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Piecewise Rational Quadratic Interpolation to Monotonic Data

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Abstract

An explicit representation of a piecewise rational quadratic function is developed which produces a monotonic interpolant to given monotonic data. The explicit representation means that the piecewise monotonic interpolant is easily constructed and numerical experiments indicate that the method produces visually pleasing curves. Furthermore, the use of the method is justified by an $0(h^4)$ convergence result.

1. Introduction

The work of this paper is motivated by a method of Fritsch and Carlson (1980) for the construction of a C^1 monotonic piecewise cubic polynomial which interpolates given monotonic data. Fritsch and Carlson use the piecewise cubic Hermite representation and show that the necessary condition that the derivative parameters should all be of a certain constant sign is not sufficient to ensure monotonicity. The Fritsch—Carlson method thus involves a derivative modification process designed so that necessary and sufficient conditions for monotonicity of a piecewise cubic are met.

In this paper we construct a piecewise rational quadratic function for which the necessary derivative condition for monotonicity is also sufficient. We thus have a closed form solution to the monotonic interpolation problem. The application of this piecewise rational quadratic on a monotonic data set gives a C^1 monotonic Interpolant which produces visually pleasing curves and for which an $0(h^4)$ convergence result can be obtained.

2, The Rational Quadratic interpolant

Let $(x_i, f_j.)$, $i = 1,..,n$ be given real data, where $a = X_1 < X_2 < \ldots < x_n = b$ is a partition of the interval $[a,b]$ and $f_i.$, $i = 1,..,n$ is a monotonic set. Thus either

$$f_i. \leq f_{i+1}, \quad i = 1,..,n^{-1} \quad \text{(monotonic increasing)}$$

or (2,1)

$$f_i \geq f_{i+1}, \quad i = 1,\ldots,n^{-1} \text{ (monotonic decreasing).}$$

Following the notation of Fritsch and Carlson (1980), let $h_i. = x_{i+1}. - x_i.$, $\Delta_i = (f_{i+1}, - f_i.)/h_i$ and let $d_i.$, $i = 1,\ldots,n$ denote derivative values given at the points (knots) $x.$, $i = 1,.,.,n$. We seek a monotonic function $s(x) \in C^1[a,b]$ such that $s(x_i) = f.$and $s^{(1)}(x_i) = d_i$, $i = 1, \ldots,n$, where we assume that the derivative values satisfy the necessary conditions for monotonicity, namely

$$d_i. = d_{i+1}. = 0 \text{ for } \Delta_\iota. = 0 \quad ,$$

(2.2)

$$\operatorname{sgn}(d_i) = \operatorname{sgn}(d_{i+1}) = \operatorname{sgn}(\Delta_{i)} \quad \text{for} \quad \Delta_i \neq 0 \quad .$$

(In (2.2) we use the convention that $\operatorname{sgn}(0)$ can equal $\operatorname{sgn}(\Delta_\iota)$.)

The necessary conditions (2.2) are not sufficient to ensure monotonicity of a piecewisecubic function . We thus consider a piecewise rational quadratic function $s(x)$ for which conditions (2.2) are sufficient to ensure monotonicity. This function $s(x)$ is constructed as follows:
For $x \in [x_i,.x_{i+1}]$ let

$$\theta = (x-x_i)/h_i \tag{2.3}$$

so that $\theta \in [0, 1]$. Then for $x \in [x_{i,}x_{i+1}]$, $i = 1,\ldots,n-1$, we define

$$s(x) = \begin{cases} p_i(\theta)/Q_i(\theta) \text{ if } \Delta_i \neq 0, \\ f_i \quad \text{if} \quad \Delta_i = 0, \end{cases} \tag{2.4}$$

where

$$P_i(\theta) = \Delta_i.\ f_{i+1}\ \theta^2 + (f_i\ d_{i.+1} + f_{i+1}\ d_i.)\ \theta\ (1-\theta) + \Delta_i\ f_i\ (1-\theta)^2 \tag{2.5}$$

$$Q_i.(\theta) = \Delta_i\ \theta^2 + (d_{i+1} + d_i)\ \theta(1-\theta) + \Delta_i\ (1-\theta)^2 \tag{2.6}$$

It should be noted that (2.6) can be written as

$$Q_i\ (\theta) = \Delta_i + (d_{i+1} + d_i - 2\Delta_i.)\ \theta(1-\theta) \tag{2.7}$$

and we can write

$$p_i(\theta)/Q_i(\theta) = f_i + \frac{f_{(i+1)} - f_i)[\Delta_i\theta^2 + d_i\,\theta(1-\theta)]}{\Delta_i + d_{i+1} + d_i - 2\Delta_i)\theta\,(1-\theta)} \tag{2.8}$$

which is a more appropriate form for numerical calculation, in particular, for small Δ_i

The rational quadratic defined by (2.4) - (2.6) has the following properties:

(i) If $\Delta_i. \neq 0$, then $Q_i.(\theta) \neq 0$ for all $0 \leq \theta \leq 1$, since $Qi(\theta)$ is a convex combination of either positive or negative values.

(ii) (Interpolation)

$s(x_i) = f_i.$, $s(x_{i.+1}) = f_{i+1}$,

$s^{(1)}(x_i) = d_i.$. $s^{(1)}(x_{i.+1}) = d_{i+1}$,

where $s^{(1)}$ represents differentiation with respect to x.

(iii (Monotonicity) s(x) is monotonic on [a,b.].

<u>Proof.</u> For $x \in [x_i., x_{i+1}..]$, $\Delta_i. \neq 0$, we have

$^{(1)}(x)$ - $[p_i(\theta)\ Q_i - P_i(\theta)\ Q_i((\theta)]\ /\ [h_i\ Q_i(\theta)^2]$

where P_j and Q_i represent differentiation with, respect to θ. Now it can be shown that

$$P_i(\theta)\ Q_i((\theta)-p_i(\theta)\ Q_i(\theta) - h_i\Delta_i^2 D_i[d_{i+1}\ \theta^2+2D_i\theta\ (l-\theta)+d_i(l-\theta)^2]$$

and hence, using the necessary conditions (2.2),

$$\text{sgn}\ (s^{(1)}(s)) = \text{sgn}\ (d_{i+1}\ \theta^2+2\Delta_i\ \theta(1-\theta)+d_i(1-\theta)^2) = \text{sgn}\ (_i)$$

(iv) $\lim_{\Delta_i \to 0} P_i(\theta)/Q_i(\theta)=f_i$ which follows directly from the monotonicity property, since

$$\min\{f_i,f_{i+1}\} \leq P_i(\theta)/Q_i(\theta) \leq \max\{f_i,f_{i+1}\}$$

(v) $s(x) \in C^1[a.b]$

3. Convergence Analysis

Given the monotonic function $f(x) \in C[a,b]$, let $f_i = f(x_i)$; $i = 1,\ldots,n$ and let $s(x)$ be the piecewise rational quadratic function defined by (2.4). Then a simple convergence result is that

$$|f(x) - s(x)| \leq |f(x_{i+1}) - f(x_i)| \quad , \quad x \in [x_i, x_{i+1}] \quad , \tag{3.1}$$

which follows from the monotonicity property. For $f(x) \in C^1[a,b]$ this implies that

$$|f(x) - s(x)| \leq h_i \ |f^{(1)}(\zeta_i)| \quad , \quad x \in [x_i, x_{i+1}] \tag{3.2}$$

where $\zeta_i(x) \in (x_i, x_{i+1})$.

The bounds (3.1) and (3.2) hold for any monotonic interpolant. However, the rational quadratic interpolant satisfies a higher order convergence result stated in the following theorem.

Theorem 3. 1. Let $f(x) \in C^4[a.b]$ and suppose $|f^{(1)}(x)| > 0$ on a compact set $K \subset [a,b]$ (i.e. f is strictly monotonic on K) . Then for $x \in [x_i, x_{i+1}]$ and $[x_i\text{-}x_{i+1}] \subset K$, $i \in \{1,\ldots,n\text{-}1\}$.

(i) There exists a constant c, independent of i, such that

$$\min_{0\leq\theta\leq 1} |Q_i.(\theta)| \geq c > 0 \quad . \tag{3,3}$$

(ii)

$$|f(x) - s(x)| \leq h_i \ A_i(f) \ \max\left\{\left|f_i^{(1)} - d_i\right|, \left|f_{i+1}^{(1)} - d_{i+1}\right|\right\}$$

$$+ \ h_i^4 \ B_i(f) \quad , \tag{3.4}$$

where

$$A_i(f) = \frac{1}{c}\left[\frac{1}{4}\| f^{(1)} \| + \| \frac{h_i}{16}\| f^{(2)} \| + \frac{h_i^2}{48}\| f^{(3)} \| + \frac{h_i^3}{768}\| f^{(4)} \|\right]$$

$$B_i(f) = \frac{1}{384c}\left[\| f^{(4)} \| \| f(1) \| + \frac{2h_i}{3}\| f^{(3)} \|^2 + 2\| f^{(2)} \| \| f^{(3)} \|\right].$$

(3.5)

and $\|.\|$ denotes the uniform norm on [a,b].

Proof Let

Then $k > 0$ and, by the Mean Value Theorem, $|D_i.| \geq k$.

Thus $|Q_i(\theta)| = |D_i| \theta^2 + (|d_{i+1}| + |d_i|)\theta(1-\theta) + |D_i| (1-\theta)^2$

$$\geq k[\theta^2 + (1-\theta)^2]$$

$$\geq k/2 \text{ on } 0 \leq \theta \leq 1$$

which completes the proof of (i). (For certain choices of d_i and d_{i+1}. we can have $|Q_i(\theta)| \geq k$.)

Let

$$F_i(\theta) - f(x_i(\theta)), \quad S_i(\theta) = s(x_i(\theta))$$

where $x_i(\theta) = x_i + \theta h$. Then we wish to find a bound on $|F_i(\theta) - S_i(\theta)|$ on $0 \leq \theta \leq 1$. Now

$$Q_i(\theta)[F_i(\theta) - S_i(\theta)] = F_i(\theta) Q_i(\theta) - P_i(\theta)$$

$$= F_i(\theta) Q_i(\theta) - H_i(\theta) + H_i(\theta) - P_i(\theta) \quad (3.6)$$

where $H_i(\theta)$ is defined as the cubic Hermite interpolant to $F_i(\theta) Q_i(\theta)$ on $0 \leq \theta \leq 1$. It can be shown that

$$|H_i(\theta) - P_i(\theta)| = |(\theta-1)^2 \theta h_i \quad \Delta i \, (f_i^{(1)} - d_i) + \quad \theta^2(\theta-1) h_i \quad \Delta_I \, f_{i+1}^{(1)} - d_{i+1})|$$

$$\leq \frac{h_i}{4} \left\| f^{(1)} \right\| \max\{ \left| f_i^{(1)} - d_i \right|, \left| f_{i+1}^{(1)} - d_{i+1} \right| \} \quad \textbf{(3.7)}$$

Also, the error bound or cubic Hermite interpolation on $0 \leq \theta \leq 1$ gives

$$F_i(\theta) Q_i(\theta) - H_i(\theta) \Big| \leq \frac{1}{384} \max_{0 \leq \theta \leq 1} \left| \frac{d^4}{d\theta^4} F_i(\theta) Q_i(\theta) \right. \quad \textbf{(3.8)}$$

Let

$$Q_i(\theta) = q_i(\theta) + (d_{i+1} - f_{i+1}^{(1)} + d_i. - f_i^{(1)}) \theta(1-\theta)$$

where

$$q_i(\theta) = \Delta_i\, \theta^2 + (f_{i+1}^{(1)} + f_i^{(1)})\, \theta(1-\theta) + \Delta_i\, (1-\theta)^2$$

This can then be substituted in (3.8), where it is a relatively simple task to bound q.(9) and its derivatives. Finally, combining these results with (3.6) and using (3.3) gives the desired result (ii).

Theorem 3.1 is similar to a lemma in Behforooz and Papamichael (1979) for cubic polynomial interpolation, but is complicated by the non-linearity of the rational quadratic function. Bounds for $|f^{(r)}(x) - s^{(r)}(x)|$, r = 1,2,3, can be obtained by differentiating (3.6) and using the optimal error bounds for the derivatives of cubic Hermite interpolants due to Birkhoff and Priver (1967). Since the non-linearity introduced by the multiplicative term $Q_i(\theta)$ makes such bounds rather involved, we do not quote them here. As would be expected, however, the bounds are reduced by an order of h_i for each derivative taken.

It can be seen from (3.4) that the order of convergence is dependent on the accuracy of d_i and d_{i+1} as approximations to the derivatives $f_i^{(1)} = f^{(1)}(x_i.)$ and $f_{i+1}^{(1)} = f^{(1)}(x_{i+1}.)$. In particular, if we can choose $d_i\, f_{i+1}^{(1)} =$ and $d_{i+1} = f_{i+1}^{(1)}$. then we obtain the best order of bound possible, namely $o(h_i^4)$. In fact in this case $P_i(\theta) \equiv H_i(\theta)$ which was an observation which motivated the development of the theorem.

In practice the $f_i^{(1)}$; i = 1,....n are usually unknown and hence the d_i ; i = 1,. . .,n must be chosen by some method consistent with the necessary conditions (2.2). Two choices for the d_i are discussed in the following section.

Remark. For particular values of θ, the inequality used to obtain (3.7) may be too weak. Thus, for example, with $\theta = \frac{1}{2}$

$$\left|H_i(\tfrac{1}{2}) - P_i(\tfrac{1}{2})\right| \le \frac{h_i}{8} \| f^{(1)} \| \left| f_i^{(1)} - d_i - f_{i+1}^{(1)} + d_{i+1} \right|$$

and $f_i^{(1)} - d_i - f_{i+1}^{(1)} + d_{i+1}$ may have a bound of higher order than the individual bounds on $f_i^{(1)} - d_i$ and $f_{i+1}^{(1)} - d_{i+1}$. This is confirmed by the numerical results which follow.

4. Numerical Results and Discussion

We first consider the two sets of monotonic data used by Fritsch and Carlson (1980). The first set of data was originally used by Akima (1970) and is shown in Table 1. The second set of data is shown in Table 2.

x	0	2	3	5	6	8	9	11	12	14	15
y	10	10	10	10	10	10	10.5	15	50	60	85

Table 1. Monotonic Data Set 1

x	7.99	8.09	8.19	8.7	9.2	. 10	12	15	20
y	0	2.76429×10^{-5}	4.37498×10^{-2}	0. 169183	0.469428	0.943740	0.998636	0.999919	0.999994

Table 2. Monotonic Data Set 2

Application of the piecewise rational quadratic interpolation scheme to each of these data sets requires some method for choosing the derivative parameters d_i ; $i = 1, \ldots, n$, and we consider two possible methods as follows.

Method 1. This method is based on three point difference approximations for the d_i, subject to modification if the necessary conditions (2.2) are not satisfied. Such approximations are used in the initialization process of the Fritsch-Carlson method and are defined by

$$d_i = \begin{cases} 0 \text{ if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0 \\ (h_i \Delta_{i-1} + h_{i-1}\Delta_i)/(h_i + h_{i-1}) \text{ otherwise}, i = 2, \ldots, n-1, \end{cases} \tag{4.1}$$

with end conditions

$$d_1 = \begin{cases} 0 \ \ if \ \ \Delta_1 = 0 \ \ or \ \ d_1^* < 0\,, \\ d_1^* \ \ otherwise \quad , \end{cases} \qquad d_n = \begin{cases} 0 \ \ if \ \ \Delta_{n-1} = 0 \ \ or \ \ d_n^* < 0 \\ d_n^* \ \ otherwise \quad , \end{cases} \tag{4.2}$$

where

$$d_1^* = \Delta_1 + (\Delta_1 - \Delta_2)\, h_1/(h_1 + h_2) \quad ,$$

$$d_n^* = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})\ h_{n-1} \ /(h_{n-2} + h_{n-1} \ .) \tag{4.3}$$

Since $f_i^{(1)}$ - d_i = $0(h^2)$, h = max (h_i) , for these approximations, the error bound defined by (3.4) will, in general, be $0(h^3)$.

Method 2. The d_i. defined by method 1 are not continuous functionals on C [a,b]. A non-linear construction which avoids this problem is given by

$$d_i = \begin{cases} 0 \text{ if } f_{i+1} - f_{i-1} = 0 \\ \Delta_i \ \Delta_{i-1}/[(f_{i+1} - f_{i-1})/(x_{i+1} - x_{i-1})] \text{ otherwise} \end{cases} ,i = 2,..., \ n-1, \tag{4.4}$$

with end conditions

$$d_1 = \begin{cases} 0 \text{ if } f_3 - f_1 = 0\,, \\ \Delta_1^2/[f_3 - f_1)/(x_3 - x_1)] \text{ otherwise,} \end{cases}$$

$$d_n = \begin{cases} 0 \ \ if \ \ f_n - f_{n-2} = 0\ , \\ \Delta_{n-1}^2/[\,f_n - f_{n-2})/(x_n - x_{n-2})] \text{ otherwise} \end{cases} \tag{4.5}$$

Equation (4.4) was suggested by fitting a rational linear function to (x_{i-1},f_{i-1}) , (x_i,,f_i) and (x_{i+1},f_{i+1}) , in analogy with (4. 1) , which can be derived by fitting a quadratic function to the data. A Taylor expansion analysis shows that $f_i^{(1)}$ - d_i = $O(h^2)$ in (4.4) and that $f_1^{(1)}$ - d_1 , $f_n^{(1)}$ - d_n are, in general, 0(h) but are $0(h^2)$ in the case of equal intervals.

The result of applying the piecewise rational quadratic scheme to the two given data sets, with the two choices of the d_i described above, is shown in Figures1and 2. The $\mathcal{J}_2$ monotonicity region method recommended by Fritsch and Carlson is also shown for purposes of comparison. We conclude from these figures that the rational quadratic scheme using method 2 appears to produce more "visually pleasing" curves than method 1 and esemble the curves given by the Fritsch-carlson method.

Our second set of results concerns the order of convergence of the interpolation scheme discussed in Section 3. Tables 3 and 4 show the interpolation errors which result from the application of the rational quadratic scheme to f (x) = exp (x) for various choices of the derivative parameters d_i. The knots x_i are taken to be equally spaced with interval lengths h = 0.2, 0.1, and 0.05 respectively, centred about x = 0.6. The errors are evaluated for two choices of θ in the intervals containing 0.6,namely $\theta = \frac{1}{2}$ *and* $\theta = \frac{1}{3}$.

method	error e_1 (h = 0.2)	error e_2 (h = 0.1)	error e_3 (h - 0.05)	e_1/e_2	e_2/e_3
d_i. exact	$-.75770\text{x}10^{-5}$	$-.47427\times10^{-6}$	$-.29653\text{x}10^{-7}$	15.98	15.99
d_i method I	$.22701 \times 10^{-4}$	$.14223\times10^{-5}$	$.88953\text{x}10^{-7}$	15.96	15.99
d_i method 2 .	$-.22701 \times 10^{-4}$	$-.14223 \text{ x } 10^{-5}$	$-.88952\text{x}10^{-7}$	15.96	15.99

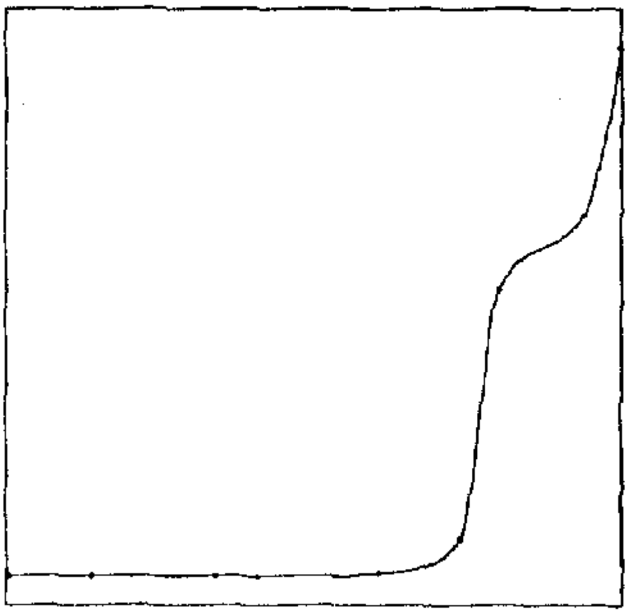
<u>Table 3</u>. Rational quadratic interpolation errors at $\theta = \frac{1}{2}$, $f(x) = \exp(x)$.

method	error e_1 (h = 0.2)	error e_2 (h = 0.1)	error e_3 (h = 0.05)	e_1/e_2	e_2/e_3
d_i exact	$-.58956\text{x}10^{-5}$	$-.37185\text{X}10^{-6}$	$-.23339\text{x}10^{-7}$	15.85	15.93
d_i. method 1	$-.15612\text{x}10^{-3}$	$-.21000\text{x}10^{-4}$	$-.27183\times10^{-5}$	7.43	7.73
d_i method 2	$.69103\text{x}10^{-4}$	$.99380 \times10^{-5}$	$.13240\text{x}10^{-5}$	6.95	7.51

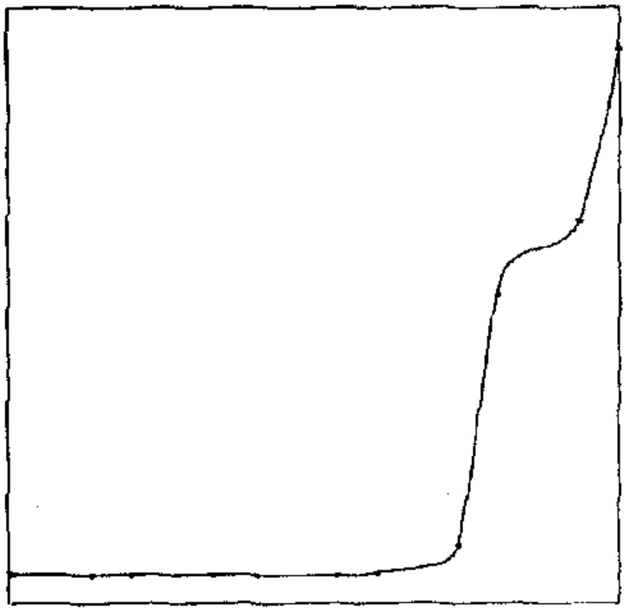
<u>Table 4</u>. Rational quadratic interpolation errors at $\theta=\frac{1}{3}$, $f(x) = \exp(x)$.

For the case $\theta=\frac{1}{3}$, the ratios of the errors confirm the expected convergence rates, namely $0(h^4)$ for the true derivative scheme and $0(h^3)$ for each of methods 1 and 2. The $0(h^4)$ convergence of methods 1 and 2 for the case $\theta=\frac{1}{2}$ is explained by the remark,at the end of Section 3, since it can he shown that $f_i^{(1)}-d_i-f_{i+1}^{(1)}+d_{i+1}=0(h^3)$ for these methods in the case of equally spaced knots.

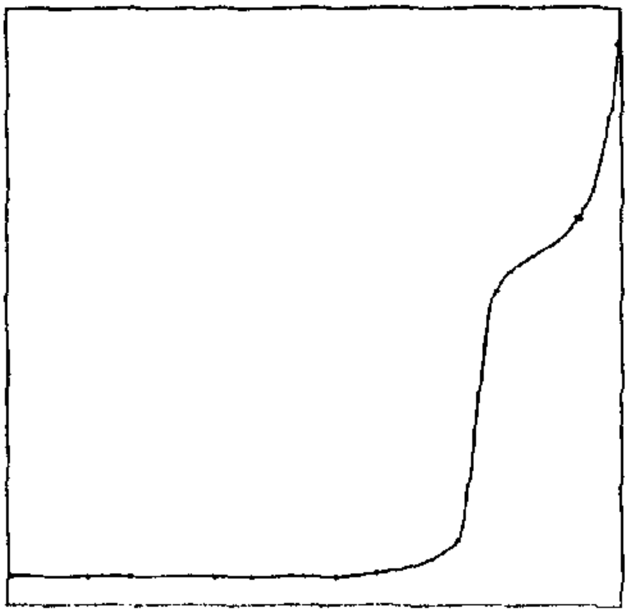
Conclusion. An explicit representation of a piecewise rational quadratic function has been developed, which produces a monotonic interpolant to given monotonic data. The numerical results indicate that, in the absence of derivative data , the derivative parameters chosen by method 2 produce the best monotonic curves.



(i) Fritsch-Carlson

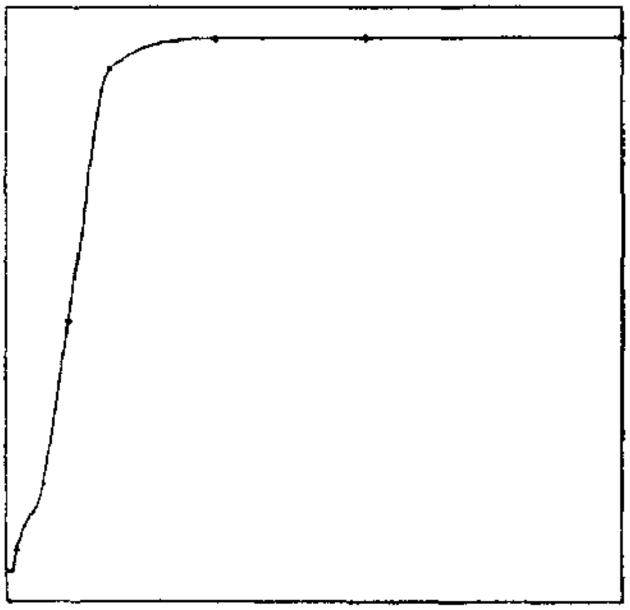


(ii) Rational Spline Method1

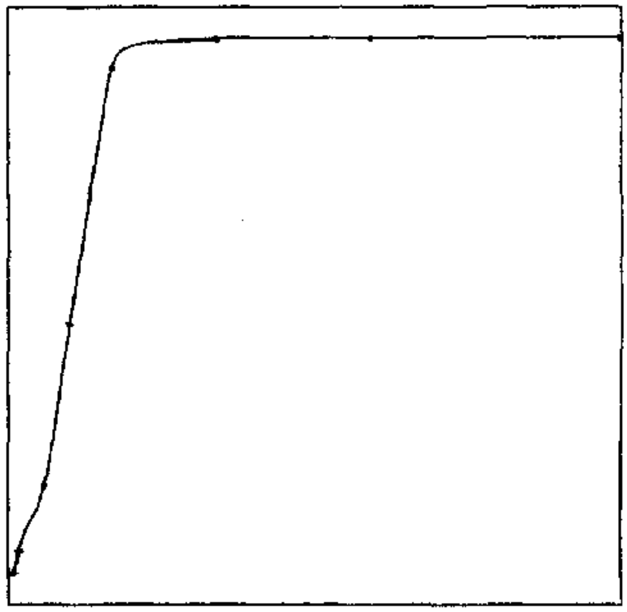


(iii) Rational Spline Method 2

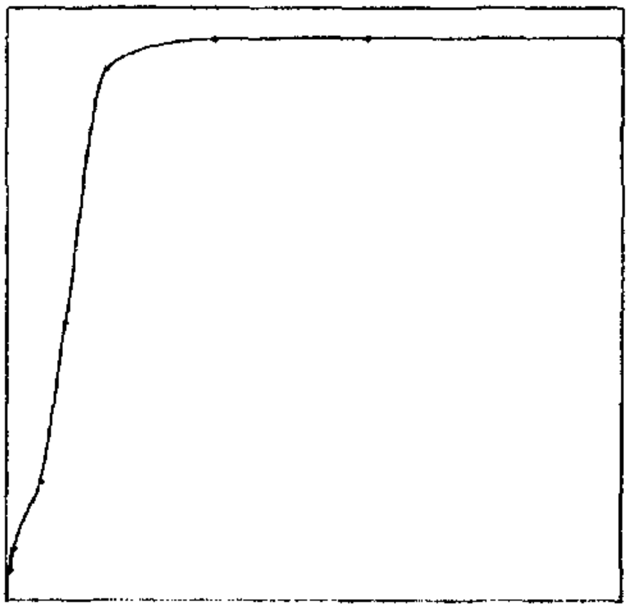
Fig 1. Results for Monotonic Data.Set 1



(i) Fritsch-Carlson



(ii) Rational Spline Method 1



(iii) Rational Spline Method 2

Fig. 2. Results for Monotonic Data Set 2

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