Network-Based Robust H_∞ Stabilization of Uncertain Systems

Arash Farnam
Electrical Engineering Department
Sahand University of Technology
Tabriz, Iran
a_farnam@sut.ac.ir

Reza Mahboobi Esfanjani
Electrical Engineering Department
Sahand University of Technology
Tabriz, Iran
mahboobi@sut.ac.ir

Abstract: **This paper deals with the problem of the stability analysis and controller gain synthesis for networked control systems with the network-induced delay, data packet dropout, parameters' uncertainties and disturbance input. To achieve less conservative results compared with existing methods in the literature, a novel Lyapunov-Krasovskii functional is constructed and new free-weighting matrices are introduced to increase degrees of freedom in the sufficient robust stability conditions. The maximum allowable delay bound, minimum attenuation level and the gain of memoryless controller is obtained by solving a set of linear matrix inequalities (LMIs). Finally an illustrative example is given to reveal the effectiveness of the proposed approach.**

Keywords: Networked Control Systems, Robust Control, Stabilization, Lyapunov-Krasovskii Theorem

I. Introduction

A networked control system (NCS) is a closed-loop system that all of the ingredients are connected to each other through a communication network. Reliability, easy maintenance and low cost made these systems impressive in practical issues. But two important challenges, network-induced delay and data dropout are unfavorable phenomenon in these systems. Nonetheless NCSs have gathered many researchers' attentions in the resent years [1-9].

The prevalent method to investigate stability and synthesis controller gain is based on Lyapunov-Krasovskii theorem [3-6]. [3] surveyed the problem of stability and controller design according to using Lyapunov-Krasovskii functional, and the results of [3] were improved in [4] by utilizing new Lyapunov-Krasovskii functional. For the first time, augmented Lyapunov-Krasovskii functional to obtain sufficient conditions for designing controller gain to satisfy robust stability for NCSs was introduced in [5]. Robust stability analysis for NCSs was improved in [6] by introducing new weighting matrices to increase the degree of freedom. [7] and [8] investigated the robust stability problem for NCSs with considering the closed-loop system as discrete time model with binary random delay and Markovian jumping parameters, respectively.

This paper is organized as follows: A continues time model for closed-loop system is presented in section 2. In section 3, sufficient conditions are extracted for robust H_∞ stability analysis and controller gain synthesis. In section 4, an illustrative example reveals the reliability of the proposed method. Finally section 5 concludes the paper.

Notation: In this paper, * denotes block in the symmetric matrix. I is identity matrix of appropriate dimension. The notation $P > 0$ (respectively $P \geq 0$) means that P is real symmetric and positive define (respectively, positive semiefinite). The superscrip ^{T} stands for matrix transposition.

II. System description and preliminaries

The controlled system is described as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + E\omega(t),\ A = A_0 + \Delta A(t),\ \ B = B_0 + \Delta B(t)$$
$$Z(t) = Cx(t) + Du(t) \tag{1}$$

where $x(t) \epsilon R^n$, $u(t) \epsilon R^m$, $\omega(t) \epsilon R^n$ and $Z(t) \epsilon R^q$ are the state vector, control input vector, disturbance vector and controlled output, respectively; A_0, B_0, E, C and D are known system matrices with appropriate dimensions. It's assumed that the pair (A,B) is completely controllable. $\Delta A(t)$ and $\Delta B(t)$ denotes the norm-bounded parameter uncertainties in plant satisfying :

$$[\Delta A(t) \quad \Delta B(t)] = J\Delta(t)[H_1 \quad H_2] \tag{2}$$

where J, H_1 and H_2 are known constant matrices with appropriate dimensions and $\Delta(t)$ is unknown time-varying matrix satisfying $\Delta^T(t)\Delta(t) \leq I$. Without loss of generality, we mention the following assumption:

Assumption 1 : A and E are matrices with same dimensions.

The considered NCS structure is shown in Fig. 1, where the controller and actuator are event-driven and sampler is clock-driven.The sampling period is assumed to be h, that is a constant value. The transmission delay may not be necessarily integer multiplies of the sampling period so zero order hold (Z.O.H) device's information may be updated between sampling instants. Since the controller is a constant gain, the feedback and forward delays are combined together at each sampling time. The updating instant of Z.O.H are t_k experience signal transmission delay η_k, where $\eta_k = \eta_{sc_k} + \eta_{ca_k}$ (η_{sc_k} and η_{ca_k} are delays from the sampler to the controller and from the controller to the Z.O.H at the updating instant t_k, respectively). Therefore, the state feedback with considering the behavior of the Z.O.H takes the following form:

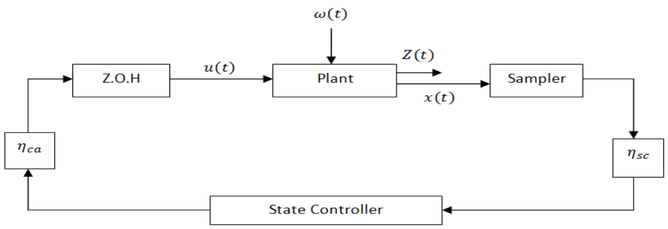

Figure 1. Networked Control System

$$u(t_k) = Kx(t_k - \eta_k)t_k \le t < t_{k+1} \quad (3)$$

in which t_{k+1} is next updating state after t_k . The network-induced delay η_k is bounded as the following inequality:

$$\eta_m \le \eta_k \le \eta_M \quad (4)$$

where η_M and η_m are the lower and upper bounds of the network-induced delay, respectively. Then, the closed-loop system in Fig. 1 is described by:

$$\dot{x}(t) = Ax(t) + BKx(t_k - \eta_k) + E\omega(t)$$

$$Z(t) = Cx(t) + DKx(t_k - \eta_k)t_k \le t < t_{k+1} \quad (5)$$

which is in the form of sampled-data system. Moreover, at the updating instant t_k, the number of accumulated data packet dropout since the last updating instant t_{k-1} is denoted by τ_k, where $0 \le \tau_k \le \tau_M$. Combining the above-mentioned facts, yields to:

$$t_{k+1} - t_k = \eta_{k+1} - \eta_k + (\tau_{k+1} + 1)h \quad (6)$$

Now, let $\eta(t) = t - t_k + \eta_k$ is replaced in (5), then the following continuous time model is obtained for the closed-loop NCS in Fig. 1:

$$\dot{x}(t) = Ax(t) + BKx\big(t - \eta(t)\big) + E\omega(t$$

$$Z(t) = Cx(t) + DKx\big(t - \eta(t)\big) \quad (7)$$

in which,

$$\eta_m \le \eta(t) \le \eta \quad (8)$$

with $\eta = \eta_M + (\tau_M + 1)h$. It's evident that η is related to the maximum number of accumulated data packet dropouts τ_M, the upper bound of network-induced delay η_M and the sampling period h of the sampler device.

***H_∞* control problem:** System (7) is said robustly asymptotically stable with H_∞ norm bound $\gamma > 0$ if the following conditions are satisfied:

1) The closed-loop system (7) is asymptotically stable when $\omega(t) = 0$ for all uncertainties $\Delta A(t)$ and $\Delta B(t)$.

2) Under the zero conditions, the controlled output $Z(t)$ satisfies $\|Z(t)\|_2 \le \gamma \|\omega(t)\|_2$ for all nonzero $\omega(t) \epsilon L_2[0, \infty)$.

Before further proceeding, the following lemma is introduced to handle the norm-bounded parameter uncertainties:

Lemma 1: Given real matrices Σ , Σ_1 and Σ_2 with appropriate dimensions, with $\Sigma^T = \Sigma$,

$$\Sigma + \Sigma_2 \Delta(t) \Sigma_1 + \Sigma_1^T \Delta^T(t) \Sigma_2^T < 0 \quad (9)$$

holds if and only if for all $\Delta^T(t)\Delta(t) \le I$ and there exist some $\epsilon > 0$ such that the following inequality holds:

$$\Sigma + \epsilon \Sigma_2 \Sigma_2^T + \epsilon^{-1} \Sigma_1^T \Sigma_1 < 0$$

which can be modified by schur complement to the following matrix inequality:

$$\begin{bmatrix} \Sigma & \Sigma_1^T & \epsilon\Sigma_2 \\ \Sigma_1 & -\epsilon I & 0 \\ \epsilon\Sigma_2^T & 0 & -\epsilon I \end{bmatrix} < 0 \quad (10)$$

III. Main results

In this section, a new robust H_∞ delay-dependent stability condition is derived in Theorem 1 to analyze robust H_∞ stability of closed-loop system (7) for all delays satisfying (8). Then, controller synthesis approach is given in Theorem 2.

Theorem 1: For given η_m, η, J, H_1, H_2 and K, the closed-loop system (7) is robustly asymptotically stable with the H_∞ norm bound γ if there exist matrices N_z, $L_z (z = 0,1,2)$, M , R, S, F , symmetric matrices $P = [P_{ij}]_{5\times5}$, $Q_1 = \left[Q_{1_{ij}}\right]_{2\times2} > 0$, $Q_2 = \left[Q_{2_{ij}}\right]_{2\times2} > 0$, $T_1 = \left[T_{1_{ij}}\right]_{2\times2} > 0$, $T_2 = \left[T_{2_{ij}}\right] > 0$, $Z_1 > 0$, $Z_2 > 0$, X_0, X_1, X_2 , U_1, U_2, $V_z = \left[V_{z_{ij}}\right]_{8\times8}$, and $W_z = \left[W_{z_{ij}}\right]_{8\times8}$ $(z = 0,1,2)$ with appropriate dimensions and scalar ϵ, satisfying (11-16)

$$\begin{bmatrix} P & R & S \\ * & U_1 & F \\ * & * & U_2 \end{bmatrix} > 0 \quad (11)$$

$$\begin{bmatrix} \Sigma_i & \Sigma_1^T & \epsilon\Sigma_2 \\ * & -\epsilon I & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0 \quad ,\forall i = 1,2 \quad (12)$$

$$\begin{bmatrix} V_0 & L_0 + \psi_0 & N_0 \\ * & T_{1_{11}} & T_{1_{12}} + X_0 \\ * & * & T_{1_{22}} \end{bmatrix} \ge 0 \quad (13)$$

$$\begin{bmatrix} V_i & L_i+\psi_1 & N_i \\ * & T_{2_{11}} & T_{2_{12}}+X_i \\ * & * & T_{2_{22}} \end{bmatrix} \geq 0, \ \forall i=1,2 \tag{14}$$

$$\begin{bmatrix} W_0 & L_0+\varphi_0 \\ * & Z_1 \end{bmatrix} \geq 0 \tag{15}$$

$$\begin{bmatrix} W_i & L_i+\varphi_1 \\ * & Z_2 \end{bmatrix} \geq 0, \ \forall i=1,2 \tag{16}$$

Where $\hat{\eta}=\eta-\eta_m, \bar{\eta}=\frac{1}{2}(\eta^2-\eta_m^2)$ and

$\Sigma_i=\pi_1+\pi_{2i}+\pi_{2i}^T+\pi_{3_0}+\pi_{3_0}^T+\eta_m V_0+\hat{\eta} \ \ V_i+\dfrac{\eta_m^2}{2}W_0+\bar{\eta} \ \ W_i$ $,\forall i=1,2$

$\pi_{2i}=[N_0+\eta_m L_0+\hat{\eta} \ \ L_i-N_0+N_1-N_2 \ \ 0 \ \ 0 \ \ 0 \ \ -N_1+N_2 \ \ 0]$

$\pi_1=\begin{bmatrix} (1,1) & (1,2) \\ * & (2,2) \end{bmatrix}, \quad (1,1)=\begin{bmatrix} 11 & 12 & 13 & 14 \\ * & 22 & 23 & 24 \\ * & * & 33 & 34 \\ * & * & * & 44 \end{bmatrix}$

$_{11}=P_{14}-R_1+P_{14}^T-R_1^T+Q_{1_{11}}+\eta_m T_{1_{11}}+\hat{\eta} \ \ T_{2_{11}}+X_0+C^TC$

$_{12}=-P_{14}+P_{15}+R_1-S_1+P_{24}^T-R_2^T$

$_{13}=-P_{15}+S_1+P_{34}^T-R_3^T$

$_{14}=P_{11}+\eta_m R_1+\hat{\eta} \ \ S_1+Q_{1_{12}}+\eta_m T_{1_{12}}+\hat{\eta} \ \ T_{2_{12}}$

$_{22}=-P_{24}+P_{25}+R_2-S_2-P_{24}^T+P_{25}^T+R_2^T-S_2^T-Q_{1_{11}}+Q_{2_{11}}-X_0+X_1,$

$_{23}=-P_{25}+S_2-P_{34}^T+P_{35}^T+R_3^T-S_3^T$

$_{24}=P_{12}^T+\eta_m R_2+\hat{\eta} \ \ S_2, \qquad {}_{34}=P_{13}^T+\eta_m R_3+\hat{\eta} \ \ S_3$

$_{33}=-P_{35}+S_3-P_{35}^T+S_3^T-Q_{2_{11}}-X_2,$

$_{44}=Q_{1_{22}}+\eta_m T_{1_{22}}+\hat{\eta} \ \ T_{2_{22}}+\dfrac{\eta_m^2}{2}Z_1+\bar{\eta}Z_2$

$$(1,2)=\begin{bmatrix} P_{12} & P_{13} & C^TDK & 0 \\ P_{22}-Q_{1_{12}}+Q_{2_{12}} & P_{23} & 0 & 0 \\ P_{23}^T & P_{33}-Q_{2_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(2,2)=\begin{bmatrix} -Q_{1_{22}}+Q_{2_{22}} & 0 & 0 & 0 \\ * & -Q_{2_{22}} & 0 & 0 \\ * & * & -X_1+X_2+K^TD^TDK & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}$$

$\Sigma_2=[-J^TM_1^T,-J^TM_2^T,-J^TM_3^T,-J^TM_4^T,-J^TM_5^T,-J^TM_6^T,-J^TM_7^T,-J^TM_8^T]^T$

$\Sigma_1=[H_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad H_2K \quad 0]$

$\pi_{3_0}=[-M \quad A_0 \quad 0 \quad 0 \quad M \quad 0 \quad 0 \quad -MB_0K \quad -ME]$

$$\psi_0=\begin{bmatrix} -P_{44}+R_4^T \\ P_{44}-P_{45}^T-R_4^T+S_4^T \\ P_{45}^T-S_4^T \\ -P_{14}-\eta_m R_4^T-\hat{\eta}S_4^T \\ -P_{24} \\ -P_{34} \\ 0 \\ 0 \end{bmatrix}, \quad \psi_1=\begin{bmatrix} -P_{45}+R_5^T \\ P_{45}-P_{55}-R_5^T+S_5^T \\ P_{55}-S_5^T \\ -P_{15}-\eta_m R_5^T-\hat{\eta}S_5^T \\ -P_{25} \\ -P_{35} \\ 0 \\ 0 \end{bmatrix}$$

$$\varphi_0=\begin{bmatrix} -R_4+U_1 \\ -R_4-R_5-U_1+F^T \\ R_5-F^T \\ -R_1-\eta_m U_1-\hat{\eta}F^T \\ -R_2 \\ -R_3 \\ 0 \\ 0 \end{bmatrix}, \varphi_1=\begin{bmatrix} -S_4+F \\ S_4-S_5-F+U_2 \\ S_5-U_2 \\ -S_1-\eta_m F-\hat{\eta}U_2 \\ -S_2 \\ -S_3 \\ 0 \\ 0 \end{bmatrix}$$

Proof: Consider a Lyaponuv-Krasovskii functional as follows:

$$V(x_t)=V_1(x_t)+V_2(x_t)+V_3(x_t)+V_4(x_t) \tag{17}$$

$$V_1(x_t)=\xi^T(t)\begin{bmatrix} P & R & S \\ * & U_1 & F \\ * & * & U_2 \end{bmatrix}\xi(t) \tag{18}$$

$$V_2(x_t)=\int_{t-\eta_m}^{t}\tau^T(\alpha)Q_1\tau(\alpha)d\alpha+\int_{t-\eta}^{t-\eta_m}\tau^T(\alpha)Q_2\tau(\alpha)d\alpha \tag{19}$$

$$V_3(x_t)=\int_{-\eta_m}^{0}\int_{t+\beta}^{t}\tau^T(\alpha)T_1\tau(\alpha)d\alpha d\beta + \int_{-\eta}^{-\eta_m}\int_{t+\beta}^{t}\tau^T(\alpha)T_2\tau(\alpha)d\alpha d\beta \tag{20}$$

$$V_4(x_t)=\int_{-\eta_m}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t}\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha d\theta d\beta + \int_{-\eta}^{-\eta_m}\int_{\beta}^{0}\int_{t+\theta}^{t}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha d\theta d\beta \tag{21}$$

wherein, $\xi(t)=col[x(t), \ \ x(t-\eta_m), \ \ x(t-\eta), \ \ \int_{t-\eta_m}^{t}x(\alpha)d\alpha, \ \ \int_{t-\eta}^{t-\eta_m}x(\alpha)d\alpha, \ \ \int_{-\eta_m}^{0}\int_{t+\beta}^{t}\dot{x}(\alpha)d\alpha d\beta, \ \ \int_{-\eta}^{-\eta_m}\int_{t+\beta}^{t}\dot{x}(\alpha)d\alpha\, d\beta]$ and $\tau(\alpha)=col[x(\alpha) \quad \dot{x}(\alpha)]$.

Now consider the following equation:

$$j_{z\omega}=\int_0^{\infty}[z^T(t)z(t)-\gamma^2\omega^T(t)\omega(t)]dt \tag{22}$$

Under zero-initial conditions, we have $V(x_0)=0$ and $V(x_\infty)\geq 0$, so (22) can be rewritten to the following inequality:

$$j_{z\omega}=\int_0^{\infty}\left[z^T(t)z(t)-\gamma^2\omega^T(t)\omega(t)+\dot{V}(x_t)\right]dt-V(x_\infty)\leq \int_0^{\infty}[z^T(t)z(t)-\gamma^2\omega^T(t)\omega(t)+\dot{V}(x_t)]\mathrm{d}t \tag{23}$$

So the closed-loop system (7) is robustly asymptotically stable with disturbance attenuation level γ if and only if satisfying:

$$z^T(t)z(t)-\gamma^2\omega^T(t)\omega(t)+\dot{V}(x_t)<0 \tag{24}$$

The time derivative of $V(x_t)$ along the trajectories of (7) is obtained as follows:

$$\dot{V}_1(x_t)=2\xi^T(t)\begin{bmatrix} P & R & S \\ * & U_1 & F \\ * & * & U_2 \end{bmatrix}\dot{\xi}(t) \tag{25}$$

$$\dot{V}_2(x_t)=\tau^T(t)Q_1\tau(t)-\tau^T(t-\eta_m)Q_1\tau(t-\eta_m) + \tau^T(t-\eta_m)Q_2\tau^T(t-\eta_m)-\tau^T(t-\eta)Q_2\tau \ \ (t-\eta) \tag{26}$$

$$\dot{V}_3(x_t)=\tau^T(t)(\eta_m T_1+\hat{\eta}T_2)\tau(t)-\int_{t-\eta_m}^{t}\tau^T(\alpha)T_1\tau(\alpha)d\alpha - \int_{t-\eta(t)}^{t-\eta_m}\tau^T(\alpha)T_2\tau \ \ (\alpha)d\alpha-\int_{t-\eta}^{t-\eta(t)}\tau^T(\alpha)T_2\tau(\alpha)d\alpha \tag{27}$$

$$\dot{V}_4(x_t) = \dot{x}^T(t)\left(\frac{\eta_m^2}{2}Z_1 + \bar{\eta}Z_2\right)\dot{x}(t) - \int_{-\eta_m}^{0}\int_{t+\beta}^{t}\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)\,d\alpha d\beta - \int_{-\eta(t)}^{-\eta_m}\int_{t+\beta}^{t}\dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha d\beta - \int_{-\eta}^{-\eta(t)}\int_{t+\beta}^{t}\dot{x}^T(\alpha)\,Z_2\dot{x}(\alpha)d\alpha d\beta \quad (28)$$

For any matrices N_0, N_1, N_2, M, L_0 , L_1 and L_2 and symmetric matrices V_0, V_1, V_2, W_0, W_1, W_2, X_0, X_1 and X_2 with appropriate dimensions, the following equalities hold:

$$\varepsilon_1(t) = 2\zeta^T(t)N_0(x(t) - x(t-\eta_m) - \int_{t-\eta_m}^{t}\dot{x}(\alpha)d\alpha) = 0 \quad (29)$$

$$\varepsilon_2(t) = 2\zeta^T(t)N_1\left(x(t-\eta_m) - x(t-\eta(t)) - \int_{t-\eta(t)}^{t-\eta_m}\dot{x}\,(\alpha)d\alpha\right) = 0 \quad (30)$$

$$\varepsilon_3(t) = 2\zeta^T(t)N_2\left(x\big(t-\eta(t)\big) - x(t-\eta) - \int_{t-\eta}^{t-\eta(t)}\dot{x}(\alpha)d\alpha\right) = 0 \quad (31)$$

$$\varepsilon_4(t) = 2\zeta^T(t)M\left(\dot{x}(t) - Ax(t) - BKx\big(t-\eta(t)\big) - E\omega(t)\right) = 0 \quad (32)$$

$$\varepsilon_5(t) = 2\zeta^T(t)L_0(\eta_m x(t) - \int_{t-\eta_m}^{t}x(\alpha)d\alpha - \int_{-\eta_m}^{0}\int_{t+\beta}^{t}\dot{x}\,(\alpha)d\alpha d\beta = 0 \quad (33)$$

$$\varepsilon_6(t) = 2\zeta^T(t)L_1((\eta(t)-\eta_m)x(t) - \int_{t-\eta(t)}^{t-\eta_m}x(\alpha)d\alpha - \int_{-\eta(t)}^{-\eta_m}\int_{t+\beta}^{t}\dot{x}(\alpha)d\alpha d\beta) = 0 \quad (34)$$

$$\varepsilon_7(t) = 2\zeta^T(t)L_2(\big(\eta-\eta(t)\big)x(t) - \int_{t-\eta}^{t-\eta(t)}x(\alpha)d\alpha - \int_{-\eta}^{-\eta(t)}\int_{t+\beta}^{t}\dot{x}(\alpha)d\alpha d\beta) = 0 \quad (35)$$

$$\varepsilon_8(t) = \eta_m\zeta^T(t)V_0\zeta(t) - \int_{t-\eta_m}^{t}\zeta^T(t)V_0\zeta(t)d\alpha = 0 \quad (36)$$

$$\varepsilon_9(t) = (\eta(t)-\eta_m)\zeta^T(t)V_1\zeta(t) - \int_{t-\eta(t)}^{t-\eta_m}\zeta^T(t)V_1\zeta(t)d\alpha = 0 \quad (37)$$

$$\varepsilon_{10}(t) = \big(\eta-\eta(t)\big)\zeta^T(t)V_2\zeta(t) - \int_{t-\eta}^{t-\eta(t)}\zeta^T(t)V_2\zeta(t)d\alpha = 0 \quad (38)$$

$$\varepsilon_{11}(t) = \frac{\eta_m^2}{2}\zeta^T(t)W_0\zeta(t) - \int_{-\eta_m}^{0}\int_{t+\beta}^{t}\zeta^T(t)W_0\zeta(t)d\alpha d\beta = 0 \quad (39)$$

$$\varepsilon_{12}(t) = \frac{(\eta^2(t)-\eta_m^2)}{2}\zeta^T(t)W_1\zeta(t) - \int_{-\eta(t)}^{-\eta_m}\int_{t+\beta}^{t}\zeta^T(t)W_1\zeta(t)d\alpha d\beta = 0 \quad (40)$$

$$\varepsilon_{13}(t) = \frac{\big(\eta^2-\eta^2(t)\big)}{2}\zeta^T(t)W_2\zeta(t) - \int_{-\eta}^{-\eta(t)}\int_{t+\beta}^{t}\zeta^T(t)W_2\zeta(t)d\alpha d\beta = 0 \quad (41)$$

$$\varepsilon_{14}(t) = x^T(t)X_0x(t) - x^T(t-\eta_m)X_0x(t-\eta_m) - 2\int_{t-\eta_m}^{t}\dot{x}^T(\alpha)X_0x(\alpha)d\alpha = 0 \quad (42)$$

$$\varepsilon_{15}(t) = x^T(t-\eta_m)X_1x\,(t-\eta_m) - x^T\big(t-\eta(t)\big)X_1x(t-\eta(t)) - 2\int_{t-\eta(t)}^{t-\eta_m}\dot{x}^T(\alpha)X_1x(\alpha)d\alpha = 0 \quad (43)$$

$$\varepsilon_{16}(t) = x^T(t-\eta(t))X_2x(t-\eta(t)) - x^T(t-\eta)X_2x(t-\eta) - 2\int_{t-\eta}^{t-\eta(t)}\dot{x}^T(\alpha)X_2x(\alpha)d\alpha = 0 \quad (44)$$

where, $\zeta(t) = col[x(t), x(t-\eta_m), x(t-\eta), \dot{x}(t), \dot{x}(t-\eta_m), \dot{x}(t-\eta), x(t-\eta(t), \omega(t)]$. Now based on (25-28) and combining (29-44), $z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(x_t)$ can be stated as follows:

$$z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(x_t) = \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t) + \dot{V}_4(x_t) + \textstyle\sum_{i=1}^{i=16}\varepsilon_i(t) + (Cx(t) + DKx(t-\eta(t))^T(Cx(t) + DKx\big(t-\eta(t)\big) - \gamma^2\omega^T(t)\omega(t) \quad (45)$$

The $\dot{V} + Z^TZ - \gamma^2\omega^T\omega$ in (45) can be rewritten as:

$$\dot{V}(x_t) + z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) = \zeta^T(t)\pi(t)\zeta(t) + \textstyle\sum_{i=1}^{i=6}\Omega_i(t) \quad (46)$$

Where

$$\pi(t) = \pi_1 + \pi_2(t) + \pi_2^T(t) + \pi_3 + \pi_3^T + \eta_m V_0 + (\eta(t)-\eta_m)V_1 + \frac{\eta_m^2}{2}W_0 + \frac{(\eta^2(t)\ -\eta_m^2)}{2}W_1 + \frac{(\eta^2-\eta^2(t))}{2}W_2$$

$$\pi_2(t) = [(1,1)(t) \quad -N_0+N_1 \quad -N_2 \quad 0 \quad 0 \quad 0 \quad -N_1+N_2 \quad 0]$$

$$(1,1)(t) = N_0 + \eta_m L_0 + (\eta(t)-\eta_m)L_1 + (\eta-\eta(t))L_2$$

$$\Omega_1(t) = -\int_{t-\eta_m}^{t}\begin{bmatrix}\zeta(t)\\x(\alpha)\\\dot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}V_0 & L_0+\psi_0 & N_0\\ * & T_{1_{11}} & T_{1_{12}}+X_0\\ * & * & T_{1_{22}}\end{bmatrix}\begin{bmatrix}\zeta(t)\\x(\alpha)\\\dot{x}(\alpha)\end{bmatrix}d\alpha$$

$$\Omega_2(t) = -\int_{t-\eta(t)}^{t-\eta_m}\begin{bmatrix}\zeta(t)\\x(\alpha)\\\dot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}V_1 & L_1+\psi_1 & N_1\\ * & T_{2_{11}} & T_{2_{12}}+X_1\\ * & * & T_{2_{22}}\end{bmatrix}\begin{bmatrix}\zeta(t)\\x(\alpha)\\\dot{x}(\alpha)\end{bmatrix}d\alpha$$

$$\Omega_3(t) = -\int_{t-\eta}^{t-\eta(t)}\begin{bmatrix}\zeta(t)\\x(\alpha)\\\dot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}V_2 & L_2+\psi_1 & N_2\\ * & T_{2_{11}} & T_{2_{12}}+X_2\\ * & * & T_{2_{22}}\end{bmatrix}\begin{bmatrix}\zeta(t)\\x(\alpha)\\\dot{x}(\alpha)\end{bmatrix}d\alpha$$

$$\Omega_4(t) = -\int_{-\eta_m}^{0}\int_{t+\beta}^{t}\begin{bmatrix}\zeta(t)\\\dot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}W_0 & L_0+\varphi_0\\ * & Z_1\end{bmatrix}\begin{bmatrix}\zeta(t)\\\dot{x}(\alpha)\end{bmatrix}d\alpha d\beta$$

$$\Omega_5(t) = -\int_{-\eta(t)}^{-\eta_m}\int_{t+\beta}^{t}\begin{bmatrix}\zeta(t)\\\dot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}W_1 & L_1+\varphi_1\\ * & Z_2\end{bmatrix}\begin{bmatrix}\zeta(t)\\\dot{x}(\alpha)\end{bmatrix}d\alpha d\beta$$

$$\Omega_6(t) = -\int_{-\eta}^{-\eta(t)}\int_{t+\beta}^{t}\begin{bmatrix}\zeta(t)\\\dot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}W_2 & L_2+\varphi_1\\ * & Z_2\end{bmatrix}\begin{bmatrix}\zeta(t)\\\dot{x}(\alpha)\end{bmatrix}d\alpha d\beta$$

Provided $\pi(t) < 0$, and $\Omega_i \geq 0\ (i = 1, \ldots, 6)$, the Theorem1 ensures that the system (7) is asymptotically stable. It is obvious that the condition $\pi(t) < 0$ is equivalent with conditions in (12), one has been obtained for $\eta(t) = \eta_m$ and another for $\eta(t) = \eta$. ■

Theorem 1 reveals the sufficient conditions for analysis of robust H_∞ stability for system (7), but this theorem cannot give the gain of controller for stabilization of system (7). So, Theorem 2 is introduced to acquire appropriate controller gain K by solving a set of Linear Matrix Inequalities.

Theorem 2: For given constants η_m, η and γ and scalars $\rho_i (i = 2, \ldots, 8)$, the closed-loop system (7) is robustly asymptotically stable for H_∞ level γ with the control gain

$K = YX^{-T}$ if there exist nonsingular matrix X, matrices $\bar{N}_z$, $\bar{L}_z$, $(z = 0,1,2)\bar{R}$, $\bar{S}$, $\bar{F}$, Y and symmetric matrices $\bar{P} = [\bar{P}_{ij}]_{5\times5}$, $\bar{Q}_1 = [\bar{Q}_{1_{ij}}]_{2\times2} > 0$, $\bar{Q}_2 = [\bar{Q}_{2_{ij}}]_{2\times2} > 0$, $\bar{T}_1 = [\bar{T}_{1_{ij}}]_{2\times2} > 0, \bar{T}_2 = [\bar{T}_{2_{ij}}]_{2\times2} > 0$, $\bar{Z}_1 > 0$, $\bar{Z}_2 > 0$, $\bar{U}_1$, $\bar{U}_2$, $\bar{V}_z = [\bar{V}_{z_{ij}}]_{8\times8}$, $\bar{W}_z = [\bar{W}_{z_{ij}}]_{8\times8}$, $\bar{X}_z$ $(z = 0,1,2)$ with appropriate dimensions and scalar $\epsilon > 0$ such that the following LMIs hold (47-52):

$$\begin{bmatrix} \bar{P} & \bar{R} & \bar{S} \\ * & \bar{U}_1 & \bar{F} \\ * & * & \bar{U}_2 \end{bmatrix} > 0 \tag{47}$$

$$\begin{bmatrix} \bar{\tilde{\Sigma}}_i & \bar{\Sigma}_1^T & \epsilon\bar{\Sigma}_2 & \bar{\square}_1 & \bar{\square}_2 \\ \bar{\Sigma}_1 & -\epsilon I & 0 & 0 & 0 \\ \epsilon\bar{\Sigma}_2^T & 0 & -\epsilon I & 0 & 0 \\ \bar{\square}_1^T & 0 & 0 & -I & 0 \\ \bar{\square}_2^T & 0 & 0 & 0 & I \end{bmatrix} < 0,\ \forall i = 1,2 \tag{48}$$

$$\begin{bmatrix} \bar{V}_0 & \bar{L}_0 + \bar{\psi}_0 & \bar{N}_0 \\ * & \bar{T}_{1_{11}} & \bar{T}_{1_{12}} + \bar{X}_0 \\ * & * & \bar{T}_{1_{22}} \end{bmatrix} \geq 0 \tag{49}$$

$$\begin{bmatrix} \bar{V}_i & \bar{L}_i + \bar{\psi}_1 & \bar{N}_i \\ * & \bar{T}_{2_{11}} & \bar{T}_{2_{12}} + \bar{X}_i \\ * & * & \bar{T}_{2_{22}} \end{bmatrix} \geq 0,\ \forall i = 1,2 \tag{50}$$

$$\begin{bmatrix} \bar{W}_0 & \bar{L}_0 + \bar{\varphi}_0 \\ * & \bar{Z}_1 \end{bmatrix} \geq 0 \tag{51}$$

$$\begin{bmatrix} \bar{W}_i & \bar{L}_i + \bar{\varphi}_1 \\ * & \bar{Z}_2 \end{bmatrix} \geq 0, \quad \forall i = 1,2 \tag{52}$$

Where

$$\bar{\tilde{\Sigma}}_i = \bar{\tilde{\pi}}_1 + \bar{\pi}_{2i} + \bar{\pi}_{2i}^T + \bar{\pi}_{3_0} + \bar{\pi}_{3_0}^T + \eta_m\bar{V}_0 + \hat{\eta}\ \ \bar{V}_i + \frac{\eta_m^2}{2}\bar{W}_0 + \bar{\eta}\ \ \bar{W}_i\ ,$$

$\forall i = 1,2$

$$\bar{\pi}_{2i} = [\bar{N}_0 + \eta_m\bar{L}_0 + \hat{\eta}\ \ \bar{L}_i \quad -\bar{N}_0 + \bar{N}_1 \quad -\bar{N}_2 \quad 0 \quad 0 \quad 0 \quad -\bar{N}_1 + \bar{N}_2 \quad 0]$$

$$\bar{\tilde{\pi}}_1 = \begin{bmatrix} \overline{\widetilde{(1,1)}} & \overline{\widetilde{(1,2)}} \\ * & \overline{\widetilde{(2,2)}} \end{bmatrix}, \qquad \overline{\widetilde{(1,1)}} = \begin{bmatrix} \bar{\tilde{\square}}_{11} & \bar{\square}_{12} & \bar{\square}_{13} & \bar{\square}_{14} \\ * & \bar{\square}_{22} & \bar{\square}_{23} & \bar{\square}_{24} \\ * & * & \bar{\square}_{33} & \bar{\square}_{34} \\ * & * & * & \bar{\square}_{44} \end{bmatrix}$$

$$\bar{\tilde{\square}}_{11} = \bar{P}_{14} - \bar{R}_1 + \bar{P}_{14}^T - \bar{R}_1^T + \bar{Q}_{1_{11}} + \eta_m\bar{T}_{1_{11}} + \hat{\eta}\ \ \bar{T}_{2_{11}} + \bar{X}_0$$

$$\overline{\widetilde{(1,2)}} = \begin{bmatrix} \bar{P}_{12} & \bar{P}_{13} & 0 & 0 \\ \bar{P}_{22} - \bar{Q}_{1_{12}} + \bar{Q}_{2_{12}} & \bar{P}_{23} & 0 & 0 \\ \bar{P}_{23}^T & \bar{P}_{33} - \bar{Q}_{2_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{\widetilde{(2,2)}} = \begin{bmatrix} -\bar{Q}_{1_{22}} + \bar{Q}_{2_{22}} & 0 & 0 & 0 \\ * & -\bar{Q}_{2_{22}} & 0 & 0 \\ * & * & -\bar{X}_1 + \bar{X}_2 & 0 \\ * & * & * & 0 \end{bmatrix}$$

$$\bar{\Sigma}_2 = [-J^T, -\rho_2 J^T, -\rho_3 J^T, -\rho_4 J^T, -\rho_5 J^T, -\rho_6 J^T, -\rho_7 J^T, -\rho_8 J]^T$$

$$\bar{\Sigma}_1 = [H_1 X^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad H_2 Y \quad 0]$$

$$\bar{\pi}_{3_0} = \begin{bmatrix} -A_0X^T & 0 & 0 & X^T & 0 & 0 & -B_0Y & -E\ X^T \\ -\rho_2A_0X^T & 0 & 0 & \rho_2X^T & 0 & 0 & -\rho_2B_0Y & -\rho_2E\ X^T \\ -\rho_3A_0X^T & 0 & 0 & \rho_3X^T & 0 & 0 & -\rho_3B_0Y & -\rho_3E\ X^T \\ -\rho_4A_0X^T & 0 & 0 & \rho_4X^T & 0 & 0 & -\rho_4B_0Y & -\rho_4E\ X^T \\ -\rho_5A_0X^T & 0 & 0 & \rho_5X^T & 0 & 0 & -\rho_5B_0Y & -\rho_5E\ X^T \\ -\rho_6A_0X^T & 0 & 0 & \rho_6X^T & 0 & 0 & -\rho_6B_0Y & -\rho_6E\ X^T \\ -\rho_7A_0X^T & 0 & 0 & \rho_7X^T & 0 & 0 & -\rho_7B_0Y & -\rho_7E\ X^T \\ -\rho_8A_0X^T & 0 & 0 & \rho_8X^T & 0 & 0 & -\rho_8B_0Y & -\rho_8E\ X^T \end{bmatrix}$$

$$\bar{\square}_1 = [CX^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad DY \quad 0]^T$$

$$\bar{\square}_2 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \gamma X^T]^T$$

and the rest of the elements ($\bar{\square}_{12}, \dots, \bar{\square}_{44}$) is equivalent to $\square_{12}, \cdots, \square_{44}$.

Proof: By schur complement (12) is equivalent to

$$\begin{bmatrix} \tilde{\Sigma}_i & \Sigma_1^T & \epsilon\Sigma_2 & \square_1 & \square_2 \\ \Sigma_1 & -\epsilon I & 0 & 0 & 0 \\ \epsilon\Sigma_2^T & 0 & -\epsilon I & 0 & 0 \\ \square_1^T & 0 & 0 & -I & 0 \\ \square_2^T & 0 & 0 & 0 & I \end{bmatrix} < 0,\ \forall i = 1,2 \tag{53}$$

Where

$$\tilde{\Sigma}_i = \tilde{\pi}_1 + \pi_{2i} + \pi_{2i}^T + \pi_{3_0} + \pi_{3_0}^T + \eta_m V_0 + \hat{\eta}\ \ V_i + \frac{\eta_m^2}{2}W_0 + \bar{\eta}\ \ W_i\ ,$$

$$\tilde{\pi}_1 = \begin{bmatrix} \widetilde{(1,1)} & \widetilde{(1,2)} \\ * & \widetilde{(2,2)} \end{bmatrix}, \quad \widetilde{(1,1)} = \begin{bmatrix} \tilde{\square}_{11} & \square_{12} & \square_{13} & \square_{14} \\ * & \square_{22} & \square_{23} & \square_{24} \\ * & * & \square_{33} & \square_{34} \\ * & * & * & \square_{44} \end{bmatrix}$$

$$\tilde{\square}_{11} = P_{14} - R_1 + P_{14}^T - R_1^T + Q_{1_{11}} + \eta_m T_{1_{11}} + \hat{\eta}\ \ T_{2_{11}} + X_0$$

$$\widetilde{(1,2)} = \begin{bmatrix} P_{12} & P_{13} & 0 & 0 \\ P_{22} - Q_{1_{12}} + Q_{2_{12}} & P_{23} & 0 & 0 \\ P_{23}^T & P_{33} - Q_{2_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{(2,2)} = \begin{bmatrix} -Q_{1_{22}} + Q_{2_{22}} & 0 & 0 & 0 \\ * & -Q_{2_{22}} & 0 & 0 \\ * & * & -X_1 + X_2 & 0 \\ * & * & * & 0 \end{bmatrix}$$

$$\square_1 = [C \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad DK \quad 0]^T$$

$$\square_2 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \gamma I]^T$$

Let $M = [M_1^T \quad M_2^T \quad M_3^T \quad M_4^T \quad M_5^T \quad M_6^T \quad M_7^T \quad M_8^T]^T$. Replace $M_1 = M_0$, $M_2 = \rho_2 M_0$, $M_3 = \rho_3 M_0$, $M_4 = \rho_4 M_0$, $M_5 = \rho_5 M_0$, $M_6 = \rho_6 M_0$, $M_7 = \rho_7 M_0$, $M_8 = \rho_8 M_0$.Feasibility of inequality (53) implies that M_0 is nonsingular. Let $X = M_0^{-1}$ then pre and postmultiply simultaneously the two side of (53) with diag$[X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad I \quad I \quad I \quad I]$, (13-14) with diag$[X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X]$, (15-16) with diag$[X \quad X \quad X \quad X \quad X \quad X \quad X \quad X \quad X]$ and (11) with diag$[X \quad X \quad X \quad X \quad X \quad X \quad X]$ and its transpose, respectively. Therefore the inequalities (11-16) leads to inequalities (47-52) with $Y = KX^T$. ■

IV. Illustrative example

An illustrative example is presented to attest the effectiveness of our propose approach in comparing with previous methods in the literatures. The YALMIP Toolbox is utilized to solve the LMI feasibility problems [10].

Example 1: Consider the following system with norm-bounded uncertainty controlled over a network [5-6]:

$$\dot{x}(t) = \left[\begin{bmatrix} -1 & 0 & -0.5 \\ 1 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} + \Delta A(t)\right] x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \omega(t)$$

$$Z(t) = [1 \quad 0 \quad 1]x(t) + 0.1u(t) \qquad (54)$$

where $\|\Delta A(t)\| \leq 0.01$.

Choose $= 0.1I,\ H_1 = 0.1I, H_2 = 0,\quad \rho_2 = 0.01,\ \rho_3 = 0.01,\ \rho_4 = 152,\ \rho_5 = 0.01,\ \rho_6 = 0.01,\ \rho_7 = 0.01$ and $\rho_8 = 0.1$.

In Table 1, the minimum disturbance attenuation level corresponding to the rival design methods are compared for different values of η_m and η .

Table I: Minimum disturbance attenuation level corresponding to the different design methods for different values of $\eta_m$and η.

η_m	η	γ		
		[5]	[6]	Proposed Method
0.1	0.5	1.843	1.714	1.475
0.3	0.7	2.642	2.455	2.175

Figure 2 and 3 show the simulation results of system (54) with state feedback controller $K = -[0.856 \ \ 0.00234 \ \ 1.8562]$ and $0.1 \leq \eta(t) \leq 0.5$. The initial values of the states are $x_1(0) = 0.1,\ x_2(0) = -0.1$ and $x_3(0) = 0.8$ and the disturbance signal $\omega(t)$ is as follows:

$$\omega(t) = \begin{cases} 0.3, & 2 \leq t \leq 4 \\ 0, & otherwise \end{cases}$$

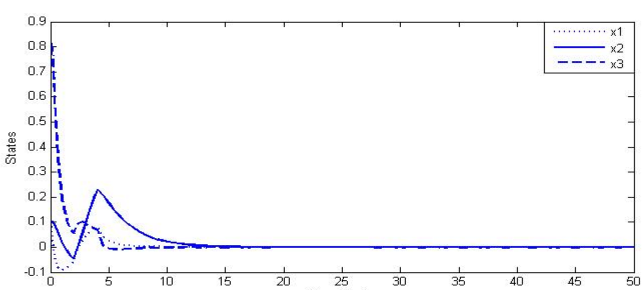

Figure.2: Simulation Results (States' Trajectory)

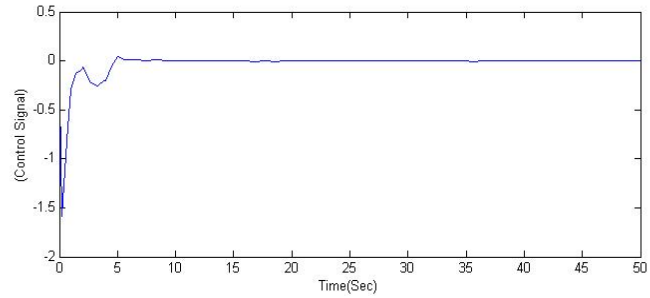

Figure.3: Simulation results (Control signal)

V. Conclusion

This paper proposed a new approach to synthesis robust H_∞ state feedback controller for the linear time invariant system which is controlled via communication network, considering interval time delay, data packet dropout, disturbance input and parameter uncertainties effects. A novel augmented Lyapunov-Krasovskii functional and free weight matrices are introduced to achieve less conservative results compared with previous methods in the literature. A set of linear matrix inequalities (LMIs) was derived to design controller gain. An illustrative example demonstrated the reliability of the proposed method.

References

[1] J. P. Hespanha, P. Naghshtabrizi and Y. Xu, "A Survey of recent results in Networked Control Systems," Proceedings of IEEE,vol. 95, pp . 138-162, 2007.

[2] M. Yu, L. Wang, T. Chu and F. Hao, "Stabilization of networked control systems with data packet dropout and transmission delays: continuous-time case," European Journal of Control, vol.11, pp. 40-49, 2005.

[3] D. Yue, Q.L. Han, and C. Peng, "State feedback controller design of networked control systems," *IEEE Transactions. Circuits and Systems II* : Express Briefs, vol. 51, pp. 640-644, 2004.

[4] Bin Tang, Guo-Ping Liu and Wei-HuaGui, "Improvement of state feedback controller design for networked control systems," *IEEE Transaction. Circuits and Systems II : Express Briefs*, vol. 55, pp. 464–468, 2008.

[5] D. Yue, Q.L. Han and J. Lam, "Netwrok-based robust H_∞ control of systems with uncertainty, ''*Automatica*, vol. 41,pp. 999-1007, 2005.

[6] S.H. Kim, P. Park and C. Jeong, "Robust H_∞ stabilization of networked control systems with packet analyser,"*IET Control Theory and Applications*, vol. 4, pp. 1828-1837, 2010.

[7] F. Yang, Z. Wang, Y.S. Hung and M. Gani, "H_∞ Control for networked control systems with random communication delays,"*IEEE Transactions . Automatic Control* ,vol. 51,pp. 511-518, 2006.

[8] P. Siler and R. Sengupta, "An H_∞ approach to networked control systems, *IEEE Transaction. Automatic Control*, " vol. 50,pp. 356-364, 2005.

[9] H. Gao, Member IEEE and T. Chen, Fellow IEEE, "Network-based H_∞ output tracking control,"*IEEE Transactions on Automatic Control*, vol. 53, pp. 655-666, 2008.

[10] J. Lofberg, "Yalmip: A toolbox for modeling and optimization in MATLAB,"*Proceeding of the CACSD Conference*, *Taipei, Taiwan*, 2004.

[11] P. Gahinet, A. Nemirovskii, A.J. Laub and M. Chilali, " *LMI Control Toolbox User's Guide*, " Natick, MA: Math Works, 1995.