/2}, & |s| \geq 1, \end{cases}$$

while, for $h \geq 1$,

(11) $$|\kappa_h(s)| \leq \frac{Chexp(-Im\, kh)}{(s^2+h^2)^{3/4}}, s \in R.$$

Since the Hankel function, $H_1^{(1)}(z)$, is continuous in the quadrant Im $z \geq 0$, **R**e $z > 0$, it follows that, for $h > 0$, $K_h \in L_1(\mathbf{R})$, and depends continuously in norm on h, and

(12) $$\|\kappa_h\|_1 = O(1), h \to 0, \ \|\kappa_h\|_1 = O(h^{1/2} exp(-Im\, k\, h)),\ h \to \infty.$$

Since $u_h \xrightarrow{s} u_0 = f \ as \ h \to 0$, it follows from (9) and (3) that

(13) $$u_h = K_h * f, \ h > 0,$$
i.e. that

(14) $$u(x) = 2\int_\Gamma \frac{\partial \Phi(x,y)}{\partial y_2} f(y) ds(y), \ x \in U.$$

We have shown that if u satisfies BVPl and Im $k > 0$ then u is given by (14). The following converse result holds for all k with Im $k \geq 0$, Re $k > 0$.

THEOREM 3.1. *If* $f \in BC(\mathbf{R})$ *then u, defined by (14), satisfies BVP1.*

Proof. We have observed already that K_h depends continuously in $L_1(\mathbf{R})$ on h. Thus, and from (13), (2), and (12), we have that $u \in C(U)$ and, for some constant C independent of f,

(15) $$\sup_{x \in U} \left|(1 + x_2)^{-1/2} u(x)\right| \leq C\|f\|_\infty.$$

Further, from (10) and the standard jump relation for double-layer potentials [7], it is easy to see that u can be continuously extended from U to $\overline{U}$, with limiting value $u_0 = f$.

It remains to show that $u \in C^2(U)$ and satisfies the Helmholtz equation. But choose $\{f_n\} \subset BC(\mathbf{R})$ such that each f_n is compactly supported and $f_n \xrightarrow{s} f$, and define $u^{(n)}$ by (14) with f replaced by f_n, so that

(16) $$u_h^{(n)} = \kappa_h * f_n, \ h > 0.$$

Then, clearly, $u^{(n)}$, a standard double-layer potential, satisfies $u^{(n)} \in C^2(U)$ and $\Delta\, u^{(n)} + k^2\, u^{(n)} = 0$ in U. Further, using (10) and (11), we can see that $u^{(n)}$ converges to u uniformly on compact subsets of u, so that also $u \in C^2(U)$ and $\Delta u + k^2\, u = 0$ in U. □

We have shown in the case Im $k > 0$ that BVPl has precisely one solution, given by (14). In the case $k > 0$ in which (14) is not the unique solution of BVPl, it is sensible to select it as the "physically correct" solution since it satisfies the *limiting absorption principle* given in the next theorem. Temporarily, for the duration of this theorem, let $u^{(\lambda)}$ denote the solution of BVPl given by (14) when $k = \lambda$.

THEOREM 3.2. *For* $k > 0$ and *all* $x \in U$, $u^{(k+i\epsilon)}(x) \to u^{(k)}(x)$ *as* $\epsilon \to 0^+$.

Proof. Temporarily denote K_h by $K_h^{(k)}$ to indicate its dependence on k. Then $K_h^{(k)} \in L_1(\mathbf{R})$ depends continuously on k in Im $k \geq 0$, Re $k > 0$ (note that (10) and (11) hold with the constant C independent of k provided that k is restricted to a compact subset of the first quadrant). But, from (13), $\left\|u_h^{(k+i\epsilon)} - u_h^{(k)}\right\|_\infty \leq \left\|\kappa_h^{(k+i\epsilon)} - \kappa_h^{(k)}\right\|_1 \|f\|_\infty$, and the result follows. □

Although it satisfies the above limiting absorption condition, the solution (14) for $k > 0$ does not have all the characteristics we associate with a radiating wave. Specifically, the bound (15) suggests that $u(x)$ may increase in magnitude as $x_2 \to \infty$. We now show that the bound (15) is sharp and construct boundary data f such that the solution u satisfies, for some $C > 0$, $\|u_h\|_\infty \geq Ch^{1/2}$, for all $h \in$ N.

For $h > 0$ define $g_h \in$ BC(**R**) with $\|g_h\|_\infty = 1$ by $g_h(s) = \overline{K_h(s)}/|K_h(s)|$, $s \in \mathbf{R}$. Then

$$|\kappa_h * g_h(0)| = \int_{-\infty}^{+\infty} |K_h(s)| ds,$$

and, since $|H_1^{(1)}(r)| \sim \sqrt{\frac{2}{\pi r}}$, $r \to +\infty$,, it follows from (8) that, as $h \to \infty$,

(17) $$|\kappa_h * g_h(0)| \sim \sqrt{\frac{k}{2\pi}}\, h \int_{-\infty}^{+\infty} \frac{ds}{(s^2+h^2)^{3/4}} = ch^{1/2},$$

where $c := \sqrt{\frac{k}{2\pi}} \int_{-\infty}^{+\infty} (1+t^2)^{-3/4}\, dt$. For $a > 0$ deiine $X_a \in BC(\mathbf{R})$ by

$$x_\alpha(s) = \begin{cases} 1, & |s| \le a, \\ 1+a-|s|, a < |s| < a+1, \\ 0, & |s| \ge a+1, \end{cases}$$

and, for $n \in$ N, define $G_n \in BC(\mathbf{R})$ by $G_n = g_n X a_n$, where $a_n > 0$ is chosen large enough so that

(18) $$\int_{R/[-a_n, a_n]} |\kappa_n(s)| ds \le \frac{1}{4} cn^{1/2}.$$

Clearly, each G_n has compact support and

(19) $$|\kappa_n * G_n(0)| \ge \frac{1}{2} cn^{1/2},$$

for all sufficiently large n.

Now define $f_n \in BC(\mathbf{R})$ by $f_n(s) = G_n(s - b_n)$, s $\in$ **R**, where the constants $0 < b_1 < b_2 < \ldots$ are chosen so that the supports of the functions f_n do not overlap. Define $f \in BC(\mathbf{R})$ by $f = \sum_{n=1}^{\infty} f_n$. (Note that the method of construction makes $\|f\|_\infty = 1$.) Then u, defined by (14) or (13) with this choice of f, satisfies (4) only for $\alpha \le -\frac{1}{2}$. For u_n, the restriction of u to Γ_n, satisfies

$$\begin{aligned} \|u_n\|_\infty = \|\kappa_n * f\|_\infty &\ge |\kappa_n * f(b_n)| \\ &\ge |\kappa_n * f(b_n)| - |\kappa_n * (f - f_n)(b_n)| \\ &= |\kappa_n * G_n(0)| - |\kappa_n * (f - f_n)(b_n)|. \end{aligned}$$

Since $\|f - f_n\|_\infty = 1$ and the support of $f - f_n$ lies outside the interval $[b_n - a_n, b_n + a_n]$, we have

$$|\kappa_n * (f - f_n)(b_n)| \le \int_{R \setminus [b_n - a_n, b_n + a_n]} |\kappa_n(b_n - t)| dt \le \frac{1}{4} cn^{1/2},$$

by (18). Thus, and by (19), $\|u_n\|_\infty \ge \frac{1}{4} cn^{1/2}$ for all sufficiently large $n \in$ N.

We finish this section by pointing out that an expression for the solution (14) as a discrete spectrum of upward propagating plane waves and evanescent waves can be given in the case when f is quasi-periodic (as defined below).

Let $\mathcal{F}$ denote the operation of Fourier transformation on R, defined, for $\psi \in L_1(\mathbf{R})$, by

$$\mathcal{F}\psi\xi) = \int_{-\infty}^{+\infty} \psi(s) e^{-is} ds, \xi \in \mathrm{R}.$$

From (8) and standard tables of Fourier transforms we can calculate $\hat{K}_h := \mathcal{F}\kappa_h$ as

(20) $$\hat{\kappa}_h(\xi) = exp(ih\sqrt{k^2-\xi^2}),\ \ \xi \in R,$$

where Re $\sqrt{k^2-\xi^2}),$ Im $\sqrt{k^2-\xi^2}),\ \geq 0.$

Given $k > 0$, we say that ψ, defined on $\overline{U}$ or on some subset of $\overline{U}$, is quasi-periodic with index α ∈ R and period L if $\psi(x+Le_1) = \psi(x)e^{iakL},$ for every x, where $e_1 = (1,0)$ is a unit vector in the x_1-direction. Clearly ψ is quasi-periodic if and only if $\psi(x)e^{-iakx_1}$ is periodic in x_1.

Suppose now that $f \in BC(\mathbf{R})$ is quasi-periodic with index a and period L and u is defined by (14). Then $f(s)e^{-iaks}$ has a Fourier series convergent in at least an L_2 sense,

$$f(s)e^{-i\alpha ks} = \sum_{m\in z} c_m e^{2\pi ims/L},$$

and, from (13), for $h > 0$,

$$u_h(s) = \sum_{m\in z} c_m \int_{-\infty}^{+\infty} \kappa_h(s-t)exp(2\pi imt/L + i\alpha kt)dt$$

$$= \sum_{m\in z} c_m \hat{\kappa}_h(2\pi m/L + ak)exp(2\pi ims/L + iaks),\ s \in R.$$

Thus, from (20),

(21) $$u(x) = \sum_{m\in z} c_m exp(i(\alpha_m x_1 + \beta_m x_2)), u \in U,$$

where

(22) $$\alpha_m := 2\pi m/L + \alpha k,\ \beta_m := \sqrt{k^2 - \alpha_m^2},$$

with Im $\beta_m \geq 0$. The series (21) is convergent absolutely and uniformly on compact subsets of U.

Conversely, if the series (21) is uniformly convergent in $\overline{U}$ (in which case it is also absolutely convergent in U) then it is easy to see that the above derivation can be reversed to write u, given by (21), in the form (13), with $f = u_0$ continuous and quasi-periodic.

While, for arbitrary $f \in BC(\mathbf{R})$, (4) need not hold for $a > -\frac{1}{2}$, we see from (21) that, if f is continuous and quasi-periodic, then (4) holds with $a = 0$, i.e. u is bounded in $\overline{U}$.

4 The Impedance Boundary Value Problem

We consider next the boundary value problem in the half-plane U with the impedance boundary condition

(23) $$\frac{\partial u}{\partial x_2} + ik\beta u = f,$$

on Γ. We consider in this section only the case $k > 0$, for which a radiation condition is required. To obtain a radiation condition we point out that, in each half-plane U_h, u satisfies a Dirichlet problem with boundary data u_h. It makes sense then to require that, for some $h > 0$ and $\varphi \in BC(\mathbf{R})$,

(24) $$u(x) = \int_{\Gamma_h} \frac{\partial \Phi(x,y)}{\partial y_2} \varphi(y)ds(y),\quad x \in U_h,$$

since, as shown in Section 3, with the choice $\varphi = u_h$, (24) is the unique solution of the Dirichlet problem in U_h satisfying the limiting absorption principle given in Theorem 3.2.

The radiation condition (24) is a generalisation of the usual radiation condition utilised in the study of plane wave diffraction by one-dimensional periodic gratings [11, 10], when u is quasi-periodic. In the case when u is quasi-periodic, the solution (23) can be rewritten (see (21)) to show that, for some set of coefficients $\{c_m : m \in Z\}$,

$$u(x) = \sum_{m\in Z} c_m \exp(i(\alpha_m x_1 + \beta_m x_2)),\ u \in U_{h,} \tag{25}$$

with α_m, β_m given by (22), which is precisely the usual radiation condition. Conversely, if the radiation condition (25) holds for some $h = h^* > 0$, then the series (25) converges uniformly in U_h, for all $h > h^*$ and (see Section 3) (24) holds, with φ quasiperiodic, for all $h > h^*$.

Let $\mathcal{R}(U) := \{u \in C(\overline{U}) \cap C^2(U) : \partial u/\partial x_2 \in C(\overline{U})\}$. The following is the impedance boundary value problem that will be considered:

BVP2. *Given* $f, \beta \in BC(\mathbf{R})$ *and* $k > 0$, *find* $u \in \mathcal{R}(U)$ *satisfying*

(i) $\Delta u + k^2 u = 0$ in U;

(ii) *for some* $\alpha \in \mathbf{R}$,

$$\sup_{x\in U} (1 + x_2)^{\alpha} (|u(x)| + |\partial u(x)/\partial x_2|) < \infty; \tag{26}$$

(iii) $\partial u(x)/\partial x_2 + ik\beta(x)u(x) = f(x)$, *for all* $x \in \Gamma$;

(iv) *the radiation condition (24), for some* $h > 0$ *and* $\varphi \in BC(\mathbf{R})$.

4.1 An Integral Equation Formulation

To prove uniqueness and existence of solution of BVP2, and as a tool for numerical computation, we reformulate BVP2 as a boundary integral equation. The fundamental solution of the Helmholtz equation which satisfies BVP2 with $f \equiv 0$ and $\beta \equiv 1$ (and a Dirac delta function inhomogeneity in the Helmholtz equation) is given by [5]

$$G(x,y) = \Phi(x,y) + \Phi(x,y') + \hat{p}(x - y'), \tag{27}$$

where

$$\hat{P}(x) := \frac{e^{ik|x|}}{\pi} \int_0^\infty \frac{t^{-1/2} e^{-k|x|t} (1 + \gamma(1 + it))}{\sqrt{t - 2i}\,(t - i(1 + \gamma))^2} dt,\quad x \in \overline{U}, \tag{28}$$

with $\gamma = x_2/|x|$.

It is shown in [5] that $\hat{P} \in C(\overline{U})$ and it is easy to see that $\hat{P} \in C^\infty(\overline{U} \setminus \{0\})$ and satisfies the Sommerfeld radiation and boundedness conditions in $\overline{U}$ [2, Lemma 3.6.5]. Further [6], given $C > 0$,

$$\mathrm{grad}_y G(x,y),\ G(x,y) = 0(|x-y|^{-3/2}) \mathrm{as}\, |x-y| \to \infty, \tag{29}$$

uniformly in $x, y \in \overline{U}$, with $0 \le x_2, y_2 \le C$.

This rapid rate of decrease in (29) is very important in the arguments which follow and holds only provided the vertical coordinates, x_2 and y_2, are restricted as indicated. (If x and y are unrestricted then $G(x,y) = O(|x-y|^{-1/2})$ *as* $|x-y| \to \infty$, the same behaviour as that of Φ.) In physical terms the rapid rate of decay (29) is due to the energy-absorbing nature of the boundary condition (23) when Re $\beta > 0$.

To derive the boundary integral equation, suppose that u satisfies BVP2 (in particular (24) for some $h > 0$) and take $x \in U$. Choose h_1, h_2 such that $0 < h_1 < x_2 < h_2$ and $h_2 > h$, and apply Green's second theorem in the bounded region $S_{A,\in} := \{x \in U_{h1} \setminus \overline{U}_{h2}: |x_1| < A\} \setminus \overline{B_\in(x)}$ to $G(x,.)$ and u to obtain

$$0 = \int_{\partial S_{A,c}} \left(\frac{\partial u}{\partial n}(y) G(x,y) - u(y) \frac{\partial G(x,y)}{\partial n(y)} \right) ds(y),$$

where n is the outward-directed normal on $\partial S_{A,\in}$. Letting $\in \to 0$ and $A \to \infty$ (note that u and (see Remark 3.1) grad u are bounded in $U_{h,1} \setminus U_{h\,2}$ so that the integrals over the vertical sides of $\partial S_{A,\in}$ vanish as $A \to \infty$) we obtain that

$$u(x) = \int_{\Gamma_{h,1} \cup \Gamma_{h,2}} \left(\frac{\partial u}{\partial n}(y) G(x,y) - u(y) \frac{\partial G(x,y)}{\partial n(y)} \right) ds(y), \tag{30}$$

Note that (26) and (29) ensure that the integrand in (30) is absolutely integrable and, since also $G(x,.) \in C^\infty(\overline{U} \setminus \{x\})$ and $u \in \mathcal{R}(U)$, it follows that the integral over $\Gamma_{h,1}$ in (30) depends continuously on h_1 for $0 \le h_1 < x_2$. Thus we may set $h_1 = 0$ in (30) and, utilising the impedance boundary conditions satisfied by u and G, we obtain that

$$\begin{aligned} u(x) = {} & \int_\Gamma G(x,y)(ik(\beta(y)-1)u(y) - f(y)ds(y) + \\ & \int_{\Gamma_{h,2}} \left(\frac{\partial u}{\partial n}(y) G(x,y) - u(y) \frac{\partial G(x,y)}{\partial n(y)} \right) ds(y). \end{aligned} \tag{31}$$

To complete the derivation we use the radiation condition to show that the integral over Γ_{h_2} vanishes. As in the proof of Theorem 3.1, we choose $\{\varphi_n\} \subset BC(\mathbf{R})$ such that each φ_n is compactly supported and $\varphi_n \xrightarrow{s} \varphi$, and define $u^{(n)}$ by (24) with φ replaced by φ_n. Then, for each n, the double-layer potential $u^{(n)} \in C^2(U_h)$ and satisfies the Helmholtz equation and Sommerfeld radiation and boundedness conditions, so that, applying Green's second theorem to $G(x,.)$ and u in $U_{h2} \cap B_R(0) \subset U_h$, and letting $R \to \infty$ we obtain

$$\int_{\Gamma_{h,2}} \left(\frac{\partial u^{(n)}}{\partial n}(y) G(x,y) - u^{(n)}(y) \frac{\partial G(x,y)}{\partial n(y)} \right) ds(y) = 0. \tag{32}$$

Now, by (15), the functions $u^{(n)}$ are uniformly bounded on $U_h \setminus U_H$ for every $H > h_2$, and therefore so are the functions grad $u^{(n)}$, $n \in \mathrm{N}$, on Γ_{h_2}. Further, $u^{(n)}$ converges to u uniformly on compact subsets of U_h (and therefore so also does grad $u^{(n)}$ converge to grad u). Thus, and bearing in mind (29), it follows that the integral in (32) converges to the same integral with $u^{(n)}$ replaced by u as $n \to \infty$, and thus the intergral over Γ_{h_2} in (31) vanishes and

$$u(x) = \int_\Gamma G(x,y)(ik(\beta(y)-1)u(y) - f(y))ds(y), \quad x \in U. \tag{33}$$

Since, for $y \in \Gamma$, $G(x,y) = 2\Phi(x,y) + \widehat{P}(x-y)$, and $\widehat{P} \in C(\overline{U})$, it is easy to see from standard properties of single-layer potentials, and bearing in mind (29), that the right hand side of (33) is continuous in $\overline{U}$, so that (33) holds also for $x \in \Gamma$. We have shown the following result:

THEOREM 4.1. *If u satisfies BVP2 then*

(34) $$u(x) = \int_\Gamma G(x,y)(ik(\beta(y)-1)u(y) - f(y)ds(y), x \in \overline{U}.$$

To establish the converse result, note that we have already observed that, if u is defined by (34) with $u_0 \in BC(\mathbf{R}\}$ then u $\in C(\overline{U})$. To see most of the rest of the result, note that (34)can be written as

(35) $$u_h = \lambda_h * (ik(\beta-1)u_0 - f),\ h > 0,$$

where $\lambda_h \in L_1(\mathbf{R})$ is defined, for $h \geq 0$, by $\lambda_h(s) = G((s,h),0)$, $s \in \mathbf{R}$. Now, (see [5, equations (21) and (25)]), $\hat{\lambda}_h := \mathcal{F}\lambda_h$ is given by

(36) $$\hat{\lambda}_h(\xi) = \frac{i\exp(ih\sqrt{k^2-\xi^2})}{\sqrt{k^2-\xi^2}+k}, \xi \in \mathrm{R},$$

with $\sqrt{k^2-\xi^2} = i\sqrt{\xi^2-k^2}\ for |\xi| \geq k, so that\ \hat{\lambda}_h = k_h\hat{\lambda}_0, and$

(37) $$\lambda_h = \kappa_h * \lambda_{0,}\ \ h > 0.$$

Thus, from (35),

(38) $$u_h = \kappa_h * \lambda_0 * (ik(\beta-1)u_0 - f) = \kappa_h * u_{o,} h > 0.$$

It follows from Theorem 3.1 that u, defined by (34), satisfies the Dirichlet BVP1 with boundary data $u_0 \in BC(\mathbf{R})$, so that we have shown that u satisfies all the conditions of BVP2, except that $\partial u/\partial x_2 \in C(\overline{U})$ and the impedance boundary condition.

To show the impedance condition we make use of the impedance condition satisfied by G, contained in the relationship [5, equation (67)],

(39) $$\frac{\partial \hat{P}(y)}{\partial y_2} + ikG(0,y) = 0, y \in \overline{U}, y \neq 0.$$

Differentiating (34) and noting that, for $x \in U$, $y \in \Gamma$, using (39),

$$\frac{\partial G(x,y)}{\partial x_2} = \frac{\partial \Phi(x,y)}{\partial x_2} + \frac{\partial \hat{P}(x-y)}{\partial x_2}$$
$$= -2\frac{\partial \Phi(x,y)}{\partial y_2} - ikG(x,y),$$

we obtain, in convolution form, that

$$\partial u_h / \partial h = -(\kappa_h + ik\lambda_h) * (ik(\beta-1)u_0 - f),\ h > 0, ,$$

(40) $$= \kappa_h * (f - ik\beta u_0),\ h > 0,$$

by (37) and (35). Thus, by Theorem 3.1, $\partial u/\partial x_2$ satisfies the Dirichlet BVP1 with boundary data $f - ik\beta u_0$, and so, in particular, $\partial u/\partial x_2 \in C(U)$ and satisfies (23). We have shown the following converse of Theorem 4.1:

THEOREM 4.2. *If u satisfies (34)* and $u_0 \in BC(\mathbf{R})$ *then u satisfies BVP2.*

As part of the proof of the above theorem we have shown that u, given by (34), satisfies BVP1 with boundary data u_0. It follows that u satisfies the bound (15) so that we have also the following result:

COROLLARY 4.1. *If u satisfies BVP2 then, for some constant C > 0 independent of β and f,*

$$\sup_{x \in U} \left|(1+x_2)^{-1/2} u(x)\right| \leq C \left\|u_0\right\|_\infty.$$

4.2 Uniqueness and Existence Results

In a future paper we will establish uniqueness and existence of solution in the general case for BVP2 (see [3] for a proof of existence given uniqueness in the case of L_∞ boundary data). For the present we note that, from Theorems 4.1 and 4.2, BVP2 and (34) are equivalent, and that if u satisfies (34) then u_0 satisfies the boundary integral equation, in operator form,

(41) $$u_0 = F + Ku_0,$$

where $F \in BC(\mathbf{R})$ is denned by $F := \lambda_0 * f$, and $K : BC(\mathbf{R}) \to BC(\mathbf{R})$ is defined by

$$K\psi = ik\lambda_0 * ((\beta - 1)\psi\psi), \in BC(\mathrm{R}).$$

Now, by a standard Neumann series argument [7], (41) has exactly one solution $u_0 \in BC(\mathbf{R})$ if $\|K\| < 1$, and we have $\|u_0\| \le (1 - \|K\|)^{-1} \|F\|_\infty$. Since, by (2), $\|F\|_\infty \le \|\lambda_0\|_1 \|f\|_\infty$ and $\|K\| \le \|\lambda_0\|_1 \|\beta - 1\|_\infty$ have, combining this result with Theorems 4.1 and 4.2 and Corollary 4.1, the following existence and uniqueness result.

THEOREM 4.3. *If* $\|\lambda_0\|_1 \|\beta - 1\|_\infty < 1$ *then BVP2 has exactly one solution and, for some constant* $C > 0$ *independent of* β *and* f,

$$\sup_{x \in \overline{U}} \left|(1 + x_2)^{-1/2} u(x)\right| \le \frac{C\|f\|_\infty}{1 - \|\lambda_0\|_1 \|\beta - 1\|_\infty}.$$

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