Decomposition algebras and axial algebras

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Preface

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Introduction

Historical context

Non-associative algebras play an important role in many areas of mathematics. The study of Lie algebras is the most well-known example. They were introduced to study infinitesimal transformations but they are crucial in almost every branch of mathematics and even in theoretical physics. Also other classes of non-associative algebras have been proven to be very fruitful tools in other areas. Jordan algebras, for instance, formalize the notion of observables in quantum mechanics but they also played a crucial role in Zel'manov's solution to the restricted Burnside problem in group theory. Although we will encounter Lie algebras and Jordan algebras, our main goal is to explore axial algebras, a new type of non-associative algebra.

Perhaps one of the most spectacular connections between finite group theory and non-associative algebras is the construction of the monster group as the automorphism group of the Griess algebra. This algebra has dimension 196884 which is, as John McKay observed, also the first non-trivial coefficient in the Fourier series expansion of the j-function, a modular form. This unexpected connection between the monster group and modular functions led to a series of conjectures referred to as the monstrous moonshine conjectures. These conjectures were made by John H. Conway and Simon P. Norton [CN79]. They were solved by Richard E. Borcherds earning him the Fields medal in 1998 [Bor92]. Borcherds' solution used the theory of vertex operator algebras (VOA's). These algebras were originally studied, in a less formal way, by physicists, in the area of conformal field theory. In fact, the Griess algebra can be interpreted in such a vertex operator algebra, called the moonshine module VOA [FLM84, FLM88].

In 2009, Alexander A. Ivanov introduced Majorana algebras in an attempt to study the Griess algebra in an axiomatic way without the enveloping VOA [Iva09]. His approach is based on the study of idempotents in the Griess algebra and motivated by the following observation by John H. Conway [Con85]. A certain conjugacy class of involutions of the monster group is in one-to-one correspondence with a class of idempotents of the Griess algebra A. The adjoint action $\mathrm{ad}_e \colon A \to A \colon a \mapsto ea$ of such an idempotent $e \in A$ is diagonalizable with eigenvalues 1, 0, $\frac{1}{4}$

and $\frac{1}{32}$. Moreover the multiplication of eigenvectors is restricted by certain rules, called fusion rules. By definition, Majorana algebras are real algebras generated by such idempotents, called axes.

Using Majorana algebras, Alexander A. Ivanov, Dmitrii V. Pasechnik, Àkos Seress and Sergey Shpectorov reproved a result of Shinya Sakuma, originally formulated within the context of vertex operator algebras [Sak07, IPSS10]. It states that every Majorana algebra generated by two axes is contained in one out of nine isomorphisms classes, called the Norton-Sakuma algebras. Each of these is isomorphic to a 2-generated subalgebra of the Griess algebra. Majorana theory has also been used to describe other subalgebras of the Griess algebra [CRI14, Dec14, FIM16a, FIM16b, IPSS10, IS12b, IS12a, Iva11a, Iva11b].

It is common in mathematics that, after an important theorem is proven for the first time, a lot of work is done to refine its hypotheses. Many times this leads to interesting new definitions and insights. Axial algebras find their origin in a generalization of Sakuma's theorem. They were introduced by Jonathan I. Hall, Felix Rehren and Sergey Shpectorov [HRS15a, HRS15b]. On the one hand, they are generalizations of Majorana algebras relaxing some of its axioms and defined over an arbitrary field. On the other hand, they can be seen as generalizations of commutative, associative algebras and Jordan algebras as well. Their defining property is that they are generated by idempotents that lead to decompositions into eigenspaces. The multiplication of eigenvectors is restricted by a fusion law. The Peirce decompositions of associative and Jordan algebras are important instances of such decompositions.

Axial algebras have been further developed since then and new examples were constructed. Matsuo algebras, a class of algebras arising from 3-transposition groups fit nicely into the framework of axial algebras. Their fusion laws resemble the one from Jordan algebras and a systematic study of axial algebras with these fusion laws has been carried out [HRS15a, HSS18, DMR17]. Computer algorithms were also developed to construct new examples and to test conjectures [Ser12, PW18, MS20].

However, the study of axial algebras has just begun and is waiting for a whole new theoretical framework. One of the goals of this dissertation, is to provide such a theoretical framework. We will introduce decomposition algebras and axial decomposition algebras as a means to study all algebras reminiscent of axial algebras. The usefulness of this approach is further emphasized by the fact that these decomposition algebras form a nice category.

Throughout the story of axial algebras, the connection with group theory has always been very important. On the level of VOA's, Masahiko Miyamoto observed the existence of involutions corresponding to conformal vectors in vertex operator algebras [Miy96]. For the moonshine module VOA these involutions generate the monster group. This connection also exists for Majorana algebras, axial algebras and our decomposition algebras. We exploit the fact that decomposition algebras form a category to strengthen this connection between groups and axial algebras.

We introduce modules over (axial) decomposition algebras. Also for modules, there exists a deep connection with groups. Our results become especially interesting for Matsuo algebras.

Within the theory of axial algebras, it is an important question whether or not a group is connected with an interesting axial algebra. Moreover, new examples can provide interesting new insights. We suggest a few general ways to construct new examples. We also provide explicit constructions for algebras for the complex Chevalley groups of simply laced type. This includes the description of a 3876-dimensional algebra on which the complex Chevalley group of type E_8 acts by automorphisms. The existence of such an algebra was proven by Skip Garibaldi and Robert M. Guralnick in 2015 [GG15], unrelated to the theory of axial algebras. However, their proof is not constructive and, to the best of our knowledge, no explicit construction for this algebra was known.

Although we have gained a better understanding of the structure of axial (decomposition) algebras, many questions remain unsolved. We present some ideas for further investigation. We discuss how to "build" new axial decomposition algebras from other ones and also provide suggestions for the construction of algebras for the Lyons group and third Janko group.

Outline

The first chapter provides the reader with some necessary background and recalls some important definitions and facts that will be used throughout the text.

We give some more information on Peirce decompositions, the Griess algebra and axial algebras in Chapter 2. We introduce *fusion laws* and *decomposition algebras* as a natural generalization of axial algebras. They remedy three limitations of axial algebras. (1) They separate fusion laws from specific values in a field, thereby allowing repetition of eigenvalues. (2) They allow for decompositions that do not arise from multiplication by idempotents. (3) They admit a natural notion of homomorphisms, making them into a nice category. We explain how axial algebras fit into this new categorical framework by defining *axial decomposition algebras* and homomorphisms between them. In Sections 2.8 and 2.9 we present an important source of examples of (axial) decomposition algebras. This is very closely related to *representation theory* and to the theory of *association schemes* via *Norton algebras*. At the end of this chapter, we thoroughly explore the category of fusion laws and decomposition algebras. Most of the results of this chapter are based on [DMPSVC20].

The important connection between decomposition algebras and groups is discussed in Chapter 3. The results in this chapter are based on the two articles [DMVC20a, DMPSVC20]. We explain how *gradings* of fusion laws can be naturally interpreted as morphisms within the category of fusion laws. If the fusion law of a decomposition algebra is graded, then this gives rise to a group, the

Miyamoto group. We explain why this connection between decomposition algebras and groups is not functorial. However, we also introduce a more universal connection (the *universal Miyamoto group*) which turns out to be functorial under some mild conditions. This is the subject of Proposition 3.7.2 and Theorems 3.7.8 and 3.7.10. We discuss the Miyamoto group of the general examples introduced in Chapter 2. As another illustration of the theory, we explain how the universal Miyamoto group of a *Matsuo algebra* can be naturally interpreted as the universal 3-transposition group connected with the Matsuo algebra.

Chapter 4 is devoted to the study of modules over (axial) decomposition algebras. We introduced modules over axial algebras in [DMVC20a]. In this chapter we generalize the concept of a *module* to the framework of decomposition algebras. We vitalize the connection between decomposition algebras and groups by relating well-behaved modules over decomposition algebras to representations of the universal Miyamoto group. This connection is especially interesting for modules over Matsuo algebras.

In Chapter 5, we aim to shed light on the structure of the algebra predicted by Skip Garibaldi and Robert M. Guralnick. More precisely, we will give an explicit construction of this algebra and show that we can give it the structure of an axial decomposition algebra. Moreover, this algebra turns out to be part of a larger class of algebras. In fact, the construction can not only be applied to the simple group of type E_8 but to any simple group of Lie type of type ADE. Each of these will be given the structure of a decomposition algebra.

Chapter 6 presents some unfinished work and provides some ideas for further investigation. We introduce *expansions* as a natural generalization of split extensions. This concept helps to describe the structure of some axial decomposition algebras and non-associative algebras in general. In Section 6.2, we suggest a way to construct an axial decomposition algebra for two other sporadic simple groups: the Lyons group and third Janko group.

Appendix A gives an overview of the most important definitions and results. A dutch summary of this dissertation can be found in Appendix B.

1

Preliminaries

In this chapter, we provide the reader with the necessary background that we will need in this dissertation. We recall some standard definitions and facts and provide references for them.

1.1 Category theory

One of the main goals of this dissertation is to provide a categorical framework to study axial algebras. In this section, we provide the necessary background and notation on category theory. A standard reference on the topic is [ML98], though we will usually refer to [Lei14]. The latter is an excellent introduction for anyone unacquainted with category theory.

We start by giving the definition of a category itself. A category formalizes the idea of studying objects and the connections between them.

Definition 1.1.1. A *category* A consists of:

- a collection of objects;
- for any two objects X and Y a collection $\mathsf{Hom}_A(X,Y)$ of morphisms from X to Y;
- for any three objects X, Y and Z, a map
$$\mathsf{Hom}_A(X,Y) \times \mathsf{Hom}_A(Y,Z) \to \mathsf{Hom}_A(X,Z)\colon (f,g) \mapsto g \circ f,$$
called the *composition* of morphisms;
- a morphism $\mathsf{id}_X \in \mathsf{Hom}_A(X,X)$ for any object X, called the *identity on* X;

and satisfies the following two axioms:

- associativity of composition, i.e. $(f \circ g) \circ h = f \circ (g \circ h)$ for each $f \in \mathsf{Hom}_A(Y,Z)$, $g \in \mathsf{Hom}_A(X,Y)$ and $h \in \mathsf{Hom}_A(W,X)$;

- left and right unital, i.e. $(f \circ \mathsf{id}_X) = f$ and $(\mathsf{id}_X \circ g) = g$ for all $f \in \mathsf{Hom}_A(X,Y)$ and $g \in \mathsf{Hom}_A(Y,X)$.

By $X \in A$ we mean that X is an object of A. If the category A is clear from the context we write $\mathsf{Hom}(X,Y)$ instead of $\mathsf{Hom}_A(X,Y)$. We also write $\mathsf{End}(X)$ instead of $\mathsf{Hom}(X,X)$. If $f \in \mathsf{Hom}(X,Y)$ is a morphism from X to Y we will usually denote it by $f\colon X \to Y$ or $X \xrightarrow{f} Y$.

Notation 1.1.2. Note that the composition $g \circ f$ (sometimes written as gf) means that we apply f before g. This is consistent with our choice to use use functional notation for our maps and morphisms, i.e., when $\varphi\colon A \to B$ is a map between sets, we denote the image of an element $a \in A$ by $\varphi(a)$ or φa. Consequently, we will denote conjugation of group elements on the left:

$$^{g}h := ghg^{-1}.$$

Remark 1.1.3. We deliberately speak of collections of objects and morphisms since they do not have to be sets. For most categories that we will encounter, the objects will not form a set. However, if they do, then we call the category *small*. On the other hand, the collections $\mathsf{Hom}(X,Y)$ will usually be sets. Such a category is called *locally small*.

Let us give a few well-known examples of categories.

Example 1.1.4. (i) The collection of all sets together with all maps between them form a category that we will denote by Set. Composition of morphisms is simply composition of maps between sets.

(ii) The category Grp consists of all groups and the group homomorphisms between them.

(iii) Let R be a commutative ring with identity. Then the R-modules form a category with their respective morphisms. We denote this category by R-Mod.

Definition 1.1.5. A category A whose objects and morphisms are subcollections of those of another category B is called a *subcategory*. It is a *full subcategory* if for any two objects of A, the morphisms between them are the same in A and B.

Example 1.1.6. The category AbGrp of abelian groups together with the group homomorphisms between them is a full subcategory of Grp.

The categorical definition of epimorphisms, monomorphisms and isomorphisms is given as follows. For the categories of the previous example they coincide with their usual definitions.

Definition 1.1.7. Let $X \xrightarrow{f} Y$ be a morphism in a category.

(i) We say that f is *monic* or a *monomorphism* if it is left-cancellative. This means that for all objects Z and all morphisms $g_1, g_2 \in \mathsf{Hom}(Z, X)$ such that $f \circ g_1 = f \circ g_2$, we must have $g_1 = g_2$.

(ii) Similarly, we say that f is *epic* or an *epimorphism* if it is right-cancellative, i.e. $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all $g_1, g_2 \in \mathsf{Hom}(Y, Z)$.

(iii) If there exists a morphism $Y \xrightarrow{g} X$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$, then f is called an *isomorphism*. If an isomorphism between two objects X and Y exists then we say that X and Y are *isomorphic* and we write $X \cong Y$. The isomorphisms contained in $\mathsf{End}(X)$ form a group, the *automorphism group* $\mathsf{Aut}(X)$ of X.

If morphisms are in particular maps between sets, then we also have the notion of injective and surjective maps. Often, but not always, this coincides with the definition of monomorphisms and epimorphisms. Also, a morphism that is both monic and epic does not have to be an isomorphism; see Lemma 2.10.2 for a counterexample.

Definition 1.1.8. An object I is called *initial* if for any other object X there exists a unique morphism $I \longrightarrow X$. Dually, an object T is a *terminal object* if there always exists a unique morphism $X \longrightarrow T$. An object that is both initial and terminal is said to be a *zero object*.

It is an easy exercise to show that, if an initial or terminal object exists, then it must be unique up to isomorphism.

Example 1.1.9. In the category Set, the empty set $\emptyset$ is initial and any singleton (a set only containing one element) is terminal.

Functors are the essential concept to study connections between categories.

Definition 1.1.10. A functor $F\colon A \to B$ between two categories A and B associates to any object X of A an object $F(X)$ of B and to any morphism $X \xrightarrow{f} Y$ in A a morphism $F(X) \xrightarrow{F(f)} F(Y)$ in B such that:

- composition of morphisms is preserved: $F(f \circ g) = F(f) \circ F(g)$;
- the identity morphisms are preserved: $F(\mathsf{id}_X) = \mathsf{id}_{F(X)}$.

Usually, it will be obvious where morphisms should be mapped once we fix the image of the objects. If so, then we define a functor as

$$F\colon A \to B\colon X \rightsquigarrow F(X)$$

meaning that F maps the object X to the object $F(X)$ and maps morphisms $X \xrightarrow{f} Y$ to their naturally corresponding morphism $F(X) \xrightarrow{F(f)} F(Y)$.

Example 1.1.11. Each of the categories from Example 1.1.4 has a *forgetful functor* to Set mapping each object to its underlying set and each morphism to the corresponding set map.

We can define a functor $F\colon A \to B$ to be an isomorphism of categories if there exists a functor $G\colon B \to A$ such that both $F \circ G$ and $G \circ F$ fix all objects and morphisms. However, this condition is often too strong to be useful. The weaker notion of equivalence is more interesting. Its definition also allows us to introduce some more terminology about functors.

Definition 1.1.12. Let $F\colon A \to B$ be a functor.

(i) We say that F is *essentially surjective* if for every object Y of B there exists an object X of A such that $F(X) \cong Y$.

(ii) The functor F is said to be *full* (resp. *faithful*) if, for all objects X and Y of A, the map

$$F\colon \mathsf{Hom}(X,Y) \to \mathsf{Hom}(F(X), F(Y))\colon f \mapsto F(f)$$

is surjective (resp. injective).

(iii) A functor that is full, faithful and essentially surjective is called an *equivalence* of categories. Two categories are said to by *equivalent* if there exists an equivalence between them.

In the same way that functors fulfill the role of morphisms between categories, natural transformations are the morphisms between functors.

Definition 1.1.13. Let $F, G\colon A \to B$ be functors. A *natural transformation* $\eta\colon F \to G$ is a collection of morphisms $\eta_X\colon F(X) \to G(X)$ for all objects $X \in A$ such that for all morphisms $X \xrightarrow{f} Y$ of A the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd} .$$

For example, for any functor $F\colon A \to B$, we can define the *identity transformation* $\mathsf{id}_F\colon F \to F$ by $(\mathsf{id}_F)_X := \mathsf{id}_{F(X)}$ for all $X \in F$. If $F \xrightarrow{\theta} G$ and $G \xrightarrow{\theta} H$ are natural transformations then we can define the composition $\theta \circ \eta$ by $(\theta \circ \eta)_X = \theta_X \circ \eta_X$. The collection of functors between categories A and B, together with the natural transformations between them, therefore forms a category. In particular, it allows us to define isomorphisms between functors.

Remark 1.1.14. We can also define an equivalence of categories as a functor $F\colon A \to B$ for which there exists a functor $G\colon B \to A$ such that $G \circ F \cong \mathsf{id}_A$ and $F \circ G \cong \mathsf{id}_B$ [Lei14, Proposition 1.3.18, p. 34].

Adjunction between functors is an essential tool in category theory and in mathematics in general. Adjoint functors arise everywhere. A mathematician can benefit greatly from knowing its definition in order to spot connections between different constructions.

Definition 1.1.15. Let $F\colon A \to B$ and $G\colon B \to A$ be functors. We say that F is a *left adjoint* to G and G is a *right adjoint* to F, and write $F \dashv G$, if

$$\mathsf{Hom}(F(X), Y) \cong \mathsf{Hom}(X, G(Y))$$

naturally in $X \in A$ and $Y \in B$. Let us explain what we mean by "naturally". If we denote the bijection $\mathsf{Hom}(F(X), Y) \to \mathsf{Hom}(X, G(Y))$ by θ, then we want for morphisms $F(X) \xrightarrow{g} Y \xrightarrow{q} Y'$ that $\theta(q \circ g) = G(q) \circ \theta(g)$. And similarly, for the inverse θ^{-1} of θ and morphisms $X' \xrightarrow{p} X \xrightarrow{f} G(Y)$, we want $\theta^{-1}(f \circ p) = \theta^{-1}(f) \circ F(p)$. This condition can also be formulated as a natural isomorphism between functors [Lei14, Chapter 4].

Not every functor has a left or right adjoint, but if it does then it is unique up to isomorphism.

Example 1.1.16. (i) Forgetful functors usually have a left adjoint corresponding to a "free" construction. For example, consider the forgetful functor $F\colon \mathsf{Grp} \to \mathsf{Set}$ mapping each group to its underlying set. This functor has a left adjoint that sends the set X to the free group over X.

(ii) Let R be a commutative ring. Note that for every two R-modules M and N we can give $\mathsf{Hom}_R(M, N)$ the structure of an R-module by setting $(rf)(m) \coloneqq r(f(m))$ for all $r \in R$, $f \in \mathsf{Hom}(M, N)$ and $m \in M$. Moreover, if $\varphi\colon N_1 \to N_2$ is a morphism of R-modules then this induces a morphism $\mathsf{Hom}(M, N_1) \to \mathsf{Hom}(M, N_2)\colon f \mapsto \varphi \circ f$ of R-modules. Therefore, for every R-module M, we have a functor $F\colon R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$ such that $F(N) = \mathsf{Hom}(M, N)$ for all $N \in R\text{-}\mathsf{Mod}$. This functor has a left adjoint $G\colon R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$ that sends the R-module N to the tensor product $M \otimes N$. In fact, this defines the tensor product uniquely up to isomorphism. This adjunction is called the *tensor-hom adjunction.*

We finish this section by introducing limits and colimits. They provide a general framework to describe objects with some kind of universal property. Many mathematical constructions can be viewed as a limit or colimit in a category. First, we introduce diagrams.

Definition 1.1.17. Let A be any category and I a small category. We will usually represent I by a graph with a vertex for each object of I and an arrow for each morphism of I that is not an identity morphism. For example the graph $\bullet \longrightarrow \bullet$ represents the category with two objects X and Y and one morphism $X \to Y$ apart from the identity morphisms id_X and id_Y. A *diagram of shape I* is a functor $D\colon I \to A$.

We are now ready to give the general definition of a limit.

Definition 1.1.18. Let A be a category, I a small category and $D\colon I \to A$ a diagram.

(i) A *cone on D* is an object $X \in A$ together with a family

$$\big(X \xrightarrow{f_i} D(i)\big)_{i\in I}$$

of morphisms in A such that for all morphisms $i \xrightarrow{u} j$ in I the triangle

$$\begin{tikzcd} & D(i) \arrow[dd, "D(u)"] \\ X \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & \\ & D(j) \end{tikzcd}$$

commutes.

(ii) A *limit of D* is a cone $\big(L \xrightarrow{p_i} D(i)\big)_{i\in I}$ that is universal in the following way. For any other cone $\big(X \xrightarrow{f_i} D(i)\big)_{i\in I}$ there exists a unique morphism $X \xrightarrow{\bar{f}} L$ such that $p_i \circ \bar{f} = f_i$ for all $i \in I$. The morphims p_i are called the *projections* of the limit.

The dual notion of a colimit can be obtained from the definition of a limit "by reversing the arrows". This can be formalized by introducing the opposite category.

Definition 1.1.19. Let A be a category. We define the *opposite* or *dual* category A^{op} as the category with the same objects as A but with the arrows inverted. Namely for every two objects $X, Y \in A^{\mathrm{op}}$ we let

$$\mathsf{Hom}_{A^{\mathrm{op}}}(X,Y) := \mathsf{Hom}_A(Y,X)$$

and define composition as

$$\mathsf{Hom}_{A^{\mathrm{op}}}(X,Y) \times \mathsf{Hom}_{A^{\mathrm{op}}}(Y,Z) \to \mathsf{Hom}_{A^{\mathrm{op}}}(X,Z)\colon (f,g) \mapsto f \circ g$$

where $f \circ g$ is the composition of f and g in A. If $F\colon A \to B$ is a functor then it also defines a functor $F^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$ such that $F^{\mathrm{op}}(X) = F(X)$ for all $X \in A^{\mathrm{op}}$ and $F^{\mathrm{op}}(f) = F(f)$ for all $f \in \mathsf{Hom}_{A^{\mathrm{op}}}(X,Y)$.

Now we can dualize the notion of a limit.

Definition 1.1.20. Let A be a category, I a small category and $D\colon I \to A$ a diagram. Consider the functor $D^{\mathrm{op}}\colon I^{\mathrm{op}} \to A^{\mathrm{op}}$. A *cocone on* D is cone on D^{op}. A *colimit* of D is a limit of D^{op}.

It is not hard to show that, if a (co)limit exists, it must be unique up to isomorphism. Let us give a few important examples of limits and colimits.

Example 1.1.21 (Products and coproducts)**.** Let I be a small category without any morphisms apart from the identity morphisms. A diagram $D\colon I \to A$ is then defined by a family of objects $\big(D(i)\big)_{i\in I}$. A limit (resp. colimit) of D is called the product (resp. coproduct) of $\big(D(i)\big)_{i\in I}$. For example, in the category Set the product of a family $(X_i)_{i\in I}$ of sets is the cartesian product $\prod_{i\in I} X_i$ and the coproduct is the disjoint union $\bigsqcup_{i\in I} X_i$.

Example 1.1.22 (Equalizers and coequalizers)**.** Consider a diagram of shape $\bullet \rightrightarrows \bullet$. It is defined by two parallel morphisms

$$X \underset{t}{\overset{s}{\rightrightarrows}} Y \ .$$

A cone of this diagram consists of objects and maps

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, shift left, "s"] \arrow[rr, shift right, "t"'] & & Y \end{tikzcd}$$

such that $s \circ f = g$ and $t \circ f = g$. A limit of this diagram is called an *equalizer*. We can view an equalizer as a morphism $f\colon L \to X$ such that $s \circ f = t \circ f$ and such that for every such morphism $f'\colon L' \to X$ there exists a unique map $\bar{f}\colon L' \to L$ such that $f \circ \bar{f} = f'$. It is not hard to show that in this case $f\colon L \to X$ has to be monic.

Dually, a *coequalizer* is a colimit of this diagram.

For example, the kernel $\ker(\theta) \to X$ of group homomorphism $X \xrightarrow{\theta} Y$ is the equalizer in Grp of θ and the group homomorphism $X \to Y\colon x \mapsto 1$.

Example 1.1.23 (Pullbacks and pushouts)**.** A diagram of shape $\bullet \longrightarrow \bullet \longleftarrow \bullet$ consists of objects and morphisms

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"] & Z \end{tikzcd}$$

in A. A cone of this diagram is a pair of morphisms $A \xrightarrow{f} X$, $A \xrightarrow{g} Y$ for which the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"] & Z \end{tikzcd}$$

The dual concept of a pullback is a *pushout*. It is the colimit of a diagram of shape $\bullet \longleftarrow \bullet \longrightarrow \bullet$.

It is an important question whether limits and colimits exist in a given category. We introduce the following terminology.

Definition 1.1.24. A category is called *complete* if it has all limits. We say that it is *cocomplete* if it has all colimits.

It is not practical to check whether a category has all limits. Luckily, it suffices to check whether it has all products and equalizers.

Proposition 1.1.25 (Existence theorem for limits and colimits)**.** *Let A be a category.*

(i) *If A has all products and equalizers, then it is complete.*

(ii) *If A has all coproducts and coequalizers, then it is cocomplete.*

Proof. See [Lei14, Proposition 5.1.26, p. 121]. □

We finish this section by stating two facts about the interaction between functors and limits. First, we define what it means for a functor to preserve limits.

Definition 1.1.26. Let I be small category and $F\colon A \to B$ a functor. Then F *preserves limits of shape I* if for all diagrams $D\colon I \to A$ and all cones $\big(X \to D(i)\big)_{i\in I}$ on D we have that

$$\big(F(L) \xrightarrow{F(p_i)} F(D(i))\big)_{i\in I}$$

is a limit of $F \circ D$ when

$$\big(L \xrightarrow{p_i} D(i)\big)_{i\in I}$$

is a limit of D. A functor is said to *preserve limits* if it preserves limits of shape I for all small categories I. Dually, a functor F *preserves colimits* if F^{op} preserves limits.

Proposition 1.1.27 ([Lei14, Theorem 6.3.1, p. 158])**.** *Let $F \dashv G$ be an adjunction. Then F preserves colimits and G preserves limits.*

If $F\colon A \to B$ is a functor and f is a monomorphism in A, then $F(f)$ need not be monic. The following proposition gives a sufficient condition for when it does.

Proposition 1.1.28. *Let $F\colon A \to B$ be a functor. If F is faithful and preserves pullbacks (resp. pushouts), then $F(f)$ is a monomorphism (resp. epimorphism) for every monomorphism (resp. epimorphism) f of A.*

Proof. This is a corollary to [Lei14, Lemma 5.1.32, p. 123]. □

1.2 Algebras

In this section we will fix our terminology concerning algebras. We will define them as R-modules endowed with a bilinear multiplication. Throughout this section we will denote by R a commutative ring with identity $\mathbf{1}_R$. Since we assume R to be commutative, there is no need to make a distinction between left or right R-modules.

Let us recall a few important constructions of R-modules.

Definition 1.2.1. Let M be an R-module.

(i) Let $M^{\otimes n}$ be the n-th tensor power of M. Then we define the *n-th symmetric power* $S^n(M)$ of M as the quotient of $M^{\otimes n}$ by the submodule generated by the elements $m_1 \otimes \cdots \otimes m_n - m_{\pi(1)} \otimes \cdots \otimes m_{\pi(n)}$ for every permutation π of $\{1, \dots, n\}$ and every $m_1, \dots, m_n \in M$. We refer to $S^2(M)$ as the *symmetric square* of M.

(ii) Similarly, we define the *n-th alternating power* $\bigwedge^n(M)$ as the quotient by the submodule generated by the elements $m_1 \otimes \cdots \otimes m_n$ where two of the m_i are equal. We call $\bigwedge^2(M)$ the *alternating square* of M.

Although we assume all rings to be associative, an algebra, in general, does not have to be.

Definition 1.2.2. (i) We call an R-module A an *algebra* if it is equipped with an R-bilinear product

$$\theta\colon A \times A \to A.$$

We usually write $\theta(a, b)$ as $a \cdot b$ or simply as ab. Note that, since θ is bilinear, we can view it as a map $A \otimes A \to A$. If the bilinear map θ is not clear from the context, we will explicitly denote the algebra by the tuple (A, θ).

We call the algebra A *associative* if $(ab)c = a(bc)$ for all $a, b, c \in A$. Note that we will explicitly state when an algebra is associative. In fact, most of the algebras that we will encounter will not be associative. To emphasize this, we will sometimes say that an algebra is *non-associative* meaning that it is not necessarily associative.

We say that A is *commutative* if $ab = ba$ for all $a, b \in A$. In that case, the map θ can be seen as a morphism $S^2(A) \to A$ of R-modules.

A *unital* algebra is an algebra A for which there exists an element $\mathbf{1} \in A$ such that $\mathbf{1}a = a\mathbf{1} = a$ for all $a \in A$.

(ii) A *morphism* of algebras is a morphism $\varphi\colon A \to B$ of R-modules that preserves the product, i.e. $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. This defines a category R-Alg of R-algebras.

A subalgebra of an algebra A is a *submodule* S of A that is closed under the multiplication, i.e. $ab \in S$ for all $a, b \in S$. If S is a subset of A, then we denote by $\langle\!\langle S \rangle\!\rangle$ the smallest subalgebra containing S. This should not be confused with $\langle S \rangle$ which is the submodule generated by S.

A *left ideal* of A is a submodule I such that $ai \in I$ for all $a \in A$ and $i \in I$. Similarly, we define a *right ideal* as a submodule I for which $ia \in I$ for all $a \in A$ and $i \in I$. We say that a submodule is a *(two-sided) ideal* if it is both a left and right ideal. An algebra is called *simple* if it has no proper, non-zero, two-sided ideal.

(iii) For any element a of an algebra A we can define its *(left) adjoint*

$$\mathrm{ad}_a \colon A \to A \colon b \mapsto ab.$$

If A is an associative algebra then the map $A \to \mathsf{End}(A) \colon a \mapsto \mathrm{ad}_a$ is a morphism of algebras. However, for arbitrary algebras, this is not true.

The book [Sch66] by Richard D. Schafer is an excellent reference on (non-associative) algebras. Two important examples of non-associative algebras are Jordan algebras and Lie algebras.

Example 1.2.3. (i) A *Jordan algebra* is a commutative algebra A over a field k for which the Jordan identity holds:

$$(ab)(aa) = a(b(aa))$$

for all $a, b \in A$.

A typical example of a Jordan algebra is constructed as follows. Given an associative algebra A over a field k with characteristic $\mathrm{char}(k) \neq 2$, we can construct a Jordan algebra A^+ by equipping the k-vector space A with the *Jordan product*:

$$a \bullet b := \frac{1}{2}(ab + ba)$$

for all $a, b \in A$. More information about Jordan algebras can be found in the standard references [Jac68] and [McC04].

(ii) *Lie algebras* form another important class of non-associative algebras. It is customary to denote the product of a Lie algebra by square brackets [,]. We call an algebra over a field k a Lie algebra if the product is alternative, i.e. $[a, a] = 0$ for all $a \in A$, and satisfies the Jacobi identity:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all $a, b, c \in A$. Note that, since the product is alternative, we have $[a, b] = -[b, a]$ for all $a, b \in A$.

Also here there is a standard way to construct a Lie algebra out of an associative algebra A by defining the new product:

$$[a, b] := ab - ba$$

for $a, b \in A$. We will devote some more attention to Lie algebras in Section 1.3.

Definition 1.2.4. If $\big((A_i, \theta_i) \mid i \in I\big)$ is a family of algebras, then we define their sum as the R-module $A = \bigoplus_{i \in I} A_i$ together with the bilinear map θ that is the trivial extension of all the θ_i, this is $\theta(a, b) = \theta_i(a, b)$ for all $a, b \in A_i$ and $i \in I$ and $\theta(A_i, A_j) = 0$ for all $i \neq j$. Note that each of the A_i is an ideal of A.

Remark 1.2.5. (i) Most of the direct sums that we will consider will *not* be direct sums of algebras but will be direct sum decompositions of an algebra $A = \bigoplus_{i \in I} M_i$ as R-module. This means that every element of A can be uniquely written as a sum $\sum_{i \in I} m_i$ where $m_i \in M_i$ for all $i \in I$. In this case the submodules M_i of A need not be ideals or subalgebras of A.

(ii) We make the usual convention that the empty sum $\bigoplus_{i \in \emptyset} M_i$ of R-modules M_i is the zero module.

The study of a special type of bilinear form is crucial in the study of non-associative algebras (see e.g. Definition 1.3.22 or [Sch66, § II.4]). Associative algebras admitting such a bilinear bilinear form are called *Frobenius algebras*. We generalize this notion to non-associative algebras.

Definition 1.2.6. A *Frobenius algebra* is an algebra A endowed with a bilinear form $\langle\ ,\ \rangle \colon A \times A \to R$ for which

$$\langle a, bc \rangle = \langle ab, c \rangle$$

for all $a, b, c \in A$. Moreover, we assume that this form is non-degenerate, i.e.

$$A \to \mathsf{Hom}(A, R) \colon a \mapsto [b \mapsto \langle a, b \rangle]$$

and

$$A \to \mathsf{Hom}(A, R) \colon a \mapsto [b \mapsto \langle b, a \rangle]$$

are isomorphisms of R-modules. Such a form is called a *Frobenius*, *trace* or *associative form*. Note that some authors do not assume such a form to be non-degenerate. If the algebra product θ and the Frobenius form $\langle\ ,\ \rangle$ are not clear from the context, we will denote a Frobenius algebra as a triple $(A, \theta, \langle\ ,\ \rangle)$.

A *morphism* $\varphi \colon A \to B$ of Frobenius algebras is a morphism of algebras that preserves the bilinear form, this is $\langle \varphi(a), \varphi(b) \rangle = \langle a, b \rangle$ for all $a, b \in A$.

A Frobenius form for a unital, commutative algebra is automatically symmetric.

Proposition 1.2.7. *If $\langle\ ,\ \rangle$ is a Frobenius form for a unital, commutative algebra A, then this form has to be symmetric, i.e. $\langle a,b\rangle = \langle b,a\rangle$ for all $a,b \in A$.*

Proof. For all $a,b \in A$ we have

$$\langle a,b\rangle = \langle \mathbf{1}a,b\rangle = \langle \mathbf{1},ab\rangle = \langle \mathbf{1},ba\rangle = \langle \mathbf{1}b,a\rangle = \langle b,a\rangle. \qquad \square$$

Remark 1.2.8. Equivalently, we can view a unital commutative Frobenius algebra as an R-module A endowed with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle\colon A \times A \to R$ and a symmetric trilinear form $f\colon A \times A \times A \to R$ defined by $f(a,b,c) = \langle ab,c\rangle$ for all $a,b,c \in A$.

Example 1.2.9. (i) Let V be a finite-dimensional vector space over a field k. Let $\mathsf{End}(V)$ be the associative algebra of endomorphisms of V. Denote the trace map by $\mathrm{tr}\colon \mathsf{End}(V) \to k$. Then the bilinear form

$$B\colon \mathsf{End}(V) \times \mathsf{End}(V)\colon fg \mapsto \mathrm{tr}(fg),$$

is a Frobenius form for $\mathsf{End}(V)$. From the associativity of $\mathsf{End}(V)$ we immediately have $\mathrm{tr}((fg)h) = \mathrm{tr}(f(gh))$ for all $f,g,h \in \mathsf{End}(V)$. The proof of the non-degeneracy of this form is an easy exercise, see for example [Lam99, Example 16.57, p. 443].

(ii) If $\mathrm{char}(k) \neq 2$, then the form from above is also a Frobenius form for the Jordan algebra $\mathsf{End}(V)^+$. This follows readily from the well-known identity $\mathrm{tr}(fgh) = \mathrm{tr}(ghf)$ for all $f,g,h \in \mathsf{End}(V)$.

(iii) Suppose that, in addition, V itself is equipped with a non-degenerate bilinear form κ and that $\mathrm{char}(k) \neq 2$. Then we call an operator $f \in \mathsf{End}(V)$ *symmetric* if $\kappa(f(a),b) = \kappa(a,f(b))$ for all $a,b \in V$ and *antisymmetric* if $\kappa(f(a),b) = -\kappa(a,f(b))$ for all $a,b \in V$. Let S (resp. A) be the subspace of $\mathsf{End}(V)$ consisting of all symmetric (resp. antisymmetric) operators; then $\mathsf{End}(V) = S \oplus A$ as vector spaces. Then S is a subalgebra of the Jordan algebra $\mathsf{End}(V)^+$. Moreover $B(S,A) = 0$. Hence the restriction of B to S is non-degenerate. Therefore S is a Frobenius subalgebra of $\mathsf{End}(V)^+$.

1.3 Lie algebras

In Example 1.2.3 (ii), we already gave the definition of a Lie algebra. From now on we will always tacitly assume that Lie algebras are finite-dimensional as vector spaces. The goal of this section is to give a classification of the complex semisimple Lie algebras. We refer to [Hum72], [Ser01] or [Jac79] for more information on the subject. Another extensive standard reference on Lie algebras is the three part book [Bou89, Bou02, Bou05] by the Bourbaki group.

1.3.1 Root systems

As we will see in Section 1.3.2, root sytems play an essential role in the classification of complex semisimple Lie algebras. We introduce them here and state a few important facts about them. Throughout this section we will let E be a real Euclidean space, this is a finite-dimensional $\mathbb{R}$-vector space endowed with a positive definite symmetric bilinear form $\langle\ ,\ \rangle$.

Definition 1.3.1. (i) For each vector $\alpha \in E$, we write s_α for the reflection in the hyperplane orthogonal to α:

$$s_\alpha \colon E \to E \colon v \mapsto v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Note that these are orthogonal transformations of E.

(ii) A *root system* is a subset Φ of E such that the following axioms hold.

(R1) The set Φ is a finite set of non-zero vectors that spans E.

(R2) If $\lambda\alpha \in \Phi$ for some $\alpha \in \Phi$ and $\lambda \in \mathbb{R}$ then $\lambda \in \{\pm 1\}$.

(R3) For any $\alpha \in \Phi$, we have $s_\alpha(\Phi) = \Phi$.

(R4) For all $\alpha, \beta \in \Phi$, we have $\frac{2\langle \alpha,\beta\rangle}{\langle \beta,\beta\rangle} \in \mathbb{Z}$.

The elements of Φ are called *roots* and $\dim(E)$ is called the *rank* of the root system.

(iii) To each root $\alpha \in \Phi$, we associate the *coroot* $h_\alpha := \frac{2\alpha}{\langle \alpha,\alpha\rangle}$. By (R4), we have $\langle \beta, h_\alpha\rangle \in \mathbb{Z}$ for all $\beta \in \Phi$.

(iv) If Φ is a root system then we call the subgroup $W(\Phi)$ of $\mathsf{GL}(E)$ generated by the reflections s_α for $\alpha \in \Phi$ the *Weyl group* of Φ.

(v) An isomorphism of root systems Φ_1 and Φ_2 with corresponding Euclidean spaces E_1 and E_2 is an isomorphism $\varphi \colon E_1 \to E_2$ of vector spaces sending Φ_1 onto Φ_2 and such that $\langle \varphi(\alpha), \varphi(h_\beta)\rangle = \langle \alpha, h_\beta\rangle$ for all $\alpha, \beta \in \Phi_1$. Note that φ need not be an isomorphism of Euclidean spaces, i.e. need not preserve the inner product.

Example 1.3.2. Fig. 1.1 depicts the root systems of rank 2.

Lemma 1.3.3 ([Hum72, § 9.4])**.** *Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. If $\langle \alpha, \beta\rangle > 0$ (resp. $\langle \alpha, \beta\rangle < 0$) then $\alpha - \beta \in \Phi$ (resp. $\alpha + \beta \in \Phi$).*

Root systems have a special type of basis.

Proposition 1.3.4 ([Hum72, § 10.1 and § 10.3])**.** *For every root system Φ there exists a subset $\Delta \subseteq \Phi$ such that the following conditions hold:*

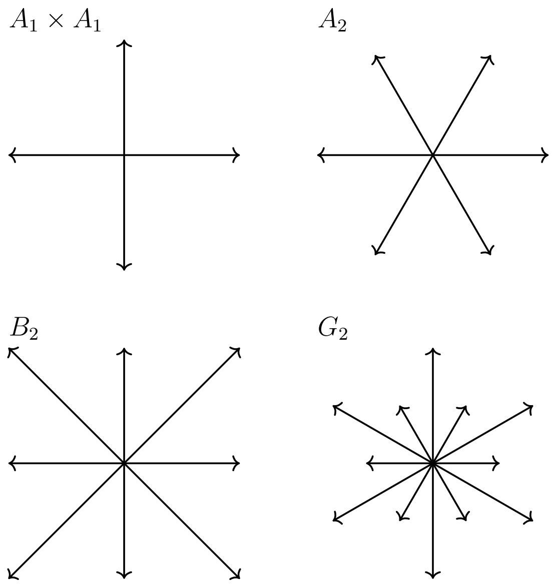


Figure 1.1: The root systems of rank 2.

(B1) Δ *is a basis of* E*,*

(B2) each root $\beta \in \Phi$ *can be written as* $\beta = \sum_{\alpha \in \Delta} c_\alpha \beta$ *with coefficients* c_α *all non-positive or non-negative integers.*

Such a subset Δ *is called a* base *for* Φ*. The Weyl group* $W(\Phi)$ *acts simply transitively on bases for* Φ*.*

Definition 1.3.5. Let Δ be a base for a root system Φ.

(i) Then we call a root $\beta \in \Phi$ positive (resp. negative) if all the coefficients c_α from Proposition 1.3.4 are non-negative (resp. non-positive). We denote the set of positive (resp. negative) roots by Φ^+ (resp. Φ^-). Note that for every root $\beta \in \Phi$ either $\beta \in \Phi^+$ or $\beta \in \Phi^-$. So Φ is the disjoint union of Φ^+ and Φ^-.

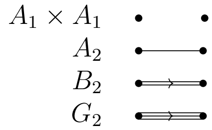
(ii) The base Δ induces a partial order $\preccurlyeq$ on E by defining $\lambda \preccurlyeq \mu$ if and only if $\mu - \lambda$ is a linear combination of the $\alpha \in \Delta$ with nonnegative coefficients. For example, we have $\Phi^+ = \{\alpha \in \Phi \mid 0 \preccurlyeq \alpha\}$.

A typical way to represent a root system is by giving its Dynkin diagram.

Definition 1.3.6. Let Φ be a root system and Δ a base for Φ. Then we can associate to Φ a graph, called its *Dynkin diagram*, having a vertex for each $\alpha \in$

Δ. We connect the vertices corresponding to $\alpha, \beta \in \Delta$ by $\langle \alpha, h_\beta \rangle \langle \beta, h_\alpha \rangle$ edges. We put an arrow on the edge from the vertex corresponding to α to the vertex corresponding to β if $\langle \alpha, \alpha \rangle > \langle \beta, \beta \rangle$. By Proposition 1.3.4, this diagram does not depend on the choice of Δ.

Example 1.3.7. The Dynkin diagrams for the examples from Example 1.3.2 are the following.



Definition 1.3.8. (i) A subset $\Psi \subseteq \Phi$ is called a *subsystem* if $s_\alpha(\Psi) \subseteq \Psi$ for all $\alpha \in \Psi$.

(ii) A *reflection subgroup* of a Weyl group $W(\Phi)$ is a subgroup generated by reflections s_α for $\alpha \in \Phi$. The roots $\alpha \in \Phi$ for which s_α is contained in this subgroup form a subsystem of Φ.

(iii) If $E = E_1 \oplus \cdots \oplus E_n$ is a direct sum of Euclidean spaces and Φ_i is a root system in E_i, then $\bigcup_{i=1}^n \Phi_i$ is a root system in E. We write it as $\Phi_1 \times \cdots \times \Phi_n$ and call this the *direct sum* of the root systems Φ_i for $1 \leq i \leq n$.

(iv) A root system Φ is called *irreducible* if Φ can not be partitioned into the union of two proper subsets Ψ_1 and Ψ_2 such that $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \Psi_1$ and $\beta \in \Psi_2$. Every root system that is not irreducible uniquely decomposes as a direct sum of irreducible root systems [Hum72, § 11.3].

(v) For any $\alpha \in \Phi$ we call $\langle \alpha, \alpha \rangle$ the *length* of α. We say that a root system Φ is *simply laced* if its Dynkin diagram has no multiple edges. For irreducible root systems this means that all roots must have the same length by the following proposition.

Proposition 1.3.9 ([Hum72, § 10.4])**.** *Let Φ be an irreducible root system and Δ a base of Φ.*

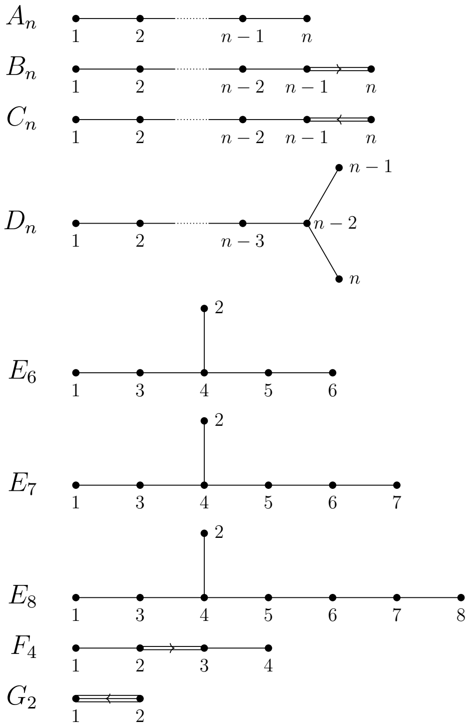
(i) *There exists a unique root $\omega \in \Phi$ such that $\alpha \preccurlyeq \omega$ for all $\alpha \in \Phi$. This root ω is called the* highest root *with respect to Δ.*

(ii) *The Weyl group $W(\Phi)$ acts transitively on roots with the same length. In particular it acts transitively on Φ if Φ is simply laced.*

Definition 1.3.10. The *extended Dynkin diagram* of a root system Φ is obtained in the same manner as the ordinary Dynkin diagram but with one extra vertex that corresponds to the negative of the highest root of Φ.

We are now ready to state the classification of root systems.

Proposition 1.3.11 ([Hum72, § 11.1 and § 11.4])**.** *A root system Φ is (up to isomorphism) completely determined by its Dynkin diagram. The possible Dynkin diagrams for irreducible root systems are the following.*

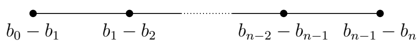


We give a construction for the irreducible simply laced root systems.

Example 1.3.12. (i) Consider a Euclidean space E of dimension $n+1$ and pick an orthonormal basis $b_0, b_1, \ldots, b_n$ for E. Then

$$\Phi = \{b_i - b_j \mid 0 \leq i, j \leq n, i \neq j\rangle\}$$

is a root system in the subspace consisting of the vectors $\sum_{i=0}^{n} \varepsilon_i b_i$ for which $\sum_{i=0}^{n} \varepsilon_i = 0$. The following vectors form a base for Φ.

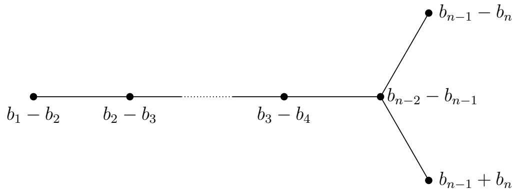


With respect to this base, we have $\Phi^+ = \{b_i - b_j \mid 0 \leq i < j \leq n\}$. This root system is of type A_n. The action of its Weyl group can be extended to E so that it permutes the basis elements $\{b_0, \ldots, b_n\}$. This defines an isomorphism with the symmetric group S_{n+1} on $n+1$ elements.

(ii) Let $b_1, \ldots, b_n$ be an orthonormal basis for a Euclidean space of dimension n. Then

$$\Phi = \{\pm b_i \pm b_j \mid 1 \leq i, j \leq n, i \neq j\}$$

forms a root system of type D_n. A base for Φ is given by the following vectors.



We have $\Phi^+ = \{b_i \pm b_j \mid 1 \le i < j \le n\}$.

Consider the subgroup W_n of $\mathsf{GL}(E)$ generated by the elements θ for which there exists a permutation π of $\{1, \ldots, n\}$ such that $\theta(b_i) = \pm b_{\pi(i)}$ for all $1 \le i \le n$. Then W_n is a split extension of S_n by an elementary abelian 2-group of order 2^n. The Weyl group of Φ is an index 2 subgroup of W_n. It consists of those elements of W_n that involve an even number of sign changes. We denote this group by W_n'. It is an extension of S_n by an elementary abelian 2-group of order 2^{n-1}.

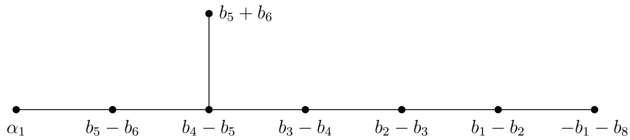
(iii) Let E be a Euclidean space of dimension 8 and $\{b_1, \ldots, b_n\}$ an orthonormal basis for E. Then

$$\Phi = \{\pm b_i \pm b_j \mid 1 \le i, j \le 8\} \cap \{\tfrac{1}{2}\textstyle\sum_{i=1}^{8} \epsilon_i b_i \mid \epsilon_i = \pm 1, \prod_{i=1}^{8} \epsilon_i = -1\}$$

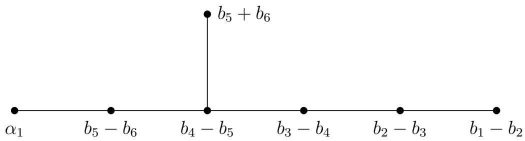
is a root system of type E_8. Consider the roots:

$$\alpha_1 := \tfrac{1}{2}(-b_1 - b_2 - b_3 - b_4 - b_5 + b_6 + b_7 + b_8),$$
$$\alpha_2 := \tfrac{1}{2}(-b_1 - b_2 - b_3 + b_4 + b_5 + b_6 + b_7 + b_8).$$

Then following roots form a base for Φ.

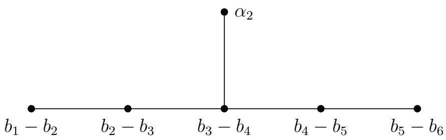


The roots contained in the subspace of vectors $\sum_{i=1}^{8} \varepsilon_i b_i$ satisfying $\varepsilon_7 = \varepsilon_8$ form a root system of type E_7. We have the following base for this root system.



Next, we consider the roots contained in the subspace of vectors $\sum_{i=1}^{8} \varepsilon_i b_i$ satisfying $\varepsilon_7 = \varepsilon_8$ and $\sum_{i=1}^{6} \varepsilon_i = 0$. They form a root system of type E_6.

A base for this root system is given by the following roots.



This is not the most comprehensive description of these root systems. We have chosen these bases because they lead to more elegant formulas in Sections 5.6 and 5.7.

The automorphisms of a root system can be described as follows.

Proposition 1.3.13 ([Hum72, § 12.2]). *Let Φ be a root system and Δ a fixed base for Φ. Let $\Gamma := \{\varphi \in \mathsf{Aut}(\Phi) \mid \varphi(\Delta) = \Delta\}$. Then $\mathsf{Aut}(\Phi)$ is the semi-direct product of the Weyl group $W(\Phi)$ and Γ. Moreover, Γ can be identified with the group of all automorphisms of the Dynkin diagram of Φ. We call the elements of Γ the* graph automorphisms *of Φ.*

We end this section by introducing some weight theory. This theory is essential to describe representations of semisimple Lie algebras; see Section 1.5.

Definition 1.3.14. Let Φ be a root system and Δ a fixed base for Φ.

(i) The set $\{v \in E \mid \langle v, \alpha\rangle \geq 0$ for all $\alpha \in \Delta\}$ is called the *fundamental domain* of Φ with respect to Δ.

(ii) The *root lattice* $\mathbb{Z}[\Phi]$ is the lattice consisting of all $\mathbb{Z}$-linear combinations of elements of Φ.

(iii) The lattice of all $v \in E$ such that $\langle v, h_\alpha\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$ is called the *weight lattice* of Φ. Its elements are called *(integral) weights*. Observe that the root lattice $\mathbb{Z}[\Phi]$ is contained in the weight lattice. The weights contained in the fundamental domain are called *dominant*.

(iv) Write $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Let $\lambda_1, \dots, \lambda_n$ be the basis dual to $\{h_{\alpha_i} \mid 1 \leq i \leq n\}$ with respect to the inner product on E. This means that $\langle \lambda_i, h_{\alpha_j}\rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Then $\lambda_1, \dots, \lambda_n$ are dominant weights and they form a basis for the weight lattice. We call them the *fundamental dominant weights*.

Every $W(\Phi)$-orbit of weights contains exactly one dominant weight.

Proposition 1.3.15 ([Hum72, § 10.3 and § 13.2]). (i) *If λ and μ are contained in the fundamental domain of Φ and $w(\lambda) = \mu$ for some $w \in W(\Phi)$, then w can be written as a product of reflections s_α with $\alpha \in \Delta$ and $\langle \lambda, \alpha\rangle = 0$. In particular $\lambda = \mu$.*

(ii) *Each weight is conjugate under $W(\Phi)$ to one and only one dominant weight.*

(iii) *If λ is a dominant weight, then $w(\lambda) \preccurlyeq \lambda$ for all $w \in W(\Phi)$. Moreover, the number of dominant weights μ for which $\mu \preccurlyeq \lambda$ is finite.*

Example 1.3.16. If Φ is irreducible and simply laced, then $W(\Phi)$ acts transitively on Φ by Proposition 1.3.9. The only root contained in the fundamental domain is the highest root.

1.3.2 Classification of complex semisimple Lie algebras

The goal of this section is to classify complex semisimple Lie algebras and introduce Chevalley bases. First, we introduce some standard terminology.

Definition 1.3.17. Let $\mathcal{L}$ be a Lie algebra.

(i) The *center* $Z(\mathcal{L})$ of $\mathcal{L}$ is the subalgebra $\{\ell \in \mathcal{L} \mid [\ell, \ell'] = 0 \text{ for all } \ell' \in \mathcal{L}\}$. The Lie algebra $\mathcal{L}$ is called *abelian* if $\mathcal{L} = Z(\mathcal{L})$.

(ii) If I_1 and I_2 are ideals of $\mathcal{L}$ then we denote by $[I_1, I_2]$ the ideal consisting of linear combinations of elements $[\ell, \ell']$ for $\ell \in I_1$ and $\ell' \in I_2$. We call $[\mathcal{L}, \mathcal{L}]$ the *derived subalgebra* of $\mathcal{L}$.

(iii) Now let $\mathcal{L}^{(0)} = \mathcal{L}^{[0]} = \mathcal{L}$. For all $n \in \mathbb{N}$, let $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}]$ and $\mathcal{L}^{[n+1]} = [\mathcal{L}, \mathcal{L}^{[n+1]}]$. Then we call $\mathcal{L}$ *solvable* (resp. *nilpotent*) if $\mathcal{L}^{(n)} = 0$ (resp. $\mathcal{L}^{[n]} = 0$) for some $n \in \mathbb{N}$.

(iv) For a subalgebra $\mathcal{L}'$ of $\mathcal{L}$, we call the subalgebra

$$N_{\mathcal{L}}(\mathcal{L}') = \{\ell \in \mathcal{L} \mid [\ell, \mathcal{L}'] \subseteq \mathcal{L}'\}$$

the *normalizer* of $\mathcal{L}'$. The *centralizer* of $\mathcal{L}'$ is the subalgebra

$$C_{\mathcal{L}}(\mathcal{L}') = \{\ell \in \mathcal{L} \mid [\ell, \mathcal{L}'] = 0\}.$$

(v) We say that a subalgebra $\mathcal{L}'$ of $\mathcal{L}$ is *self-normalizing* if $N_{\mathcal{L}}(\mathcal{L}') = \mathcal{L}'$. A maximal nilpotent self-normalizing subalgebra of $\mathcal{L}$ is called a *Cartan subalgebra* of $\mathcal{L}$. A maximal solvable subalgebra is called a *Borel subalgebra.*

Proposition 1.3.18 ([Hum72, § 3.1])**.** *Any Lie algebra $\mathcal{L}$ has a unique maximal solvable ideal called the* radical *of $\mathcal{L}$ and is denoted by* $\mathrm{rad}(\mathcal{L})$.

Definition 1.3.19. A Lie algebra $\mathcal{L}$ is called *semisimple* (resp. *reductive*) if $\mathrm{rad}(\mathcal{L}) = 0$ (resp. $\mathrm{rad}(\mathcal{L}) = Z(\mathcal{L})$).

From now on, we will restrict to Lie algebras over the field $\mathbb{C}$ of complex numbers.

Example 1.3.20. Let V be an n-dimensional $\mathbb{C}$-vector space and equip $\mathsf{End}(V)$ with the Lie bracket from Example 1.2.3 (ii), i.e. $[f,g] = fg - gf$. Then we obtain a reductive Lie algebra denoted by $\mathfrak{gl}(V)$. Its center is the 1-dimensional space of endomorphisms $f\colon V \to V\colon v \mapsto \lambda v$ for some $\lambda \in \mathbb{C}$. Its derived subalgebra consists of those endomorphism that have trace zero. We denote its derived subalgebra by $\mathfrak{sl}(V)$. If we choose a basis for V then we can identify $\mathsf{End}(V)$ and therefore $\mathfrak{gl}(V)$ with the space of all $n \times n$ matrices over $\mathbb{C}$. We denote the corresponding Lie algebra by $\mathfrak{gl}_n(\mathbb{C})$. The diagonal matrices form its center. Its derived subalgebra, which is denoted by $\mathfrak{sl}_n(\mathbb{C})$ contains the matrices with trace zero.

Proposition 1.3.21 ([Hum72, § 19.1])**.** *If $\mathcal{L}$ is a complex reductive Lie algebra, then*

$$\mathcal{L} = \operatorname{rad}(\mathcal{L}) \oplus [\mathcal{L}, \mathcal{L}]$$

and $[\mathcal{L}, \mathcal{L}]$ is semisimple.

A semisimple Lie algebra can be endowed with a Frobenius form.

Definition 1.3.22. Let $\mathcal{L}$ be a complex Lie algebra. Consider the symmetric bilinear form

$$\kappa\colon \mathcal{L} \times \mathcal{L} \to \mathbb{C}\colon (\ell_1, \ell_2) \mapsto \operatorname{tr}(\operatorname{ad}_{\ell_1} \operatorname{ad}_{\ell_2}).$$

This form is called the *Killing form* of $\mathcal{L}$ and satisfies $\kappa([\ell_1, \ell_2], \ell_3) = \kappa(\ell_1, [\ell_2, \ell_3])$ for all $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$.

Proposition 1.3.23 ([Hum72, § 5.1 and § 5.2])**.** *Let $\mathcal{L}$ be a complex Lie algebra.*

(i) *The Killing form κ of $\mathcal{L}$ is non-degenerate (and therefore a Frobenius form) if and only if $\mathcal{L}$ is semisimple.*

(ii) *If $\mathcal{L}$ is semisimple, then there exist ideals $\mathcal{L}_1, \dots, \mathcal{L}_r$ which are simple (as Lie algebras) such that $\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$. The Killing form of $\mathcal{L}_i$ is the restriction of κ to $\mathcal{L}_i \times \mathcal{L}_i$.*

Cartan subalgebras are essential to understand the structure of a semisimple Lie algebra.

Proposition 1.3.24. *Let $\mathcal{L}$ be a complex semisimple Lie algebra and $\mathcal{H}$ a Cartan subalgebra.*

(i) *We have $C_{\mathcal{L}}(\mathcal{H}) = \mathcal{H}$. In particular, $\mathcal{H}$ is abelian.*

(ii) *The operators ad_h for $h \in \mathcal{H}$ are simultaneously diagonalizable.*

(iii) *The restriction of the Killing form κ to $\mathcal{H} \times \mathcal{H}$ is non-degenerate.*

Proof. This is proven in [Hum72, § 8.1 and § 8.2] for maximal toral subalgebras. But since $\mathcal{L}$ is a complex semisimple Lie algebra, the Cartan subalgebras are the maximal toral subalgebras by [Hum72, § 15.3]. □

Item (ii) allows us to introduce the following definition.

Definition 1.3.25. Let $\mathcal{L}$ be a complex semisimple Lie algebra and $\mathcal{H}$ a Cartan subalgebra.

(i) For any linear map $\alpha \in \mathcal{H}^* = \mathsf{Hom}(\mathcal{H}, \mathbb{C})$, denote

$$\mathcal{L}_\alpha = \{\ell \in \mathcal{L} \mid [h, \ell] = \alpha(h)\ell \text{ for all } h \in \mathcal{H}\}.$$

Let Φ be the set of non-zero $\alpha \in \mathcal{H}^*$ for which $\mathcal{L}_\alpha \neq 0$. Since $C_{\mathcal{L}}(\mathcal{H}) = \mathcal{H}$, we have

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha$$

as vector spaces. We call Φ the set of *roots* and $\mathcal{L}_\alpha$ the *root space* of $\mathcal{L}$ with respect to $\alpha \in \Phi$.

(ii) Since the restriction of κ to $\mathcal{H} \times \mathcal{H}$ is non-degenerate, it allows us to identify $\mathcal{H}$ with its dual $\mathcal{H}^*$. In particular, we can view Φ is a subset of $\mathcal{H}$. We denote by $\mathbb{R}[\Phi]$ the subset of $\mathcal{H}$ of all $\mathbb{R}$-linear combinations of the elements of Φ. It has the structure of an $\mathbb{R}$-vector space.

Proposition 1.3.26. (i) *The set Φ spans $\mathcal{H}$.*

(ii) *For every $\alpha \in \Phi$, we have* $\dim(\mathcal{L}_\alpha) = 1$.

(iii) *The restriction of the Killing form κ defines an inner product on* $\mathbb{R}[\Phi]$*, which turns it into a Euclidean space with dimension* $\dim(\mathcal{H})$.

(iv) *The set Φ is a root system for the Euclidean space* $\mathbb{R}[\Phi]$*. It is irreducible if and only if $\mathcal{L}$ is simple.*

Proof. Item (i) is proven in [Hum72, § 8.3], Item (ii) in [Hum72, § 8.4]. Item (iii) and the fact that Φ forms a root system follows from [Hum72, § 8.5]. The final assertion is proven in [Hum72, § 14.1]. □

Example 1.3.27. Consider the complex semisimple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ from Example 1.3.20. The diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$ form a Cartan subalgebra $\mathcal{H}$ of $\mathfrak{sl}_n$. Denote by e_{ij} the matrix with 1 in the (i, j) entry and 0 elsewhere. Then the spaces of the form $\langle e_{ij} \rangle$ for $i \neq j$ are the root spaces of $\mathfrak{sl}_n(\mathbb{C})$. The corresponding roots are the matrices $\frac{1}{2n}(e_{ii} - e_{jj})$ for $i \neq j$. They form a root system of type A_{n-1}.

Bases for Φ determine a Borel subalgebra of $\mathcal{L}$ containing $\mathcal{H}$.

Proposition 1.3.28 ([Hum72, § 16.3])**.** *Let Δ be a base for Φ and let Φ^+ be the corresponding set of positive roots. Then*

$$B(\Delta) \coloneqq \mathcal{H} \oplus \bigoplus_{\alpha \in \Phi^+} \mathcal{L}_\alpha$$

is a Borel subalgebra of $\mathcal{L}$.

In fact every Borel subalgebra containing $\mathcal{H}$ is of this form and therefore determines a base Δ of Φ. The following proposition also assures that the root system Φ is independent of the choice of the Cartan subalgebra $\mathcal{H}$.

Proposition 1.3.29 ([Hum72, § 16])**.** *Let $\mathcal{L}$ be a complex semisimple Lie algebra. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Borel subalgebras of $\mathcal{L}$ containing respective Cartan subalgebras $\mathcal{H}_1$ and $\mathcal{H}_2$. Then there exists an automorphism φ of $\mathcal{L}$ such that $\varphi(\mathcal{B}_1) = \mathcal{B}_2$ and $\varphi(\mathcal{H}_1) = \mathcal{H}_2$.*

We will discuss the automorphism group of $\mathcal{L}$ in more detail in Section 1.6. We end this section by introducing a special type of basis for $\mathcal{L}$.

Proposition 1.3.30 ([Hum72, § 25.2])**.** *Let $\mathcal{L}$ be a complex semisimple Lie algebra, $\mathcal{H}$ a Cartan subalgebra and Δ a base for the root system Φ relative to $\mathcal{H}$. Denote the coroot corresponding to the root $\alpha \in \Phi$ by h_α. Then there exists a basis $\{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ for $\mathcal{L}$ such that $e_\alpha \in \mathcal{L}_\alpha$ for all $\alpha \in \Phi$ and, for all $\alpha, \beta \in \Phi$,*

$$\begin{aligned}
[h_\alpha, h_\beta] &= 0, \\
[h_\alpha, e_\beta] &= \kappa(h_\alpha, \beta) e_\beta, \\
[e_\alpha, e_{-\alpha}] &= h_\alpha, \\
[e_\alpha, e_\beta] &= 0 && \textit{if } \alpha + \beta \notin \Phi \cup \{0\}, \\
[e_\alpha, e_\beta] &= \pm(r+1) e_{\alpha+\beta} && \textit{if } \alpha + \beta \in \Phi,
\end{aligned}$$

where r is the greatest integer for which $\alpha - r\beta$ is a root. Such a basis is called a Chevalley basis *for $\mathcal{L}$ (relative to $\mathcal{H}$).*

Remark 1.3.31. The importance of the Chevalley basis lies in the fact that the structure constants of $\mathcal{L}$ relative to this basis are integers. This allows to reduce the Lie algebra to an arbitrary field by extension of scalars. See [Hum72, § 25.4] for more information.

Proposition 1.3.32 ([Car72, Theorem 4.1.2])**.** *Consider a Chevalley basis as in Proposition* 1.3.30*. For $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$, let $c_{\alpha,\beta} \in \mathbb{C}$ such that $[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}$. Then, for all $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$, we have*

(i) $c_{\alpha,\beta} = -c_{\beta,\alpha}$,

(ii) $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$,

(iii) $\frac{c_{\alpha,\beta}}{\kappa(\gamma,\gamma)} = \frac{c_{\beta,\gamma}}{\kappa(\alpha,\alpha)} = \frac{c_{\gamma,\alpha}}{\kappa(\beta,\beta)}$ *for* $\gamma := -(\alpha + \beta) \in \Phi$.

1.4 Linear representation theory of finite groups

This section aims to give the reader a quick overview on representation and character theory of finite groups. We restrict ourselves to representation theory over the field $\mathbb{C}$ of complex numbers. Proofs and more information can be found in [Isa94].

Definition 1.4.1. (i) A *representation* of a group G on a finite-dimensional complex vector space V is a group homomorphism $\rho\colon G \to \mathsf{GL}(V)$. If there is no confusion about ρ, then we will simply say that V is a representation of G or a G-*module* and denote $\rho(g)(v)$ by $g \cdot v$. We say that $\dim(V)$ is the *degree* of the representation.

(ii) A representation $\rho\colon G \to \mathsf{GL}(V)$ is said to be *faithful* if it is a monomorphism.

(iii) Let $\rho\colon G \to \mathsf{GL}(V)$ and $\rho'\colon G \to \mathsf{GL}(W)$ be representations of G. A map $\varphi\colon V \to W$ is a *morphism* of representations if it is a morphism of vector spaces for which the following diagram commutes for every $g \in G$:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd} .$$

We denote the space of morphisms between V and W by $\mathsf{Hom}_G(V, W)$. The collection of representations of a group G together with the morphisms between them defines a category denoted by $\mathsf{Rep}(G)$.

(iv) A *subrepresentation* of a representation V is a subspace $W \leq V$ that is G-invariant: $g \cdot W \subseteq W$ for all $g \in G$. We call V *irreducible* if its only subrepresentations are 0 and V. If $W \leq V$ is a subrepresentation of W, then the *quotient* V/W inherits the action of G on V, making it a representation of G.

(v) The *direct sum* $V \oplus W$ can uniquely be given the structure of a representation such that the natural projections are morphisms of representations. Also the *tensor product* $V \otimes W$ is a representation for G by setting $g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$ for all $g \in G$, $v \in V$ and $w \in W$.

(vi) We can equip $\mathbb{C}$ (as a 1-dimensional $\mathbb{C}$-vector space) with a trivial G-action by setting $g \cdot v = v$ for all $g \in G$ and $v \in \mathbb{C}$. We call this the *trivial representation* for G.

(vii) The dual space $V^* := \mathsf{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of the underlying vector space of a representation V can be given the structure of a representation for G by setting $(g \cdot f)(v) = f(g^{-1} \cdot v)$. This is the *dual representation* of V. The natural pairing $V \otimes V^* \to \mathbb{C}\colon v \otimes f \mapsto f(v)$ between V and its dual is then a morphism of G-representations.

Example 1.4.2. (i) Let G be a group acting on a set X. Consider the vector space V with basis $\{e_x \mid x \in X\}$ indexed by the set X. Then we can turn V into a G-representation by setting $g \cdot e_x := e_{g \cdot x}$ for all $g \in G$ and $x \in X$. We call this the *permutation representation* with respect to the action of G on X.

(ii) Let V be a representation of G. The subspace of $V^{\otimes n}$ generated by the elements
$$v_1 \otimes \cdots \otimes v_n - v_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)}$$
for all $v_1, \ldots, v_n \in V$ and permutations π of $\{1, \ldots, n\}$, is a subrepresentation of $V^{\otimes n}$. Therefore the symmetric power $S^n(V)$, the factor space of this subspace, also has the structure of a G-representation. The same holds for the alternating power.

Definition 1.4.3. Let A be an algebra. Then we say that it is an *algebra for G* if A is also a representation for G and the multiplication map
$$A \otimes A \to A \colon a \otimes b \mapsto ab$$
is *G-equivariant*, i.e. a morphism of representations for G. If A is a Frobenius algebra then we call it a *Frobenius algebra for G* if, in addition to the multiplication being G-equivariant, the Frobenius form
$$A \otimes A \to \mathbb{C} \colon a \otimes b \mapsto \langle a, b\rangle$$
is *G-equivariant*, i.e. $\langle g \cdot a, g \cdot b\rangle = \langle a, b\rangle$ for all $g \in G$ and $a, b \in A$.

From now on we will restrict to representations of finite groups.

Proposition 1.4.4 (Maschke)**.** *Let G be a finite group and V a representation for G. Then for every subrepresentation $W \leq V$ there exists a subrepresentation $W' \leq V$ such that $V = W \oplus W'$. In particular, we can write every representation as a direct sum of irreducible representations.*

Proof. See [Isa94, Theorem 1.9, p. 4]. □

Morphisms of irreducible representations are easy.

Proposition 1.4.5 (Schur's lemma)**.** *If V and W are non-isomorphic irreducible representations then $\mathsf{Hom}_G(V, W) = 0$. For every irreducible representation V we have $\mathsf{Hom}_G(V, V) = \{\lambda \cdot \mathsf{id}_V \mid \lambda \in \mathbb{C}\}$ with $\mathsf{id}_V \colon V \to V$ the identity map.*

Proof. The proof is an easy exercise and can be found in [Isa94, Lemma 1.5, p. 4]. □

The character of a representation carries its essential information in a more condensed form.

Definition 1.4.6. (i) Let $\rho\colon G \to \mathsf{GL}(V)$ be a representation for G. The *character of G afforded by V* is the function
$$\chi\colon G \to \mathbb{C}\colon g \mapsto \mathrm{tr}(\rho(g))$$
where tr denotes the trace map. Since $\chi(1)$ equals $\dim(V)$, the degree of the representation, we call it the *degree* of the character. A *linear character* is a character of degree one. It is a group homomorphism $G \to \mathbb{C}^\times$.

(ii) A *class function* of G is a map $G \to \mathbb{C}$ that is constant on the conjugacy classes of G. The class functions of G form a ring, denoted by $C(G)$, under pointwise multiplication and addition. It is immediately verified that all characters are class functions. If χ_V and χ_W are the characters afforded by V and W, then $\chi_V + \chi_W$ (resp. $\chi_V\chi_W$) is the character afforded by $V \oplus W$ (resp. $V \otimes W$).

Example 1.4.7. (i) The character afforded by the trivial representation is the class function with constant value 1. We denote it by $\mathbf{1}_G$.

(ii) The character of the permutation representation of a group G acting on a finite set X is given by $\chi(g) = |\mathrm{Fix}(g)|$ where $|\mathrm{Fix}(g)|$ denotes the number of fixed points of $g \in G$.

(iii) If χ is the character afforded by a representation V. Then

$$G \to \mathbb{C}\colon g \mapsto \tfrac{1}{2}(\chi(g)^2 + \chi(g^2))$$

and

$$G \to \mathbb{C}\colon g \mapsto \tfrac{1}{2}(\chi(g)^2 - \chi(g^2))$$

are the characters afforded by $S^2(V)$ and $\bigwedge^2(V)$ respectively; see the proof of [Isa94, Theorem 4.5, p. 50].

Proposition 1.4.8 ([Isa94, Corollary 2.9, p. 17])**.** *Two representations of G are isomorphic if and only if they afford the same character.*

Definition 1.4.9. We call a character *irreducible* if it is afforded by an irreducible representation. By [Isa94, Lemma 1.14, p. 7], the number of irreducible representations of a finite group (up to isomorphism) is finite. We denote by $\mathsf{Irr}(G)$ the finite set of irreducible characters of G.

Proposition 1.4.10 ([Isa94, Corollary 2.6, p. 16])**.** *A group G is abelian if and only if all its irreducible characters have degree 1. In particular, if $\chi \in \mathsf{Irr}(G)$ is afforded by V then we have $g \cdot v = \chi(g)v$ for all $g \in G$.*

Example 1.4.11. The irreducible characters of a cyclic group $G = \langle g \rangle$ of order n are of the form $\chi(g^k) = \zeta^k$ for some complex n-th root of unity ζ. In particular, they form a group, under the product of $C(G)$, isomorphic to G.

Proposition 1.4.12 ([Isa94, Theorem 2.8,p. 16])**.** *Every class function $\psi \in C(G)$ can be uniquely written as*

$$\psi = \sum_{\chi \in \mathsf{Irr}(G)} a_\chi \chi$$

for $a_\chi \in \mathbb{C}$. The characters of G are precisely those class functions ψ for which the a_χ are non-negative integers and $\psi \neq 0$.

Definition 1.4.13. Let $\psi = \sum_{\chi \in \mathsf{Irr}(G)} a_\chi \chi$ be a character of G. Then we call $\chi \in \mathsf{Irr}(G)$ an *irreducible constituent* of ψ if $a_\chi > 0$ and we call a_χ its multiplicity. In general, we say that a character χ is *constituent* of ψ if $\psi - \chi$ is zero or a character of G.

Next, we state a few relations of the irreducible characters of a finite group.

Proposition 1.4.14 ([Isa94, Lemma 2.14 and Definition 2.16, p. 20])**.** *The bilinear map*

$$C(G) \times C(G) \to \mathbb{C}\colon (\psi, \psi') \mapsto \langle \psi, \psi' \rangle := \frac{1}{|G|} \sum_{g \in G} \psi(g)\overline{\psi'(g)}$$

defines an inner product on the space of class function of G (where the bar stands for complex conjugation). The irreducible characters form an orthonormal basis of $C(G)$ with respect to this inner product.

Definition 1.4.15. If χ is a class function of G, then we call the function $\bar{\chi}$ defined by $\bar{\chi}(g) := \overline{\chi(g)}$ the *conjugate* of χ. By [Isa94, Lemma 2.15, p. 20] we have $\bar{\chi}(g) = \chi(g^{-1})$ for all $g \in G$. If χ afforded by V, then $\bar{\chi}$ is afforded by V^*. We have $\langle \chi_1\chi_2, \chi_3 \rangle = \langle \chi_1, \bar{\chi}_2\chi_3 \rangle$ for all $\chi_1, \chi_2, \chi_3 \in C(G)$.

We also have the following orthogonality relation for irreducible characters.

Proposition 1.4.16 ([Isa94, Theorem 2.18, p. 21])**.** *For all $g, h \in G$ we have*

$$\sum_{\chi \in \mathsf{Irr}(G)} \chi(g)\overline{\chi(h)} = 0$$

if g is not conjugate to h. Otherwise, this sum equals $|C_G(g)|$ where $C_G(g)$ is the centralizer of g.

The degrees of irreducible characters satisfy the following identities.

Proposition 1.4.17 ([Isa94, Corollary 2.7 and Theorem 3.11, p. 16 and 38])**.** *For all $\chi \in \mathsf{Irr}(G)$ we have $\chi(1) \mid |G|$. Also*

$$\sum_{\chi \in \mathsf{Irr}(G)} \chi(1)^2 = |G|.$$

From Proposition 1.4.4 we know that every representation of G can be written as a direct sum of irreducible representations. However, such a decomposition is not unique. Therefore, we introduce a decomposition which is unique.

Definition 1.4.18. Let V be a representation for G. For any $\chi \in \mathsf{Irr}(G)$, we define the *G-isotypic component* V_χ of V as the sum of all subrepresentations of V that afford the irreducible character χ. Of course we have $V = \bigoplus_{\chi \in \mathsf{Irr}(G)} V_\chi$ and we call this the *isotypic decomposition* of V. It is unique (up to reordering of the terms). If all V_χ are irreducible, then we say that V is *multiplicity-free.*

From the results above, we understand the character theory of a fixed group reasonably well. Next, we discuss the connections between characters of different groups.

Definition 1.4.19. (i) Let V and W be representations of respective finite groups G and H. Then we can give $V \otimes W$ the structure of a representation for $G \times H$ by setting $(g, h) \cdot (v \otimes w) := (g \cdot v) \otimes (h \cdot w)$ for all $g \in G$, $h \in H$, $v \in V$ and $w \in W$. If χ_V and χ_W are the characters afforded by V and W, then we denote the character of $G \times H$ afforded by $V \otimes W$ by $\chi_V \times \chi_W$. In fact $\chi \times \chi'$ for $\chi \in \mathsf{Irr}(G)$ and $\chi' \in \mathsf{Irr}(H)$ are the irreducible characters of $G \times H$ [Isa94, Theorem 4.21, p. 59].

(ii) Let G be a finite group and $H \leq G$. If $\rho \colon G \to \mathsf{GL}(V)$ is a representation of G, then its restriction to H is a representation of H. The character of this representation is the restriction to H of the character χ afforded by V. We denote it by χ_H or $\mathsf{Res}_H^G(\chi)$.

(iii) Now let χ be a class function of H. Then we can define a class function χ^G of G, called the *induced class function*, by

$$\chi^G(g) := \frac{1}{|H|} \sum_{x \in G} \varphi({}^x g)$$

where $\varphi \colon G \to \mathbb{C}$ is defined by $\varphi(h) = \chi(h)$ if $h \in H$ and $\varphi(h) = 0$ otherwise. We also denote χ^G by $\mathsf{Ind}_H^G(\chi)$.

Example 1.4.20. From Example 1.4.11 it follows that the irreducible characters of a finite abelian group G form a group, under the product of $C(G)$, isomorphic to G.

Example 1.4.21. Let $\mathbf{1}$ be the trivial character of H. Then $\mathsf{Ind}_H^G(\mathbf{1})$ is the character afforded by the permutation representation of G corresponding to the action of G on the left cosets of H by left multiplication.

The product of a class function with an induced class function is an induced class function itself.

Proposition 1.4.22 ([Isa94, Problem 5.3, p. 74])**.** *Let $H \leq G$ and χ and ψ be class functions of H and G respectively. Then $\chi^G \psi = (\chi \psi_H)^G$.*

Restriction and induction are dual constructions in the following sense.

Proposition 1.4.23 (Frobenius reciprocity)**.** *Let $H \leq G$ and χ (resp. ψ) be a class function of G (resp. H). Then*

$$\langle \chi_H, \psi \rangle = \langle \chi, \psi^G \rangle.$$

Proof. See [Isa94, Lemma 5.2, p. 62]. □

We will now state a few important facts that we will need later on. First, we introduce some more terminology.

Definition 1.4.24. Let χ be a class function of G. Then we define

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\}.$$

We call χ *faithful* if $\ker(\chi) = 1$.

Proposition 1.4.25 (Burnside-Brauer)**.** *Let χ be a faithful character for G and suppose that $\chi(g)$ takes m different values for $g \in G$. Then every irreducible character $\psi \in \mathsf{Irr}(G)$ is a constituent of χ^j for some $0 \leq j < m$.*

Proof. See [Isa94, Theorem 4.3, p. 49]. □

Definition 1.4.26. (i) If $\varphi \in G_1 \to G_2$ is an isomorphism of groups, then we can define a class function ${}^{\varphi}\chi \in C(G_2)$ for every class function $\chi \in C(G_1)$ by

$${}^{\varphi}\chi(g) := \chi(\varphi^{-1}(g)).$$

The map

$$C(G_1) \to C(G_2)\colon \chi \mapsto {}^{\varphi}\chi$$

defines a ring isomorphism. Note that for isomorphisms $G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3$ and $\chi \in C(G_1)$, we have ${}^{\varphi_2 \circ \varphi_1}\chi = {}^{\varphi_2}({}^{\varphi_1}\chi)$.

(ii) Let $H \trianglelefteq G$. Then, for every $g \in H$, we can consider the automorphism $H \to H\colon h \mapsto {}^{g}h$ of H. Thus for every class function χ of H, the map

$${}^{g}\chi\colon H \to \mathbb{C}\colon h \mapsto \chi({}^{g^{-1}}h) = \chi(g^{-1}hg)$$

is also a class function of H. We call the class functions of this form the *G-conjugates* of χ.

Proposition 1.4.27 (Clifford)**.** *Let $H \trianglelefteq G$ and $\chi \in \mathsf{Irr}(G)$ an irreducible character of G. Let θ be an irreducible constituent of χ_H and $\theta_1, \theta_2, \ldots, \theta_t$ its distinct G-conjugates including θ itself. Then*

$$\chi_H = \langle \chi_H, \theta \rangle \sum_{i=1}^{t} \theta_i.$$

Proof. See [Isa94, Theorem 6.2, p. 79]. □

We also state two important corollaries of Clifford's theorem.

Proposition 1.4.28 ([Isa94, Corollary 6.7, p. 81])**.** *Let $H \trianglelefteq G$ and suppose that $\chi \in \mathsf{Irr}(G)$ for which $\langle \mathbf{1}_H, \chi_H \rangle \neq 0$. Then $H \leq \ker \chi$.*

Proposition 1.4.29. *Suppose that H is a central subgroup of G, i.e. a subgroup of $Z(G)$. For every $\chi \in \mathsf{Irr}(G)$, the character $\frac{\chi_H}{\chi(1)}$ is an irreducible character of H. If χ is afforded by V, then $h \cdot v = \frac{\chi(h)}{\chi(1)} v$ for all $h \in H$.*

Proof. This follows immediately from Clifford's theorem and Proposition 1.4.10. □

1.5 Representation theory of Lie algebras

In Chapter 5 we will make extensive use of the representation theory of complex semisimple Lie algebras. Here, we introduce our notation and will provide some well-known facts. Proofs of the statements can be found in [Hum72] and [Hal15]. Also [FH91] is an excellent introduction to the topic.

We start by defining representations and modules for Lie algebras. Note that we restrict to finite-dimensional representations.

Definition 1.5.1. Let $\mathcal{L}$ be a Lie algebra over a field k.

(i) A representation of $\mathcal{L}$ on a finite-dimensional k-vector space V is a morphism

$$\rho\colon \mathcal{L} \to \mathfrak{gl}(V)$$

of Lie algebras. As for representations of groups, we will simply refer to V as a representation of $\mathcal{L}$ and write $\ell \cdot v$ instead of $\rho(\ell)(v)$. In fact, we have that

$$\mathcal{L} \times V \to V\colon (\ell, v) \mapsto \ell \cdot v$$

gives V the structure of an *$\mathcal{L}$-module*. This means that this map is bilinear and

$$[\ell_1, \ell_2] \cdot v = \ell_1 \cdot (\ell_2 \cdot v) - \ell_2 \cdot (\ell_1 \cdot v)$$

for all $\ell_1, \ell_2 \in \mathcal{L}$ and $v \in V$. Conversely, if V is an $\mathcal{L}$-module, then the equation $\rho(\ell)(v) = \ell \cdot v$ defines a representation $\rho\colon \mathcal{L} \to \mathfrak{gl}(V)$ of $\mathcal{L}$.

(ii) The reader should have no difficulty coming up with the definitions for *morphisms*, $\mathsf{Hom}_{\mathcal{L}}$, *direct sums*, *subrepresentations*, *submodules*, *quotients*, *faithfulness* and *irreducibility* of $\mathcal{L}$-representations since they are analogous to the representations of groups; see Definition 1.4.1. They can also be found in [Hum72, § 6.1].

(iii) If V and W are $\mathcal{L}$-modules, then we can also give the *tensor product* $V \otimes W$ the structure of an $\mathcal{L}$-module by setting

$$\ell \cdot (v \otimes w) \coloneqq (\ell \cdot v) \otimes w + v \otimes (\ell \cdot w)$$

for all $\ell \in \mathcal{L}$, $v \in V$ and $w \in W$.

Example 1.5.2. (i) Consider k as a one-dimensional k-vector space. Define $\ell \cdot v = 0$ for all $\ell \in \mathcal{L}$ and $v \in k$. Then k is an $\mathcal{L}$-module. We call it the *trivial $\mathcal{L}$-module*.

(ii) Consider the morphism

$$\mathrm{ad}\colon \mathcal{L} \to \mathfrak{gl}(\mathcal{L})\colon \ell \mapsto \mathrm{ad}_\ell$$

with ad_ℓ as in Definition 1.2.2. Then the Jacobi identity ensures that this is a representation of $\mathcal{L}$. We call it the *adjoint representation* of $\mathcal{L}$ and the corresponding module the *adjoint module*.

(iii) As for representations of finite groups, the symmetric and alternating power of an $\mathcal{L}$-module are $\mathcal{L}$-modules.

Definition 1.5.3. Let A be a k-algebra. We call a linear map $f\colon A \to A$ a *derivation* of A if

$$f(ab) = f(a)b + af(b)$$

for all $a, b \in A$. The space $\mathsf{Der}(A)$ of derivations of A forms a Lie algebra with $[f, g] = fg - gf$ for all $f, g \in \mathsf{Der}(A)$. We call A an *algebra for the Lie algebra* $\mathcal{L}$ if A is an $\mathcal{L}$-module and the multiplication map

$$A \otimes A \to A\colon a \otimes b \mapsto ab$$

is $\mathcal{L}$*-equivariant*, i.e. a morphism of $\mathcal{L}$-modules. Equivalently, this means that

$$\mathcal{L} \to \mathsf{Der}(A)\colon \ell \mapsto \mathrm{ad}_\ell$$

is a morphism of Lie algebras. We call a Frobenius algebra a *Frobenius algebra for* $\mathcal{L}$ if, in addition, its bilinear form $\langle\ ,\ \rangle$ is also $\mathcal{L}$*-equivariant*:

$$\langle \ell \cdot a, b\rangle + \langle a, \ell \cdot b\rangle = 0$$

for all $\ell \in \mathcal{L}$ and $a, b \in A$.

From now on we will restrict to the case where $k = \mathbb{C}$ is the field of complex numbers. Statements similar to Maschke's theorem and Schur's lemma also hold for representations of complex semisimple Lie algebras.

Proposition 1.5.4 (Weyl)**.** *Let $\mathcal{L}$ be a complex semisimple Lie algebra. Then any $\mathcal{L}$-module is a direct sum of irreducible submodules.*

Proof. See [Hum72, § 6.3]. □

Proposition 1.5.5 (Schur's lemma)**.** *If V and W are irreducible $\mathcal{L}$-modules for a complex semisimple Lie algebra $\mathcal{L}$, then $\mathsf{Hom}_{\mathcal{L}}(V, W) = 0$ unless $V \cong W$. Moreover the only endomorphisms of an irreducible $\mathcal{L}$-module are the scalars $\lambda \cdot \mathsf{id}$ for $\lambda \in \mathbb{C}$.*

Proof. See [Hum72, § 6.1]. □

Let us fix some notation for the remainder of this section. Write $\mathcal{L}$ for a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$. Let Φ be the root system of $\mathcal{L}$ with respect to $\mathcal{H}$ and W its Weyl group. Let Δ be a base of Φ and $\mathcal{B} \coloneqq \mathcal{B}(\Delta)$ the corresponding Borel subalgebra of $\mathcal{L}$. Write Λ for the weight lattice of Φ and Λ^+ for the set of dominant weights. The base Δ of Φ defines a partial order $\preccurlyeq$ on the weight lattice Λ of Φ and determines a set of positive roots Φ^+; see Definition 1.3.5.

We try to classify the irreducible (finite-dimensional) $\mathcal{L}$-modules. In Section 1.3, we benefited greatly from looking at the action of the Cartan subalgebra $\mathcal{H}$ on $\mathcal{L}$. The same will be true for modules.

Definition 1.5.6. Let V be any $\mathcal{L}$-module and $\lambda \in \mathcal{H}^* = \mathsf{Hom}(\mathcal{H}, \mathbb{C})$. Recall from Definition 1.3.25 that we can use the Killing form κ of $\mathcal{L}$ to identify $\mathcal{H}$ and $\mathcal{H}^*$ so that $\kappa(\lambda, h) = \lambda(h)$ for all $h \in \mathcal{H}$. We define

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathcal{H}\}.$$

We call V_λ the *weight-λ-space* of V (with respect to $\mathcal{H}$). Its elements are called *weight vectors* of weight λ. Those $\lambda \in \mathcal{H}^*$ for which $V_\lambda \neq 0$ are called the *weights* of V. Because we restrict to finite-dimensional modules, it follows from the discussion in [Hum72, § 21.1] that the weights of V are also weights of Φ, i.e. contained in Λ.

Example 1.5.7. The weight spaces of the adjoint module $\mathcal{L}$ with respect to $\mathcal{H}$ are $\mathcal{H}$, the zero weight space, and the root spaces $\mathcal{L}_\alpha$, the weight-α-spaces, for $\alpha \in \Phi$. The weights of the adjoint module are the elements of $\{0\} \cup \Phi$.

We will frequently use the following rules for weight vectors.

Proposition 1.5.8. *Let V and W be $\mathcal{L}$-modules. Let ℓ, v and w be weight vectors of $\mathcal{L}$, V and W of respective weights $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$. Then*

(i) *the element $\ell \cdot v$ is a weight vector of weight $\lambda_1 + \lambda_2$,*

(ii) *for any morphism $\varphi\colon V \to W$ of $\mathcal{L}$-modules, $\varphi(v) \in W_{\lambda_2}$,*

(iii) *the element $v \otimes w$ is a weight vector of $V \otimes W$ of weight $\lambda_2 + \lambda_3$.*

Proof. For arbitrary $h \in \mathcal{H}$, we have

(i) $$\begin{aligned} h \cdot (\ell \cdot v) &= [h, \ell] \cdot v + \ell \cdot (h \cdot v), \\ &= \lambda_1(h)\ell \cdot v + \lambda_2(h)\ell \cdot v, \\ &= (\lambda_1 + \lambda_2)(h)\ell \cdot v, \end{aligned}$$

(ii) $$\begin{aligned} h \cdot \varphi(v) &= \varphi(h \cdot v), \\ &= \lambda_2(h)\varphi(v), \end{aligned}$$

(iii) $$\begin{aligned} h \cdot (v \otimes w) &= (h \cdot v) \otimes w + v \otimes (h \cdot v), \\ &= (\lambda_2 + \lambda_3)(h)v \otimes w. \end{aligned}$$

Since $h \in \mathcal{H}$ is arbitary, the statement follows. □

As a consequence, we have the following rules for Frobenius algebras for $\mathcal{L}$.

Corollary 1.5.9. *Let A be a Frobenius algebra for $\mathcal{L}$ and $\lambda, \mu \in \Lambda$. Then*

(i) $A_\lambda A_\mu \subseteq A_{\lambda+\mu}$,

(ii) $\langle A_\lambda, A_\mu \rangle = 0$ *unless* $\lambda + \mu = 0$.

Proof. This follows from Proposition 1.5.8 (ii) and (iii) and the fact that the multiplication and bilinear form are $\mathcal{L}$-equivariant, i.e. morphisms of $\mathcal{L}$-modules. □

In order to classify irreducible $\mathcal{L}$-modules, we introduce the notion of a highest weight vector.

Definition 1.5.10. A vector $v^+ \in V$ is called a *highest weight vector* if $\mathcal{B} \cdot v^+ \subseteq \langle v^+ \rangle$. Note that v^+ is indeed a weight vector for any Cartan subalgebra contained in $\mathcal{B}$.

The following statement classifies the irreducible $\mathcal{L}$-modules.

Proposition 1.5.11. (i) *Any irreducible $\mathcal{L}$-module has a highest weight vector $v^+ \in V$. This vector is unique up to scalar and its weight λ is dominant. We call λ the* highest weight *of V.*

(ii) *Two irreducible $\mathcal{L}$-modules are isomorphic if and only if their highest weight is the same.*

(iii) *For every dominant weight $\lambda \in \Lambda^+$ there exists an irreducible representation with highest weight λ.*

Proof. See [Hum72, § 20 and § 21]. □

Example 1.5.12. If $\mathcal{L}$ is simple, then its adjoint module is irreducible. From Definition 1.3.25, the weights of this module are zero and the roots $\alpha \in \Phi$. Its highest weight is precisely the highest root from Proposition 1.3.9.

In Definition 1.3.25 we saw that the adjoint module of a complex semisimple Lie algebra is the direct sum of its weight spaces. In fact, this is true for any $\mathcal{L}$-module.

Proposition 1.5.13. *Let V be an irreducible $\mathcal{L}$-module of highest weight $\lambda \in \Lambda^+$. Then*

(i) *for all weights μ of V we have $\mu \preccurlyeq \lambda$;*

(ii) $V = \bigoplus_{\mu \preccurlyeq \lambda} V_\mu$*;*

(iii) $\dim(V_\lambda) = 1$ *and* $\dim(V_\mu) = \dim(V_{w(\mu)})$ *for all weights μ and $w \in W$.*

Proof. See [Hum72, § 20 and § 21]. □

This allows us to introduce formal characters for $\mathcal{L}$-modules. They play the same role as the characters of group representation as they carry a lot of the structure of the module in a more condensed form.

Definition 1.5.14. Let $\mathbb{C}[\Lambda]$ be the complex vector space with basis $\{e^\lambda \mid \lambda \in \Lambda\}$ indexed by Λ. For any $\mathcal{L}$-module V, we define its *(formal) character* as

$$\mathrm{ch}_V = \sum_{\lambda\in\Lambda} \dim(V_\lambda) e^\lambda.$$

Note that $\dim(V_\lambda) \neq 0$ only for a finite number of $\lambda \in \Lambda$. We call $\dim(V_\lambda)$ the *multiplicity* of the weight λ. We denote the character of the irreducible $\mathcal{L}$-module of highest weight $\lambda \in \Lambda^+$ by ch_λ and write $\mathsf{Irr}(\mathcal{L}) = \{\mathrm{ch}_\lambda \mid \lambda \in \Lambda^+\}$.

The character of the irreducible $\mathcal{L}$-module with highest weight λ can be determined inductively using Freudenthal's formula.

Proposition 1.5.15 (Freudenthal's formula)**.** *Let V be the irreducible $\mathcal{L}$-module of highest weight λ. Let $\delta := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be the half sum of the positive roots. For $\mu \in \Lambda$ let $m_\mu := \dim(V_\mu)$. Then the m_μ is given recursively as*

$$\big(\kappa(\lambda+\delta, \lambda+\delta) - \kappa(\mu+\delta, \mu+\delta)\big) = 2 \sum_{\alpha\in\Phi^+} \sum_{i=1}^{\infty} m_{\mu+i\alpha} \kappa(\mu+i\alpha, \alpha)$$

starting from $m_\lambda = 1$.

Proof. See [Hum72, § 22.3]. □

If $\sum_{\lambda\in\Lambda} m_\lambda e^\lambda$ is the character of an $\mathcal{L}$-module, then $m_\lambda = m_{w(\lambda)}$ for all $w \in W$ by Proposition 1.5.13. Conversely, we have the following proposition.

Proposition 1.5.16 ([Hum72, § 22.5, Proposition A])**.** *Let $\mathbb{C}[\Lambda]^W$ be the subspace of the $\mathbb{C}[\Lambda]$ consisting of the elements $\sum_{\lambda\in\Lambda} m_\lambda e^\lambda \in \mathbb{C}[\Lambda]$ for which $m_\lambda = m_{w(\lambda)}$ for all $\lambda \in \Lambda$ and $w \in W$. Then any element $\psi \in \mathbb{C}[\Lambda]^W$ can uniquely be written as*

$$\psi = \sum_{\lambda\in\Lambda^+} a_\lambda \mathrm{ch}_\lambda$$

for $a_\lambda \in \mathbb{C}$. The characters of $\mathcal{L}$-modules are those $\psi \neq 0$ for which all a_λ are non-negative integers. In particular, two representations are isomorphic if and only if they afford the same character.

Definition 1.5.17. (i) For a semisimple Lie algebra, we write $\mathbb{C}[\Lambda]^W$ also as $C(\mathcal{L})$. We can define a product on $C(\mathcal{L})$ by linearly extending

$$e^\lambda e^\mu = e^{\lambda+\mu}$$

for all $\lambda, \mu \in \Lambda$. Note that, by Proposition 1.5.8 (iii), this multiplication is compatible with the tensor product of $\mathcal{L}$-modules in the sense that $\mathrm{ch}_{V\otimes W} = \mathrm{ch}_V \mathrm{ch}_W$ for $\mathcal{L}$-modules V and W.

(ii) As for characters of finite groups, we can define a unique inner product $\langle\ ,\ \rangle$ on $C(\mathcal{L})$ such that $\mathsf{Irr}(\mathcal{L})$ is an orthonormal basis.

(iii) Let ch_V be the character of an $\mathcal{L}$-module V. We call a character θ a *consituent* of ch_V if $\langle \mathrm{ch}_V - \theta, \mathrm{ch}_\lambda \rangle \geq 0$ for all $\lambda \in \Lambda^+$, i.e. $\mathrm{ch}_V - \theta$ is zero or a character of $\mathcal{L}$. The module V is said to be *multiplicity-free* if $\langle \mathrm{ch}_V, \mathrm{ch}_\lambda \rangle \leq 1$ for all $\lambda \in \Lambda^+$.

(iv) As for representations of finite groups, we define the *$\mathcal{L}$-isotypic component* V_χ for an $\mathcal{L}$-module V and $\chi \in \mathsf{Irr}(\mathcal{L})$ as the sum of irreducible $\mathcal{L}$-submodules that afford the character χ.

(v) If $\varphi\colon \mathcal{L}_1 \to \mathcal{L}_2$ is an isomorphism of complex semisimple Lie algebras, then it maps any Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}_1$ to a Cartan subalgebra $\varphi(\mathcal{H})$ of $\mathcal{L}_2$. It induces an isomorphism

$$(\varphi^{-1})^*\colon \mathcal{H}^* \to \varphi(\mathcal{H})^*\colon \lambda \mapsto {}^{\varphi}\lambda := \lambda \circ \varphi^{-1}.$$

Since φ is an isomorphism of Lie algebras, this isomorphism maps the weight lattice Λ_1 of $\mathcal{L}_1$ with respect to $\mathcal{H}$ to the weight lattice Λ_2 of $\mathcal{L}_2$ with respect to $\varphi(\mathcal{H})$. Thus φ induces an isomorphism

$$C(\mathcal{L}_1) \to C(\mathcal{L}_2)\colon \chi = \sum_{\lambda \in \Lambda_1} m_\lambda e^\lambda \mapsto {}^{\varphi}\chi := \sum_{\lambda \in \Lambda_1} m_\lambda e^{{}^{\varphi}\lambda}.$$

Characters of $\mathcal{L}_1$ get mapped to characters of $\mathcal{L}_2$ under this isomorphism. This statement should be compared to the similar statement for finite groups; see Definition 1.4.26 (i).

1.6 Chevalley groups and automorphisms of Lie algebras

Chevalley groups are groups associated to complex semisimple Lie algebras. We will only define them over the complex numbers but they can in fact be defined over any field, in particular any finite field, due to the work of Claude Chevalley [Che55]. As such, they form an important class of finite simple groups. The material presented here is standard and based on [Ste68]. Another excellent source, with an emphasis on finite simple groups, is [Car72].

The adjoint complex Chevalley group acts by automorphisms on the Lie algebra. In Corollary 1.6.11 we describe the full automorphism group. We will refer to [Hum72] or [Bou05] for proofs.

Definition 1.6.1. Let $\mathcal{L}$ be a complex semisimple Lie algebra. We call an element $\ell \in \mathcal{L}$ *strongly ad-nilpotent* if it is contained in one of the root spaces of $\mathcal{L}$ with respect to some Cartan subalgebra $\mathcal{H}$. By Proposition 1.5.8 (i), $\rho(\ell)$ is nilpotent for any representation $\rho\colon \mathcal{L} \to \mathfrak{gl}(V)$ of $\mathcal{L}$. This means that $\rho(\ell)^k = 0$ for some $k \in \mathbb{N} \setminus \{0\}$. Let

$$\exp(\rho(\ell)) \coloneqq \sum_{i=0}^{k-1} \frac{\rho(\ell)^i}{i!}.$$

Then $\exp(\rho(\ell)) \in \mathsf{GL}(V)$ and acts as an automorphism on the Lie algebra $\rho(\mathcal{L})$ by conjugation. We call the subgroup of $\mathsf{GL}(V)$ generated by the $\exp(\rho(\ell))$ for all strongly ad-nilpotent $\ell \in \mathcal{L}$ the *Chevalley group* of $(\mathcal{L}, \rho)$ and we denote it by $\mathsf{Int}(\mathcal{L}, \rho)$.

Remark 1.6.2. If $\{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ is a Chevalley basis for $\mathcal{L}$ (cf. Proposition 1.3.30) then the elements te_α for $t \in \mathbb{C}$ and $\alpha \in \Phi$ are stronlgy ad-nilpotent. The elements

$$x_\alpha(t) \coloneqq \exp(\rho(te_\alpha))$$

generate $\mathsf{Int}(\mathcal{L}, \rho)$. This follows from the constructions in [Jac79, § IX.2].

We want to compare the Chevalley groups $\mathsf{Int}(\mathcal{L}, \rho)$ when ρ varies amongst the representations of $\mathcal{L}$. First, we state the following lemma.

Lemma 1.6.3 ([Ste68, Lemma 27, p. 42])**.** *Fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$. Let Λ be the weight lattice of $\mathcal{L}$ with respect to $\mathcal{H}$ and $\mathbb{Z}[\Phi]$ the sublattice spanned by the roots of $\mathcal{L}$. For a faithful representation ρ of $\mathcal{L}$, let $\Lambda(\rho)$ the lattice spanned by the weights of ρ. Then $\mathbb{Z}[\Phi] \leq \Lambda(\rho) \leq \Lambda$ and $\Lambda/\mathbb{Z}(\Phi)$ is finite.*

The Chevalley groups of $\mathcal{L}$ only differ up to a finite central subgroup.

Proposition 1.6.4 ([Ste68, Corollary 5, p. 44])**.** *Let ρ and ρ' be representations of $\mathcal{L}$. If $\Lambda(\rho) \leq \Lambda(\rho')$ then there exists an epimorphism defined by*

$$\theta\colon \mathsf{Int}(\mathcal{L}, \rho') \to \mathsf{Int}(\mathcal{L}, \rho)\colon \exp(\rho'(\ell)) \to \exp(\rho(\ell))$$

for any strongly ad-nilpotent $\ell \in \mathcal{L}$. The kernel $\ker\theta$ *is contained in the center of $\mathsf{Int}(\mathcal{L}, \rho')$ and isomorphic to $\Lambda(\rho')/\Lambda(\rho)$. In particular, if $\Lambda(\rho) = \Lambda(\rho')$, then θ is an isomorphism.*

By [Ste68, Lemma 27 (c), p. 42], any lattice $\Lambda \leq \Lambda' \leq \mathbb{Z}[\Phi]$ can be realized as $\Lambda' = \Lambda(\rho)$ for some representation ρ of $\mathcal{L}$. The Chevalley groups $\mathsf{Int}(\mathcal{L}, \rho)$ for which $\Lambda(\rho)$ is equal to the root lattice $\mathbb{Z}[\Phi]$ or the weight lattice Λ deserve their own name.

Definition 1.6.5. We call the Chevalley groups corresponding to the lattices $\mathbb{Z}[\Phi]$ and Λ the *adjoint Chevalley group* and *universal Chevalley group* of $\mathcal{L}$. We write $\mathsf{Int}(\mathcal{L})$ and $\widehat{\mathsf{Int}}(\mathcal{L})$ respectively.

The adjoint Chevalley groups of simple Lie algebras account for a large number of simple groups.

Proposition 1.6.6 ([Ste68, Theorem 5, p. 47])**.** *If $\mathcal{L}$ is simple, then so is its adjoint Chevalley group.*

Example 1.6.7. The universal Chevalley group of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is the group $\mathsf{SL}_n(\mathbb{C})$. Its adjoint Chevalley group is $\mathsf{PSL}_n(\mathbb{C})$ which is a simple group.

For any representation ρ of $\mathcal{L}$ there exists a natural map $\widehat{\mathsf{Int}}(\mathcal{L}) \to \mathsf{Int}(\mathcal{L}, \rho)$ by Proposition 1.6.4. This allows us to view ρ as a representation of $\widehat{\mathsf{Int}}(\mathcal{L})$ as we will illustrate now. In fact, when we view $\widehat{\mathsf{Int}}\,\mathcal{L}$ as a complex simply connected Lie group then representations of $\widehat{\mathsf{Int}}\,\mathcal{L}$ and $\mathcal{L}$ are equivalent [Hal15, Theorem 5.6, p. 119].

Definition 1.6.8. (i) Let $\rho \colon \mathcal{L} \to \mathfrak{gl}(V)$ be a representation of the complex semisimple Lie algebra $\mathcal{L}$. By Proposition 1.6.4, there is a natural epimorphism

$$\theta \colon \widehat{\mathsf{Int}}(\mathcal{L}) \to \mathsf{Int}(\mathcal{L}) \leq \mathsf{GL}(V).$$

Therefore θ defines a representation of $\widehat{\mathsf{Int}}(\mathcal{L})$ on V. We will often write $g \cdot v$ instead of $\theta(g)(v)$ for $v \in V$ and $g \in G$. Note that, in particular, $\widehat{\mathsf{Int}}(\mathcal{L})$ acts on the adjoint module $\mathcal{L}$. Moreover, a calculation shows that

$$\exp(\rho(\ell))(\ell' \cdot v) = (\exp(\mathrm{ad}_\ell)(\ell')) \cdot (\exp(\rho(\ell)) \cdot v)$$

for $\ell, \ell' \in \mathcal{L}$ and ℓ strongly ad-nilpotent. Therefore

$$g \cdot (\ell \cdot v) = (g \cdot \ell) \cdot (g \cdot v)$$

for all $g \in \widehat{\mathsf{Int}}(\mathcal{L})$, $\ell \in \mathcal{L}$ and $v \in V$.

(ii) In particular $\widehat{\mathsf{Int}}(\mathcal{L})$ acts by automorphisms on the Lie algebra $\mathcal{L}$. The weight lattice of the adjoint representation of $\mathcal{L}$ is, by definition, equal to the root lattice. Hence the kernel of this action is the center of $\widehat{\mathsf{Int}}(\mathcal{L})$ and we have an induced action of the corresponding factor group on $\mathcal{L}$. This factor group is isomorphic to the adjoint Chevalley group $\mathsf{Int}(\mathcal{L})$ and we retrieve the usual action of $\mathsf{Int}(\mathcal{L})$ on $\mathcal{L}$. We call the automorphisms of this type the *inner automorphisms* of $\mathcal{L}$ and say that two subalgebras of $\mathcal{L}$ are *conjugate* if they are conjugate by some inner automorphism.

We have the following stronger version of Proposition 1.3.29.

Proposition 1.6.9 ([Hum72, § 16])**.** *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are Borel subalgebras of $\mathcal{L}$ containing Cartan subalgebra $\mathcal{H}_1$ and $\mathcal{H}_2$ then there exists an inner automorphism of $\mathcal{L}$ mapping $\mathcal{B}_1$ onto $\mathcal{B}_2$ and $\mathcal{H}_1$ onto $\mathcal{H}_2$.*

The automorphisms of the root system Φ of $\mathcal{L}$ is another source of automorphisms of $\mathcal{L}$.

Proposition 1.6.10 (Isomorphism theorem)**.** *Let Φ be the root system of $\mathcal{L}$ with respect to a Cartan subalgebra $\mathcal{H}$. Let $\varphi\colon \mathcal{H} \to \mathcal{H}$ be an automorphism of the root system Φ. Fix a base Δ for Φ and, for each $\alpha \in \Delta$, pick nonzero $x_\alpha \in \mathcal{L}_\alpha$ and $y_\alpha \in \mathcal{L}_{\varphi(\alpha)}$. Then there exists a unique automorphism θ of $\mathcal{L}$ extending φ and such that $\theta(x_\alpha) = y_\alpha$ for all $\alpha \in \Delta$. The automorphism θ is inner if and only if φ is contained in the Weyl group of Φ.*

Proof. See [Hum72, § 14.2] for the existence and uniqueness of θ. The second assertion follows from [Bou05, § VIII.5.2, Proposition 4]. □

Let us now give a detailed description of the automorphism group of $\mathcal{L}$.

Corollary 1.6.11. *The automorphism group $\mathsf{Aut}(\mathcal{L})$ of a complex semisimple Lie algebra is the semidirect product of $\mathsf{Int}(\mathcal{L})$ and the group of graph automorphisms of the Dynkin diagram of $\mathcal{L}$.*

Proof. Consider any automorphism τ of the Lie algebra $\mathcal{L}$. Pick any Cartan subalgebra $\mathcal{H}$ and Borel subalgebra $\mathcal{B} \supseteq \mathcal{H}$ of $\mathcal{L}$. Then, by Proposition 1.6.9, there exists an inner automorphism σ of $\mathcal{L}$ mapping $(\tau(\mathcal{H}), \tau(\mathcal{B}))$ to $(\mathcal{H}, \mathcal{B})$. Therefore $\sigma\tau$ is an automorphism of $\mathcal{L}$ leaving both $\mathcal{H}$ and $\mathcal{B}$ invariant. Let Φ be the root system with base Δ corresponding to $\mathcal{H}$ and $\mathcal{B}$. Then $\sigma\tau$ induces an automorphism of $\mathcal{L}$ leaving both Φ and Δ invariant, i.e. a *graph automorphism.* By Proposition 1.6.10, two such automorphisms that correspond to the same graph automorphism only differ by a *diagonal automorphism*, i.e. an automorphism fixing every element of $\mathcal{H}$ and acting by scalar multiplication on each of the root spaces. By Proposition 1.6.10, we also know that these diagonal automorphisms are inner. This proves that $\mathsf{Aut}(\mathcal{L})$ is an extension of $\mathsf{Int}(\mathcal{L})$ by the group of graph automorphism. The proof that this extension is split, is given in [Bou05, § VIII.5.3, Corollary 1]. □

From the isomorphism theorem, we know that the subgroup of $\mathsf{Int}(\mathcal{L})$ leaving a Cartan subalgebra $\mathcal{H}$ invariant, induces on $\mathcal{H}$ the action of the Weyl group W of $\mathcal{L}$ with respect to $\mathcal{H}$. In fact it is true for any representation of $\mathcal{L}$ that there is a natural action of W on the zero weight space.

Definition 1.6.12. Let $\{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ be a Chevalley basis for $\mathcal{L}$ as in Proposition 1.3.30 and ρ a representation such that $\Lambda(\rho) = \Lambda$. Then $\widehat{\mathsf{Int}}(\mathcal{L}) = \mathsf{Int}(\mathcal{L}, \rho)$ is generated by the elements

$$x_\alpha(t) := \exp(\rho(te_\alpha))$$

for $\alpha \in \Phi$ and $t \in \mathbb{C}$. For any root $\alpha \in \Phi$ and $t \in \mathbb{C}^\times$ define

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$$

$$h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1}.$$

Then each $h_\alpha(t)$ acts as a diagonal automorphism on $\mathcal{L}$. Each $w_\alpha(t)$ leaves the Cartan subalgebra $\mathcal{H} = \langle h_\alpha \mid \alpha \in \Delta \rangle$ invariant and acts by the reflection $s_\alpha \in W$ on it [Ste68, Lemma 19, p. 27]. Now let

$$\begin{aligned} H &:= \langle h_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{C}^\times \rangle \quad \text{and,} \\ N &:= \langle w_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{C}^\times \rangle. \end{aligned}$$

From the discussion above, it is clear that $N/H \cong W$. By [Ste68, Lemma 19, p. 27], the group H acts trivially on the zero weight space V_0 of any representation V of $\mathcal{L}$ and N leaves V_0 invariant. Therefore this induces an action of $W \cong N/H$ on V_0. For the adjoint representation, this is the well-known action of the Weyl group on the Cartan subalgebra.

1.7 Association schemes

Association schemes will be used in Section 2.9 to construct Norton algebras. The study of association schemes is an important topic in algebraic combinatorics. There is also a strong connection with finite permutation groups; see Example 1.7.2.

Our use of association schemes is limited to the basic definitions and a few examples that we will introduce here. We use the approach from [BI84], a standard reference on the topic.

Definition 1.7.1. Let X be a finite set and let $R_i \subseteq X \times X$ be a binary relation for $i = 0, \dots, d$. Assume:

(AS1) $X \times X = R_0 \cup \cdots \cup R_d$ and $R_i \cap R_j = \emptyset$ for all $i \neq j$; that is, the sets R_i form a partition of $X \times X$;

(AS2) $R_0 = \{(x, x) \mid x \in X\}$;

(AS3) for each i, ${}^tR_i := \{(x, y) \mid (y, x) \in R_i\} = R_{i'}$ for some i';

(AS4) for any $(x, y) \in R_k$, the number of $z \in X$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p_{ij}^k only depending on i, j, k;

(AS5) $p_{ij}^k = p_{ji}^k$ for all i, j, k.

Then $(X, \{R_i\}_{0 \le i \le d})$ is called a *(commutative) association scheme*. If ${}^tR_i = R_i$ for all i, then we call the association scheme *symmetric*. The parameters p_{ij}^k are called the *intersection parameters* of the association scheme.

Example 1.7.2 ([BI84, § II.2, Example 2.1])**.** Let G be a transitive permutation group acting on a finite set Ω. Denote the orbits of G on $\Omega \times \Omega$ by $O_0, \ldots, O_d$ where $O_0 = \{(x, x) \mid x \in \Omega\}$. Then $(\Omega, \{O_i\}_{0 \leq i \leq d})$ satisfies (AS1) to (AS4). Condition (AS5) is satisfied if and only if the corresponding permutation representation is multiplicity-free.

This association scheme is symmetric if and only if for any $x, y \in \Omega$ there exists a $g \in G$ such that ${}^g x = y$ and ${}^g y = x$. If this is true, we say that G acts *generously transitively* on Ω. If G acts generously transitively on Ω, then the corresponding permutation representation is automatically multiplicity-free [BI84, § II.1, Lemma 1.5].

Distance-regular graphs are another important source of examples. Let us recall their definition.

Definition 1.7.3 ([BI84, § 3.1])**.** (i) Let Γ be an undirected connected graph with finite vertex set V. Then Γ is called *distance-regular* if the following two conditions are satisfied.

- Every vertex is adjacent to the same number of vertices, i.e. Γ is regular.
- For any two vertices v and w at distance k, the number of vertices at distance i from v and at distance j from w only depends on i, j and k.

(ii) We call a graph *strongly regular with parameters* (v, k, λ, μ) if it has v vertices, every vertex has k neighbors, any adjacent vertices have λ common neighbors and any non-adjacent vertices have μ common neighbors. If $\mu \geq 1$, then this graph is also distance-regular.

Example 1.7.4. Consider a distance-regular graph with finite vertex set V. Denote the distance between two vertices x and y by $d(x, y)$ and let m be the maximal distance between two vertices. For all $0 \leq i \leq m$ write

$$R_i \coloneqq \{(x, y) \in V \times V \mid d(x, y) = i\}.$$

Then $(V, \{R_i\}_{0 \leq i \leq m})$ is a symmetric association scheme.

We finish this section with some more definitions about association schemes.

Definition 1.7.5. Let $S = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme.

(i) For each i, let A_i be the matrix whose rows and columns are indexed by X and such that

$$(A_i)_{xy} = \begin{cases} 0 & \text{if } (x, y) \notin R_i, \\ 1 & \text{if } (x, y) \in R_i. \end{cases}$$

Then $A_0 = I$ is the identity matrix and $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$ for all i, j. Hence, by (AS5), these matrices span a commutative subalgebra $\mathcal{A}(S)$ of the full matrix algebra. This algebra is called the *Bose-Mesner algebra* or the *adjacency algebra* of the association scheme S. This algebra is also closed under the entry-wise or *Hadamard matrix product* which we denote by $\circ$: $(A \circ B)_{ij} = (A_{ij} B_{ij})$.

(ii) Let V be the Hermitian space with orthonormal basis $\{e_x \mid x \in X\}$ indexed by the set X. Then the A_i act naturally on V and because they pairwise commute, they can be diagonalized simultaneously by a unitary matrix U. Let $V = V_0 \oplus V_1 \oplus \cdots \oplus V_r$ be the decomposition of V into common eigenspaces. It is readily verified that we can pick $V_0 = \langle (1, \ldots, 1) \rangle$. Denote the matrix form, with respect to the basis $\{e_x \mid x \in X\}$, of the projection π_i of V onto V_i by E_i. Then $r = d$ and $E_0, \ldots, E_d$ form a basis of *primitive idempotents* for the adjacency algebra $\mathcal{A}(S)$ of S [BI84, § 2.3, Theorem 3.1]. Since $\mathcal{A}(S)$ is also closed under the Hadamard product, there exist constants q_{ij}^k such that $E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k$. We call q_{ij}^k the *Krein parameters* of S.

2

Axial and decomposition algebras

In mathematics it is very common for a new definition to undergo many stages until it finds its resting place. The definition of an axial algebra is no exception. Since the study of this type of algebras is still in an early stage, it is very beneficial to keep these different perspectives in mind. In this chapter, we give a historical overview of the different variations of axial algebras and introduce some new definitions of our own. We also give a motivation for each of them.

2.1 Peirce decompositions

The definition of an axial algebra tries to axiomatize a property observed in different kinds of algebras. This property originates from the study of idempotents. For example, the existence of an idempotent in an associative algebra leads to a *Peirce decomposition* of the algebra. This decomposition was introduced by Benjamin Peirce in 1870 [Pei81, p. 109–111].

Proposition 2.1.1. *Let A be an R-algebra that is both associative and unital. If $e \in A$ is an idempotent of A, then A decomposes (as an R-module) as*

$$eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e).$$

These four spaces are the common eigenspaces of the linear operators $A \to A \colon a \mapsto ea$ and $A \to A \colon a \mapsto ae$. Moreover, we have

$$(e_1Ae_2)(e_3Ae_4) \subseteq \begin{cases} \{0\} & \text{if } e_2 = 1 - e_3, \\ e_1Ae_4 & \text{if } e_2 = e_3, \end{cases}$$

where $e_1, e_2, e_3, e_4 \in \{e, 1-e\}$.

Proof. Because e is an idempotent, the element $1-e$ is an idempotent as well. Moreover $e(1-e) = (1-e)e = 0$. The statement can be derived from these identities. $\square$

Remark 2.1.2. Note that if A is also commutative, then $eA(1-e) = (1-e)Ae = 0$.

Also for idempotents in Jordan algebras there exists a similar Peirce decomposition. This was first observed by Adrian A. Albert in 1947 [Alb47, §10, pp. 555–556].

Proposition 2.1.3. *Let $e \in A$ be an idempotent of a Jordan algebra A over a ring containing $\frac{1}{2}$. Then A decomposes as*

$$A_1 \oplus A_0 \oplus A_{\frac{1}{2}}$$

where A_1, A_0 and $A_{\frac{1}{2}}$ are the eigenspaces of the linear operator $A \to A \colon a \mapsto ea$ corresponding to the respective eigenvalues 1, 0 and $\frac{1}{2}$. Moreover

$$(A_0)^2 \subseteq A_0, (A_1)^2 \subseteq A_1, A_0A_1 \subseteq \{0\},$$

and

$$(A_{\frac{1}{2}})^2 \subseteq A_1 \oplus A_0, A_1A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}, A_0A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}}.$$

Proof. A proof can be found in any standard textbook on Jordan algebras. See for example [McC04, Chapter 8, pp. 235–240] or [Jac68, § III.1, pp. 117–119]. □

The theory of axial algebras embraces the study of algebras which have similar decompositions. This means that there exist idempotents that lead to decompositions into eigenspaces. The multiplication of these eigenspaces is restricted by certain rules.

2.2 The Griess algebra

The classification of finite simple groups is one of the most important mathematical accomplishments of the twentieth century. According to this classification, every finite simple group is cyclic of prime power order, an alternating group, a finite simple group of Lie type, or one of the twenty-six sporadic finite simple groups [GLS94, p. 6].

The largest of these sporadic groups is referred to as the monster group. Its existence was predicted by Bernd Fischer, John G. Thompson and Robert L. Griess as a group containing a double cover of the baby monster group and an extension of Conway's simple group Co_1 (two other sporadic simple groups) as centralizers of involutions [Gri76].

Through the work of John G. Thompson, Simon P. Norton, Robert L. Griess, Ulrich Meierfrankenfeld and Yoav Segev, we know that such a group must be unique [Tho79, Nor85, GMS89].

In 1982, Robert L. Griess constructed such a simple group as a group acting by automorphisms on a 196884-dimensional non-associative algebra. Moreover, this algebra comes equipped with a non-degenerate associative form which gives it the structure of a Frobenius algebra.

Proposition 2.2.1 ([Gri82])**.** *There exists a 196884-dimensional non-associative, commutative, unital, Frobenius algebra A over $\mathbb{R}$ whose automorphism group contains an isomorphic copy of the monster group. In particular, there exists an involutive automorphism τ of A whose centralizer contains a double cover of the baby monster group.*

We will refer to the algebra A as the Griess algebra. Griess' construction was independently simplified by Jacques Tits [Tit84] and John H. Conway [Con85]. As a result of Tits' approach, it follows that the monster group is in fact the full automorphism group of the algebra A.

Proposition 2.2.2 ([Tit84, p. 497])**.** *The monster group is the full automorphism group of the Griess algebra (preserving its bilinear form).*

Also Conway's approach leads to new insights that make the Griess algebra a prominent example within the theory of axial algebras.

Proposition 2.2.3. *Let A and τ be as in Proposition 2.2.1. The centralizer of τ within the monster group fixes a particular idempotent of A. We call any conjugate of this idempotent an axis of A. Let e be such an axis.*

(i) *The operator $\mathrm{ad}_e\colon A \to A\colon a \mapsto ea$ is diagonalizable with eigenvalues $1, 0, \frac{1}{4}$ and $\frac{1}{32}$. The corresponding eigenspaces have respective dimensions 1, 96256, 4371 and 96256.*

(ii) *Denote the λ-eigenspace of this operator by A^e_λ. Then we have*

$$A^e_\lambda A^e_\mu \subseteq \bigoplus_{\chi \in \lambda \star \mu} A^e_\chi$$

for $\lambda, \mu \in \{1, 0, \frac{1}{4}, \frac{1}{32}\}$. Here $\lambda \star \mu$ is the set given by the entry at position (λ, μ) in the following table.

	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	$\{1\}$	$\emptyset$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
0	$\emptyset$	$\{0\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
$\frac{1}{4}$	$\{\frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{1, 0\}$	$\{\frac{1}{32}\}$
$\frac{1}{32}$	$\{\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{1, 0, \frac{1}{4}\}$

Proof. The first part follows from [Con85, §14 and §15]. A proof for the second part can be found in [Iva09, Lemma 8.5.1, p. 209]. $\square$

Remark 2.2.4. The history of the Griess algebra presented here is a simplified version of its true history. As Jonathan I. Hall pointed out us, the algebras constructed by Robert L. Griess, Jacques Tits and John H. Conway are in fact slightly different. Griess' original construction does not contain axes while Conway's construction does and he calls them transposition vectors. Therefore, we refer to Conway's algebra as the Griess algebra. However, Propositions 2.2.1 and 2.2.2 are also true for this algebra.

In his book [Iva09, Chapter 8], Alexander A. Ivanov provides a more general framework to study the properties of the Griess algebra and the monster group. He introduces *Majorana involutions* and *Majorana algebras*. Such a Majorana algebra is generated by idempotents that lead to a decomposition similar to the one from Proposition 2.2.3. This framework has been used with great success to reprove a theorem of Sakuma [Sak07], originally formulated in the context of vertex operator algebras. This theorem classifies the Majorana algebras generated by two idempotents [IPSS10, Section 2]. There are only nine isomorphism classes of such algebras and they are called the *Norton-Sakuma algebras*. They can all be found as subalgebras of the Griess algebra generated by two of its axes.

2.3 Axial algebras

Axial algebras try to capture the concept of Peirce decompositions and Majorana algebras into one framework. The definition of an axial algebra was given by Jonathan I. Hall, Felix Rehren ans S. Shpectorov in [HRS15a, HRS15b]. We want to study algebras generated by a set of idempotents. These idempotents lead to decompositions into eigenspaces and we impose restrictions on the multiplication of these eigenspaces. We start by providing a way to describe these restrictions by introducing fusion laws.

Definition 2.3.1. Let R be a commutative ring with identity.

(i) A *fusion law*[1] is a pair $(\Phi, \star)$ where $\Phi \subseteq R$ and $\star$ is a map that associates to any two elements of Φ a subset of elements of Φ, this is $\star\colon \Phi \times \Phi \to 2^{\Phi}$. We will call Φ the set of *eigenvalues*.

(ii) We will usually write down a fusion law as a table, called a *fusion table*. The rows and columns of this table will be labeled by the elements of Φ. At position (λ, μ) for $\lambda, \mu \in \Phi$, we write the elements of the set $\lambda \star \mu$.

We are now ready to state the definition of an axial algebra.

[1] In earlier papers on axial algebras, this was referred to as "the fusion rules", leading to singular/plural problems. It has also been referred to as a "fusion table".

Definition 2.3.2. (i) Let R be a commutative ring with identity, A an R-algebra and $e \in A$ an idempotent. For each $\lambda \in R$ we denote the eigenspace of ad_e with eigenvalue λ by

$$A^e_\lambda = \{a \in A \mid ea = \lambda a\}.$$

For each subset $\Lambda \subseteq R$, let

$$A^e_\Lambda = \bigoplus_{\lambda \in \Lambda} A^e_\lambda,$$

with the convention that $A^e_\emptyset = \{0\}$.

(ii) Let $(\Phi, \star)$ be a fusion law. We call $e \in A$ a $(\Phi, \star)$-*axis* if

$$A = \bigoplus_{\lambda \in \Phi} A^e_\lambda$$

and

$$A^e_\lambda A^e_\mu \subseteq A^e_{\lambda \star \mu}$$

for all $\lambda, \mu \in \Phi$. This means that the product of a λ-eigenvector and a μ-eigenvector is a sum of χ-eigenvectors where χ runs through $\lambda \star \mu$.

(iii) A $(\Phi, \star)$-*axial algebra* is a pair (A, Ω) where:

(a) A is a commutative (not necessarily associative) R-algebra and,

(b) $\Omega \subset A$ is a generating set of $(\Phi, \star)$-axes for A.

We will often omit the set Ω in our notation.

We give three important examples of fusion laws and axes, based on the properties from Sections 2.1 and 2.2.

Example 2.3.3 (Associative algebras)**.** Let A be an associative, commutative, unital algebra and $e \in A$ an idempotent. Then, by Proposition 2.1.1, the idempotent e is a $(\{1, 0\}, \star)$-axis where $(\{1, 0\}, \star)$ is the fusion law whose fusion table is given as follows. We have $A^e_1 = eA$ and $A^e_0 = (1 - e)A$.

$\star$	1	0
1	1	
0		0

Example 2.3.4 (Jordan algebras)**.** Every idempotent in a Jordan algebra is an axis for the following fusion law; see Proposition 2.1.3.

$\star$	1	0	$\frac{1}{2}$
1	1		$\frac{1}{2}$
0		0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$0, 1$

Example 2.3.5 (Griess algebra)**.** Let Ω be the set of axes of the Griess algebra A from Proposition 2.2.3. Then (A, Ω) is an axial algebra for the fusion law given by the following fusion table.

$\star$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

The study of axial algebras has received a lot of attention since its definition was given in 2015. Connections were developed as far afield as the regularity theory of some minimal classes of elliptic PDEs and algebraic solutions of eiconal and minimal surface equations [Tka19a, Tka19b]. See also the earlier book [NTV14].

2.4 Fusion laws

In May 2018, a specialized workshop on axial algebras took place at the University of Bristol funded by the Heilbronn Institute for Mathematical Research. It became apparent at this workshop that there was a need for a more general framework to study axial algebras. New observations forced us to generalize the definition even further and to separate fusion laws from their ring. At the same time, we noticed that the crucial aspect of an axial algebra is the existence of the corresponding decompositions, and not so much the fact that these arise from idempotents.

In [DMPSVC20], we introduce (axial) decomposition algebras to provide a natural generalization of axial algebras that takes all these facts into account. Furthermore, these decomposition algebras form a nice category. They will provide the necessary language to state our results.

We start this section by defining (general) fusion laws. In contrast to the previous definition, these will no longer depend on a ring or a field. From now on, we will always use the following definition of a fusion law.

Definition 2.4.1. (i) A *fusion law* is a pair $(X, \star)$ where X is a set[2] and $\star$ is a map from $X \times X$ to 2^X, where 2^X denotes the power set of X. We will often identify $(X, \star)$ with the set X if the map $\star$ is clear from the context.

(ii) A fusion law $(X, \star)$ is called *symmetric* if $x \star y = y \star x$ for all $x, y \in X$.

(iii) Once again, we will represent fusion laws as tables, called *fusion tables*, whose rows and columns are indexed by the set X and whose entry at position (x, y), for $x, y \in X$, lists the elements of $x \star y$.

[2] The set X is often, but not always, a finite set.

Definition 2.4.2. Let $(X, \star)$ be a fusion law and let $e \in X$.

(i) We call e a *unit* if $e \star x \subseteq \{x\}$ and $x \star e \subseteq \{x\}$ for all $x \in X$.

(ii) We call e *annihilating* if $e \star x = \emptyset$ and $x \star e = \emptyset$ for all $x \in X$.

(iii) We call e *absorbing* if $e \star x \subseteq \{e\}$ and $x \star e \subseteq \{e\}$ for all $x \in X$.

Lemma 2.4.3. *Let $(X, \star)$ be a fusion law. If $e, f \in X$ are units with $e \neq f$, then $e \star f = \emptyset$.*

Proof. We have both $e \star f \subseteq \{e\}$ and $e \star f \subseteq \{f\}$. □

We repeat the examples from Section 2.3 as the elements of our fusion law need not be ring elements anymore.

Example 2.4.4 (Fusion law for commutative associative algebras)**.** Consider the fusion law $(X, \star)$ with $X = \{e, z\}$ and the following fusion table.

$\star$	e	z
e	e	
z		z

Both the elements e and z are units and accordingly $e \star z = \emptyset$. They also are absorbing elements for this fusion law.

Example 2.4.5 (Jordan fusion law)**.** Consider the set $X = \{e, z, h\}$ with the following symmetric fusion law.

$\star$	e	z	h
e	e		h
z		z	h
h	h	h	e, z

Again, both e and z are units.

Example 2.4.6 (Ising fusion law)**.** Consider the set $X = \{e, z, q, t\}$ with the following symmetric fusion law.

$\star$	e	z	q	t
e	e		q	t
z		z	q	t
q	q	q	e, z	t
t	t	t	t	e, z, q

Once more, both e and z are units.

This fusion law is named after physicist Ernst Ising. If we merge e with z in this fusion law, then we obtain precisely the fusion law of the Virasoro algebra with central charge $\frac{1}{2}$ (see [BPZ84, Equation (6.18) on page 364] and [DMZ94, p. 302]). This algebra is equivalent to the 2-dimensional Ising model and describes the free Majorana fermion (see [GNT98, Chapter 12]).

Example 2.4.7 (Full and empty fusion law)**.** Let X be an arbitrary set.

(i) For any two elements $x, y \in X$, let $x \star y = X$. Then we call $(X, \star)$ the *full fusion law on* X.

(ii) On the other hand, the *empty fusion law on* X is the fusion law $(X, \star)$ such that $x \star y = \emptyset$ for all $x, y \in X$.

Definition 2.4.8. Let $(X, \star)$ be a fusion law and $Y \subseteq X$. We call the fusion law $(Y, *)$ determined by
$$x * y = (x \star y) \cap Y$$
for all $x, y \in Y$, the *sublaw on* Y.

Definition 2.4.9. (i) Let $(X, \star)$ be a fusion law. A relation of the form $z \in x \star y$ or $z \notin x \star y$, for $x, y, z \in X$, is called a *fusion rule*. A fusion law can thus be seen as a collection of fusion rules.

(ii) A fusion law $(X, \star)$ can also be viewed as a map $\omega\colon X \times X \times X \to \{0, 1\}$, where we define $\omega(x, y, z) = 1$ if and only if $z \in x \star y$. As such, it is clear that there is an action of $\mathrm{Sym}(3)$ on the set of all fusion laws. We call the fusion law *Frobenius* if the fusion law is invariant under this action. See Corollary 2.7.2 for the connection with Frobenius algebras.

Example 2.4.10. The fusion laws from Examples 2.4.4 to 2.4.6 are Frobenius.

Now, we define morphisms between fusion laws.

Definition 2.4.11. Let $(X, \star)$ and $(Y, \star)$ be two fusion laws. A *morphism* from $(X, \star)$ to $(Y, \star)$ is a map $\xi\colon X \to Y$ such that
$$\xi(x_1 \star x_2) \subseteq \xi(x_1) \star \xi(x_2)$$
for all $x_1, x_2 \in X$, where we have denoted the obvious extension of ξ to a map $2^X \to 2^Y$ also by ξ.

This makes the set of all fusion laws into a category Fus. In Section 2.10, we explore this category in more detail.

Example 2.4.12. Let $(X, \star)$ be an arbitrary fusion law. The identity map $X \to X\colon x \mapsto x$ defines a morphism from $(X, \star)$ to the full fusion law on X and from the empty fusion law on X to $(X, \star)$.

Definition 2.4.13. Let $(X, \star)$ and $(Y, \star)$ be two fusion laws.

(i) We define the *product* of $(X, \star)$ and $(Y, \star)$ to be the fusion law $(X \times Y, \star)$ given by
$$(x_1, y_1) \star (x_2, y_2) := \{(x, y) \mid x \in x_1 \star x_2, y \in y_1 \star y_2\}.$$

(ii) We define the *union* of $(X, \star)$ and $(Y, \star)$ to be the fusion law $(X \sqcup Y, \star)$, where $\star$ extends the given fusion laws on X and Y and is defined by
$$x \star y := \emptyset$$
for all $x \in X$ and all $y \in Y$.

An important class of fusion laws are the group fusion laws.

Definition 2.4.14. Let Γ be a group. Then the map
$$\star \colon \Gamma \times \Gamma \to 2^\Gamma \colon (g, h) \mapsto \{gh\}$$
defines a *group fusion law*. The identity element of Γ is the unique unit of the fusion law $(\Gamma, \star)$.

Remark 2.4.15. The category Grp of groups is a *full* subcategory of Fus: if Γ and Δ are groups, then the fusion law morphisms from $(\Gamma, \star)$ to $(\Delta, \star)$ are precisely those arising from homomorphisms from Γ to Δ.

Two further examples of fusion laws arising in group theory and representation theory are given in the following examples.

Example 2.4.16 (Class fusion law)**.** Let G be a group and let X be its set of conjugacy classes. Then we can define a fusion law on X by declaring
$$E \in C \star D \iff E \cap CD \neq \emptyset,$$
where CD is the setwise product of C and D inside G. The trivial conjugacy class $\{1\} \subseteq G$ is a unit for this fusion law. If G is an abelian group, this fusion law coincides with the group fusion law introduced in Definition 2.4.14.

Example 2.4.17 (Representation fusion law)**.** (i) Let G be a finite group and let $X \subseteq \mathsf{Irr}(G)$ be a set of irreducible (complex) characters of G. Then we can define a fusion law on X by declaring
$$\chi \in \chi_1 \star \chi_2 \iff \chi \text{ is a constituent of } \chi_1\chi_2.$$
If the trivial character is contained in X, then it is a unit for this fusion law. If $X = \mathsf{Irr}(G)$, then we call $(X, \star)$ the *representation fusion law* of G.

(ii) Similarly, if $\mathcal{L}$ is a complex semisimple Lie algebra, then we can define a fusion law on any subset X of irreducible characters of $\mathcal{L}$ in the same way as for finite groups (see Definition 1.5.14 for the definition of characters in this case). Once again, if the trivial character is contained in X, then it is a unit for this fusion law.

2.5 Decomposition algebras

We are now ready to introduce decomposition algebras as in [DMPSVC20, § 4]. We believe that they provide the right axiomatic framework to study all algebras reminiscent of axial algebras. This definition allows for an interesting definition of homomorphisms. For each choice of a base ring and a fusion law, this will give rise to a corresponding *category of decomposition algebras*. We refer to Section 2.11 for further categorical properties.

Definition 2.5.1. Let R be a commutative ring and let $\mathcal{F} = (X, \star)$ be a fusion law.

(i) An *$\mathcal{F}$-decomposition* of an R-algebra A (not assumed to be commutative, associative or unital) is a direct sum decomposition $A = \bigoplus_{x \in X} A_x$ (as R-modules) such that $A_x A_y \subseteq A_{x \star y}$ for all $x, y \in X$, where $A_Y := \bigoplus_{y \in Y} A_y$ for all $Y \subseteq X$.

(ii) An *$\mathcal{F}$-decomposition algebra* is a triple (A, I, Ω) where A is an R-algebra, I is an index set and Ω is a tuple[3] of $\mathcal{F}$-decompositions of A indexed by I. We will usually write the corresponding decompositions as $A = \bigoplus_{x \in X} A^i_x$, so

$$\Omega = \big((A^i_x)_{x \in X} \mid i \in I \big);$$

we sometimes use the shorthand notation $\Omega[i] := (A^i_x)_{x \in X}$. Notice that we do not require the decompositions to be distinct.

We will often omit the explicit reference to $\mathcal{F}$ if it is clear from the context and simply talk about decompositions and decomposition algebras.

Example 2.5.2 (Commutative associative algebras)**.** Consider the fusion law $\mathcal{F}$ from Example 2.4.4. Let A be any *commutative, associative, unital* algebra over a ring R. Let $\{a_i \mid i \in I\} \subseteq A$ be any collection of idempotents in A, indexed by some set I. For each $i \in I$, the algebra A decomposes as $A = a_i A \oplus (1 - a_i) A$. Write $A^i_e := a_i A$ and $A^i_z := (1 - a_i) A$. Then each decomposition $A = A^i_e \oplus A^i_z$ is indeed an $\mathcal{F}$-decomposition. If we write Ω for the I-tuple of all those decompositions, then (A, I, Ω) is an $\mathcal{F}$-decomposition algebra.

Example 2.5.3 (Jordan algebras)**.** Consider the Jordan fusion law $\mathcal{F}$ from Example 2.4.5. Let A be any *Jordan* algebra over a commutative ring R containing $\frac{1}{2}$. Let $\{a_i \mid i \in I\} \subseteq A$ be any collection of idempotents in A, indexed by some set I. For each $i \in I$, let

$$A = A_1 \oplus A_0 \oplus A_{\frac{1}{2}}$$

[3] Formally, we could define Ω as a set and define this "tuple" as a map from I to Ω, but we will not do so in order not to make our notation unnecessarily complicated.

be the Peirce decomposition with respect to a_i from Proposition 2.1.3. Let $A_e^i := A_1$, $A_z^i := A_0$ and $A_h^i := A_{\frac{1}{2}}$. Then $A = A_e^i \oplus A_z^i \oplus A_h^i$ is indeed an $\mathcal{F}$-decomposition of A. If we write Ω for the I-tuple of those decompositions, then (A, I, Ω) is an $\mathcal{F}$-decomposition algebra.

Example 2.5.4. Let $\mathcal{F}$ be the Ising fusion law from Example 2.4.6. Let A be the Griess algebra from Section 2.2. Let $\{a_i \mid i \in I\}$ be its set of axes as defined in Proposition 2.2.3. For each $i \in I$, let

$$A = A_1 \oplus A_0 \oplus A_{\frac{1}{4}} \oplus A_{\frac{1}{32}}$$

be the decomposition of A into eigenspaces of ad_{a_i} from Proposition 2.2.3. Write $A_e^i := A_1$, $A_z^i := A_0$, $A_q^i = A_{\frac{1}{4}}^i$ and $A_t^i := A_{\frac{1}{32}}$ and let Ω be the I-tuple of these decompositions. Then (A, I, Ω) is an $\mathcal{F}$-decomposition algebra.

Remark 2.5.5. Let $\mathcal{F}$ be a fusion law and let (A, I, Ω) be an $\mathcal{F}$-decomposition algebra. If $e \in \mathcal{F}$ is annihilating (see Definition 2.4.2), then each subspace A_e^i is annihilating for the algebra A, in the sense that $A_e^i \cdot A = 0 = A \cdot A_e^i$. Similarly, if $e \in \mathcal{F}$ is absorbing (see Definition 2.4.2), then each A_e^i is an ideal: $A_e^i \cdot A \subseteq A_e^i \supseteq A \cdot A_e^i$.

We introduce some terminology that allows us to compare different decompositions.

Definition 2.5.6. Let R be a ring and M an R-module. Let $M = \bigoplus_{x \in X} M_x$ and $M = \bigoplus_{y \in Y} M_y$ be two decompositions of M.

(i) We call these decompositions *compatible* if $M = \bigoplus_{x \in X} \bigoplus_{y \in Y} (M_x \cap M_y)$ is another decomposition of M.

(ii) We say that $\bigoplus_{y \in Y} M_y$ is a *refinement* of $\bigoplus_{x \in X} M_x$ if there is a map $\varphi \colon Y \to X$ such that $M_y \subseteq M_{\varphi(y)}$ for all $y \in Y$. Note that such a map, if it exists, must be unique.

(iii) Conversely, we call $\bigoplus_{y \in Y} M_y$ a *coarsening* of $\bigoplus_{x \in X} M_x$ if $\bigoplus_{x \in X} M_x$ is a refinement of $\bigoplus_{y \in Y} M_y$.

Proposition 2.5.7. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fusion laws and let A be an R-algebra.*

(i) *Let $\bigoplus_{x \in \mathcal{F}_1} A_x$ be an $\mathcal{F}_1$-decomposition of A and $\bigoplus_{y \in \mathcal{F}_2} A_y$ a compatible $\mathcal{F}_2$-decomposition. Write $A_{(x,y)} := A_x \cap A_y$ for all $x \in \mathcal{F}_1$ and $y \in \mathcal{F}_2$. Then*

$$\bigoplus_{x \in \mathcal{F}_1} \bigoplus_{y \in \mathcal{F}_2} A_{(x,y)}$$

is an $(\mathcal{F}_1 \times \mathcal{F}_2)$-decomposition of A where $\mathcal{F}_1 \times \mathcal{F}_2$ is the direct product of the fusion laws $\mathcal{F}_1$ and $\mathcal{F}_2$. It is a refinement of both the decompositions $\bigoplus_{x \in \mathcal{F}_1} A_x$ and $\bigoplus_{y \in \mathcal{F}_2} A_y$.

(ii) *Let $\xi\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of fusion laws and $\bigoplus_{x\in\mathcal{F}_1} A_x$ an $\mathcal{F}_1$-decomposition of A. For each $y \in \mathcal{F}_2$, write*

$$A_y \coloneqq A_{\xi^{-1}(y)} = \bigoplus_{x\in\xi^{-1}(y)} A_x$$

where $\xi^{-1}(y) \coloneqq \{x \in \mathcal{F}_1 \mid \xi(x) = y\}$. Then $\bigoplus_{y\in\mathcal{F}_2} A_y$ is a coarsening of $\bigoplus_{x\in\mathcal{F}_1} A_x$ and an $\mathcal{F}_2$-decomposition of A.

Proof. (i) Since the decompositions are compatible, we have that

$$\bigoplus_{x\in\mathcal{F}_1}\bigoplus_{y\in\mathcal{F}_2} A_{(x,y)}$$

is indeed a decomposition of A. Since $A_{(x,y)} \subseteq A_x$ and $A_{(x,y)} \subseteq A_y$ for all $x \in \mathcal{F}_1$ and all $y \in \mathcal{F}_2$, it is clear that this decomposition is a refinement of both $\bigoplus_{x\in X} A_x$ and $\bigoplus_{y\in\mathcal{Y}} A_y$. Now, for all $x_1, x_2 \in \mathcal{F}_1$, we have

$$A_{x_1}A_{x_2} \subseteq A_{x_1\star x_2}.$$

Therefore, for all $y_1, y_2 \in \mathcal{F}_2$,

$$(A_{x_1} \cap A_{y_1})(A_{x_2} \cap A_{y_2}) \subseteq A_{x_1\star x_2}.$$

Similarly, we have

$$(A_{x_1} \cap A_{y_1})(A_{x_2} \cap A_{y_2}) \subseteq A_{y_1\star y_2}$$

for all $x_1, x_2 \in \mathcal{F}_1$ and $y_1, y_2 \in \mathcal{F}_2$. Thus

$$A_{(x_1,y_1)}A_{(x_2,y_2)} \subseteq A_{x_1\star x_2} \cap A_{y_1\star y_2} = A_{(x_1,y_1)\star(x_2,y_2)}$$

where the last equality follows from Definition 2.4.13.

(ii) We have to verify that for all $y, z \in \mathcal{F}_2$, we have $A_yA_z \subseteq A_{y\star z}$. By the definition of a fusion law morphism, we have

$$\xi^{-1}(y) \star \xi^{-1}(z) \subseteq \xi^{-1}(y \star z),$$

and hence indeed

$$\begin{aligned} A_yA_z &= A_{\xi^{-1}(y)}A_{\xi^{-1}(z)} \\ &\subseteq A_{\xi^{-1}(y)\star\xi^{-1}(z)} \\ &\subseteq A_{\xi^{-1}(y\star z)} = A_{y*z}, \end{aligned}$$

proving the proposition. □

We end this section by illustrating that the decomposition algebras with respect to a fixed fusion law form a nice category.

Definition 2.5.8. Let R be a commutative ring with identity and $\mathcal{F}$ a fusion law. We define a category $\mathcal{F}\text{-}\mathsf{Dec}_R$ having as objects the $\mathcal{F}$-decomposition algebras over R. If (A, I, Ω_A) and (B, J, Ω_B) are two objects, with

$$\Omega_A = \big((A_x^i)_{x \in X} \mid i \in I\big), \qquad \Omega_B = \big((B_x^j)_{x \in X} \mid j \in J\big),$$

then the *morphisms* between (A, I, Ω_A) and (B, J, Ω_B) are defined to be pairs (φ, ψ) where $\varphi\colon A \to B$ is an R-algebra morphism and $\psi\colon I \to J$ is a map (of sets) such that

$$\varphi(A_x^i) \subseteq B_x^{\psi(i)}$$

for all $x \in \mathcal{F}$ and all $i \in I$.

Proposition 2.5.9. *If $\xi\colon \mathcal{F}_1 \to \mathcal{F}_2$ is a fusion law morphism and (A, I, Ω) is an $\mathcal{F}_1$-decomposition algebra, then A can also be viewed as an $\mathcal{F}_2$-decomposition algebra (A, I, Σ) by considering the corresponding coarsening from Proposition* 2.5.7*. More precisely, we declare*

$$A_y^i := A_{\xi^{-1}(y)}^i = \bigoplus_{x \in \xi^{-1}(y)} A_x^i$$

for each $i \in I$ and each $y \in \mathcal{F}_2$. This induces a functor

$$F_\xi\colon \mathcal{F}_1\text{-}\mathsf{Dec}_R \to \mathcal{F}_2\text{-}\mathsf{Dec}_R.$$

Proof. This follows from Proposition 2.5.7 (i). □

In Section 2.11, we study the category $\mathcal{F}\text{-}\mathsf{Dec}_R$ in some more detail.

2.6 Axial decomposition algebras

Next, we explain how axial algebras fit into the framework of decomposition algebras.

Definition 2.6.1. Let $\mathcal{F} = (X, \star)$ be a fusion law with a distinguished unit $e \in X$. For each $x \in X$, let $\lambda_x \in R$. An $\mathcal{F}$-decomposition algebra (A, I, Ω) will be called *left-axial* (with *parameters* λ_x) if for each $i \in I$, there is some non-zero $a_i \in A_e^i$ (called a *left axis*) such that:

$$a_i \cdot b = \lambda_x b \quad \text{for all } x \in X \text{ and for all } b \in A_x^i. \tag{2.1}$$

Similarly, (A, I, Ω) is a *right-axial* decomposition algebra (with *parameters* λ_x) if for each $i \in I$, there is some non-zero $a_i \in A_e^i$ (called a *right axis*) such that:

$$b \cdot a_i = \lambda_x b \quad \text{for all } x \in X \text{ and for all } b \in A_x^i. \tag{2.2}$$

Of course, if A is commutative, then we drop the prefix "left" or "right" and simply talk about axial decomposition algebras. We call a (left- or right-)axial decomposition algebra *primitive* if $A_e^i = Ra_i$ for each $i \in I$.

Remark 2.6.2. Recall from Definition 2.3.2 that an *axial algebra* is an R-algebra A generated by a set E of idempotents (called *axes*), such that for each axis $c \in E$, the left multiplication operator $\operatorname{ad}_c\colon A \to A\colon x \mapsto cx$ is semisimple and its eigenspaces multiply according to a given fusion law $(X, \star)$ with $X \subseteq R$.

Every axial algebra is an axial decomposition algebra. Indeed, if (A, E) is an axial algebra, then for each $c \in E$, there is a corresponding decomposition $A = \bigoplus_{x \in X} A_x^c$, so certainly (A, E, Ω) with $\Omega = \{(A_x^c)_{x\in X} \mid c \in E\}$ is a decomposition algebra. It is indeed axial, with $a_c = c$ for each $c \in E \subseteq A$ and $\lambda_x = x$ for each $x \in X \subseteq R$.

On the other hand, axial decomposition algebras are more general objects than axial algebras, in four ways:

- The elements $a_c \in A$ are not required to be idempotents. If the corresponding parameter $\lambda_e \neq 0$ is a unit in R (for example when R is a field), then we can rescale a_c to an idempotent. If $\lambda_e = 0$, then $a_c^2 = 0$, i.e., a_c is nilpotent.
- The algebra A is not assumed to be generated by the axes. However, this will often be the case.
- By distinguishing between $x \in X$ and $\lambda_x \in R$, we allow the possibility that some of the $\lambda_x \in R$ coincide.
- The algebra A is not assumed to be commutative.

Example 2.6.3 (Associative algebras)**.** Let $X = \{e, z\}$ and consider the fusion law on $X \times X$ given by

$$(x_1, y_1) \star (x_2, y_2) = \begin{cases} \{(x_1, y_2)\} & \text{if } y_1 = x_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose that A is a unital, associative (but not necessarily commutative) algebra and let $e \in A$ be an idempotent. Then, by Proposition 2.1.1, the algebra A decomposes as

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e).$$

Let

$$\begin{aligned} A_{(e,e)} &\coloneqq eAe, \\ A_{(e,z)} &\coloneqq eA(1-e), \\ A_{(z,e)} &\coloneqq (1-e)Ae, \end{aligned}$$

$$A_{(z,z)} \coloneqq (1-e)A(1-e).$$

Then $A = \bigoplus_{x \in X\times X} A_x$ is an $(X \times X, \star)$-decomposition of A, again by Proposition 2.1.1. The idempotent $e \in A$ is a left axis for this decomposition with parameters $\lambda_{(e,e)} = \lambda_{(e,z)} = 1$ and $\lambda_{(z,e)} = \lambda_{(z,z)} = 0$. It is also a right axis with parameters $\lambda_{(e,e)} = \lambda_{(z,e)} = 1$ and $\lambda_{(e,z)} = \lambda_{(z,z)} = 0$.

Example 2.6.4. Let us recall the decomposition algebras from Examples 2.5.2 to 2.5.4. Remember that their index set is in one-to-one correspondence with a set of idempotents of the algebra. Each of these idempotents is an axis for its corresponding decomposition. Its parameters are $\lambda_e = 1$, $\lambda_z = 0$, $\lambda_h = \frac{1}{2}$, $\lambda_q = \frac{1}{4}$, $\lambda_t = \frac{1}{32}$.

Remark 2.6.5. Suppose that (A, I, Ω) is a *commutative* axial $\mathcal{F}$-decomposition algebra with parameters λ_x for $x \in \mathcal{F}$. If $\lambda_e \in R^\times$ for the distinguished unit $e \in \mathcal{F}$, then the axis a_i for $i \in I$ is uniquely determined. Indeed, if both a_i and b_i would be axes for the decomposition $\bigoplus_{x\in\mathcal{F}} A^i_x$, then

$$\lambda_e a_i = b_i \cdot a_i = \lambda_e b_i$$

because $a_i, b_i \in A^i_e$. Since $\lambda_e \in R^\times$ we have $a_i = b_i$.

We now make the class of (left) axial decomposition algebras into a category.

Definition 2.6.6. Let $\mathcal{F} = (X, \star)$ be a fusion law with a distinguished unit $e \in X$ and let $\lambda\colon X \to R\colon x \mapsto \lambda_x$ be an arbitrary map, called the *evaluation map*. We define a category $(\mathcal{F}, \lambda)$-AxDec_R with as objects the axial $\mathcal{F}$-decomposition algebras together with the collection of left axes, for the choice of parameters λ_x given by the evaluation map. In other words, the objects are quadruples (A, I, Ω, α), where (A, I, Ω) is a $\mathcal{F}$-decomposition algebra and $\alpha\colon I \to A\colon i \mapsto a_i$ is a map such that $a_i \in A^i_e$ and (2.1) holds.

The morphisms in this category are the morphisms

$$(\varphi, \psi)\colon (A, I, \Omega_A, \alpha) \to (B, J, \Omega_B, \beta)$$

of decomposition algebras such that $\varphi \circ \alpha = \beta \circ \psi$, this is, φ maps each axis a_i to the corresponding axis $b_{\psi(i)}$.

2.7 Axial decomposition algebras with a Frobenius form

We now explore some additional properties that come from the existence of a Frobenius form on an axial decomposition algebra. Let us establish some notation. We will assume that our base ring is a field k and that $(A, I, \Omega, i \mapsto a_i)$ is a (commutative) axial decomposition algebra for some fusion law $\mathcal{F}$. Denote its evaluation map by $\lambda\colon \mathcal{F} \to k\colon x \mapsto \lambda_x$. Assume that $\langle\ ,\ \rangle\colon A \times A \to k$ is a symmetric Frobenius form for A.

First, we prove that the eigenspaces of each axis a_i are orthogonal with respect to the Frobenius form. This property was already observed by Jonathan I. Hall, Felix Rehren en Sergey Shpectorov in the context of axial algebras [HRS15b, Proposition 3.6].

Proposition 2.7.1. *The eigenspaces of* $\mathrm{ad}_{a_i}\colon A \to A\colon x \mapsto a_i x$, *for an axis* a_i *with* $i \in I$, *are orthogonal with respect to the Frobenius form* $\langle\ ,\ \rangle$.

Proof. Suppose that $i \in I$, $b \in A^i_x$, $c \in A^i_y$ with $\lambda_x \neq \lambda_y$, then

$$\lambda_x \langle b, c\rangle = \langle a_i b, c\rangle = \langle b, a_i c\rangle = \lambda_y \langle b, c\rangle.$$

Hence it follows that $\langle b, c\rangle = 0$. □

As a consequence, if the evaluation map λ is injective, the decomposition $\bigoplus_{x\in\mathcal{F}} A^i_x$ is a decomposition into orthogonal subspaces. Since the bilinear form $\langle\ ,\ \rangle$ is non-degenerate, we have the following corollary.

Corollary 2.7.2. *Suppose that the fusion law* $\mathcal{F}$ *is minimal for the axial decomposition algebra* $\mathcal{A} = (A, I, \Omega, i \mapsto a_i)$. *This means that for all* $x, y \in \mathcal{F}$, $x \star y$ *is the smallest subset* Z *for which* $A^i_x A^i_y \subseteq A^i_Z$ *for all* $i \in I$. *If, moreover, the evaluation map of* $\mathcal{A}$ *is injective, then the fusion law* $\mathcal{F}$ *is Frobenius.*

Proof. Since we assume that $\langle\ ,\ \rangle$ is non-degenerate and the evaluation map is injective, we have that $\langle\ ,\ \rangle$ is non-degenerate on each A^i_x by Proposition 2.7.1. Thus the projection of $A^i_x A^i_y$ onto A^i_z is zero if and only if $\langle A^i_x A^i_y, A^i_z\rangle = 0$. If $z \in x \star y$ then the projection of $A^i_x A^i_y$ onto A^i_z is non-zero for some $i \in I$. But then $\langle A^i_x A^i_y, A^i_z\rangle \neq 0$ and thus $\langle A^i_{\pi(x)} A^i_{\pi(y)}, A^i_{\pi(z)}\rangle \neq 0$ for any permutation π of $\{x, y, z\}$. So we have that the projection of $A^i_{\pi(x)} A^i_{\pi(y)}$ onto $A^i_{\pi(z)}$ is non-zero and thus $\pi(z) \in \pi(x) \star \pi(y)$ for any permutation π of $\{x, y, z\}$. □

2.8 Decomposition algebras from representations

In this section we will see how representation theory directly gives rise to interesting decomposition algebras. We will assume that our base ring is the field $\mathbb{C}$ of complex numbers to avoid some technicalities.

Theorem 2.8.1. *Let* G *be a finite group or a complex semisimple Lie algebra and let* A *be a finite-dimensional* $\mathbb{C}$*-algebra for* G. *Let* H *be a subgroup (or subalgebra) of* G *and* $\mathcal{F} = (\mathsf{Irr}(H), \star)$ *its representation fusion law, as in Example* 2.4.17. *Then the* H*-isotypic decomposition* $A = \bigoplus_{\chi\in\mathsf{Irr}(H)} A_\chi$ *of* A *is an* $\mathcal{F}$*-decomposition.*

Proof. Let $V_1 \oplus \cdots \oplus V_n$ be a decomposition of A into irreducible G-representations. Denote the irreducible character of V_i by χ_i. By Schur's lemma (Propositions 1.4.5 and 1.5.5) and the complete reducibility of G-representations (Propositions 1.4.4 and 1.5.4), we have $\mathsf{Hom}_G(V_i \otimes V_j, V_k) = 0$ whenever V_k is not isomorphic to a subrepresentation of $V_i \otimes V_j$. By Propositions 1.4.8 and 1.5.16 this happens if and only if χ_k is not a constituent of $\chi_i\chi_j$. Since the algebra product on A is G-equivariant, V_iV_j is isomorphic to a subrepresentation of $V_i \otimes V_j$. Thus also $\mathsf{Hom}_G(V_iV_j, V_k) = 0$ if $\chi_k \notin \chi_i \star \chi_j$. Hence the projection of V_iV_j onto V_k is zero. Since $A_\chi = \bigoplus_{\chi_i=\chi} V_i$ for each $\chi \in \mathsf{Irr}(H)$, the assertion follows. □

Sometimes we can use this new decomposition to make a further refinement of an existing decomposition. The following proposition gives a sufficient condition for compatibility.

Proposition 2.8.2. *A decomposition $A = \bigoplus_{x\in X} A_x$ is compatible with the decomposition $A = \bigoplus_{\chi\in\mathsf{Irr}(H)} A_\chi$ of A into H-isotypic components if each A_x is an H-subrepresentation of A. In particular, the decomposition of A into isotypic components with respect to two subgroups (or subalgebras) H_1 and H_2 is compatible if $H_1 \leq H_2$.*

Proof. If A_x is an H-subrepresentation of G, then $\bigoplus_{\chi\in\mathsf{Irr}(H)} A_x \cap A_\chi$ is its decomposition into H-isotypic components. Thus it is a decomposition of A_x. □

Example 2.8.3. Let G be the monster group and let A be the Griess algebra. Let $e \in A$ be an axis and τ its corresponding involution as in Proposition 2.2.3. Let H be the centralizer of τ, which is a double cover of the baby monster group. We know that H stabilizes the axis $e \in A$. Therefore, it must leave the eigenspaces of ad_e invariant. Thus this decomposition into eigenspaces is compatible with the decomposition into H-isotypic components. More precisely, we have that the 1-, $\frac{1}{4}$- and $\frac{1}{32}$-eigenspace are irreducible H-representations while the 0-eigenspace is the direct sum of a trivial H-representation and an irreducible representation M. Denote the respective characters of the 1-eigenspace, M, the $\frac{1}{4}$-eigenspace and $\frac{1}{32}$-eigenspace by χ_1, χ_2, χ_3 and χ_4. Then they have respective degrees 1, 96255, 4371 and 96256. A character computation shows that the representation fusion law on $\{\chi_1, \chi_2, \chi_3, \chi_4\}$ is given as follows.

$\star$	χ_1	χ_2	χ_3	χ_4
χ_1	χ_1	χ_2	χ_3	χ_4
χ_2	χ_2	χ_1, χ_2	χ_3	χ_4
χ_3	χ_3	χ_3	χ_1, χ_2	χ_4
χ_4	χ_4	χ_4	χ_4	χ_1, χ_2, χ_3

Note that this fusion law is almost identical to the usual Ising fusion law for the Griess algebra.

If φ is an automorphism of G, then φ induces a natural isomorphism

$$C(H) \to C(\varphi(H)) \colon \chi \mapsto {}^{\varphi}\chi$$

between the character rings of H and $\varphi(H)$; see Definition 1.4.26 (i) and Definition 1.5.17 (v). This leads to the following corollary.

Corollary 2.8.4. *Let $(\varphi_i \mid i \in I)$ be a tuple of automorphisms of G indexed by some set I. For each $i \in I$ and $\chi \in \mathsf{Irr}(H)$ let A^i_χ be the $\varphi_i(H)$-isotypic component of A with respect to the irreducible character ${}^{\varphi_i}\chi$ of $\varphi_i(H)$. Now, for each $i \in I$, let*

$$\Omega[i] \coloneqq (A^i_\chi \mid \chi \in \mathsf{Irr}(H)).$$

Then (A, I, Ω) is an $\mathcal{F}$-decomposition algebra for the representation fusion law $\mathcal{F}$ of H.

Proof. This follows immediately from Theorem 2.8.1 and the natural isomorphism between $C(H)$ and $C(\varphi_i(H))$ that maps $\mathsf{Irr}(H)$ to $\mathsf{Irr}(\varphi_i(H))$. □

Remark 2.8.5. If the φ_i are inner automorphisms of G, then, for every $i \in I$ there exists some $g_i \in G$ (or $g_i \in \widehat{\mathsf{Int}}(G)$ if G is a Lie algebra) such that $\varphi_i(H) = {}^{g_i}H$. In that case we have that $A^i_\chi = g_i \cdot A_\chi$.

The following generic type of examples will be essential in Sections 5.6 and 5.7.

Example 2.8.6. Let G be a finite group or a complex semisimple Lie algebra. Let I be an index set and $(H_i \mid i \in I)$ an I-tuple of conjugate subgroups or conjugate semisimple subalgebras respectively. Let A be an algebra for G.

(i) Write $A = \bigoplus_{\chi \in \mathsf{Irr}(G)} A_\chi$ for the decomposition of A into G-isotypic components. Due to Theorem 2.8.1, this is an $(\mathsf{Irr}(G), \star)$-decomposition of the algebra A. Let $(X_g, \star)$ be the sublaw of $(\mathsf{Irr}(G), \star)$ on those irreducible characters χ that have a non-trivial isotypic component, i.e. $A_\chi \neq 0$. Then $\bigoplus_{\chi \in X_g} A_\chi$ is an $(X_g, \star)$-decomposition of A. We call this the *global decomposition* of A with respect to G and we call $(X_g, \star)$ the *global fusion law.* (The subscript "g" stands for "global".)

(ii) Let $H \coloneqq H_i$ for some $i \in I$. Since the H_i are conjugate, there exists a family of inner automorphisms $(\varphi_i \mid i \in I)$ of G such that $\varphi_i(H) = H_i$. Apply Corollary 2.8.4 to obtain an $(\mathsf{Irr}(H), \star)$-decomposition algebra (A, I, Ω_l). Note that this decomposition algebra depends on the choice of the φ_i. However, this choice will be irrelevant to us.

Of course, we can again restrict to the characters $\chi \in \mathsf{Irr}(H)$ for which $A^i_\chi \neq 0$ for some and hence all $i \in I$. We write $(X_l, \star)$ for the sublaw of $(\mathsf{Irr}(H), \star)$ of those characters and we view (A, I, Ω_l) as an $(X_l, \star)$-decomposition algebra. We call them the *local decompositions* of A with respect to $(H_i \mid i \in I)$ and $(X_l, \star)$ the *local fusion law.* (Here, the subscript "l" stands for "local".)

(iii) Since for each $i \in I$ we have $H_i \leq G$, it follows from Proposition 2.8.2, that each of the local decompositions is compatible with the global decomposition. Let $\bigoplus_{\chi \in X_g} A_\chi$ be the global decomposition and $\bigoplus_{\psi \in X_l} A^i_\psi$ be the local decomposition for $i \in I$. For each $\chi \in X_g$, $\psi \in X_l$ and $i \in I$ we write $A^i_{\chi,\psi} := A_\chi \cap A^i_\psi$. Let $\mathcal{F}$ be the direct product of the fusion laws $(X_g, \star)$ and $(X_l, \star)$. Then $\bigoplus_{x \in \mathcal{F}} A^i_x$ is an $\mathcal{F}$-decomposition of A. If we write Ω for the I-tuple of these decompositions, then (A, I, Ω) is an $\mathcal{F}$-decomposition algebra by Proposition 2.5.7 (i).

If we make the additional assumption that A is multiplicity-free as an H-representation, then any non-zero element of the trivial isotypic component is an axis for the decomposition into H-isotypic components.

Theorem 2.8.7. *Consider the situation from Theorem* 2.8.1 *and let* $\mathbf{1} \in \mathsf{Irr}(H)$ *denote the trivial character of* H*. Then any non-zero element* $a \in A_{\mathbf{1}}$ *is an axis for the* $\mathcal{F}$*-decomposition* $A = \bigoplus_{\chi \in \mathsf{Irr}(H)} A_\chi$.

Proof. Since A is multiplicity-free, every A_χ is an irreducible H-representation. Schur's lemma implies that $\mathsf{Hom}_G(A_\chi, A_\chi) = \langle \lambda \mathsf{id} \mid \lambda \in \mathbb{C} \rangle$. Observe that the restriction of ad_a to A_χ induces an element of $\mathsf{Hom}_G(A_\chi, A_\chi)$. Hence the element a is an axis for $\bigoplus_{\chi \in \mathsf{Irr}(H)} A_\chi$. □

Corollary 2.8.8. *Let* $(\varphi_i \mid i \in I)$ *be a tuple of inner automorphisms of* G*. Denote by* g_i *the element of* G *(or* $\widehat{\mathsf{Int}}(G)$*) corresponding to* φ_i*. Let* (A, I, Ω) *be the corresponding* $\mathcal{F}$*-decomposition algebra from Corollary* 2.8.4 *and let* a *be as in Theorem* 2.8.7*. For each* $i \in I$ *let* $a_i := g_i \cdot a$*. Then* (A, I, Ω, α) *is an axial decomposition algebra where* $\alpha\colon I \to A\colon i \mapsto a_i$*.*

Proof. The element a is an axis for the decomposition $\bigoplus_{\chi \in \mathsf{Irr}(H)} A_\chi$ of A into H-isotypic components by Theorem 2.8.7. Since $A^i_\chi = g_i \cdot A_\chi$ and g_i acts as an automorphism on A, we have that $a_i = g_i \cdot a$ is an axis for $\Omega[i]$ with the same evaluation map. □

2.9 Norton algebras

This section aims to construct an axial decomposition algebra from an association scheme. More precisely, we will show that Norton algebras are axial decomposition algebras. Norton algebras, in the sense of this section, were first introduced by Peter J. Cameron, Jean-Marie Goethals and Johan J. Seidel in [CGS78] starting from association schemes.

Definition 2.9.1. Let $S = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme. Denote its Krein parameters by q^k_{ij} for $0 \leq i, j, k \leq d$. As in Definition 1.7.5, let V be the Hermitian space with orthonormal basis $\{e_x \mid x \in X\}$ indexed by the set X

on which the adjacency algebra $\mathcal{A}(S)$ acts naturally. Let $V = V_1 \oplus \cdots \oplus V_d$ be the orthogonal decomposition of V where each V_i is a common eigenspace of the elements of $\mathcal{A}(S)$. Write $\pi_i \colon V \to V_i$ for the orthogonal projection onto V_i.

For each i, j and k we can define a bilinear map $\sigma_{ij}^k \colon V_i \times V_j \to V_k$ as pointwise multiplication with respect to the basis $\{e_x \mid x \in X\}$ composed with projection onto V_k. That is,

$$\sigma_{ij}^k(v, w) := \sum_{x \in X} \langle v, e_x \rangle \langle w, e_x \rangle \pi_k(e_x).$$

In particular, σ_{ii}^i gives V_i the structure of a commutative non-associative algebra, which is called a *Norton algebra*. We denote its product by $*$.

Remark 2.9.2. Recall that the association scheme S is symmetric if $(x, y) \in R_i$ if and only if $(y, x) \in R_i$ for all $0 \le i \le d$ and $x, y \in X$. If S is a symmetric association scheme then all the matrices of $\mathcal{A}(S)$ will be symmetric and hence simultaneously diagonalizable by a real orthogonal matrix. In that case the projections π_i and therefore the Norton algebras can be defined over $\mathbb{R}$.

Remark 2.9.3. Consider the symmetric bilinear form on V defined by $\langle e_x, e_y \rangle = \delta_{xy}$ for all $x, y \in X$, where δ is the Kronecker delta. Then $\langle\ ,\ \rangle$ is a Frobenius form for the pointwise product on V. If S is symmetric, then each of the projections π_i is orthogonal with respect to this form. Hence the restriction of $\langle\ ,\ \rangle$ to V_i is a Frobenius form for the Norton algebra on V_i (if S is symmetric).

Proposition 2.9.4. *We have*

(i) $\sigma_{ij}^k = 0$ *if and only if* $q_{ij}^k = 0$*;*

(ii) $\sigma_{ij}^k(\pi_i(e_x), \pi_j(e_x)) = \frac{1}{|X|} q_{ij}^k \pi_k(e_x)$.

Proof. This is readily verified from the identities

$$E_i \circ E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_{ij}^k E_k$$
$$E_i E_j = \delta_{ij} E_i$$

for the matrix representations E_i of the π_i; see Definition 1.7.5. □

Norton algebras provide a rich source of examples of axial decomposition algebras. First, we prove the following lemma.

Lemma 2.9.5. *Let $S = (X, \{R_i\}_{0 \le i \le d})$ be a symmetric association scheme. Let e_x and π_i be as in Definition* 2.9.1*. Let $N := V_i$ be one of its Norton algebras and let $a_x := \pi_i(e_x)$. Then*

$$\mathrm{ad}_{a_x} \colon N \to N \colon v \mapsto a_x * v$$

is diagonalizable for all $x \in X$ with real eigenvalues.

Proof. Consider the linear operator

$$\theta\colon V \to V : v \mapsto \sum_{y\in X} \langle \pi_i(e_x), e_y\rangle \langle \pi_i(v), e_y\rangle\, \pi_i(e_y).$$

Its restriction to V_i equals $\iota \circ \mathrm{ad}_{\pi_i(e_x)}$, where $\iota\colon V_i \to V$ is the natural embedding. Since V_i is an invariant subspace of θ, it suffices to prove that θ is diagonalizable. Since S is symmetric, the projection π_i is a real hermitian operator. Hence θ is a real hermitian operator on V and therefore θ is unitarily diagonalizable over the reals. □

Let G be a group acting generously transitively on a set X. Then the orbits of G on $X \times X$ form a symmetric association scheme (Example 1.7.2). We can use this group action to give the Norton algebra the structure of an axial decomposition algebra by Theorem 2.8.1.

Definition 2.9.6. Let G be a finite group acting generously transitively on a set X and let S be the corresponding symmetric association scheme. Let $N \coloneqq V_i$ be one of its Norton algebras.

(i) Let $x_0 \in X$ be arbitrary. Let H be the stabilizer of x_0 in G. Since G acts transitively on X, there exists for each $x \in X$, a $g \in G$ such that ${}^g x_0 = x$. Let $\Omega[x] \coloneqq (N^x_\chi \mid \chi \in \mathsf{Irr}(H))$ be the decomposition of N into ${}^g H$-isotypic components as in Corollary 2.8.4. Then (A, X, Ω) is an $\mathcal{F}$-decomposition algebra where $\mathcal{F}$ is the representation fusion law of H.

(ii) For each $x \in X$, let $a_x \coloneqq \pi_i(e_x)$. By Lemma 2.9.5, we can consider the decomposition of N into eigenspaces for ad_{a_x}:

$$N = \bigoplus_{\lambda\in\Lambda} N^x_\lambda$$

where $\Lambda \subseteq \mathbb{R}$ is the set of eigenvalues of ad_{a_x} and N^x_λ is its λ-eigenspace. Note that since ${}^g\mathrm{ad}_{a_x} = \mathrm{ad}_{g\cdot a_x}$ and $g \cdot a_x = a_{{}^g x}$, the eigenvalues do not depend on $x \in X$.

(iii) By Proposition 2.9.4, we have that $a_x * a_x = \frac{1}{|X|} q^i_{ii} a_x$. In particular $\lambda_0 \coloneqq \frac{1}{|X|} q^i_{ii} \in \Lambda$. Consider the following fusion law on Λ:

$$\lambda \star \mu = \begin{cases} \{\mu\} & \text{if } \lambda = \lambda_0,\\ \{\lambda\} & \text{if } \mu = \lambda_0,\\ \Lambda & \text{otherwise.} \end{cases}$$

(This is the terminal object in the full subcategory of Fus of fusion laws with underlying set Λ and such that $\lambda_0 \in \Lambda$ is a unit.)

Theorem 2.9.7. *Consider the situation of Definition* 2.9.6. *For each $x \in X$, we have*

$$N = \bigoplus_{\chi \in \mathsf{Irr}(H)} \bigoplus_{\lambda \in \Lambda} N^x_\chi \cap N^x_\lambda.$$

Let Σ be the X-tuple of these decompositions. Then $(N, X, \Sigma, x \mapsto a_x)$ is a primitive axial $(\mathcal{F} \times \Lambda)$-decomposition algebra for the unit $(\mathbf{1}_H, \lambda_0) \in \mathcal{F} \times \Lambda$. The evaluation map is given by $(\chi, \lambda) \mapsto \lambda$.

Proof. Recall that H is the stabilizer of $x_0 \in X$ in G. Its action on N therefore commutes with the action of $\mathrm{ad}_{a_{x_0}}$. Thus H leaves the eigenspace $N^{x_0}_\lambda$ invariant for all $\lambda \in \Lambda$. The first statement now follows from Proposition 2.8.2 and the transitivity of G on X.

Suppose that $a_x = 0$ for some $x \in X$. Since G acts transitively on X and therefore on $\{a_y \mid y \in X\}$, we have $\pi_i(e_y) = a_y = 0$ for all $y \in X$. But then $\pi_i = 0$, a contradiction.

Next, we show that $N^x_{\mathbf{1}_H} = \langle a_x \rangle$ for all $x \in X$ and the trivial character $\mathbf{1}_H \in \mathsf{Irr}(H)$. Of course, we have $a_x \in N^x_{\mathbf{1}_H}$. Now let ψ be the permutation character for the action of G on X. Since G acts generously transitively on X, the character ψ is multiplicity-free; see Example 1.7.2. Since $(\mathbf{1}_H)^G = \psi$, it follows from Frobenius reciprocity (Proposition 1.4.23) that $\mathbf{1}_H$ occurs with multiplicity one in every irreducible constituent of the restriction ψ_H.

By definition of N^x_λ we have $a_x * v = \lambda v$ for all $v \in N^x_\lambda$. This concludes the proof. □

Example 2.9.8. The Higman-Sims group HS is a sporadic finite simple group that can be realized as a subgroup of index 2 of the automorphism group of graph on 100 vertices [HS68]. This graph is strongly regular with parameters $(100, 22, 0, 6)$ and HS acts generously transitively on its vertices. The corresponding association scheme has rank three and multiplicities 1, 22 and 77. We can verify that the Norton algebra on the subspace of dimension 77 is non-zero. The operator ad_{a_x} is diagonalizable with eigenvalues $\frac{3}{5}$, $\frac{1}{20}$ and $\frac{-3}{100}$ and respective multiplicities 1, 21 and 55. The stabilizer in HS of a vertex is isomorphic to the Mathieu group M_{22}. It acts irreducibly on each of the eigenspaces. Denote the corresponding characters by χ_1, χ_2 and χ_3. The representation fusion law on $\{\chi_1, \chi_2, \chi_3\}$ is given as follows.

$\star$	χ_1	χ_2	χ_3
χ_1	χ_1	χ_2	χ_3
χ_2	χ_2	χ_1, χ_2, χ_3	χ_2, χ_3
χ_3	χ_3	χ_2, χ_3	χ_1, χ_2, χ_3

Therefore this Norton algebra is an axial decomposition algebra for this fusion law and parameters $\lambda_{\chi_1} = \frac{3}{5}$, $\lambda_{\chi_2} = \frac{1}{20}$, $\lambda_{\chi_3} = \frac{-3}{100}$.

2.10 The category of fusion laws

In this section, we study the category of fusion laws. We use the terminology and concepts from Section 1.1.

Let us start by making the following observation.

Remark 2.10.1. (i) The unique fusion law with $\emptyset$ as underlying set is the unique initial object for Fus. The full fusion law on a set $\{*\}$ with one element is a terminal object.

(ii) The category Fus admits a faithful forgetful functor to Set that maps every fusion law $(X, \star)$ to X and every morphism to the corresponding morphism between the underlying sets. This functor has a left adjoint that sends every set X to the empty fusion law on X. It also has a right adjoint that maps every set X to the full fusion law on X.

We prove that a morphism ξ of fusion laws is monic (resp. epic) if its image under the forgetful functor to Set is monic (resp. epic). However, it is not sufficient that its image is bijective to conclude that ξ is an isomorphism.

Lemma 2.10.2. *Let $\xi\colon (X, \star) \to (Y, \star)$ be a morphism of fusion laws.*

(i) *Then ξ is a monomorphism (resp. epimorphism) if and only if ξ is injective (resp. surjective) as a map from X to Y.*

(ii) *The morphism ξ is an isomorphism if and only if it is bijective as a map from X to Y and $\xi(x_1 \star x_2) = \xi(x_1) \star \xi(x_2)$ for all $x_1, x_2 \in X$.*

Proof. (i) Since the forgetful functor from Fus to Set has both a left and right adjoint, it preserves all limits and colimits by Proposition 1.1.27. Since this functor is also faithful, it must preserve monomorphism and epimorphisms by Proposition 1.1.28.

The converse follows because this forgetful functor is faithful.

(ii) If ξ is an isomorphism then it is both monic and epic and therefore bijective as map from X to Y by (i). Its inverse has to be the inverse of ξ as a map from X to Y, once again because the forgetful functor from Fus to Set is faithful. Since this has to be a morphism, we have

$$\xi(x_1) \star \xi(x_2) \subseteq \xi(\xi^{-1}(\xi(x_1) \star \xi(x_2))) \subseteq \xi(x_1 \star x_2) \subseteq \xi(x_1) \star \xi(x_2)$$

for all $x_1, x_2 \in X$. Conversely, suppose that ξ is bijective and satisfies

$$\xi(x_1 \star x_2) = \xi(x_1) \star \xi(x_2)$$

for all $x_1, x_2 \in X$. Let ξ^{-1} be the inverse map of ξ. For $y_1, y_2 \in Y$, let $x_1 \coloneqq \xi^{-1}(y_1)$ and $x_2 \coloneqq \xi^{-1}(y_2)$. Now

$$\xi^{-1}(y_1 \star y_2) = \xi^{-1}(\xi(x_1) \star \xi(x_1)) = \xi^{-1}(\xi(x_1 \star x_2)) = \xi^{-1}(y_1) \star \xi^{-1}(y_2).$$

In particular, the inverse ξ^{-1} of ξ is a morphism of fusion laws. We conclude that ξ is an isomorphism. □

Next, we prove that the definition of the product and union of fusion laws are naturally interpreted as product and coproduct in the category Fus.

Proposition 2.10.3. *The product and coproduct in the category* Fus *are given by the product and union of fusion laws, respectively, as defined in Definition* 2.4.13.

Proof. This follows easily from the definitions. Notice, in particular, that for given fusion laws $(X, \star)$ and $(Y, \star)$, the projection maps $X \times Y \to X$ and $X \times Y \to Y$ and the inclusion maps $X \to X \sqcup Y$ and $Y \to X \sqcup Y$ indeed induce morphisms in Fus as in Definition 2.4.11. □

In fact, the category Fus has all (small) limits and colimits.

Proposition 2.10.4. *The category* Fus *is complete.*

Proof. From the existence theorem (Proposition 1.1.25) if suffices to show that Fus has equalizers and all products.

First, we prove the existence of products. Let $(\mathcal{F}_i = (X_i, \star) \mid i \in I)$ be a family of fusion laws indexed by a set I. Observe that the forgetful functor of Remark 2.10.1 preserves limits by Proposition 1.1.27. Thus, if the product $\prod_{i \in I} \mathcal{F}_i$ exists, its underlying set must be $\prod_{i \in I} X_i$. Now define a fusion law on $\prod_{i \in I} X_i$ as follows:

$$(x_i)_{i \in I} \star (y_i)_{i \in I} = \{(z_i)_{i \in I} \mid z_i \in x_i \star y_i \text{ for all } i \in I\}.$$

Then it is easily verified that $(\prod_{i \in I} X_i, \star)$ is the product of the fusion laws $(\mathcal{F}_i \mid i \in I)$.

Next, we prove the existence of equalizers. Let ξ_1 and ξ_2 be two morphisms between the fusion laws $\mathcal{F}_1 = (X_1, \star)$ and $\mathcal{F}_2 = (X_2, \star)$. Let $\mathcal{X}$ be the sublaw of $\mathcal{F}_1$ on the set $Z = \{x \in X \mid \xi_1(x) = \xi_2(x)\}$. Then $\mathcal{X}$ with its natural embedding into the fusion law $\mathcal{F}_1$ is the equalizer for ξ_1 and ξ_2. Because, if $\xi_3 \colon \mathcal{F}_3 \to \mathcal{F}_1$ is a morphism such that $\xi_1 \circ \xi_3 = \xi_2 \circ \xi_3$, then for all $x \in \mathcal{F}_3$, we have $\xi_1(\xi_3(x)) = \xi_2(\xi_3(x))$. Thus $\xi_3(x) \in Z$ and ξ_3 factors through the embedding of X into $\mathcal{F}_1$. □

Proposition 2.10.5. *The category* Fus *is cocomplete.*

Proof. Again, from the existence theorem (Proposition 1.1.25), it suffices to prove that Fus has coequalizers and coproducts.

Let $(\mathcal{F}_i = (X_i, \star) \mid i \in I)$ be a family of fusion laws. Consider the set $\bigsqcup_{i \in I} X_i$ together with the fusion law defined by

$$x_i \star x_j = \begin{cases} \emptyset & \text{if } i \neq j \\ x_i \star x_j & \text{if } i = j \end{cases}$$

for all $x_i \in X_i$, $x_j \in X_j$ and $i, j \in I$. Then $\bigsqcup_{i \in I} \mathcal{F}_i \coloneqq (\bigsqcup_{i \in I} X_i, \star)$ is easily verified to be the coproduct of $(\mathcal{F}_i \mid i \in I)$.

Now let ξ_1 and ξ_2 be two morphism between the fusion laws $\mathcal{X} = (X, \star)$ and $\mathcal{Y} = (Y, \star)$. Consider the equivalence relation on Y such that $y_1 \sim y_2$ if and only if $(y_1, y_2) \in \{(\xi_1(x), \xi_2(x)), (\xi_2(x), \xi_1(x)) \mid x \in X\}$. Let $Z \subseteq 2^Y$ be the corresponding set of equivalence classes. Define a fusion law on Z by

$$z_1 \star z_2 = \{z_3 \mid z_3 \cap (y_1 \star y_2) \neq \emptyset \text{ for some } y_1 \in z_1 \text{ and } y_2 \in z_2\}.$$

Then $(Z, \star)$ together with the map $Y \to Z$ that maps every element of y to its equivalence class, is the coequalizer of ξ_1 and ξ_2. □

2.11 The category of decomposition algebras

We now explore some more advanced categorical properties of decomposition algebras.

Fix a commutative ring R and a fusion law $\mathcal{F} = (X, \star)$ and let the category $\mathcal{F}\text{-}\mathsf{Dec}_R$ be as in Definition 2.5.8.

Remark 2.11.1. The category $\mathcal{F}\text{-}\mathsf{Dec}_R$ has an initial object $(0, \emptyset, \emptyset)$ and a terminal object $(0, \{*\}, (0))$. This category admits two obvious forgetful functors, namely

$$\begin{aligned} &\mathcal{F}\text{-}\mathsf{Dec}_R \to \mathsf{Alg}_R \colon (A, I, \Omega) \rightsquigarrow A \qquad \text{and} \\ &\mathcal{F}\text{-}\mathsf{Dec}_R \to \mathsf{Set} \colon (A, I, \Omega) \rightsquigarrow I. \end{aligned}$$

These functors have corresponding left adjoints given by

$$\begin{aligned} \mathsf{Alg}_R \to \mathcal{F}\text{-}\mathsf{Dec}_R &\colon A \rightsquigarrow (A, \emptyset, \emptyset) \qquad \text{and} \\ \mathsf{Set} \to \mathcal{F}\text{-}\mathsf{Dec}_R &\colon I \rightsquigarrow \big(0, I, (0 \mid i \in I)\big), \end{aligned}$$

respectively. For the second forgetful functor, its left adjoint is also its right adjoint.

Proposition 2.11.2. *Let $(\varphi, \psi) \colon (A, I, \Omega) \to (B, J, \Sigma)$ be a morphism of $\mathcal{F}$-decomposition algebras.*

(i) *Then (φ, ψ) is monic (resp. epic) if and only if both φ and ψ are injective (resp. surjective).*

(ii) *The morphism (φ, ψ) is an isomorphism if and only if both φ and ψ are bijective.*

Proof. (i) Since the forgetful functor $\mathcal{F}\text{-}\mathsf{Dec}_R \to \mathsf{Set} \colon (A, I, \Omega) \mapsto I$ has both a left and right adjoint, it preserves both monomorphisms and epimorphism by Propositions 1.1.27 and 1.1.28. Similarly, the forgetful functor to Alg_R preserves monomorphisms. Now suppose that (φ, ψ) is an epimorphism. Let

φ_1, φ_2 be algebra morphisms such that $\varphi_1 \circ \varphi = \varphi_2 \circ \varphi$ where $\varphi_1, \varphi_2 \colon C \to A$ for some algebra C. Then consider the morphisms

$$(\varphi_1, 0), (\varphi_2, 0) \colon (C, \emptyset, \emptyset) \to (A, I, \Omega).$$

We have $(\varphi_1, 0) \circ (\varphi, \psi) = (\varphi_2, 0) \circ (\varphi, \psi)$. Since (φ, ψ) is epic, it follows that $(\varphi_1, 0) = (\varphi_2, 0)$. In particular, $\varphi_1 = \varphi_2$ which proves that φ is surjective.

Conversely, suppose that φ and ψ are injective and let (φ_1, ψ_1) and (φ_2, ψ_2) be morphisms in $\mathcal{F}\text{-}\mathsf{Dec}_R$ such that $(\varphi, \psi) \circ (\varphi_1, \psi_1) = (\varphi, \psi) \circ (\varphi_2, \psi_2)$. Then $\varphi_1 \circ \varphi = \varphi_2 \circ \varphi$ (resp. $\psi_1 \circ \psi = \psi_2 \circ \psi$) and thus, since φ (resp. ψ) is injective, $\varphi_1 = \varphi_2$ (resp. $\psi_1 = \psi_2$). If both φ and ψ are surjective, then it follows similarly that (φ, ψ) is epic.

(ii) If (φ, ψ) is an isomorphism, then it has an inverse (φ', ψ'). Then φ' is an inverse of φ and ψ' is an inverse of ψ.

Conversely, suppose that φ and ψ have respective inverses φ' and ψ'. We prove that (φ', ψ') is a morphism of decomposition algebras. If so, then it is clearly an inverse of (φ, ψ). Thus, we need to show that for all $j \in J$ and for all $x \in \mathcal{F}$, we have $\varphi'(B_x^j) \subseteq A_x^{\psi'(j)}$. Now φ is an isomorphism that maps distinct components of the direct sum $\bigoplus_{x \in \mathcal{F}} A_x^{\psi'(j)}$ into distinct components of the direct sum $\bigoplus_{x \in \mathcal{F}} B_x^j$. Since φ is surjective, we must have $\varphi(A_x^{\psi'(j)}) = B_x^j$. Apply φ' to conclude that $\varphi'(B_x^j) = A_x^{\psi'(j)}$. $\square$

Proposition 2.11.3. *The category $\mathcal{F}\text{-}\mathsf{Dec}_R$ is complete.*

Proof. From the existence theorem for limits it is sufficient to show that $\mathcal{F}\text{-}\mathsf{Dec}_R$ has equalizers and all products; see Proposition 1.1.25.

We begin by showing the existence of products. Let $(A_j, I_j, \Omega_j)_{j \in J}$ be a family of decomposition algebras indexed by some set J. The forgetful functors of Remark 2.11.1 preserve limits and hence if the product of $(A_j, I_j, \Omega_j)_{j \in J}$ exists, it must consist of the algebra $\prod_{j \in J} A_j$ and the index set $\prod_{j \in J} I_j$. Let Π be the tuple of decompositions indexed by $\prod_{j \in J} I_j$, where

$$\Pi[(i_j)_{j \in J}] = \Big(\prod_{j \in J} (A_j)_x^{i_j} \;\Big|\; x \in X \Big)$$

Let $\pi_k \colon \prod_{j \in J} A_j \to A_k$ and $\psi_k \colon \prod_{j \in J} I_j \to I_k$ be the natural projections of algebras and sets respectively. We will show that

$$\Big(\prod_{j \in J} A_j, \prod_{j \in J} I_j, \Pi \Big)$$

together with the morphisms (π_k, ψ_k) for $k \in I$ is the product of the family of decomposition algebras $(A_j, I_j, \Omega_j)_{j \in J}$.

Firstly, if $\mathbf{i} = (i_j)_{j\in J} \in \prod_{j\in J} I_j$ and $x \in X$ then

$$\pi_k\big(\Pi[\mathbf{i}]_x\big) = \pi_k\Big(\prod_{j\in J} (A_j)_x^{i_j}\Big) = (A_k)_x^{i_k} = (A_k)_x^{\psi_k(\mathbf{i})}$$

and so (π_k, ψ_k) is a morphism in $\mathcal{F}\text{-}\mathsf{Dec}_R$.

Next we need to show that for any cone $(\varphi_j, \theta_j)\colon (B, K, \Sigma) \to (A_j, I_j, \Omega_j)$ there is a unique morphism from (B, K, Σ) to the product making the following diagram commute.

$$\begin{tikzcd}
 & (B,K,\Sigma) \arrow[ddl, "{(\varphi_{j_1},\theta_{j_1})}"'] \arrow[d, dashed] \arrow[ddr, "{(\varphi_{j_2},\theta_{j_2})}"] & \\
 & \text{Product} \arrow[dl, "{(\pi_{j_1},\psi_{j_1})}"] \arrow[dr, "{(\pi_{j_2},\psi_{j_2})}"'] & \\
(A_{j_1}, I_{j_1}, \Omega_{j_1}) & \cdots & (A_{j_2}, I_{j_2}, \Omega_{j_2})
\end{tikzcd}$$

If $b \in B_x^k$ then $\varphi_j(b) \in (A_j)_x^{\theta_j(k)}$ for all $j \in J$ and hence

$$(\varphi_j(b))_{j\in J} \in \prod_{j\in J} (A_j)_x^{\theta_j(k)}.$$

This shows that the obvious map from (B, K, Σ) to the product is actually a morphism in $\mathcal{F}\text{-}\mathsf{Dec}_R$. This map clearly makes the diagram commute and the uniqueness is a consequence of the uniqueness of π_j and ψ_j in their respective categories. This completes the proof of the existence of products.

We now show that equalizers exist in $\mathcal{F}\text{-}\mathsf{Dec}_R$. Let (φ_1, ψ_1) and (φ_2, ψ_2) be two morphisms of $\mathcal{F}\text{-}\mathsf{Dec}_R$:

$$(\varphi_1, \psi_1), (\varphi_2, \psi_2)\colon (A, I, \Omega) \to (B, J, \Theta).$$

Let $\varphi\colon E \to A$ be the equalizer of φ_1 and φ_2 in Alg_R, let $\psi\colon K \to I$ be the equalizer of ψ_1 and ψ_2 in Set and let Σ be the tuple of decompositions given by

$$\Sigma[k] = \Big(\varphi^{-1}\Big(A_x^{\psi(k)}\Big) \Bigm| x \in X\Big) \text{ for } k \in K.$$

To see that this is indeed a tuple of decompositions: firstly, if $e \in E_x^k \cap \sum_{y\neq x} E_y^k$ then $\varphi(e) \in A_x^{\psi(k)} \cap \sum_{y\neq x} A_y^{\psi(k)} = 0$. Now since equalizers are monic we must have $e = 0$. Secondly, if $e \in E$ and $k \in K$ then $\varphi(e) = \sum_{x\in X} a_x$ for some $a_x \in A_x^{\psi(k)}$. It is sufficient to show that each a_x is in the image of φ. As $e \in E$ we know that $\varphi_1(e) = \varphi_2(e)$ and hence $\sum_{x\in X} (\varphi_1(a_x) - \varphi_2(a_x)) = 0$. However $k \in K$ implies that each term $\varphi_1(a_x) - \varphi_2(a_x)$ is in a distinct component of a direct sum and hence each is zero. Now since φ_1 and φ_2 act equally on a_x for each $x \in X$, each a_x must have a preimage in E.

It is clear from the definition that (φ, ψ) is a morphism of $\mathcal{F}\text{-}\mathsf{Dec}_R$ so we need only check that it is the equalizer of (φ_1, ψ_1) and (φ_2, ψ_2). Let $(\gamma, \tau)\colon (F, L, \Phi) \to (A, I, \Omega)$ be a morphism such that $(\varphi_1, \psi_1) \circ (\gamma, \tau) = (\varphi_2, \psi_2) \circ (\gamma, \tau)$. Define (δ, σ) by

$$\begin{aligned} \delta\colon F &\to E & \sigma\colon L &\to K \\ f &\mapsto \varphi^{-1}(\gamma(f)) & l &\mapsto \psi^{-1}(\tau(l)). \end{aligned}$$

Then (φ, ψ) is a morphism of decomposition algebras and $(\varphi, \psi) \circ (\delta, \sigma) = (\gamma, \tau)$. Uniqueness again follows from the uniqueness of φ and ψ in Alg and Set respectively. This completes the proof that equalizers exist in $\mathcal{F}\text{-}\mathsf{Dec}_R$ and hence that $\mathcal{F}\text{-}\mathsf{Dec}_R$ is complete. □

We now turn our attention to ideals and quotients of decomposition algebras.

Definition 2.11.4. (i) Let (A, I, Ω) be a decomposition algebra and let $K \trianglelefteq A$ be an algebra ideal. For each $i \in I$ and each $x \in X$, let $K^i_x := A^i_x \cap K$ and let $\Omega \cap K := \big((K^i_x)_{x\in X} \mid i \in I\big)$. We call K a *decomposition ideal* of (A, I, Ω) if for each $i \in I$, we have $K = \bigoplus_{x\in X} K^i_x$. Notice that this implies that $(K, I, \Omega \cap K)$ is an object in $\mathcal{F}\text{-}\mathsf{Dec}_R$.

(ii) If K is a decomposition ideal of (A, I, Ω) and $B = A/K$, then (B, I, Σ) is again a decomposition algebra (which we then call the *quotient* decomposition algebra) obtained by setting

$$B^i_x := (A^i_x + K)/K$$

for all $i \in I$ and all $x \in X$, and then letting $\Sigma = \big((B^i_x)_{x\in X} \mid i \in I\big)$. Notice that the condition $K = \bigoplus_{x\in X} K^i_x$ ensures that the sum $\sum_{x\in X} B^i_x$ is a direct sum.

Proposition 2.11.5. *Let $(\varphi, \psi)\colon (A, I, \Omega_A) \to (B, J, \Omega_B)$ be a morphism of decomposition algebras. Then $K = \ker\varphi$ is a decomposition ideal of (A, I, Ω_A).*

Conversely, if K is a decomposition ideal of (A, I, Ω) and $\pi\colon A \to A/K$ is the natural projection of algebras, then $(K, I, \Omega\cap K)$ is the equalizer of the epimorphism

$$(\pi, \mathsf{id})\colon (A, I, \Omega) \to (A/K, I, \Sigma)$$

and the morphism $(0, \mathsf{id})$.

Proof. We begin by showing that $K = \ker\varphi$ is a decomposition ideal. Fix some $i \in I$ and let $K^i_x = K \cap A^i_x$. It is clear that $K^i_x \cap \sum_{y\neq x} K^i_y = 0$ for all $x \in X$ and that $K \supseteq \sum_{x\in X} K^i_x$, thus we only need to show the opposite inclusion. For any $k \in K$ we may write $k = \sum_{x\in X} a^i_x$, where each $a^i_x \in A^i_x$. It is sufficient to show that $a^i_x \in K$, but

$$\sum_{x\in X} \varphi(a^i_x) = \varphi(k) = 0$$

where each $\varphi(a_x^i) \in B_x^{\psi(i)}$ is in a different component of a direct sum. Hence $\varphi(a_x^i) = 0$ for all x.

The second part follows directly from the first part once we note that K is the algebra kernel of π. □

Remark 2.11.6. Recall that the categorical definition of a kernel of a morphism is the equalizer of the given morphism and a zero morphism. We would like to be able to refer to the decomposition ideal $(K, I, \Omega \cap K)$ in Proposition 2.11.5 as the kernel of the projection, however since the category $\mathcal{F}\text{-}\mathsf{Dec}_R$ does not contain zero morphisms the definition of kernel does not make sense. Instead, in Proposition 2.11.5, we use $(0, \mathsf{id})$ in place of the zero morphism and in this sense the decomposition ideals (as equalizers of these morphisms) are as close to kernels as we can realistically achieve.

3

Miyamoto groups

In Section 2.2, we discussed the Griess algebra and its connection to the monster group. We saw how a conjugacy class of involutions of the monster group is in one-to-one correspondence with a set of axes for the Griess algebra. It is possible to recover the involution corresponding to an axis by looking at its decomposition. More precisely, this involution acts trivially on every 1-, 0- or $\frac{1}{4}$-eigenvector of its axis and it negates the $\frac{1}{32}$-eigenvectors.

This connection between the monster group and the Griess algebra is reflected in the fusion law. If we merge the 1, 0 and $\frac{1}{4}$ in the fusion law, then we obtain the group fusion law of the cyclic group of order 2. Such a property is called a grading of a fusion law.

In Section 3.2 we will see that if the fusion law of a decomposition algebra is graded, then we can always associate a group to it. We call this group the Miyamoto group of the decomposition algebra. As an example, we will discuss the Miyamoto group corresponding to the decomposition algebras from Sections 2.8 and 2.9. Moreover, we will explain how 3-transposition groups can be realized as Miyamoto groups.

Next, we will review whether the connection between decomposition algebras and their corresponding Miyamoto groups is functorial. This will lead to the concept of universal Miyamoto groups.

Most content of this chapter is based on joint work with Tom De Medts, Simon F. Peacock and Sergey Shpectorov [DMVC20a, DMPSVC20].

3.1 Gradings

This section introduces the necessary preparations for the important connection between axial algebras and groups as in [DMPSVC20, § 3]. On the level of fusion laws, this connection boils down to a morphism from a given fusion law to a group fusion law. We illustrate how to get the strongest possible connection by introducing the finest (abelian) grading of a fusion law.

Definition 3.1.1. (i) Let $(X, \star)$ be a fusion law and let $(\Gamma, \star)$ be a group fusion law. A Γ-*grading* of $(X, \star)$ is a morphism $\xi\colon (X, \star) \to (\Gamma, \star)$. We call the grading *abelian* if Γ is an abelian group and we call it *adequate* if $\xi(X)$ generates Γ.

(ii) Every fusion law admits a Γ-grading where Γ is the trivial group; we call this the *trivial* grading.

(iii) Let $(X, \star)$ be a fusion law. We say that a Γ-grading ξ of $(X, \star)$ is a *finest grading* of $(X, \star)$ if every grading of $(X, \star)$ factors uniquely through $(\Gamma, \star)$, in other words, if for each Λ-grading ζ of $(X, \star)$, there is a unique group homomorphism $\rho\colon \Gamma \to \Lambda$ such that $\zeta = \rho \circ \xi$. (In categorical terms, this can be rephrased as the fact that ξ is an initial object in the category of gradings of $(X, \star)$.) Similarly, we say that an abelian Γ-grading ξ of $(X, \star)$ is a *finest abelian grading* of $(X, \star)$ if every abelian grading of $(X, \star)$ factors uniquely through $(\Gamma, \star)$.

Proposition 3.1.2. *Every fusion law $(X, \star)$ admits a unique finest grading, given by the group with presentation*

$$\Gamma_X := \langle \gamma_x, x \in X \mid \gamma_x \gamma_y = \gamma_z \textit{ whenever } z \in x \star y \rangle,$$

with grading map $\xi\colon (X, *) \to (\Gamma_X, *)\colon x \mapsto \gamma_x$. Similarly, there is a unique finest abelian grading, given by the abelianization $\Gamma_X/[\Gamma_X, \Gamma_X]$ of Γ_X. Both gradings are adequate.

Proof. In order to verify that the map $\xi\colon (X, \star) \to (\Gamma_X, \star)\colon x \mapsto \gamma_x$ is a morphism of fusion laws, we have to check that $\xi(z) \in \xi(x) \star \xi(y)$ for all $z \in x \star y$. This is clear from the definition of Γ_X, since $\xi(z) = \gamma_z$ and $\xi(x) \star \xi(y) = \{\gamma_x \gamma_y\}$. Clearly, ξ is then an adequate grading since Γ_X is generated by the elements γ_x.

Assume now that $\zeta\colon (X, \star) \to (\Lambda, \star)$ is another grading of $(X, \star)$. If $x, y, z \in X$ satisfy $z \in x \star y$, then $\zeta(z) \in \zeta(x) \star \zeta(y) = \{\zeta(x)\zeta(y)\}$, so the elements $\zeta(x)$ satisfy the defining relations of the generators γ_x in the presentation for Γ_X. This implies that the map $\rho\colon \Gamma_X \to \Lambda\colon \gamma_x \mapsto \zeta(x)$ is a well defined group homomorphism, with $\zeta = \rho \circ \xi$. Since ξ is adequate, the identity $\zeta = \rho \circ \xi$ also uniquely determines the group homomorphism ρ.

The proof of the remaining statement is similar. $\square$

Remark 3.1.3. There is a lot of "collapsing" in the group Γ_X.

(a) If $y \in x \star y$ for some $y \in X$, then $\gamma_x = 1$ in Γ_X. In particular, $\gamma_x = 1$ for each non-annihilating unit $x \in X$.

(b) All γ_z, where z runs through some fixed set $x \star y$, are equal to each other in Γ_X.

(c) If z belongs to $x \star y$ and to $x \star y'$, then $\gamma_y = \gamma_{y'}$. Similarly, if z belongs to $x \star y$ and to $x' \star y$, then $\gamma_x = \gamma_{x'}$.

From this it is clear that Γ_X is trivial for most fusion laws $(X, \star)$, i.e., they only admit the trivial grading. We call a fusion law $(X, \star)$ *graded* if $\Gamma_X \neq 1$ and *ungraded* otherwise. It will turn out that graded fusion laws are more interesting for our purposes.

Example 3.1.4. The Jordan fusion law in Example 2.4.5 is $\mathbb{Z}/2\mathbb{Z}$-graded. Indeed, the map $\xi\colon X \to \mathbb{Z}/2\mathbb{Z}$ mapping e and z to 0 and h to 1 is a fusion law morphism. Notice that this is the finest grading of the Jordan fusion law.

Similarly, the Ising fusion law in Example 2.4.6 admits a $\mathbb{Z}/2\mathbb{Z}$-grading: the map $\xi\colon X \to \mathbb{Z}/2\mathbb{Z}$ mapping e, z and q to 0 and t to 1 is a fusion law morphism. Again, this is the finest grading of the Ising fusion law.

In the remainder of this section, we describe the finest grading of two special types of fusion laws: class fusion laws and representation fusion laws.

The class fusion law of a group G was introduced in Example 2.4.16. For $g \in G$, let $\overline{g}$ denote the image of g in $G/[G,G]$.

Proposition 3.1.5. *Let $(X, \star)$ be the class fusion law of a group G. Then the finest grading of $(X, \star)$ is given by the group $\Gamma = G/[G,G]$ with grading map $X \to \Gamma\colon {}^G g \mapsto \overline{g}$.*

Proof. By Proposition 3.1.2, the finest grading of $(X, \star)$ is the group

$$\Gamma_X \coloneqq \langle \gamma_C, C \in X \mid \gamma_C \gamma_D = \gamma_E \text{ whenever } CD \cap E \neq \emptyset \rangle.$$

Consider the map $\varphi\colon G \to \Gamma_X\colon g \mapsto \gamma_{({}^G g)}$ and notice that φ is a group morphism, precisely by the defining relations of Γ_X. It is clearly surjective; moreover, $\varphi({}^g h) = \varphi(h)$ for all $g, h \in G$. It follows that for each commutator $[g,h] = ghg^{-1}h^{-1}$, we have $\varphi([g,h]) = \varphi({}^g h)\varphi(h)^{-1} = 1$; hence $[G,G] \leq \ker\varphi$. Hence φ induces a group epimorphism $\tilde{\varphi}\colon \Gamma \to \Gamma_X$.

Finally, the map $\Gamma_X \to \Gamma\colon \gamma_{({}^G g)} \to g[G,G]$ is well defined because it kills each relator of Γ_X, and this map provides an inverse of $\tilde{\varphi}$, showing that it is an isomorphism from Γ_X to Γ. □

Recall the definition of the representation fusion law from Example 2.4.17.

Proposition 3.1.6. *Let G be a finite group and let $(X, \star)$ be the representation fusion law of G. Then the finest grading of $(X, \star)$ is given by $\Gamma_X = \mathsf{Irr}(Z(G))$ with grading map $X \to \mathsf{Irr}(Z(G))\colon \chi \mapsto \frac{\chi_{Z(G)}}{\chi(1)}$.*

[1]Thanks to David Craven and Frieder Ladisch for providing the central argument in this proof. As Frieder Ladisch pointed out to us, this result also follows from [GN08, Example 3.2 and Corollary 3.7].

Proof[1]. Consider an arbitrary adequate grading $f\colon X \to \Gamma$ and define

$$K = \{\chi \in \mathsf{Irr}(G) \mid f(\chi) = 1\}\,.$$

Let $H = \bigcap_{\chi \in K} \ker \chi$. If $\chi \in K$ then it is clear that $H \leq \ker \chi$; we aim to show the opposite inclusion. Consider $\theta = \sum_{\chi \in K} \chi$, which may be considered as a character of G/H. Since θ is faithful as a character of G/H, by the Burnside-Brauer theorem (Proposition 1.4.25) every irreducible character of G/H is a constituent of some power of θ. Now since f is trivial on each constituent of θ, it also is trivial on all irreducible characters of G/H. Thus if $H \leq \ker \chi$ then $\chi \in K$. We have now established that $K = \{\chi \in \mathsf{Irr}(G) \mid H \leq \ker \chi\}$.

Note that $f(\overline{\chi}) = f(\chi)^{-1}$. Indeed, $\mathbf{1}_G$ is a constituent of $\chi\overline{\chi}$; that is, $\mathbf{1}_G \in \chi \star \overline{\chi}$. This means that $f(\chi)f(\overline{\chi}) = f(\mathbf{1}_G) = 1$.

Now let $\psi \in \mathsf{Irr}(H)$ and let χ and η be constituents of the induced character ψ^G, so that ψ is a constituent of the restrictions χ_H and η_H by Frobenius reciprocity (Proposition 1.4.23). Thus $0 < \langle \eta_H, \chi_H \rangle = \langle \mathbf{1}_H, (\chi\overline{\eta})_H \rangle$ (where $\langle\ ,\ \rangle$ represents the inner product of class functions) and hence $\mathbf{1}_H$ is a constituent of $(\chi\overline{\eta})_H$. Since $H \trianglelefteq G$, a corollary of Clifford's theorem now implies that $\chi\overline{\eta}$ has a constituent $\theta \in \mathsf{Irr}(G)$ with $H \leq \ker \theta$ (see Proposition 1.4.28). Hence $f(\chi)f(\eta)^{-1} = f(\chi\overline{\eta}) = f(\theta) = 1$. That is, $f(\chi) = f(\eta)$. Thus, we obtain a well-defined map $f'\colon \mathsf{Irr}(H) \to \Gamma$ by setting $f'(\psi) = f(\chi)$ for any constituent χ of ψ^G.

Next, we show that H is in the center of G, so let us assume that there is some non-central $x \in H$. As x is not central, the relations from Propositions 1.4.16 and 1.4.17 imply that there must be a character $\chi \in \mathsf{Irr}(G)$ such that $|\chi(x)| < \chi(1)$ and, therefore, there is a constituent θ of $\chi\overline{\chi}$ with $\theta(x) \neq \theta(1)$. On the other hand, $f(\theta) = f(\chi)f(\overline{\chi}) = 1$, yielding $\theta \in K$. This means that $H \leq \ker \theta$ and so $\theta(x) = \theta(1)$; a contradiction.

Since H is central, by Proposition 1.4.29, the map $X \to \mathsf{Irr}(H)\colon \chi \mapsto \frac{\chi_H}{\chi(1)}$ is defined and f is the composition of this map and f'. Clearly, the map $X \to \mathsf{Irr}(H)$ factors through the similar map $X \to \mathsf{Irr}(Z(G))$, and so the claim of the proposition holds. □

Remark 3.1.7. (i) It is immediate from the definition that the finest grading of the union of fusion laws $(X, \star)$ and $(Y, \star)$ is the free product of Γ_X and Γ_Y with the obvious grading map.

(ii) The similar question about the finest grading of the product $(X \times Y, \star)$ is more difficult. It is easy to see that there is a grading of $(X \times Y, \star)$ by the group $\Gamma_X \times \Gamma_Y$. However, it is equally easy to find examples where this is not the finest grading. For instance, if $(X, \star)$ is the empty fusion law on X and $(Y, \star)$ is any fusion law, then the product $(X \times Y, \star)$ is the empty fusion law on $X \times Y$, but the finest grading of an empty fusion law is always a free group.

3.2 Definitions

Let Γ be a group fusion law. To each Γ-decomposition algebra (A, I, Ω), we will associate a subgroup of the automorphism group of A, called the Miyamoto group of (A, I, Ω).

We will, at the same time, construct subgroups of these Miyamoto groups, one for each subgroup of the character group.

Definition 3.2.1. Let $R^\times$ be the group of invertible elements of the base ring R. A *linear R-character* of Γ is a group homomorphism $\chi\colon \Gamma \to R^\times$. The *$R$-character group* of Γ is the group $\mathcal{X}_R(\Gamma)$ consisting of all linear R-characters of Γ, with group operation induced by multiplication in $R^\times$. If $R = \mathbb{C}$ and Γ is abelian then $\mathcal{X}_R(\Gamma) \cong \Gamma$; see Example 1.4.20. For non-abelian Γ, we have $\mathcal{X}_R(\Gamma) \cong \mathcal{X}_R(\Gamma/[\Gamma, \Gamma])$.

Notice that, depending on R, the group $\mathcal{X}_R(\Gamma)$ might be infinite even if Γ is finite.

Definition 3.2.2. Let (A, I, Ω) be a Γ-decomposition algebra.

(i) Let $\chi \in \mathcal{X}_R(\Gamma)$. For each decomposition $(A^i_g)_{g\in\Gamma} \in \Omega$, we define a linear map

$$\tau_{i,\chi}\colon A \to A\colon a \mapsto \chi(g)a \quad \text{for all } a \in A^i_g;$$

we call this a *Miyamoto map*. It follows immediately from the definitions that each $\tau_{i,\chi}$ is an automorphism of the R-algebra A.

(ii) Let $\mathcal{Y}$ be any subgroup of the character group $\mathcal{X}_R(\Gamma)$. We then define the *Miyamoto group* with respect to $\mathcal{Y}$ as

$$\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega) := \langle \tau_{i,\chi} \mid i \in I, \chi \in \mathcal{Y} \rangle \leq \mathsf{Aut}(A).$$

Two important special cases get their own notation:

$$\mathsf{Miy}(A, I, \Omega) := \mathsf{Miy}_{\mathcal{X}_R(\Gamma)}(A, I, \Omega);$$
$$\mathsf{Miy}_\chi(A, I, \Omega) := \mathsf{Miy}_{\langle\chi\rangle}(A, I, \Omega) \quad \text{for a given character } \chi \in \mathcal{X}_R(G).$$

Remark 3.2.3. The existence of involutions related to $\mathbb{Z}/2\mathbb{Z}$-gradings was already observed by Masahiko Miyamoto within the context of vertex operator algebras [Miy96].

If $\mathcal{F}$ is a fusion law for which there exists a Γ-grading, then we can view any $\mathcal{F}$-decomposition algebra as a Γ-decomposition algebra by Proposition 2.5.9. In particular, we can consider its Miyamoto group.

Definition 3.2.4. Let $\mathcal{F}$ be a fusion law and $\xi\colon \mathcal{F} \to \Gamma$ a grading of $\mathcal{F}$. Let $\mathcal{Y}$ be any subgroup of the character group $\mathcal{X}_R(\Gamma)$.

(i) Let (A, I, Ω) be an $\mathcal{F}$-decomposition algebra. Then we define the Miyamoto group with respect to $\mathcal{Y}$ as

$$\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega) \coloneqq \mathsf{Miy}_{\mathcal{Y}}(F_\xi(A, I, \Omega)),$$

where F_ξ is the functor from Proposition 2.5.9. More precisely, we define Miyamoto maps

$$\tau_{i,\chi} \colon A \to A \colon a \mapsto \chi(g)a \quad \text{for all } a \in A^i_{\xi^{-1}(g)} \text{ and } g \in G$$

where $\xi^{-1}(g) \coloneqq \{x \in \mathcal{F} \mid \xi(x) = g\}$. Then, we let $\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega)$ be the subgroup of $\mathsf{Aut}(A)$ generated by all $\tau_{i,\chi}$ for $i \in I$ and $\chi \in \mathcal{Y}$.

(ii) Similarly, for an axial $\mathcal{F}$-decomposition algebra (A, I, Ω, α) we define its Miyamoto group with respect to $\mathcal{Y}$ as

$$\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega, \alpha) \coloneqq \mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega).$$

Note that these Miyamoto groups depend on the grading ξ.

Example 3.2.5. The simplest non-trivial example is the case where $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{Y} = \{1, \chi\}$ where χ maps the non-trivial element of Γ to $-1 \in R$ (assuming that $-1 \neq 1$ in R). In the case of axial algebras, we recover the definition of the Miyamoto group as in [DMVC20a, Definition 2.5].

Example 3.2.6. Let (A, I, Ω) be the Griess algebra, viewed as a decomposition algebra for the Ising fusion law $\mathcal{F}$; see Example 2.5.4. The finest grading of $\mathcal{F}$ is $\xi \colon \mathcal{F} \to \mathbb{Z}/2\mathbb{Z}$ where $\xi(e) = \xi(z) = \xi(q) = 0$ and $\xi(t) = 1$. Let χ be the only non-trivial $\mathbb{R}$-character of $\mathbb{Z}/2\mathbb{Z}$. Then the Miyamoto map $\tau_{i,\chi}$ coincides with involution of the monster group corresponding to $i \in I$, in its action on A. In particular, the Miyamoto group $\mathsf{Miy}_\chi(A, I, \Omega)$ is isomorphic to the monster group.

In the previous example, we saw that we can construct the monster group as the Miyamoto group of an axial decomposition algebra. It is interesting to see whether or not a group can be constructed as a Miyamoto group and what the possible fusion laws are for the underlying axial decomposition algebra. For some groups we can take subalgebras of the Griess algebra based on an embedding into the monster group; see e.g. [FIM16a] for the Harada-Norton group. In Section 3.5 we illustrate that 3-transposition groups are the Miyamoto groups of Matsuo algebras. This class includes for example the sporadic simple groups of Fischer. In Section 5.8 we will explicitly construct an axial decomposition algebra whose Miyamoto group is the complex Chevalley group of type E_8. We suggest constructions for the Lyons group and third Janko group in Section 6.2.

There also have been attempts to classify all axial decomposition algebra for a given fusion law, evaluation map and Miyamoto group; see [IPSS10] and [FIM16b] for symmetric groups, [IS12a], [Iva11a], [Iva11b] and [CRI14] for some alternating groups and [IS12b] for $\mathsf{PSL}_3(2)$. Also computer algorithms have been developed to tackle this problem [Ser12, PW18, MS20].

3.3 Examples from representation theory and Norton algebras

As another illustration of Miyamoto groups, we consider the Miyamoto groups of the decomposition algebras from Sections 2.8 and 2.9. Let us recall the context. We consider an algebra A for G, i.e. an algebra A for which there exists a morphism $\rho\colon G \to \mathsf{Aut}(A)$. Let H be a subgroup of G and let $\{\varphi_i \mid i \in I\}$ be a family of automorphisms of G. Then for each $i \in I$, we can consider the decomposition $\Omega[i] \coloneqq (A^i_\chi)_{\chi\in\mathsf{Irr}(H)}$ of A into $\varphi_i(H)$-isotypic components. By Corollary 2.8.4, we have that (A, I, Ω) is an $\mathsf{Irr}(H)$-decomposition algebra for the representation fusion law $\mathsf{Irr}(H)$.

Theorem 3.3.1. *Consider the $\mathsf{Irr}(H)$-decomposition algebra (A, I, Ω) constructed in Theorem 2.8.1. Let $\xi\colon \mathsf{Irr}(H) \to \mathsf{Irr}(Z(H))\colon \chi \mapsto \frac{\chi_{Z(H)}}{\chi(1)}$ be the finest grading of $\mathsf{Irr}(H)$ from Proposition 3.1.6. Then $\mathcal{X}_{\mathbb{C}}(\mathsf{Irr}(Z(H)))$ is naturally isomorphic to $Z(H)$ via*

$$Z(H) \to \mathcal{X}_{\mathbb{C}}(\mathsf{Irr}(Z(H)))\colon z \mapsto [\chi \mapsto \chi(z)].$$

For the Miyamoto maps $\tau_{i,z}$, we have

$$\tau_{i,z}(a) = \varphi_i(z) \cdot a$$

for all $i \in I$, $z \in Z(H)$ and $a \in A$. In particular,

$$\mathsf{Miy}(A, I, \Omega) = \langle \rho(z) \mid z \in \varphi_i(Z(H)), i \in I\rangle,$$

where $\rho(z) \in \mathsf{Aut}(A)$ is the automorphism corresponding to the action of $z \in G$ on N.

Proof. The natural isomorphism between $Z(H)$ and $\mathcal{X}_{\mathbb{C}}(\mathsf{Irr}(Z(H)))$ follows immediately from the character theory of abelian groups; see Example 1.4.20.

Let $\chi \in \mathsf{Irr}(H)$. Then A^i_χ is the $\varphi_i(H)$-isotypic component of A corresponding to the character ${}^{\varphi_i}\chi \in \mathsf{Irr}(\varphi_i(H))$. Since $\varphi_i(Z(H))$ is central, we have, by Proposition 1.4.29, that

$$\varphi_i(z) \cdot a = \frac{{}^{\varphi_i}\chi(\varphi_i(z))}{{}^{\varphi_i}\chi(1)} a = \frac{\chi(z)}{\chi(1)} a$$

for all $z \in Z(G)$ and $a \in A^i_\chi$. The statement now follows from the definition of the Miyamoto maps and the grading from Proposition 3.1.6. □

In Theorem 2.9.7, we gave certain Norton algebras the structure of an axial decomposition algebra. Its decomposition structure is a refinement of a decomposition algebra from Theorem 2.8.1. Therefore, we can apply the previous theorem to it. Let us recall some of the notation.

Definition 3.3.2. Let G be a finite simple group and D a conjugacy class on which G acts generously transitively. By Theorem 2.9.7, we can give the corresponding Norton algebras the structure of an axial decomposition algebra. Recall, from Definition 2.9.6 and Theorem 2.9.7, that its fusion law is a direct product $\mathcal{F} \times \Lambda$ where $\mathcal{F}$ is a representation fusion law of the group $C_G(d)$ for $d \in D$.

Now consider the finest grading $\xi \colon \mathsf{Irr}(C_G(d)) \to \Gamma$ of the representation fusion law $\mathsf{Irr}(C_G(d)) = \mathcal{F}$. Then this induces a grading of the fusion law $\mathcal{F} \times \Lambda$:

$$\xi' \colon \mathcal{F} \times \Lambda \to \Gamma \colon (x, \lambda) \mapsto \xi(x).$$

Theorem 3.3.3. *Let G be a finite simple group and D a conjugacy class on which G acts generously transitively. Consider the corresponding Norton algebras as an axial decomposition algebra (N, D, Σ, α) as in Theorem 2.9.7. Let ξ' be the grading for its fusion law from Definition 3.3.2. Then the corresponding Miyamoto group is the group G in its action on N.*

Proof. Let $\mathcal{F} \times \Lambda$ be the fusion law of (N, D, Σ, α) from Definition 2.9.6. Then $\mathcal{F}$ is the representation fusion law $\mathsf{Irr}(C_G(d))$ for $d \in D$. By construction of the decomposition algebra (N, D, Σ) we know that $\Omega[\chi] := \bigoplus_{\lambda \in \Lambda} N^d_{\chi,\lambda}$ is the $C_G(d)$-isotypic component of N with respect to the character $\chi \in \mathcal{F} \subseteq \mathsf{Irr}(C_G(d))$. Apply Theorem 3.3.1 to the decomposition algebra (N, D, Ω) to conclude that $\mathsf{Miy}(N, D, \Omega)$ is the group $\langle z \mid z \in Z(C_G(d)), d \in D \rangle$ in its action on N. The elements of D are clearly contained in this group. Since D is a conjugacy class of the simple group G, we must have that the Miyamoto group of (N, D, Ω) is G in its action on N. By definition of the grading ξ', this is also the Miyamoto group of (N, D, Σ, α). □

3.4 Miyamoto-closed decomposition algebras

Later on, e.g. in the definition of the universal Miyamoto group, we will often assume that our decomposition algebras are Miyamoto-closed. We introduce this notion in this section and explain why it is no real restriction to assume it.

Let (A, I, Ω) be an $\mathcal{F}$-decomposition algebra and let φ be an automorphism of A. Then for every decomposition $\bigoplus_{x \in \mathcal{F}} A^i_x$, we have that $\bigoplus_{x \in \mathcal{F}} \varphi(A^i_x)$ is also an $\mathcal{F}$-decomposition. If $\mathcal{F}$ is graded, then we can take φ to be one of the Miyamoto automorphisms. Sometimes, it will be convenient to assume that these decomposition again lie in Ω. Therefore, we introduce the following definition.

Definition 3.4.1. Let $\mathcal{F}$ be a Γ-graded fusion law and let $\mathcal{Y}$ be any subgroup of the character group $\mathcal{X}_R(\Gamma)$.

(i) We call an $\mathcal{F}$-decomposition algebra (A, I, Ω) *Miyamoto-closed* with respect to $\mathcal{Y}$ if Ω is invariant under the Miyamoto group with respect to $\mathcal{Y}$. That is, for each $i \in I$ and each $\chi \in \mathcal{Y}$, there is a permutation[2] $\pi_{i,\chi}$ of I such that $\tau_{i,\chi}$

[2]In the situation where some of the decompositions $(A^j_x)_{x \in \mathcal{F}} \in \Omega$ coincide, there might be some freedom in the choice of the permutation $\pi_{i,\chi}$, but this choice will be irrelevant for us.

maps each decomposition $(A_x^j)_{x\in\mathcal{F}} \in \Omega$ to the decomposition $(A_x^{\pi_{i,\chi}(j)})_{x\in\mathcal{F}} \in \Omega$. Notice that in this case, each pair $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of (A, I, Ω) in the category $\mathcal{F}$-Dec_R.

(ii) Similarly, we call an axial $\mathcal{F}$-decomposition algebra (A, I, Ω, α) *Miyamoto-closed* with respect to $\mathcal{Y}$ if, for any $i \in I$ and $\chi \in \mathcal{Y}$, there exists a permutation $\pi_{i,\chi}$ of I such that $(\tau_{i,\chi}, \pi_{i,\chi})$ is a morphism in the category $(\mathcal{F}, \lambda)$-AxDec_R. This implies in particular that the Miyamoto maps permute the axes; explicitly, $\tau_{i,\chi}(\alpha(j)) = \alpha(\pi_{i,\chi}(j))$ for all $j \in I$.

If we assume that a decomposition algebra is Miyamoto-closed, then the conjugate of a Miyamoto map by a Miyamoto map is again a Miyamoto map.

Proposition 3.4.2. *If (A, I, Ω) is Miyamoto-closed with respect to $\mathcal{Y}$, then*

$$^{\tau_{i,\chi}}\tau_{j,\chi'} = \tau_{\pi_{i,\chi}(j),\chi'}$$

for all $i, j \in I$ and $\chi, \chi' \in \mathcal{Y}$.

Proof. Let ξ denote the Γ-grading of $\mathcal{F}$. If $a \in A_x^j$, then $\tau_{i,\chi}(\tau_{j,\chi'}(a)) = \chi'(g)\tau_{i,\chi}(a)$ where $g := \xi(x)$. Now $\tau_{i,\chi}(a) \in A_x^{\pi_{i,\chi}(j)}$ and hence $\tau_{\pi_{i,\chi}(j),\chi'}(\tau_{i,\chi}(a)) = \chi'(g)\tau_{i,\chi}(a)$. Since $A = \bigoplus_{x\in\mathcal{F}} A_x^j$, the statement follows. □

Later on, we will often assume that an $\mathcal{F}$-decomposition algebra is Miyamoto-closed (with respect to some subgroup $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$). The following proposition illustrates why this is not a strong restriction.

Proposition 3.4.3. *Let (A, I, Ω) be an $\mathcal{F}$-decomposition algebra for a Γ-graded fusion law $\mathcal{F}$. Denote its Miyamoto group $\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega)$ with respect to $\mathcal{Y}$ by G. Let J be the set $I \times G$. Write Σ for the J-tuple of decompositions defined by*

$$\Sigma[(i, \varphi)] := (\varphi(A_x^j) \mid x \in \mathcal{F}).$$

Then (A, J, Σ) is an $\mathcal{F}$-decomposition algebra that is Miyamoto-closed with respect to $\mathcal{Y}$. Moreover

$$\tau_{(i,\varphi),\chi} = {}^{\varphi}(\tau_{i,\chi})$$

and thus $\mathsf{Miy}_{\mathcal{Y}}(A, J, \Sigma) = \mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega)$.

Proof. If $\bigoplus_{x\in\mathcal{F}} A_x^i$ is an $\mathcal{F}$-decomposition of A, then so is $\bigoplus_{x\in\mathcal{F}} \varphi(A_x^i)$ for any automorphism φ of A. This proves that (A, J, Σ) is indeed an $\mathcal{F}$-decomposition algebra.

Next, we prove the identity

$$\tau_{(i,\varphi),\chi} = {}^{\varphi}(\tau_{i,\chi})$$

for all $i \in I$, $\varphi \in G$ and $\chi \in \mathcal{Y}$. Let $a \in A_x^{(i,\varphi)} = \varphi(A_x^i)$, then $a = \varphi(b)$ for some $b \in A_x^i$. Now

$$\begin{aligned}\tau_{(i,\varphi),\chi}(a) &= \chi(g)a,\\ &= \chi(g)\varphi(b),\\ &= \varphi(\tau_{i,\chi}(b)),\\ &= ({}^{\varphi}\tau_{i,\chi})(a).\end{aligned}$$

Since $A = \bigoplus_{x\in\mathcal{F}} A_x^{(i,\varphi)}$, this proves the identity.

For any $(i,\varphi) \in J$ and $\chi \in \mathcal{Y}$, we can consider the permutation

$$\pi_{(i,\varphi),\chi} \colon J \to J \colon (j,\psi) \mapsto (j, {}^{\varphi}(\tau_{i,\chi})\psi).$$

From the identity above, it follows that $(\tau_{(i,\varphi),\chi}, \pi_{(i,\varphi),\chi})$ is indeed an automorphism of the decomposition algebra (A, J, Σ). We conclude that (A, J, Σ) is Miyamoto-closed. $\square$

Usually, the decomposition algebra (A, J, Σ) contains every decomposition multiple times which makes it hard to control. The previous proposition is rather a proof of concept. It shows that we can simply add decompositions to make a decomposition algebra Miyamoto-closed, without changing the Miyamoto group itself. In practice, it is usually pretty clear what needs to be done to make an (axial) decomposition algebra Miyamoto-closed.

3.5 Matsuo algebras

In this section we will discuss Matsuo algebras as examples of axial decomposition algebras admitting an interesting Miyamoto group. More precisely, we show that the Miyamoto group of a Matsuo algebras over a Fischer space is a 3-transposition group. Conversely, every 3-transposition group will give rise to a Matsuo algebra.

This section is based on [DMVC20a, § 4]. Most of the results are due to Jonathan I. Hall, Felix Rehren and Sergey Shpectorov [HRS15a, §6] but are now formulated in the language of axial decomposition algebras. The work of Atsushi Matsuo and Mika Matsuo already contained a version of some of these results; see the unpublished [MM99] and [Mat05].

Definition 3.5.1. (i) A *point-line geometry* $\mathcal{G}$ is a pair $(\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is a set whose elements are called *points* and $\mathcal{L}$ is a set of subsets of $\mathcal{P}$. The elements of $\mathcal{L}$ are called *lines*.

(ii) Two distinct points x and y of a point-line geometry are said to be *collinear* if there is a line containing both and we denote this by $x \sim y$. We write $x \nsim y$ if x and y are not collinear. If $x \nsim y$ for all $y \neq x$, then we call x an *isolated point*.

(iii) A *subspace* of a point-line geometry $(\mathcal{P}, \mathcal{L})$ is a point-line geometry $(\mathcal{P}', \mathcal{L}')$ such that:

- $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L} \subseteq \mathcal{L}'$,
- if $x, y \in \mathcal{P}'$ (with $x \neq y$) and $x, y \in \ell$ for some $\ell \in \mathcal{L}$, then $\ell \in \mathcal{L}'$.

Given a subset X of $\mathcal{P}$, there is a unique minimal subspace of $(\mathcal{P}, \mathcal{L})$ containing the points of X; we call it the *subspace generated by* those points.

(iv) Two points x and y of a point-line geometry are called *connected* if there exist points $x = x_0, x_1, \ldots, x_n = y$ such that $x_{i-1} \sim x_i$ for all $1 \leq i \leq n$. This relation defines an equivalence relation on the set of points of a point-line geometry. The subspaces generated by its equivalence classes are called the *connected components* of the point-line geometry.

(v) An *isomorphism* between two point-line geometries $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ and $\mathcal{L} = (\mathcal{P}', \mathcal{L}')$ is a bijection $\theta : \mathcal{P} \to \mathcal{P}'$ that induces a bijection between $\mathcal{L}$ and $\mathcal{L}'$. We write $\mathcal{G} \cong \mathcal{G}'$ and call $\mathcal{G}$ and $\mathcal{G}'$ *isomorphic* if an isomorphism between them exists.

(vi) Let $\mathcal{G}$ be a point-line geometry such that through any two points there is at most one line and such that each line contains exactly three points. Such a point-line geometry is called a *partial triple system*. If x and y are collinear points in a partial triple system, there is a unique third point on the unique line through x and y and we will denote it by $x \wedge y$.

(vii) Let $\mathcal{G}$ be a partial triple system. We call $\mathcal{G}$ a *Fischer space* if each subspace generated by the points of two distinct intersecting lines is isomorphic to the dual affine plane of order 2 or the affine plane of order 3. A sketch of these two point-line geometries is given in Fig. 3.1.

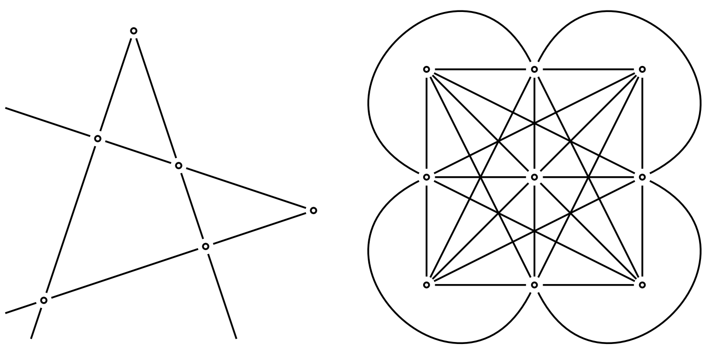


Figure 3.1: The dual affine plane of order 2 and the affine plane of order 3.

The main motivation for studying Fischer spaces is their connection with 3-transposition groups, due to Francis Buekenhout [Bue74]; see Proposition 3.5.5 below.

Definition 3.5.2 ([Asc97])**.** A *3-transposition group* is a pair (G, D) where D is a generating set of involutions of G, closed under conjugation, such that the order of the product of any two elements of D is at most 3.

Example 3.5.3. (i) All symmetric groups together with their set of transpositions form a 3-transposition group.

(ii) The Fischer groups Fi_{22}, Fi_{23} and Fi_{24} are 3-transposition groups.

In the connection between Fischer spaces and 3-transposition groups that we will need, it will be of importance to treat isolated points in Fischer spaces with some care.

Definition 3.5.4. (i) Let $(\mathcal{P}, \mathcal{L})$ be a Fischer space and let $\mathcal{P}' \subseteq \mathcal{P}$ be the set of isolated points of $(\mathcal{P}, \mathcal{L})$. Then $(\mathcal{P} \setminus \mathcal{P}', \mathcal{L})$ is clearly a Fischer space without isolated points that we will denote by $(\mathcal{P}, \mathcal{L})^\circ$.

(ii) If $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ is a Fischer space, then we associate with each point $x \in \mathcal{P}$ an automorphism $\tau_x \in \mathsf{Aut}(\mathcal{G})$ defined as

$$\tau_x \colon \mathcal{P} \to \mathcal{P} \colon y \mapsto \begin{cases} x \wedge y & \text{if } y \sim x, \\ y & \text{if } y \nsim x. \end{cases}$$

Notice that τ_x is an involution unless x is an isolated point (in which case τ_x is trivial). These involutions not only leave the set of points of the Fischer space invariant but also map lines to lines; hence they induce automorphisms of the Fischer space.

Now let $D = \{\tau_x \mid x \in \mathcal{P} \text{ and } \tau_x \neq 1\}$ and $G = \langle D \rangle \leq \mathsf{Aut}(\mathcal{G})$; then we define $f(\mathcal{G}) := (G, D)$.

(iii) Let (G, D) be a 3-transposition group. Then we write $(G, D)^\circ$ for the 3-transposition group $(G/Z(G), \{dZ(G) \mid d \in D \setminus Z(G)\})$.

(iv) Let (G, D) be a 3-transposition group. Let $\mathcal{P} = D$ and let

$$\mathcal{L} = \{\{c, d, c^d = d^c\} \mid |cd| = 3\}$$

where $|cd|$ is the order of the element $cd \in G$. Then $g(G, D) := (\mathcal{P}, \mathcal{L})$ is a point-line geometry.

Proposition 3.5.5 ([Bue74])**.** *Let $\mathcal{G}$ be a Fischer space and let (G, D) be a 3-transposition group. Then*

(i) *$g(G, D)$ is a Fischer space,*

(ii) *$f(\mathcal{G})$ is a 3-transposition group,*

(iii) *$f(g(G, D)) \cong (G, D)^\circ$,*

(iv) *$g(f(\mathcal{G})) \cong \mathcal{G}^\circ$.*

Example 3.5.6. (i) The Fischer space corresponding to the 3-transposition group S_4 is the dual affine plane of order 2.

(ii) The 3-transposition group related to the affine plane of order 3 is a semidirect product $3^2 : 2$ where the action is given by inversion.

We now present the definition of Matsuo algebras, which will be instances of axial decomposition algebras; see Proposition 3.5.8 below. When the Matsuo algebra arises from a Fischer space, its fusion law will be $\mathbb{Z}/2\mathbb{Z}$-graded; see Proposition 3.5.9 below.

Definition 3.5.7. Let k be a field with $\mathrm{char}(k) \neq 2$, let $\alpha \in k$ and let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a partial triple system. Define the Matsuo algebra $M_\alpha(\mathcal{G})$ as the k-vector space with basis $\mathcal{P}$ and multiplication defined by linearly extending

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \not\sim y, \\ \frac{\alpha}{2}(x + y - x \wedge y) & \text{if } x \sim y, \end{cases}$$

for all $x, y \in \mathcal{P}$.

The following proposition gives us a decomposition of a Matsuo algebra for any $x \in \mathcal{P}$. Moreover, the element $x \in \mathcal{P}$ is an axis for this decomposition.

Proposition 3.5.8 ([HRS15a, Theorem 6.2])**.** *For each $x \in \mathcal{P}$, the algebra $M_\alpha(\mathcal{G})$ decomposes as a direct sum into the following three spaces:*

$$\begin{aligned} A_e^x &:= \langle x \rangle, \\ A_z^x &:= \langle y + x \wedge y - \alpha x \mid y \sim s \rangle \oplus \langle y \mid y \not\sim x \rangle \textit{ and,} \\ A_h^x &:= \langle y - x \wedge y \mid y \sim x \rangle. \end{aligned}$$

These consist of 1-, 0- and α-eigenvectors of ad_x *respectively.*

Proof. This is an easy calculation. □

These decompositions satisfy the Jordan fusion law from Example 2.4.5 precisely when $\mathcal{G}$ is a Fischer space.

Proposition 3.5.9 ([HRS15a, Theorem 6.5]). *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a partial triple system, with corresponding Matsuo algebra $A = M_\alpha(\mathcal{G})$. Let Ω be the $\mathcal{P}$-tuple of decompositions from Proposition* 3.5.8*. Denote the Jordan fusion law from Example* 2.4.5 *by $\mathcal{F}$. Then $(A, \mathcal{P}, \Omega)$ is an $\mathcal{F}$-decomposition algebra if and only if $\mathcal{G}$ is a Fischer space.*

Proof. The proof from [HRS15a, Theorem 6.5] is formulated in the language of axial algebras. Therefore, they have to exclude the values $\alpha \in \{0, 1\}$. However, since only the decomposition from Proposition 3.5.8 is crucial, the proof can be followed mutatis mutandis. □

Since the components of the decomposition corresponding to $x \in \mathcal{P}$ consist of eigenvectors for $x \in A$, we also have the following corollary.

Corollary 3.5.10. *The quadruple $(A, \mathcal{P}, \Omega, \mathcal{P} \to A \colon x \mapsto x)$ is an axial $\mathcal{F}$-decomposition algebra if and only if $\mathcal{G}$ is a Fischer space. Its evaluation map is then given by*

$$\lambda \colon \mathcal{F} \to A \colon \begin{cases} e \mapsto 1, \\ z \mapsto 0, \\ h \mapsto \alpha. \end{cases}$$

Remark 3.5.11. Note that the Matsuo algebra $M_\alpha(\mathcal{G})$ is generated by the axes $x \in \mathcal{P}$. So if $\alpha \neq 0, 1$ and $\mathcal{G}$ is a Fischer space, then $M_\alpha(\mathcal{G})$ is also an axial algebra in the sense of Section 2.3.

Definition 3.5.12. Let $\mathcal{G}$ be a Fischer space. In order not to overload the notation, we will identify the Matsuo algebra $A = M_\alpha(\mathcal{G})$ with the corresponding $\mathcal{F}$-decomposition algebra $(M_\alpha(\mathcal{G}), \mathcal{P}, \Omega)$ from Proposition 3.5.9. The Jordan fusion law $\mathcal{F}$ is $\mathbb{Z}/2\mathbb{Z}$-graded with grading map $\xi \colon \mathcal{F} \to \mathbb{Z}/2\mathbb{Z}$ defined by $\xi(e) = \xi(z) = 0$ and $\xi(h) = 1$. Recall that we assume $\operatorname{char}(k) \neq 2$ so that there exists a unique non-trivial linear k-character $\chi \colon \mathbb{Z}/2\mathbb{Z} \to k^\times$. We can now consider the Miyamoto maps

$$\tau_{x,\chi} \colon A \to A \colon a \mapsto \begin{cases} a & \text{if } a \in A^x_{\{e,z\}}, \\ -a & \text{if } a \in A^x_h. \end{cases}$$

We will now study the corresponding Miyamoto group of $M_\alpha(\mathcal{G})$, this is the subgroup of $\mathsf{Aut}(M_\alpha(\mathcal{G}))$ generated by the $\tau_{x,\chi}$ for $x \in \mathcal{P}$.

Proposition 3.5.13 ([HRS15a, Theorem 6.4]). *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a Fischer space with corresponding Matsuo algebra $A = M_\alpha(\mathcal{G})$. Let $(G, D) := f(\mathcal{G})$ be the 3-transposition group associated to $\mathcal{G}$. Then we have the following.*

(i) *For each $x \in \mathcal{P}$, the Miyamoto involution $\tau_{x,\chi} \in \mathsf{Aut}(A)$ acts on $\mathcal{P}$ as the automorphism τ_x introduced in Definition* 3.5.4 (ii)*.*

(ii) *The Matsuo algebra A is Miyamoto-closed with respect to $\langle\chi\rangle$. More precisely, $(\tau_{x,\chi}, \tau_x)$ is an automorphism of A for every $x \in \mathcal{P}$.*

(iii) *The morphism defined by*

$$\theta\colon \mathsf{Miy}_\chi(A) \to G\colon \tau_{x,\chi} \mapsto \tau_x$$

is an isomorphism of groups.

Proof. (i) Let $x, y \in \mathcal{P}$. If $y \nsim x$, then we have $y \in A^x_{\{e,z\}}$. Therefore $\tau_x(y) = y$. If $y \sim x$, then $y + x \wedge y - \alpha x \in A^x_z$ and $y - x \wedge y \in A^x_h$. Thus we have

$$\begin{aligned}\tau_x(y) &= \frac{1}{2}\tau_x(y + x \wedge y - \alpha x) + \frac{1}{2}\tau_x(y - x \wedge y) + \frac{\alpha}{2}\tau_x(x)\\ &= x \wedge y.\end{aligned}$$

(ii) It follows immediately from (i) that $(\tau_{x,\chi}, \tau_x)$ is an automorphism of A as axial decomposition algebra.

(iii) Note that θ defines a morphism by (i). It is surjective since G is generated by the elements $\{\tau_x \mid x \in \mathcal{P}\}$. Since $\{x \mid x \in \mathcal{P}\}$ is a basis for $M_\alpha(\mathcal{G})$, the Miyamoto map $\tau_{x,\chi}$ is uniquely determined by its action on the elements of $\mathcal{P}$. Therefore, θ is an isomorphism. □

We now start from a 3-transposition group (G, D) and we would like to construct a Matsuo algebra with Miyamoto group isomorphic to G. The situation giving rise to isolated points of the corresponding Fischer space should be treated with some care. It turns out that this works out nicely when we assume that G is centerless, since in that case it is the 3-transposition group corresponding to some Fischer space by Proposition 3.5.5 (iii).

Proposition 3.5.14. *Let (G, D) be a 3-transposition group. Write $\mathcal{G} \coloneqq g(G, D)$ for its corresponding Fischer space and let $A = M_\alpha(\mathcal{G})$ be the Matsuo algebra for $\mathcal{G}$. Recall from Definition* 3.5.4 (iv) *that the points of $\mathcal{G}$ are the elements of D. The morphism defined by*

$$\theta\colon G \to \mathsf{Miy}_\chi(A)\colon d \mapsto \tau_{d,\chi}$$

for all $d \in D$, is an epimorphism with kernel $Z(G)$.

Proof. From Proposition 3.5.5 (iii), we know that $f(g(G, D)) = (G, D)^\circ$. The statement now follows from Proposition 3.5.13 (iii). □

Proposition 3.5.15. *Let $\mathcal{G}$ be a Fischer space. Then $\mathsf{Miy}_\chi(M_\alpha(\mathcal{G}))$ is centerless.*

Proof. This follows from Proposition 3.5.13 (iii) and Proposition 3.5.14. □

3.6 Universal Miyamoto groups

The Miyamoto group is interesting—it is a subgroup of the automorphism group of the algebra—but is not so easy to control (cf. Example 3.7.11 below). It is useful to construct a cover of this group, which we call the *universal Miyamoto group*.

Definition 3.6.1. Let $\mathcal{F}$ be a Γ-graded fusion law and (A, I, Ω) an $\mathcal{F}$-decomposition algebra. Assume that (A, I, Ω) is Miyamoto-closed with respect to $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$. We define the *universal Miyamoto group* with respect to $\mathcal{Y}$ as the group given by the following presentation. For each $i \in I$, we let $\mathcal{Y}_i$ be a copy of the group $\mathcal{Y}$ and we denote its elements by

$$\mathcal{Y}_i = \{t_{i,\chi} \mid \chi \in \mathcal{Y}\}.$$

For each $a = t_{i,\chi} \in \mathcal{Y}_i$, let us write $\overline{a}$ for the corresponding Miyamoto map $\tau_{i,\chi} \in \mathsf{Miy}(A, I, \Omega)$. Notice that for each $i \in I$, the group

$$\overline{\mathcal{Y}_i} := \{\overline{a} \mid a \in \mathcal{Y}_i\} = \{\tau_{i,\chi} \mid \chi \in \mathcal{Y}\}$$

is an *abelian* subgroup of $\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega)$.

We will define the universal Miyamoto group $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$ as a quotient of the free product $\ast_{i \in I} \mathcal{Y}_i$ by conjugation relations between the groups $\mathcal{Y}_i$ that exist "globally" between the corresponding groups $\overline{\mathcal{Y}_i}$ in $\mathsf{Miy}(A, I, \Omega)$. More precisely, let $\mathcal{U} := \bigcup_{i \in I} \mathcal{Y}_i$; for each $a \in \mathcal{U}$, we consider the set

$$R_{\overline{a}} := \{(j, k) \in I \times I \mid {}^{\overline{a}}\tau_{j,\chi} = \tau_{k,\chi} \text{ for all } \chi \in \mathcal{Y}\}. \tag{3.1}$$

We then let

$$\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega) := \left\langle \mathop{\ast}_{i \in I} \mathcal{Y}_i \;\middle|\; {}^{a}t_{j,\chi} = t_{k,\chi} \text{ for all } a \in \mathcal{U}, \text{ all } (j,k) \in R_{\overline{a}} \text{ and all } \chi \in \mathcal{Y} \right\rangle.$$

Remark 3.6.2. The reader might wonder why we only consider conjugation relations that exist globally and do not define the universal Miyamoto group as the group

$$\left\langle \mathop{\ast}_{i \in I} \mathcal{Y}_i \;\middle|\; {}^{a}b = c \text{ for all } a, b, c \in \mathcal{U} \text{ satisfying } {}^{\overline{a}}\overline{b} = \overline{c} \right\rangle.$$

instead. The problem with this definition is that some conjugation relations might hold "by coincidence" and we do not want to transfer those to the universal Miyamoto group. For instance, Theorem 3.7.10 below would become false with this seemingly easier definition.

On the other hand, since (A, I, Ω) is Miyamoto-closed with respect to $\mathcal{Y}$, we always have many conjugation relations at our disposal. Recall that we say that (A, I, Ω) is Miyamoto-closed if for every Miyamoto automorphism $\tau_{i,\chi}$ there exist a permutation $\pi_{i,\chi}$ such that $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of (A, I, Ω).

Lemma 3.6.3. *Let $i, j \in I$.*

(i) *For each $\chi, \chi' \in \mathcal{Y}$, the relation*

$$^{t_{i,\chi}}t_{j,\chi'} = t_{\pi_{i,\chi}(j),\chi'}$$

holds in $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$.

(ii) *If $\tau_{i,\chi} = \tau_{j,\chi}$ for all $\chi \in \mathcal{Y}$, then also $t_{i,\chi} = t_{j,\chi}$ for all $\chi \in \mathcal{Y}$.*

Proof. (i) Let $a = t_{i,\chi}$ for some $\chi \in \mathcal{Y}$. Since (A, I, Ω) is Miyamoto-closed with respect to $\mathcal{Y}$, we have, by Proposition 3.4.2, $^{\tau_{i,\chi}}\tau_{j,\chi'} = \tau_{\pi_{i,\chi}(j),\chi'}$ for all $\chi' \in \mathcal{Y}$. Hence $(j, \pi_{i,\chi}(j)) \in R_{\overline{a}}$. It follows that all relations of the form

$$^{t_{i,\chi}}t_{j,\chi'} = t_{\pi_{i,\chi}(j),\chi'}$$

hold in $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$.

(ii) Let $a = t_{i,\chi}$ for some $\chi \in \mathcal{Y}$. Recall that $\overline{\mathcal{Y}_i}$ is abelian, hence $\tau_{i,\chi}$ commutes with $\tau_{i,\chi'}$ for all $\chi' \in \mathcal{Y}$. Since $\tau_{i,\chi'} = \tau_{j,\chi'}$, it follows that $(i, j) \in R_{\overline{a}}$. Therefore, the relations

$$^{t_{i,\chi}}t_{i,\chi'} = t_{j,\chi'}$$

hold in $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$. Since $t_{i,\chi}$ and $t_{i,\chi'}$ both belong to the abelian group $\mathcal{Y}_i \leq \widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$, we conclude that the relation $t_{i,\chi'} = t_{j,\chi'}$ holds in $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$. $\square$

Proposition 3.6.4. *Let $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$ and let (A, I, Ω) be Miyamoto-closed with respect to $\mathcal{Y}$. Then $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$ is a central extension of $\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega)$.*

Proof. Let $\widehat{G} := \widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$, $G := \mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega)$ and $\mathcal{U} := \bigcup_{i \in I} \mathcal{Y}_i \subseteq \widehat{G}$; then $\widehat{G} = \langle \mathcal{U} \rangle$. It is immediately clear from the definition of $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega)$ that the map $\mathcal{U} \to G \colon a \mapsto \overline{a}$ extends to an epimorphism $\Phi \colon \widehat{G} \to G$; it remains to show that $\ker \Phi$ is central.

Let $z \in \ker \Phi$ be arbitrary; as each generator $a \in \mathcal{U}$ has finite order, we can write $z = a_m \cdots a_1$ with $a_i \in \mathcal{U}$. We have to show that $^{z}b = b$ for each $b = t_{j,\chi'} \in \mathcal{U}$. Fix such an element $b \in \mathcal{U}$. For each $k \in \{0, \dots, m\}$, we write

$$\begin{aligned} b_k &:= {}^{a_k \cdots a_1}b \in \widehat{G} \quad \text{and} \\ c_k &:= {}^{\overline{a}_k \cdots \overline{a}_1}\overline{b} \in G. \end{aligned}$$

By repeatedly applying Lemma 3.6.3 (i), we see that each b_k is again of the form $t_{j_k,\chi'}$ for some $j_k \in I$ (which only depends on z and j but not on χ') and that $c_k = \overline{b_k}$ for each k.

In particular, $\overline{b_m} = c_m = {}^{\Phi(z)}\overline{b} = \overline{b}$ with $b = t_{j,\chi'}$ and $b_m = t_{j_m,\chi'}$. Hence $\tau_{j_m,\chi'} = \tau_{j,\chi'}$. Because this holds for all $\chi' \in \mathcal{Y}$, Lemma 3.6.3 (ii) now implies that $t_{j_m,\chi'} = t_{j,\chi'}$ for all χ'. Varying $j \in I$ finishes the proof. □

Since the Matsuo algebras over a Fischer space $\mathcal{G}$ are Miyamoto-closed by Proposition 3.5.13 (ii), we can consider their universal Miyamoto group. We show that it can be interpreted as the universal 3-transposition group for the Fischer space $\mathcal{G}$. Recall the notation from Section 3.5, in particular the maps f and g from Definition 3.5.4.

Theorem 3.6.5. *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a Fischer space without isolated points and $A := M_\alpha(\mathcal{G})$ its Matsuo algebra. Then:*

(i) $\big(\widehat{\mathsf{Miy}}_\chi(A), \{t_{x,\chi} \mid x \in \mathcal{P}\}\big)$ *is a 3-transposition group and*

$$g\big(\widehat{\mathsf{Miy}}_\chi(A), \{t_{x,\chi} \mid x \in \mathcal{P}\}\big) \cong \mathcal{G}.$$

(ii) *Let (G, D) be a 3-transposition group such that $g(G, D) \cong \mathcal{G}$. Let $\varphi : \mathcal{P} \to D$ be an isomorphism between $\mathcal{G}$ and $g(G, D)$. The map defined by*

$$\theta : \widehat{\mathsf{Miy}}_\chi(A) \to G : t_{x,\chi} \mapsto \varphi(x)$$

is a group epimorphism and $G'/Z(G') \cong \widehat{\mathsf{Miy}}_\chi(A)/Z(\widehat{\mathsf{Miy}}_\chi(A)) \cong \mathsf{Miy}_\chi(A)$.

Proof. (i) Since all relations that hold in $\widehat{\mathsf{Miy}}_\chi(A)$ have even length and all $t_{x,\chi}$ have order at most 2, all $t_{x,\chi}$ have order exactly 2. Note that, by Proposition 3.5.13 (i), we have ${}^{\tau_{y,\chi}}\tau_{x,\chi} = \tau_{\tau_y(x),\chi}$. By Lemma 3.6.3 (i), we also have ${}^{t_{y,\chi}}t_{x,\chi} = t_{\tau_y(x),\chi}$. Therefore $\{t_{x,\chi} \mid x \in \mathcal{P}\}$ is, by definition of $\widehat{\mathsf{Miy}}_\chi(A)$, a generating set of involutions invariant under conjugation. Since $\mathcal{G}$ has no isolated points, all $\tau_{x,\chi}$ are different by Proposition 3.5.13 (i). By Lemma 3.6.3 (ii), all $t_{x,\chi}$ must also be different.

Let $x, y \in \mathcal{P}$; then, by Definition 3.5.4 (ii), either $x = y$ or $\tau_y(x) = x$ if $x \nsim y$ or $\tau_y(x) = x \wedge y$ if $x \sim y$. In the first case, $t_{x,\chi}t_{y,\chi} = 1$. In the second case, $(t_{x,\chi}t_{y,\chi})^2 = t_{x,\chi}t_{\tau_y(x),\chi} = 1$. In the third case, $\tau_x(y) = x \wedge y$ and thus $(t_{x,\chi}t_{y,\chi})^3 = t_{\tau_x(y),\chi}t_{\tau_y(x),\chi} = (t_{x\wedge y,\chi})^2 = 1$. This proves that $(\widehat{\mathsf{Miy}}_\chi(A), \{t_{x,\chi} \mid x \in \mathcal{P}\})$ is indeed a 3-transposition group.

Let $x, y, z \in \mathcal{P}$; then $t_{x,\chi}$, $t_{y,\chi}$ and $t_{z,\chi}$ lie on a line in $g(\widehat{\mathsf{Miy}}(A), \{t_{x,\chi} \mid x \in \mathcal{P}\})$ if and only if $t_{x,\chi}t_{y,\chi}$ has order 3 and ${}^{t_{y,\chi}}t_{x,\chi} = t_{z,\chi}$. By our previous arguments, this happens if and only if $x \sim y$ and $z = x \wedge y$.

(ii) For each $x \in \mathcal{P}$, $\varphi(x) \in D$ is an involution. Let $x, y \in \mathcal{P}$ be distinct. If $x \sim y$, then, by definition of $g(G, D)$, ${}^{\varphi(y)}\varphi(x) = \varphi(x \wedge y) = \varphi(\tau_y(x))$. If $x \nsim y$, then $\varphi(x)\varphi(y)$ has order 2 and ${}^{\varphi(y)}\varphi(x) = \varphi(x) = \varphi(\tau_y(x))$. This proves that θ is a group epimorphism.

The isomorphisms follow by applying Proposition 3.5.14 to the 3-transposition groups (G, D) and $(\widehat{\mathsf{Miy}}_\chi(A), \{t_{x,\chi} \mid x \in \mathcal{P}\})$. $\square$

Remark 3.6.6. Let $\mathcal{G}$ be a Fischer space without isolated points. Suppose that (G, D) is a 3-transposition group for which $g(G, D) \cong \mathcal{G}$. Then on the one hand, it has a quotient isomorphic to $\mathsf{Miy}(M_\alpha(\mathcal{G}))$ by Proposition 3.5.14. On the other hand, it is isomorphic to a quotient of $\widehat{\mathsf{Miy}}(M_\alpha(\mathcal{G}))$ by Theorem 3.6.5.

Example 3.6.7. Let $\mathcal{G}$ be the affine plane of order 3. Then $\widehat{\mathsf{Miy}}(M_\alpha(\mathcal{G}))$ is a semidirect product $3^2 : S_3$ which is a central extension of the group $3^2 : 2$ from Example 3.5.6(ii) by a cyclic group of order 3.

3.7 Morphisms of Miyamoto groups

In this section, we explain when $\mathsf{Miy}_\mathcal{Y}$ and $\widehat{\mathsf{Miy}}_\mathcal{Y}$ are functorial. This means that a morphism between decomposition algebra leads to a corresponding morphism of their (universal) Miyamoto groups. For this section, let $\mathcal{F}$ be a fusion law with a fixed Γ-grading ξ.

For *surjective* morphisms between decomposition algebras, both $\mathsf{Miy}_\mathcal{Y}$ and $\widehat{\mathsf{Miy}}_\mathcal{Y}$ are functorial. The following easy lemma is the key point.

Lemma 3.7.1. *Let $(\varphi, \psi)\colon (A, I, \Omega_A) \to (B, J, \Omega_B)$ be a morphism between two $\mathcal{F}$-decomposition algebras. Then for each $i \in I$ and $\chi \in \mathcal{X}_R(\Gamma)$, we have $\varphi \circ \tau_{i,\chi} = \tau_{\psi(i),\chi} \circ \varphi$.*

Proof. Let $a \in A^i_x$ for some $x \in \mathcal{F}$ and let $g \coloneqq \xi(x)$. Then on the one hand, $\varphi(\tau_{i,\chi}(a)) = \varphi(\chi(g)a) = \chi(g)\varphi(a)$, while on the other hand, $\varphi(a) \in B^{\psi(i)}_x$ and hence $\tau_{\psi(i),\chi}(\varphi(a)) = \chi(g)\varphi(a)$ as well. Since $A = \bigoplus_{x\in\mathcal{F}} A^i_x$, the result follows. $\square$

Proposition 3.7.2. *Let $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$. Let (φ, ψ) be a morphism between two Miyamoto-closed $\mathcal{F}$-decomposition algebras (A, I, Ω_A) and (B, J, Ω_B). Assume that φ is surjective.*

(i) *There is a corresponding morphism $\theta\colon \mathsf{Miy}_\mathcal{Y}(A, I, \Omega_A) \to \mathsf{Miy}_\mathcal{Y}(B, J, \Omega_B)$ mapping each generator $\tau_{i,\chi}$ of $\mathsf{Miy}_\mathcal{Y}(A, I, \Omega_A)$ to the corresponding generator $\tau_{\psi(i),\chi}$ of $\mathsf{Miy}_\mathcal{Y}(B, J, \Omega_B)$.*

(ii) *There is a corresponding morphism $\widehat{\theta}\colon \widehat{\mathsf{Miy}}_\mathcal{Y}(A, I, \Omega_A) \to \widehat{\mathsf{Miy}}_\mathcal{Y}(B, J, \Omega_B)$ mapping each generator $t_{i,\chi}$ of $\widehat{\mathsf{Miy}}_\mathcal{Y}(A, I, \Omega_A)$ to the corresponding generator $t_{\psi(i),\chi}$ of $\widehat{\mathsf{Miy}}_\mathcal{Y}(B, J, \Omega_B)$.*

Proof. (i) It suffices to verify that if the $\tau_{i,\chi}$ satisfy some relation

$$\tau_{i_1,\chi_1} \cdots \tau_{i_\ell,\chi_\ell} = 1$$

inside $\mathsf{Aut}(A)$, then also

$$\tau_{\psi(i_1),\chi_1} \cdots \tau_{\psi(i_\ell),\chi_\ell} = 1$$

inside $\mathsf{Aut}(B)$. This follows immediately from Lemma 3.7.1 and the fact that φ is surjective.

(ii) We have to show that each relator of $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega_A)$ is killed by $\widehat{\theta}$. Consider a relator

$$r = t_{k,\chi'}^{-1} \cdot {}^{t_{i,\chi}}t_{j,\chi'} \quad \text{with} \quad (j,k) \in R_{\tau_{i,\chi}}.$$

Then, by definition, we have $\tau_{k,\chi'} = {}^{\tau_{i,\chi}}\tau_{j,\chi'}$ in $\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega_A)$. This implies that $\tau_{\psi(k),\chi'} \circ \varphi = {}^{\tau_{\psi(i),\chi}}\tau_{\psi(j),\chi'} \circ \varphi$ by Lemma 3.7.1. Since φ is surjective, it follows that $\tau_{\psi(k),\chi'} = {}^{\tau_{\psi(i),\chi}}\tau_{\psi(j),\chi'}$ in $\mathsf{Miy}_{\mathcal{Y}}(B, J, \Omega_B)$. Because this holds for all $\chi' \in \mathcal{Y}$, we have

$$(\psi(j), \psi(k)) \in R_{\tau_{\psi(i),\chi}}.$$

Now $\widehat{\theta}$ maps the given relator r to $t_{\psi(k),\chi'}^{-1} \cdot {}^{t_{\psi(i),\chi}}t_{\psi(j),\chi'}$ and, by the definition of $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(B, J, \Omega_B)$, this element is trivial. □

The requirement that φ is surjective cannot be dropped in general, as the following generic type of examples illustrates.

Example 3.7.3. Let $\Gamma = \{1, \sigma\}$ be the group of order 2 and let $\mathcal{Y} = \{1, \chi\}$ where χ is the character that maps σ to $-1 \in R$ (assuming that $-1 \neq 1$ in R). Since there is only one non-trivial character in $\mathcal{Y}$, we will omit it from our notation and, for example, write τ_i in place of $\tau_{i,\chi}$. Let (A, I, Ω) be a Γ-decomposition algebra. The only (very weak) assumption we make, is the existence of three different $j, k, \ell \in I$ such that there is a relation ${}^{\tau_k}\tau_j = \tau_\ell$.

We will now construct another Γ-decomposition algebra (B, J, Ω') and a morphism $(\varphi, \psi)\colon (A, I, \Omega) \to (B, J, \Omega')$ such that the map $t_i \mapsto t_{\psi(i)}$ does *not* induce a group morphism between the corresponding universal Miyamoto groups.

Let $B = A \oplus M$, where M is a free R-module of rank 2 with basis $\{e, f\}$, and extend the multiplication of A to B trivially ($AM = MA = MM = 0$). Let $\varphi\colon A \to B$ be the natural inclusion. Let $J = I \times \{1, 2\}$; we will construct two decompositions of B for each decomposition of A in Ω. Define

$$\begin{aligned} \Omega'[i,1] &:= (A_1^i \oplus Re, A_\sigma^i \oplus Rf) \quad \text{and} \\ \Omega'[i,2] &:= (A_1^i \oplus Rf, A_\sigma^i \oplus Re). \end{aligned}$$

If we arbitrarily choose $c_i \in \{1, 2\}$ for each $i \in I$, then the map $\psi\colon I \to J\colon i \mapsto (i, c_i)$ will give rise to a morphism (φ, ψ) of Γ-decomposition algebras. In particular, this holds if we choose $c_j = c_k = 1$ and $c_\ell = 2$. Now consider the corresponding

Miyamoto involutions $\tau_{(j,1)}$, $\tau_{(k,1)}$ and $\tau_{(\ell,2)}$ of B; then $\tau_{(j,1)}$ and $\tau_{(k,1)}$ fix the element e whereas $\tau_{(\ell,2)}$ maps e to $-e$. In particular,

$$^{\tau_{\psi(k)}}\tau_{\psi(j)} = {}^{\tau_{(k,1)}}\tau_{(j,1)} \neq \tau_{(\ell,2)} = \tau_{\psi(\ell)}.$$

Hence the map $t_i \mapsto t_{\psi_i}$ does not induce a group morphism

$$\widehat{\mathsf{Miy}}_\chi(A, I, \Omega) \to \widehat{\mathsf{Miy}}_\chi(B, J, \Omega').$$

This behavior is caused by the fact that we can distort the map ψ. If we now restrict to decomposition algebras and morphisms that are sufficiently nice with respect to the Miyamoto maps, then this type of distortion cannot occur, and $\widehat{\mathsf{Miy}}_\mathcal{Y}$ becomes a functor.

First, we introduce a definition that tells us when a morphism is well-behaved with respect to the Miyamoto maps.

Definition 3.7.4. Let $(A, I, \Omega_A) \xrightarrow{(\varphi,\psi)} (B, J, \Omega_B)$ be a morphism of $\mathcal{F}$-decomposition algebras that are Miyamoto-closed with respect to $\mathcal{Y}$. Then we call (φ, ψ) *Miyamoto-admissable* with respect to $\mathcal{Y}$ if, for every $i \in I$ and $\chi \in \mathcal{Y}$, there exists a permutation $\pi_{i,\chi}$ of I and a permutation $\pi_{\psi(i),\chi}$ of J such that

- $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of (A, I, Ω_A),
- $(\tau_{\psi(i),\chi}, \pi_{\psi(i),\chi})$ is an automorphism of (B, J, Ω_B) and,
- the following diagram commutes:

$$\begin{tikzcd} I \arrow[r, "\psi"] \arrow[d, "\pi_{i,\chi}"'] & J \arrow[d, "\pi_{\psi(i),\chi}"] \\ I \arrow[r, "\psi"] & J \end{tikzcd}.$$

Observe that, by Lemma 3.7.1, then also

$$\begin{tikzcd} (A, I, \Omega_A) \arrow[r, "(\varphi,\psi)"] \arrow[d, "(\tau_{i,\chi},\pi_{i,\chi})"'] & (B, J, \Omega_B) \arrow[d, "(\tau_{\psi(i),\chi},\pi_{\psi(i),\chi})"] \\ (A, I, \Omega_A) \arrow[r, "(\varphi,\psi)"] & (B, J, \Omega_B) \end{tikzcd} \tag{3.2}$$

is a commuting diagram in $\mathcal{F}\text{-}\mathsf{Dec}_R$.

Next, we want to impose restrictions in such a way that we can recover the element $i \in I$ if we know the action of the Miyamoto maps $\tau_{i,\chi}$ on A for all $\chi \in \mathcal{Y}$.

Definition 3.7.5. Let $\mathcal{F}$ be a Γ-graded fusion law, let $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$ be a subgroup of the R-character group of Γ.

(i) Let (A, I, Ω) be an $\mathcal{F}$-decomposition algebra. For each $i \in I$ and each $\chi \in \mathcal{Y}$, let $\tau_{i,\chi}$ be the corresponding Miyamoto map. We call (A, I, Ω) *of unique type* with respect to $\mathcal{Y}$ if the map

$$I \to \mathsf{Hom}(\mathcal{Y}, \mathsf{Aut}(A)) \colon i \mapsto (\chi \mapsto \tau_{i,\chi})$$

is injective. In other words, for each $i \neq j$ there is at least one $\chi \in \mathcal{Y}$ such that $\tau_{i,\chi} \neq \tau_{j,\chi}$.

(ii) Let $\lambda \colon \mathcal{F} \to R$ be an evaluation map and let $(A, I, \Omega, \alpha) \in (\mathcal{F}, \lambda)\text{-}\mathsf{AxDec}_R$ be an axial decomposition algebra with axes $a_i := \alpha(i)$ for each $i \in I$. Then we say that (A, I, Ω, α) is *of unique type* with respect to $\mathcal{Y}$ if:

- the decomposition algebra (A, I, Ω) is of unique type with respect to $\mathcal{Y}$ and,
- $a_i \neq a_j$ for each $i \neq j$.

Example 3.7.6. Let k be a field with $\operatorname{char}(k) \neq 2$ and $\mathcal{Y}$ the k-character group of $\mathbb{Z}/2\mathbb{Z}$. Consider the Matsuo algebra $M_\alpha(\mathcal{G})$ from Section 3.5 with respect to a Fischer space $\mathcal{G}$ without isolated points. Then it follows from Proposition 3.5.5 (iv) and Proposition 3.5.13 (i) that $M_\alpha(\mathcal{G})$ is an axial decomposition algebra which is of unique type with respect to $\mathcal{Y}$.

Remark 3.7.7. If (A, I, Ω) is Miyamoto-closed and of unique type with respect to $\mathcal{Y}$ then there exists precisely one permutation $\pi_{i,\chi}$ of I such that $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of (A, I, Ω) for every $i \in I$ and $\chi \in \mathcal{Y}$. Thus if a morphism between such decomposition algebras is Miyamoto-admissable, then diagram (3.2) must commute for this unique choice of permutations.

If we restrict to decomposition algebras that are Miyamoto-closed and of unique type and morphisms that are Miyamoto-admissable with respect to $\mathcal{Y}$, then $\widehat{\mathsf{Miy}}_{\mathcal{Y}}$ becomes a functor.

Theorem 3.7.8. *Let $\mathcal{F}$ be a Γ-graded fusion law and let $\mathcal{Y}$ be a group of linear R-characters of Γ. Let $\mathcal{A} = (A, I, \Omega_A)$ and $\mathcal{B} = (B, J, \Omega_B)$ be $\mathcal{F}$-decomposition algebras that are Miyamoto-closed and of unique type. If $(\varphi, \psi) \colon \mathcal{A} \to \mathcal{B}$ is a morphism that is Miyamoto-admissable with respect to $\mathcal{Y}$ then*

$$\theta \colon \widehat{\mathsf{Miy}}(\mathcal{A}) \to \widehat{\mathsf{Miy}}(\mathcal{B}) \colon t_{i,\chi} \mapsto t_{\psi(i),\chi}$$

defines a group homomorphism.

Proof. Since (φ, ψ) is Miyamoto-admissable with respect to $\mathcal{Y}$, we have

$$\psi(\pi_{i,\chi}(j)) = \pi_{\psi(i),\chi}(\psi(j)). \tag{3.3}$$

for all $i, j \in I$ and $\chi \in \mathcal{Y}$. We will show that the map $t_{i,\chi} \mapsto t_{\psi(i),\chi}$ induces a group morphism $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(A, I, \Omega_A) \to \widehat{\mathsf{Miy}}_{\mathcal{Y}}(B, J, \Omega_B)$. More precisely, we show that if $(j,k) \in R_{\tau_{i,\chi}}$, then also $(\psi(j),\psi(k)) \in R_{\tau_{\psi(i),\chi}}$. So let $(j,k) \in R_{\tau_{i,\chi}}$; then by Lemma 3.6.3 (i),

$$\tau_{k,\chi'} = {}^{\tau_{i,\chi}}\tau_{j,\chi'} = \tau_{\pi_{i,\chi}(j),\chi'}$$

for all $\chi' \in \mathcal{Y}$. Because (A, I, Ω_A) is of unique type with respect to $\mathcal{Y}$, this can only happen if $k = \pi_{i,\chi}(j)$. Hence, by (3.3) and by Lemma 3.6.3 (i) again, also

$$\tau_{\psi(k),\chi'} = \tau_{\psi(\pi_{i,\chi}(j)),\chi'} = \tau_{\pi_{\psi(i),\chi}(\psi(j)),\chi'} = {}^{\tau_{\psi(i),\chi}}\tau_{\psi(j),\chi'}$$

for all $\chi' \in \mathcal{Y}$. We conclude that indeed $(\psi(j),\psi(k)) \in R_{\tau_{\psi(i),\chi}}$. □

For axial decomposition algebras the statement becomes even nicer due to the following lemma.

Lemma 3.7.9. *Let $(A, I, \Omega_A, \alpha) \xrightarrow{(\varphi,\psi)} (B, J, \Omega_B, \beta)$ be a morphism of axial decomposition algebras that are Miyamoto-closed with respect to $\mathcal{Y}$. If β is injective, then $(A, I, \Omega_A) \xrightarrow{(\varphi,\psi)} (B, J, \Omega_B)$ is Miyamoto-admissable.*

Proof. By Lemma 3.7.1, $\varphi \circ \tau_{i,\chi} = \tau_{i,\chi} \circ \varphi$ for all $i \in I$ and $\chi \in \mathcal{Y}$. For each $i \in I$ and $j \in J$, we write that $a_i := \alpha(i)$ and $b_j := \beta(j)$. Since our axial decomposition algebras are Miyamoto-closed, there exist permutations $\pi_{i,\chi}$ of I (resp. J) such that $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of (A, I, Ω_A, α) (resp. (B, J, Ω_B, β)). Then, for all $i, j \in I$ and $\chi \in \mathcal{Y}$, we have

$$b_{\psi(\pi_{i,\chi}(j))} = \varphi(a_{\pi_{i,\chi}(j)}) = \varphi_{\tau_{i,\chi}(a_j)} = \tau_{\psi(i),\chi}(\varphi(a_j)) = \tau_{\psi(i),\chi}(b_{\psi(j)}) = b_{\pi_{\psi(i),\chi}(\psi(j))},$$

and because β is assumed to be injective, we get

$$\psi(\pi_{i,\chi}(j)) = \pi_{\psi(i),\chi}(\psi(j)).$$

This proves the statement. □

Theorem 3.7.10. *Let $\mathcal{F}$ be a Γ-graded fusion law, $\mathcal{Y}$ a group of linear R-characters of Γ. Let $(A, I, \Omega_A, \alpha) \xrightarrow{(\varphi,\psi)} (B, J, \Omega_B, \beta)$ be any morphism of axial $\mathcal{F}$-decomposition algebras that are Miyamoto-closed, of unique type with respect to $\mathcal{Y}$ and with β injective. Then there exists a corresponding group homomorphism defined by*

$$\theta\colon \widehat{\mathsf{Miy}}(A, I, \Omega_A, \alpha) \to \widehat{\mathsf{Miy}}(B, J, \Omega_B, \beta)\colon t_{i,\chi} \mapsto t_{\psi(i),\chi}$$

for all $i \in I$ and $\chi \in \mathcal{Y}$.

Proof. This follows immediately from Lemma 3.7.9 and Theorem 3.7.8. □

Example 3.7.11. The previous theorem is false for the ordinary Miyamoto group $\mathsf{Miy}_{\mathcal{Y}}$, as we now illustrate. Let $n \geq 3$ be odd and consider the matrix algebra $M_n(k)$ of all $n \times n$-matrices over a field k with $\mathrm{char}(k) \neq 2$. Let $J_n := M_n(k)^+$ be the corresponding Jordan algebra; this is the commutative non-associative algebra with multiplication $A \bullet B := \frac{1}{2}(AB + BA)$.

Let E_n be the set of all primitive idempotents of J_n. These are the matrices that are diagonalizable with eigenvalues 0, with multiplicity $n-1$, and 1, with multiplicity 1. By Proposition 2.1.3, each idempotent e in a Jordan algebra J gives rise to a decomposition of J into *Peirce subspaces*, the eigenspaces of ad_e with eigenvalues 0, $\frac{1}{2}$ and 1, and moreover, this decomposition satisfies the Jordan fusion law from Example 2.4.5 (see Example 2.5.3). In the case of J_n and $e \in E_n$, these eigenspaces have dimension $(n-1)^2$, $2(n-1)$ and 1, respectively. This gives J_n the structure of a primitive axial decomposition algebra $(J_n, E_n, \Omega, \mathsf{id})$ admitting a $\mathbb{Z}/2\mathbb{Z}$-grading; it is clearly of unique type.

For each $e \in E_n$, the corresponding Miyamoto map τ_e is precisely the conjugation action of $2e-1$ on J_n; since n is odd, $2e-1 \in \mathsf{SL}_n(k)$. Hence the Miyamoto group $G = \mathsf{Miy}(J_n, E_n, \Omega)$ is isomorphic to the group generated by the elements $[2e-1] \in \mathsf{PSL}_n(k) \leq \mathsf{Aut}(J_n)$ for $e \in E_n$. Since G is a non-trivial normal subgroup of $\mathsf{PSL}_n(k)$, it is isomorphic to $\mathsf{PSL}_n(k)$ itself.

Now consider the algebra morphism

$$\varphi\colon J_n \to J_{n+2}\colon A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

and the map $\psi\colon E_n \to E_{n+2}$ given by restriction of φ to E_n. Then the pair (φ, ψ) is a morphism of axial decomposition algebras. However, the map $\tau_e \mapsto \tau_{\psi(e)}$ does not extend to a group homomorphism from $\mathsf{PSL}_n(k)$ to $\mathsf{PSL}_{n+2}(k)$: the product of the Miyamoto maps corresponding to the primitive idempotents $E_{11}, \dots, E_{nn}$ (where E_{ij} is the matrix that is zero everywhere except at position (i,j) where it has entry 1) is trivial in $\mathsf{PSL}_n(k)$, but the product of their images under ψ is equal to the element $[\mathrm{diag}(1, 1, \dots, 1, -1, -1)] \in \mathsf{PSL}_{n+2}(k)$.

Notice that, in contrast, the *universal* Miyamoto group always has a quotient isomorphic to $\mathsf{SL}_n(k)$. (Determining the precise structure of the universal Miyamoto group seems to be a challenging problem.)

4

Modules over (axial) decomposition algebras

In this chapter we present a natural definition of modules over (axial) decomposition algebras. If the fusion law of the decomposition algebra is graded, then we will show that sufficiently nice modules naturally lead to representations of the universal Miyamoto group; see Theorem 4.1.11. In this sense, our definition of modules over (axial) decomposition algebras reinforces the connection between the algebras and their Miyamoto groups.

Firstly, we direct our attention to modules over decomposition algebras. Next, we look at axial decomposition algebras. It turns out that in the latter case modules are even better behaved. As an illustration of this theory, we will discuss modules over Matsuo algebras. The specific nature of these algebras allows us to build a module for it, starting from a representation of its corresponding 3-transposition group.

The content of this chapter is based on [DMVC20a] but is now generalized to (axial) decomposition algebras.

4.1 Modules over decomposition algebras

The definition of a decomposition algebra leads to the following natural definition of modules. For this section, let $\mathcal{F}$ be a fusion law and R a commutative ring with identity.

Definition 4.1.1. Let $\mathcal{A} = (A, I, \Omega)$ be an $\mathcal{F}$-decomposition algebra.

(i) An $\mathcal{A}$-module is a pair (M, Σ) where M is an R-module equipped with a (left) R-bilinear action of A:

$$A \times M \to M \colon (a, m) \mapsto a \cdot m.$$

Moreover, Σ is an I-tuple of decompositions $M = \bigoplus_{x\in\mathcal{F}} M_x^i$ of R-modules such that

$$A_x^i \cdot M_y^i \subseteq M_{x\star y}^i := \bigoplus_{z\in x\star y} M_z^i,$$

for all $x, y \in \mathcal{F}$ and $i \in I$.

(ii) The collection of all $\mathcal{A}$-modules forms a category $\mathsf{Mod}_{\mathcal{A}}$. A morphism between two $\mathcal{A}$-modules (M, Ω_M) and (N, Ω_N) is an R-module homomorphism $\varphi\colon M \to N$ such that the following two conditions hold.

- The morphism φ must preserve the action of A, this is

$$\varphi(a \cdot m) = a \cdot \varphi(m)$$

  for all $a \in A$ and $m \in M$.
- For each $x \in \mathcal{F}$ and $i \in I$, we have $\varphi(M_x^i) \subseteq N_x^i$.

Let us give a few easy examples.

Example 4.1.2. (i) Observe that the multiplication on A turns (A, Ω) into an $\mathcal{A}$-module. We refer to it as the *adjoint module* for $\mathcal{A}$.

(ii) Let M be an arbitrary R-module. Equip M with the trivial A-action, this is $A\cdot M = 0$. Then, for every I-tuple Ω of decompositions of M, the pair (M, Ω) is an $\mathcal{A}$-module. This example illustrates that this definition might be too general to be interesting. Therefore we will, later on, put extra restrictions on our modules. See e.g. Definition 4.1.9 or Definition 4.2.1.

We also have a natural definition of submodules of $\mathcal{A}$-modules.

Definition 4.1.3. Let (M, Σ) be an $\mathcal{A}$-module. A submodule of (M, Σ) is an R-submodule N of M such that

- the submodule N is invariant under the action of A, i.e. $A \cdot N \subseteq N$, and
- for every $i \in I$, we have that $N = \bigoplus_{x\in\mathcal{F}} N \cap M_x^i$ is a decomposition of N.

Write $N_x^i = N \cap M_x^i$. Then $\Big(N, \big((N_x^i)_{x\in\mathcal{F}} \mid i \in I\big)\Big)$ is an $\mathcal{A}$-module.

We can also extend the definition of Frobenius forms to arbitrary modules over an $\mathcal{F}$-decomposition algebra.

Definition 4.1.4. Let $\mathcal{A} = (A, I, \Omega)$ be a decomposition algebra over a field k. A *Frobenius $\mathcal{A}$-module* is an $\mathcal{A}$-module (M, Σ) equipped with a non-degenerate bilinear form, called a *Frobenius form*,

$$\langle\ ,\ \rangle\colon M \times M \to k\colon (m, n) \mapsto \langle m, n\rangle$$

such that $\langle a \cdot m, n\rangle = \langle m, a \cdot n\rangle$ for all $m, n \in M$ and $a \in A$.

Example 4.1.5. If (A, I, Ω) is a commutative decomposition k-algebra for which there exists a Frobenius form, then its adjoint module (A, Ω) is a Frobenius $\mathcal{A}$-module.

The following proposition is reminiscent of Maschke's theorem for linear representations of finite groups; see Proposition 1.4.4.

Proposition 4.1.6. *Let (M, Σ) be a finite-dimensional Frobenius $\mathcal{A}$-module such that $\langle M^i_x, M^i_y\rangle = 0$ for all $i \in I$ and $x \neq y$. Let N be an $\mathcal{A}$-submodule of (M, Σ) such that the restriction of the Frobenius form on N is non-degenerate. Then there exists an $\mathcal{A}$-submodule N_0 of M such that $M = N \oplus N_0$.*

Proof. Let $N_0 = \{m \in M \mid \langle m, n\rangle = 0 \text{ for all } n \in N\}$. Then for each $a \in A$, $n_0 \in N_0$ and $n \in N$, we have $\langle a \cdot n_0, n\rangle = \langle n_0, a \cdot n\rangle = 0$ since $a \cdot n \in N$. Hence $a \cdot n_0 \in N_0$. Since we require the Frobenius form to be non-degenerate on N and $\langle M^i_x, M^i_y\rangle = 0$ for all $i \in I$ and $x \neq y$, it follows that the Frobenius form is non-degenerate on each N^i_x. Therefore, it follows from the properties of orthogonal complements in finite-dimensional vector spaces that $M^i_x = N^i_x \oplus (M^i_x \cap N_0)$. Hence N_0 is a submodule of (M, Ω) and $M = N \oplus N_0$. □

From now on, we will assume that our fusion law $\mathcal{F}$ has a grading $\xi \colon \mathcal{F} \to \Gamma$ and that $\mathcal{Y}$ is a group of linear R-characters of Γ. In the remainder of this section, we will study the connection between representations of the universal Miyamoto group and modules over decomposition algebras. Let us first observe that the definition of Miyamoto maps naturally transfers to modules.

Definition 4.1.7. Let $\mathcal{A} = (A, I, \Omega)$ be an $\mathcal{F}$-decomposition algebra and (M, Σ) an $\mathcal{A}$-module. For any $i \in I$ and any $\chi \in \mathcal{Y}$, we can define a Miyamoto map

$$\mu_{i,\chi} \colon M \to M \colon m \mapsto \chi(g)m \quad \text{for all } m \in M^i_{\xi^{-1}(g)} \text{ and } g \in \Gamma$$

where $\xi^{-1}(g) := \{x \in \mathcal{F} \mid \xi(x) = g\}$.

The Miyamoto maps of $\mathcal{A}$ and its module are compatible in the following sense.

Proposition 4.1.8. *We have*

$$\mu_{i,\chi}(a \cdot m) = \tau_{i,\chi}(a) \cdot \mu_{i,\chi}(m)$$

for all $a \in A$ and $m \in M$.

Proof. Let $g, h \in \Gamma$, $a \in A^i_{\xi^{-1}(g)}$ and $m \in M^i_{\xi^{-1}(h)}$. Since ξ is a grading, we have $\xi^{-1}(g) \star \xi^{-1}(h) \subseteq \xi^{-1}(gh)$. Since (M, Σ) is an $\mathcal{A}$-module, we have $a \cdot m \in M_{\xi^{-1}(gh)}$. Because $\chi \colon \Gamma \to R^\times$ is a group homomorphism, we have

$$\mu_{i,\chi}(a \cdot m) = \chi(gh)(a \cdot m) = \chi(g)a \cdot \chi(h)m = \tau_{i,\chi}(a) \cdot \mu_{i,\chi}(m).$$

Since $A = \bigoplus_{g \in \Gamma} A^i_{\xi^{-1}(g)}$ and $M = \bigoplus_{g \in \Gamma} M^i_{\xi^{-1}(g)}$, the result follows. □

Due to Example 4.1.2 (ii), it is easy to come up with examples where the Miyamoto maps do not preserve the decompositions of the module (M, Σ). In the same way as for decomposition algebras themselves, we introduce the concept of Miyamoto-admissable modules.

Definition 4.1.9. Let $\mathcal{A} = (A, I, \Omega)$ be an $\mathcal{F}$-decomposition algebra that is Miyamoto-closed with respect to $\mathcal{Y}$. We call an $\mathcal{A}$-module (M, Σ) *Miyamoto-admissable* with respect to $\mathcal{Y}$ if, for each $i \in I$ and $\chi \in \mathcal{Y}$, there exists a permutation $\pi_{i,\chi}$ of I such that

- $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of the decomposition algebra $\mathcal{A}$,
- $\mu_{i,\chi}\big(M_x^j\big) = M_x^{\pi_{i,\chi}(j)}$ for all $x \in \mathcal{F}$, $\chi \in \mathcal{Y}$ and $i, j \in I$.

In general $\mu_{i,\chi}$ is not a homomorphism of (M, Σ) since it does not preserve the action of $\mathcal{A}$; see Proposition 4.1.8.

If an $\mathcal{A}$-module is Miyamoto-admissable, then the conjugate of a Miyamoto map by a Miyamoto map is a Miyamoto map.

Proposition 4.1.10. *If (M, Σ) is Miyamoto-admissable then ${}^{\mu_{i,\chi}}\mu_{j,\chi'} = \mu_{\pi_{i,\chi}(j),\chi'}$.*

Proof. Let $m \in M_x^j$ and $g := \xi(x)$. Then $\mu_{i,\chi}(\mu_{j,\chi'}(m)) = \chi'(g)\mu_{i,\chi}(m)$. Since (M, Σ) is Miyamoto-admissable, there exists a permutation $\pi_{i,\chi}$ of I such that $\mu_{i,\chi}(m) \in M_x^{\pi_{i,\chi}(j)}$. Hence

$$\mu_{\pi_{i,\chi}(j),\chi'}(\mu_{i,\chi}(m)) = \chi'(g)\mu_{i,\chi}(m).$$

Since $M = \bigoplus_{x \in \mathcal{F}} M_x^j$, this proves the assertion. $\square$

If an $\mathcal{A}$-module is Miyamoto-admissable, then it induces a linear representation of the universal Miyamoto group of $\mathcal{A}$. As in Section 3.7, we need to restrict to decomposition algebras that are of unique type.

Theorem 4.1.11. *Let $\mathcal{F}$ be a Γ-graded fusion law and $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$. Suppose that $\mathcal{A} = (A, I, \Omega)$ is an $\mathcal{F}$-decomposition algebra that is Miyamoto-closed and of unique type with respect to $\mathcal{Y}$ and let (M, Σ) be an $\mathcal{A}$-module. Write $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(\mathcal{A})$ for the universal Miyamoto group of $\mathcal{A}$ (see Definition 3.6.1). Denote the Miyamoto maps of (M, Σ) by $\mu_{i,\chi}$ for all $i \in I$ and $\chi \in \mathcal{Y}$. If (M, Σ) is Miyamoto-admissable, then the map defined by*

$$\mu\colon\ \widehat{\mathsf{Miy}}_{\mathcal{Y}}(\mathcal{A}) \to \mathsf{GL}(M)\colon t_{i,\chi} \mapsto \mu_{i,\chi}$$

for all $i \in I$ and $\chi \in \mathcal{Y}$ is a group homomorphism.

Proof. Since (M, Σ) is Miyamoto-admissable we can pick a permutation $\pi_{i,\chi}$ as in Definition 4.1.9. Then, for all $x \in \mathcal{F}$, $\chi \in \mathcal{Y}$ and $i, j \in I$, we have

$$\mu_{i,\chi}\big(M_x^j\big) = M_x^{\pi_{i,\chi}(j)}. \tag{4.1}$$

We need to show that every relator of $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(\mathcal{A})$ is killed by μ. Consider such a relator:

$$r = t_{k,\chi'}^{-1} \cdot {}^{t_{i,\chi}}t_{j,\chi'} \quad \text{with} \quad (j,k) \in R_{\tau_{i,\chi}}.$$

Then we have ${}^{\tau_{i,\chi}}\tau_{j,\chi'} = \tau_{k,\chi'}$ for all $\chi' \in \mathcal{Y}$. Moreover ${}^{\tau_{i,\chi}}\tau_{j,\chi'} = \tau_{\pi_{i,\chi}(j),\chi'}$ by Proposition 3.4.2. Because (A, I, Ω) is assumed to be of unique type with respect to $\mathcal{Y}$, this can only happen if $k = \pi_{i,\chi}(j)$. Thus we need to show that

$${}^{\mu_{i,\chi}}\mu_{j,\chi'} = \mu_{\pi_{i,\chi}(j),\chi'}$$

for all $i, j \in I$ and $\chi, \chi' \in \mathcal{Y}$. This follows immediately from Proposition 4.1.10. $\square$

4.2 Modules over axial decomposition algebras

In this section we define modules over axial decomposition algebras. Although this definition still allows for interesting examples, it is more restrictive than just a module over its underlying decomposition algebra.

In this section we assume that $\mathcal{F}$ is a fusion law with a distinguished unit.

Definition 4.2.1. Let $\mathcal{A} = (A, I, \Omega, \alpha)$ be an axial $\mathcal{F}$-decomposition algebra with evaluation map $\lambda\colon \mathcal{F} \to R$. Write $\mathcal{A}' = (A, I, \Omega)$ for its underlying decomposition algebra.

(i) An $\mathcal{A}$-module is an $\mathcal{A}'$-module (M, Σ) such that

$$\alpha(i) \cdot m = \lambda(x)m \tag{4.2}$$

for all $m \in M_x^i$, $x \in \mathcal{F}$ and $i \in I$.

(ii) A map $\varphi\colon M \to N$ for $\mathcal{A}$-modules (M, Ω_M) and (N, Ω_N) is a morphism of $\mathcal{A}$-modules if it is a morphism of $\mathcal{A}'$-modules. The collection $\mathsf{Mod}_{\mathcal{A}}$ of all $\mathcal{A}$-modules is therefore a full subcategory of the category of $\mathcal{A}'$-modules.

(iii) A submodule of an $\mathcal{A}$-module (M, Σ) is defined to be a submodule of (M, Σ) as $\mathcal{A}'$-module. Note that this is automatically an $\mathcal{A}$-module since it must satisfy (4.2).

(iv) Similarly, we define a Frobenius $\mathcal{A}$-module as a Frobenius $\mathcal{A}'$-module that satisfies the extra condition (4.2).

Proposition 4.1.6 gets the following alternative form for modules over axial decomposition algebras.

Proposition 4.2.2. *Suppose that $\mathcal{A} = (A, I, \Omega_A, \alpha)$ is an axial $\mathcal{F}$-decomposition algebra over a field k. Assume that its evaluation map $\lambda\colon \mathcal{F} \to k$ is injective. Let (M, Ω_M) be a Frobenius $\mathcal{A}$-module and N a submodule on which the Frobenius form is non-degenerate. Then there exists an $\mathcal{A}$-submodule N_0 of M such that $M = N \oplus N_0$.*

Proof. Let $m \in M_x^i$ and $n \in M_y^i$, then

$$\lambda(x)\langle m, n\rangle = \langle \alpha(i) \cdot m, n\rangle = \langle m, \alpha(i) \cdot n\rangle = \lambda(y)\langle m, n\rangle.$$

Since $\lambda(x) \neq \lambda(y)$ if $x \neq y$, it follows that $\langle M_x^i, M_y^i\rangle = 0$ if $x \neq y$ for all $i \in I$. The assertion now follows from Proposition 4.1.6. □

In Lemma 4.2.6, we want to be able to recover the decomposition $\bigoplus_{x\in\mathcal{F}} M_x^i$ from the action of $\alpha(i)$ on M. To this end, we introduce the following definition.

Definition 4.2.3. Let $\mathcal{A} = (A, I, \Omega, \alpha)$ be an axial $\mathcal{F}$-decomposition algebra with evaluation map $\lambda\colon \mathcal{F} \to R$. We call an $\mathcal{A}$-module *of axial type* if, for all $x \in \mathcal{F}$ and $i \in I$, we have that $M_x^i = \{m \in M \mid \alpha(i) \cdot m = \lambda(x)m\}$. This definition reminds of the original definition of an axial algebra from Section 2.3.

The following proposition gives a sufficient condition for a module to be of axial type.

Proposition 4.2.4. *Suppose (M, Σ) is an $\mathcal{A}$-module such that for all $x, y \in \mathcal{F}$ and $m \in M$, we have that $(\lambda(x) - \lambda(y))m = 0$ if and only if $x = y$ or $m = 0$. Then (M, Σ) is of axial type.*

Proof. We have to prove that if $\alpha(i) \cdot m = \lambda(x)m$ for $i \in I$ and $x \in \mathcal{F}$ then $m \in M_x^i$. Since $M = \bigoplus_{x\in\mathcal{F}} M_x^i$ we can write $m = \sum_{y\in\mathcal{F}} m_y$ where $m_y \in M_y^i$. Then $\alpha(i) \cdot m = \sum_{y\in\mathcal{F}} \lambda(y)m_y$. Once again, because $M = \bigoplus_{x\in\mathcal{F}} M_x^i$, we have $(\lambda(x) - \lambda(y))m_y = 0$ for all $y \in \mathcal{F}$. Our condition on (M, Σ) implies that $m_y = 0$ for all $y \neq x$. We conclude that $m = m_x \in M_x^i$. □

Example 4.2.5. If $\mathcal{A}$ is an axial decomposition algebra over a field, with an injective evaluation map, then any $\mathcal{A}$-module is of axial type.

Let us now assume that the fusion law $\mathcal{F}$ is graded by a group Γ and let $\mathcal{Y}$ be a group of linear R-characters of Γ. Since any $\mathcal{A}$-module is in particular a module for its underlying decomposition algebra, we can consider its Miyamoto group. It turns out that any $\mathcal{A}$-module of axial type is automatically Miyamoto-admissable.

Lemma 4.2.6. *Let $\mathcal{A} = (A, I, \Omega, \alpha)$ be an axial $\mathcal{F}$-decomposition algebra. Assume that $\mathcal{A}$ is Miyamoto-closed with respect to $\mathcal{Y}$. Then any $\mathcal{A}$-module (M, Σ) of axial type is Miyamoto-admissable with respect to $\mathcal{Y}$ when viewed as a module for (A, I, Ω).*

Proof. Since $\mathcal{A}$ is Miyamoto-closed, for each $i \in I$ and $\chi \in \mathcal{Y}$, there exists a permutation $\pi_{i,\chi}$ of I such that $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of $\mathcal{A}$. Let (M, Σ) be an $\mathcal{A}$-module and let $\mu_{i,\chi}$ be a Miyamoto map of (M, Σ). We show that

$$\mu_{i,\chi}(M_x^j) \subseteq M_x^{\pi_{i,\chi}(j)}$$

for all $j \in I$ and $x \in \mathcal{F}$, which proves the assertion. Since $(\tau_{i,\chi}, \pi_{i,\chi})$ is an automorphism of $\mathcal{A}$, we have in particular that $\tau_{i,\chi}(\alpha(j)) = \alpha(\pi_{i,\chi}(j))$. For any $m \in M_x^j$ we now have

$$\begin{aligned}\alpha(\pi_{i,\chi}(j)) \cdot \mu_{i,\chi}(m) &= \tau_{i,\chi}(\alpha(j)) \cdot \mu_{i,\chi}(m)\\ &= \mu_{i,\chi}(\alpha(j) \cdot m)\\ &= \lambda(x)\mu_{i,\chi}(m)\end{aligned}$$

by Proposition 4.1.8. Because our module (M, Σ) is of axial type, we conclude that $\mu_{i,\chi}(m) \in M_x^{\pi_{i,\chi}(j)}$. □

Together with Theorem 4.1.11 we obtain the following result.

Theorem 4.2.7. *Let $\mathcal{F}$ be a Γ-graded fusion law with a distinguished unit and let $\mathcal{Y} \leq \mathcal{X}_R(\Gamma)$. Let $\mathcal{A} = (A, I, \Omega, \alpha)$ be an axial $\mathcal{F}$-decomposition algebra that is Miyamoto-closed with respect to $\mathcal{Y}$ and (M, Σ) an $\mathcal{A}$-module of axial type. Let $\widehat{\mathsf{Miy}}_{\mathcal{Y}}(\mathcal{A})$ be the universal Miyamoto group of $\mathcal{A}$ from Definition* 3.6.1*. Denote the Miyamoto maps of (M, Σ) by $\mu_{i,\chi}$ for all $i \in I$ and $\chi \in \mathcal{Y}$. Then the map defined by*

$$\mu \colon \widehat{\mathsf{Miy}}_{\mathcal{Y}}(\mathcal{A}) \to \mathsf{GL}(M) \colon t_{i,\chi} \mapsto \mu_{i,\chi}$$

for all $i \in I$ and $\chi \in \mathcal{Y}$ is a group homomorphism.

Proof. This follows immediately from Lemma 4.2.6 and Theorem 4.1.11. □

4.3 Modules over Matsuo algebras

In this section, we direct our attention to modules over Matsuo algebras. If $\mathcal{G}$ is a Fischer space then, by Corollary 3.5.10, we can view $M_\alpha(\mathcal{G})$ as an axial decomposition algebra for the Jordan fusion law $\mathcal{F}$. If we assume that $\alpha \neq 0, 1$ then any $M_\alpha(\mathcal{G})$-module is of axial type and leads to a linear representation of the universal Miyamoto group $\widehat{\mathsf{Miy}}_{\chi}(M_\alpha(\mathcal{G}))$ from Theorem 3.6.5 by Theorem 4.2.7. We will prove that the converse is also true. In Definition 4.3.2 we construct a module for $M_\alpha(\mathcal{G})$ from such a representation. First, we recall some terminology and introduce some notation.

Notation 4.3.1. (i) Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a Fischer space, k a field with $\mathrm{char}(k) \neq 2$ and $\alpha \in k \setminus \{0, 1\}$. Consider the Matsuo algebra $A = M_\alpha(\mathcal{G})$ over k. By Corollary 3.5.10 this is an axial decomposition algebra for the Jordan fusion law $\mathcal{F} = (\{e, z, h\}, \star)$ from Example 2.4.5. Its evaluation map $\lambda\colon \mathcal{F} \to k$ is defined by $\lambda(e) = 1$, $\lambda(z) = 0$ and $\lambda(h) = \alpha$. Since we assume that $\alpha \neq 0, 1$ we can identify $\mathcal{F}$ with its image under λ and we will simply write A_1^x, A_0^x and A_α^x instead of A_e^x, A_0^x and A_h^x. In fact, this means that we view A as an axial algebra in the sense of Section 2.3. The decompositions of A, as well as the decompositions of any A-module, will be decompositions into eigenspaces. In particular, any A-module will be of axial type.

(ii) Recall that the fusion law of A is $\mathbb{Z}/2\mathbb{Z}$-graded. Since $\mathbb{Z}/2\mathbb{Z}$ only has one non-trivial linear k-character χ we simply denote Miyamoto maps $\tau_{x,\chi}$, $t_{x,\chi}$ and $\mu_{x,\chi}$ by τ_x, t_x and μ_x. Due to Proposition 3.5.13 (i) this should never lead to any confusion. Recall from Proposition 3.5.13 (i) and Proposition 3.4.2 that ${}^{\tau_x}\tau_y = \tau_{\tau_x(y)}$. By definition of the universal Miyamoto group, see Definition 3.6.1, we have that $\widehat{\mathsf{Miy}}(A)$ is the group with presentation

$$\left\langle t_x \text{ for each } x \in \mathcal{P} \;\middle|\; \begin{array}{ll} (t_x)^2 & \text{for all } x \in \mathcal{P} \\ {}^{t_x}t_y = t_{\tau_x(y)} & \text{for all } x, y \in \mathcal{P} \end{array} \right\rangle.$$

Definition 4.3.2. Now consider a k-vector space V and a group homomorphism

$$\rho\colon \widehat{\mathsf{Miy}}(A) \to \mathsf{GL}(V).$$

Since $\rho(t_x)^2 = 1$ for every $x \in \mathcal{P}$ and $\operatorname{char}(k) \neq 2$, we can decompose V as a direct sum of the 1- and (-1)-eigenspace of $\rho(t_x)$. Let V_0^x (resp. V_α^x) be the 1-eigenspace (resp. (-1)-eigenspace) of $\rho(t_x)$ and let $V_1^x = 0$. Then $V = V_1^x \oplus V_0^x \oplus V_\alpha^x$ for every $x \in \mathcal{P}$. Let Ω_V be the $\mathcal{P}$-tuple of these decompositions.

Also define an action of $M_\alpha(\mathcal{G})$ on V by linearly extending

$$x \cdot v = \begin{cases} 0 & \text{if } \rho(t_x)(v) = v, \\ \alpha v & \text{if } \rho(t_x)(v) = -v, \end{cases}$$

for each $x \in \mathcal{P}$.

Proposition 4.3.3. *Let $V_\rho := (V, \Omega_V)$ be as in Definition 4.3.2 for a representation ρ of $\widehat{\mathsf{Miy}}(A)$. Then V_ρ is an $M_\alpha(\mathcal{G})$-module.*

We start by mimicking Proposition 4.1.8 with μ_x replaced by $\rho(t_x)$.

Lemma 4.3.4. *For every $v \in V$, $a \in M_\alpha(\mathcal{G})$ and $x \in \mathcal{P}$, we have $\rho(t_x)(a \cdot v) = \tau_x(a) \cdot \rho(t_x)(v)$.*

Proof. Since $M_\alpha(\mathcal{G})$ is spanned by the elements of $\mathcal{P}$ and the action of $M_\alpha(\mathcal{G})$ is linear by definition, we may assume that $a = y \in \mathcal{P}$. Since V is decomposable into a 1- and (-1)-eigenspace of $\rho(t_x)$, it suffices to consider the cases where v is a 1- or (-1)-eigenvector of $\rho(t_x)$. If $\rho(t_y)(v) = v$ then $y \cdot v = 0$ and $\rho(t_x)(y \cdot v) = 0$. Recall from Proposition 3.5.13 (ii) that ${}^{\tau_x}\tau_y = \tau_{\tau_x(y)}$ and therefore also ${}^{t_x}t_y = t_{\tau_x(y)}$. Now we have

$$\rho(t_{\tau_x(y)})\big(\rho(t_x)(v)\big) = \big(\rho(t_x t_y t_x)\rho(t_x)\big)(v) = \rho(t_x)(v)$$

and hence $\tau_x(y) \cdot \rho(t_x)(v) = 0$. In the second case, $\rho(t_y)(v) = -v$ and therefore $y \cdot v = \alpha v$ and $\rho(t_x)(y \cdot v) = \alpha\rho(t_x)(v)$. Now

$$\rho(t_{\tau_x(y)})\big(\rho(t_x)(v)\big) = \big(\rho(t_x t_y t_x)\rho(t_x)\big)(v) = -\rho(t_x)(v)$$

and thus $\tau_x(y)\cdot\rho(t_x)(v) = \alpha\rho(t_x)(v)$. In both cases, $\rho(t_x)(y\cdot v) = \tau_x(y)\cdot\rho(t_x)(v)$. $\square$

Proof of Proposition 4.3.3. Let $x \in \mathcal{P}$. We already know that $V = V_1^x \oplus V_0^x \oplus V_\alpha^x$. It only remains to verify that the fusion law is satisfied. Let $A = M_\alpha(\mathcal{G})$. For $a \in A^x_{\{1,0\}}$ and $v \in V_0^x$ (resp. $a \in A^x_\alpha$ and $v \in V^x_\alpha$) we have $\rho(t_x)(v) = v$ (resp. $\rho(t_x)(v) = -v$) and $\tau_x(a) = a$ (resp. $\tau_x(a) = -a$). By Lemma 4.3.4,

$$\rho(t_x)(a \cdot v) = \tau_x(a) \cdot \rho(t_x)(v) = a \cdot v$$

and thus $a \cdot v$ is a 1-eigenvector of $\rho(t_x)$. Therefore $a \cdot v$ belongs to V_0^x. If $a \in A^x_\alpha$ and $v \in V_0^x$ (resp. $a \in A^x_{\{1,0\}}$ and $v \in V^x_\alpha$), then $\tau_x(a) = -a$ (resp. $\tau_x(a) = a$) and $\rho(t_x)(v) = v$ (resp. $\rho(t_x)(v) = -v$). In both cases, by Lemma 4.3.4,

$$\rho(t_x)(a \cdot v) = \tau_x(a) \cdot \rho(t_x)(v) = -a \cdot v.$$

Therefore $a \cdot v$ is a 1-eigenvector of $\rho(t_x)$ and hence $a \cdot v \in V^x_\alpha$. $\square$

Proposition 4.3.3 has the important consequence that for a Matsuo algebra A, the study of representations for the group $\widehat{\mathsf{Miy}}(A)$ is equivalent to the study of certain A-modules, as we now point out. Recall from Definition 4.2.1 that Mod_A is the category of all A-modules, for any axial decompostion algebra A.

Corollary 4.3.5. *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a Fischer space, let A be its Matsuo algebra and let $\widehat{\mathsf{Miy}}(A)$ be its universal Miyamoto group as in Notation* 4.3.1. *Let $\mathsf{Mod}_A^{(1)}$ be the full subcategory of Mod_A of A-modules (M, Σ) for which $M_1^x = \{0\}$ for all $x \in \mathcal{P}$. Then the category $\mathsf{Mod}_A^{(1)}$ is equivalent to the category $\mathsf{Rep}(\widehat{\mathsf{Miy}}(A))$ of $\widehat{\mathsf{Miy}}(A)$-representations.*

Proof. Using Proposition 4.3.3, we can associate to each linear representation ρ of $\widehat{\mathsf{Miy}}(A)$ an A-module V_ρ with $(V_\rho)_1^x = \{0\}$ for all x. We first observe that the resulting map $F\colon \rho \rightsquigarrow V_\rho$ is a functor. Indeed, if $\theta\colon \rho \to \rho'$ is a homomorphism of $\widehat{\mathsf{Miy}}(A)$-representations, then, for each $x \in \mathcal{P}$, θ maps $\rho(t_x)$-eigenspaces to $\rho'(t_x)$-eigenspaces. Hence θ is a homomorphism of the corresponding A-modules. It is now also clear that for given $\widehat{\mathsf{Miy}}(A)$-representations ρ, ρ', the functor F induces a bijection between the morphisms from ρ to ρ' and the morphisms from M_ρ to M'_ρ; hence F is a fully faithful functor.

To show that the functor F is essentially surjective, let (V, Ω_V) be an arbitrary A-module with $V_1^x = \{0\}$ for all $x \in \mathcal{P}$. By Theorem 4.2.7, there is an associated linear representation ρ of the group $\widehat{\mathsf{Miy}}(A)$; we claim that $V_\rho \cong (V, \Omega_V)$. Indeed, since $\rho(t_x) = \mu_x$ and $M_1^x = \{0\}$, we see that $x \cdot m = 0$ (resp. $x \cdot m = \alpha m$) if and only if $\rho(t_x)(v) = v$ (resp. $\rho(t_x)(v) = -v$) for all $v \in V$ and $x \in \mathcal{P}$. By definition of V_ρ, this proves the claim.

This shows that the functor F is an equivalence. □

More generally, given any module M over the Matsuo algebra $M_\alpha(\mathcal{G})$, Theorem 4.2.7 gives us a representation ρ of the group $\widehat{\mathsf{Miy}}(M_\alpha(\mathcal{G}))$. The module constructed by Proposition 4.3.3 out of ρ resembles M, with the important difference that all 1-eigenvectors of an axis $x \in \mathcal{P}$ have become 0-eigenvectors. (The action on the 0- and α-eigenvectors remains unchanged.) This leaves of course the question what the role of a 1-eigenvector is inside a module. Since the adjoint module (see Example 4.1.2 (i)) contains non-trivial 1-eigenspaces, we definitely want to allow the existence of 1-eigenspaces in Definition 4.2.1. We will now prove that, under certain conditions, this is the only way a 1-eigenspace can turn up in a module over a Matsuo algebra.

From now on, we would like to restrict to Matsuo algebras over connected Fischer spaces without isolated points. This is not a serious restriction, as the following proposition allows to generalize results to arbitrary Fischer spaces without isolated points by looking at their connected components.

Proposition 4.3.6 ([HRS15a, Theorem 6.2])**.** *Let $\mathcal{G}$ be a Fischer space and let $\{\mathcal{G}_i \mid i \in I\}$ be the set of its connected components. Then $M_\alpha(\mathcal{G}_i) \cong \bigoplus_{i \in I} M_\alpha(\mathcal{G}_i)$.*

The connectedness of a Fischer space has the following implications on its Matsuo algebra.

Lemma 4.3.7. *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a Fischer space and let $M_\alpha(\mathcal{G})$ be its Matsuo algebra. Let $D = \{\tau_x \mid x \in \mathcal{P}\}$ and $G = \mathsf{Miy}(M_\alpha(\mathcal{G})) = \langle D \rangle$. The following statements are equivalent.*

(a) *The Fischer space $\mathcal{G}$ is connected.*

(b) *The action of G on $\mathcal{P}$ is transitive.*

(c) *The set D of Miyamoto maps is a conjugacy class of G.*

(d) *The set $\{t_x \mid x \in \mathcal{P}\}$ is a conjugacy class of $\widehat{\mathsf{Miy}}(M_\alpha(\mathcal{G}))$.*

Proof. Suppose the Fischer space $\mathcal{G}$ is connected. Let $x, y \in \mathcal{P}$. Then there exists a path $x, x_1, \dots, x_n, y$ from x to y. Now

$$(\tau_{y \wedge x_n} \tau_{x_n \wedge x_{n-1}} \cdots \tau_{x_2 \wedge x_1} \tau_{x_1 \wedge x})(x) = y$$

and therefore G acts transitively on $\mathcal{P}$. Conversely we have that, by Proposition 3.5.13 (i), all Miyamoto maps stabilize the connected components of the Fischer space. Therefore, the Fischer space is connected when G acts transitively on $\mathcal{P}$. The equivalence between (b) and (c) follows from ${}^{\tau_x}\tau_y = \tau_{\tau_x(y)}$. By definition of the universal Miyamoto group $\widehat{\mathsf{Miy}}(M_\alpha(\mathcal{G}), \mathcal{P})$, it follows that (c) and (d) are equivalent. □

From now on, let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a connected Fischer space without isolated points and $M_\alpha(\mathcal{G})$ is Matsuo algebra. Let $(G, D) := f(\mathcal{G})$ be the 3-transposition group corresponding to $\mathcal{G}$. Then we can identify G with the Miyamoto group of $M_\alpha(\mathcal{G})$ by Proposition 3.5.13. Let M be an $M_\alpha(\mathcal{G})$-module and suppose $M_1^x \neq \{0\}$ for some (and hence every) axis $x \in \mathcal{P}$. We will prove in Theorem 4.3.10 that, under certain conditions, M contains a submodule that is a quotient of $M_\alpha(\mathcal{G})$ as $M_\alpha(\mathcal{G})$-module. First, we will construct the submodule. We introduce some notation.

Definition 4.3.8. Let M be an $M_\alpha(\mathcal{G})$-module and suppose $M_1^x \neq \{0\}$ for each $x \in \mathcal{P}$.

(i) Let
$$U := \langle \mu_x \mid x \in \mathcal{P} \rangle \leq \mathsf{GL}(M).$$
Denote the universal Miyamoto group $\widehat{\mathsf{Miy}}(M_\alpha(\mathcal{G}))$ by T and consider, as in Proposition 3.6.4 and Theorem 4.2.7, the group epimorphisms defined by
$$\mu : T \to U : t_x \mapsto \mu_x,$$
$$\tau : T \to G : t_x \mapsto \tau_x.$$

(ii) For each $x \in \mathcal{P}$, we define a subgroup $U_x = \mu(C_T(t_x))$ of U, where $C_T(t_x)$ is the centralizer of t_x in T.

(iii) Fix $x \in \mathcal{P}$. Let $m \in M_1^x$ with $m \neq 0$. Let
$$m_x = \sum_{f \in U_x} f(m).$$
Because we assume that $\mathcal{G}$ is connected, the group T acts transitively on the set $\{t_x \mid x \in \mathcal{P}\}$ by conjugation. Thus for every $y \in \mathcal{P}$ there exists a $t \in T$ such that ${}^t t_x = t_y$ and we define
$$m_y = \mu(t)(m_x). \tag{4.3}$$
By Lemma 4.3.9 (iv) below, this definition of m_y is independent of the choice of t.

Lemma 4.3.9. *Let $y \in \mathcal{P}$, $t \in T$ and $\phi \in \{1, 0, \alpha\}$. Suppose $t' \in T$ such that ${}^t t_x = {}^{t'} t_x$. Then the following hold:*

(i) ${}^{t}t_y = t_{\tau(t)(y)}$,

(ii) $\mu(t)(M_\phi^y) = M_\phi^{\tau(t)(y)}$,

(iii) $\mu(t)U_x = \mu(t')U_x$,

(iv) $\mu(t)(m_x) = \mu(t')(m_x)$,

(v) $m_y \in M_1^y$,

(vi) $\mu(t)(m_y) = m_{\tau(t)(y)}$.

Proof. (i) This follows from Lemma 3.6.3 (i) and the fact that (τ_x, τ_x) is an automorphism of $M_\alpha(\mathcal{G})$ for every $x \in \mathcal{P}$.

(ii) This follows from Lemma 4.2.6 and because μ and τ are group homomorphisms.

(iii) Since $tC_T(t_x) = t'C_T(t_x)$ whenever ${}^{t}t_x = {}^{t'}t_x$, also $\mu(t)U_x = \mu(t')U_x$.

(iv) By (iii),

$$\mu(t)(m_x) = \sum_{f \in \mu(t)U_x} f(m) = \sum_{f \in \mu(t')U_x} f(m) = \mu(t')(m_x).$$

(v) Let $t \in C_T(t_x)$. Since $t_x = {}^{t}t_x = t_{\tau(t)(x)}$, we also have $\tau_x = \tau(t_x) = \tau(t_{\tau(t)(x)}) = \tau_{\tau(t)(x)}$. Because $M_\alpha(\mathcal{G})$ is of unique type (see Example 3.7.6) it follows that $x = \tau(t)(x)$. Therefore $\mu(t)(m) \in \mu(t)(M_1^x) = M_1^{\tau(t)(x)} = M_1^x$. Thus $m_x = \sum_{f \in U_x} f(m) \in M_1^x$. For $y \in \mathcal{P}$, let $t \in T$ such that ${}^{t}t_x = t_{\tau(t)(x)} = t_y$ and thus $\tau(t)(x) = y$. Now, $m_y = \mu(t)(m_x) \in \mu(t)(M_1^x) = M_1^{\tau(t)(x)} = M_1^y$.

(vi) Because ${}^{t'}t_x = t_y$, $\tau(t')(x) = y$. Since ${}^{tt'}t_x = t_{\tau(tt')(x)} = t_{\tau(t)(y)}$, it follows, by (4.3), that $m_{\tau(t)(y)} = \mu(tt')(m_x) = \mu(t)(m_y)$. □

Theorem 4.3.10. *Consider the setting of Definition* 4.3.8*. The subspace $\langle m_y \mid y \in \mathcal{P}\rangle$ is a submodule of M and the map, defined by*

$$M_\alpha(\mathcal{G}) \to \langle m_y \mid y \in \mathcal{P}\rangle : y \mapsto m_y,$$

is an epimorphism of $M_\alpha(\mathcal{G})$-modules. In particular, $\langle m_y \mid y \in \mathcal{P}\rangle$ is a quotient of the adjoint module for $M_\alpha(\mathcal{G})$.

Proof. Since $M_\alpha(\mathcal{G})$ is spanned by $\mathcal{P}$, it suffices to prove that for all axes $y, z \in \mathcal{P}$

$$z \cdot m_z = m_z; \tag{4.4}$$
$$z \cdot m_y = 0 \qquad \text{if } y \nsim z; \tag{4.5}$$
$$z \cdot (m_y + m_{y \wedge z} - \alpha m_z) = 0 \qquad \text{if } y \sim z; \tag{4.6}$$

$$z \cdot (m_y - m_{y\wedge z}) = \alpha(m_y - m_{y\wedge z}) \text{ if } y \sim z. \tag{4.7}$$

Statement (4.4) follows from Lemma 4.3.9 (v). For (4.5), let $z, y \in \mathcal{P}$ and $y \nsim z$. This implies that $z \in (M_\alpha(\mathcal{G}))_0^y$. By Lemma 4.3.9 (v), $m_y \in M_1^y$. From the fusion rule $0 \star 1 = \{0\}$ we infer $z \cdot m_y = 0$. For (4.6) and (4.7), let $z, y \in \mathcal{P}$ and $y \sim z$. Since $\mu_z(m_y - m_{y\wedge z}) = m_{y\wedge z} - m_y$ by Lemma 4.3.9 (vi), (4.7) follows. For (4.6), note that

$$(y + z \wedge y - \alpha z) \cdot m_z = 0$$

since $y + z \wedge y - \alpha z \in (M_\alpha(\mathcal{G}))_0^z$, $m_z \in M_1^z$ and $1 \star 0 = \{0\}$. Interchanging the roles of y, z and $y \wedge z$ in this relation and relation (4.7) gives us the following 5 relations:

$$(z + y \wedge z - \alpha y) \cdot m_y = 0, \tag{4.8}$$
$$(y \wedge z + y - \alpha z) \cdot m_z = 0, \tag{4.9}$$
$$(y + z - \alpha y \wedge z) \cdot m_{y\wedge z} = 0, \tag{4.10}$$
$$y \cdot (m_z - m_{y\wedge z}) - \alpha(m_z - m_{y\wedge z}) = 0, \tag{4.11}$$
$$y \wedge z \cdot (m_y - m_z) - \alpha(m_y - m_z) = 0. \tag{4.12}$$

It is easy to verify, using relation (4.4), that (4.8) − (4.9) + (4.10) + (4.11) − (4.12) gives us exactly (4.6). □

Remark 4.3.11. We do not know whether it is always possible to choose $m \in M_1^x$ such that $m_x \neq 0$.

Theorem 4.3.13 will tell us when the morphism from Theorem 4.3.10 is an isomorphism. The proof makes use of a Frobenius form for Matsuo algebras defined in Lemma 4.3.12 below. The only obstructions are the facts that m_x might be zero or that the Matsuo algebra does not admit a non-degenerate Frobenius form.

Lemma 4.3.12 ([HRS15b, Corollary 7.4])**.** *Let $\mathcal{G}$ be a Fischer space. The Matsuo algebra $M_\alpha(\mathcal{G})$ admits a bilinear form $\langle\ ,\ \rangle$ for which $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in M_\alpha(\mathcal{G})$. When $\mathcal{G}$ is connected, this form is, up to scalar, uniquely determined. It is given by*

$$\langle x, y\rangle = \begin{cases} 1 & \text{if } x = y, \\ \frac{\alpha}{2} & \text{if } x \sim y, \\ 0 & \text{if } x \nsim y, \end{cases}$$

for all $x, y \in \mathcal{P}$.

Theorem 4.3.13. *Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be a connected Fischer space without isolated points. Let (M, Ω_M) be an $M_\alpha(\mathcal{G})$-module, $x \in \mathcal{P}$, $m \in M_1^x \setminus \{0\}$ and define*

$$m_x = \sum_{f \in U_x} f(m)$$

as in Definition 4.3.8. Let $U = \langle \mu_x \mid x \in \mathcal{P}\rangle \leq \mathsf{GL}(M)$. If $m_x \neq 0$ and the form on $M_\alpha(\mathcal{G})$ from Lemma 4.3.12 is non-degenerate, then $\langle f(m_x) \mid f \in U\rangle$ is an adjoint submodule of $M_\alpha(\mathcal{G})$, i.e., it is isomorphic to $M_\alpha(\mathcal{G})$ as $M_\alpha(\mathcal{G})$-module.

Proof. It suffices to prove that the map from Theorem 4.3.10 is injective. Let m_a be the image of $a \in M_\alpha(\mathcal{G})$ under this map. We shall prove that for the stated conditions, $m_a \neq 0$ when $a \neq 0$. Since we require $\mathcal{G}$ to be connected, for every $y \in \mathcal{P}$, there exists an $f \in U$ such that $m_y = f(m_x)$. Because $m_x \neq 0$ and $f \in \mathsf{GL}(M)$, $m_y \neq 0$ for all $y \in \mathcal{P}$.

Suppose that $a \in M_\alpha(\mathcal{G})$ and $m_a = 0$. Let $y \in \mathcal{P}$ be an arbitrary axis and write $a = a_1 + a_0 + a_\alpha$ where $a_\phi \in (M_\alpha(\mathcal{G}))^y_\phi$. Then

$$m_a = m_{a_1} + m_{a_0} + m_{a_\alpha} = 0.$$

Since $0 = y \cdot m_a = m_{y \cdot a}$,

$$m_{a_1} + \alpha m_{a_\alpha} = 0.$$

We also have $0 = \mu_y(m_a) = m_{\tau_y(a)}$ and therefore

$$m_{a_1} + m_{a_0} - m_{a_\alpha} = 0.$$

From these three equations it follows that $m_{a_1} = m_{a_0} = m_{a_\alpha} = 0$.

Since the 1-eigenspace of y in $M_\alpha(\mathcal{G})$ is spanned by y, $a_1 = \lambda y$ for some $\lambda \in k$. Because $m_y \neq 0$, we conclude that a_1 must be zero. As our choice of $y \in \mathcal{P}$ was arbitrary, we infer that for every axis in $\mathcal{P}$ the component of a in its 1-eigenspace is zero.

Let $\langle\ ,\ \rangle$ be the form for $M_\alpha(\mathcal{G})$ as given by Lemma 4.3.12. Since eigenvectors corresponding to different eigenvalues are perpendicular to each other, see Proposition 2.7.1, our previous conclusion is equivalent to

$$\langle a, y \rangle = 0 \text{ for all } y \in \mathcal{P}.$$

The elements of $\mathcal{P}$ span $M_\alpha(\mathcal{G})$ and therefore $\langle a, n \rangle = 0$ for all $n \in M_\alpha(\mathcal{G})$. Because the bilinear form $\langle\ ,\ \rangle$ is non-degenerate, $a = 0$. □

Remark 4.3.14. (i) Let $\mathcal{G}$ be a connected Fischer space without isolated points and let M be a finite-dimensional Frobenius $M_\alpha(\mathcal{G})$-module. Suppose that the form from Lemma 4.3.12 is non-degenerate. In that case, $M_\alpha(\mathcal{G})$ is a Frobenius algebra. Combining Theorem 4.3.13, Lemma 4.3.12 and Proposition 4.2.2, we can decompose M as a direct sum of adjoint modules and a module M' for which $\sum_{f \in U_x} f(m) = 0$ for every $m \in (M')^x_1$ and every $x \in \mathcal{P}$.

(ii) Suppose $\mathcal{G}$ is a finite connected Fischer space without isolated points. Let A be its collinearity matrix. The condition that the form from Lemma 4.3.12 is non-degenerate can be expressed as $\det\left(I + \frac{\alpha}{2}A\right) \neq 0$. From this, it is clear that this can only fail for a finite number of choices for α.

(iii) Since the Miyamoto group of $M_\alpha(\mathcal{G})$ acts transitively on the points of a connected Fischer space $\mathcal{G}$, the number of lines through a point is a constant

$d \in \mathbb{N}$ if $\mathcal{G}$ is finite. The number of points collinear with any given point is then $2d$. The vector of all ones is therefore a $(1+\alpha d)$-eigenvector of $I + \frac{\alpha}{2}A$ where A is the collinearity matrix of $\mathcal{G}$. It is easy to verify that

$$\left(\sum_{x\in\mathcal{P}} x\right) a = (1+\alpha d)a$$

for all $a \in M_\alpha(\mathcal{G})$. If $1+\alpha d = 0$ and hence $\det\left(I + \frac{\alpha}{2}A\right) = 0$, then $\langle\sum_{x\in\mathcal{P}} x\rangle$ is a 1-dimensional $M_\alpha(\mathcal{G})$-submodule of $M_\alpha(\mathcal{G})$. The quotient module is then an $M_\alpha(\mathcal{G})$-module non-isomorphic to $M_\alpha(\mathcal{G})$ with a non-trivial 1-eigenspace for every axis $x \in \mathcal{P}$. The condition that the form must be non-degenerate can therefore not be omitted. Note that if $1+\alpha d \neq 0$, then $M_\alpha(\mathcal{G})$ is a unital algebra with unit $(1+\alpha d)^{-1}\left(\sum_{x\in\mathcal{P}} x\right)$.

5

Frobenius algebras for simply laced Chevalley groups

In 2015, Skip Garibaldi and Robert M. Guralnick observed that there exists a 3875-dimensional, commutative, non-associative, Frobenius algebra on which the simple algebraic group of type E_8 acts by automorphisms [GG15]. To the best of our knowledge, however, no explicit construction of this algebra was known.

This chapter aims to shed light on the structure of this 3875-dimensional algebra. On the one hand, we will give a very explicit construction of this algebra. On the other hand, we will be able to give it the structure of an axial decomposition algebra. Furthermore, this algebra fits into a larger class of algebras: the construction can be applied to any simple group of Lie type of type ADE and each of these algebras will have the structure of a decomposition algebra.

We start in Section 5.1 by defining a commutative product on the symmetric square of a simple Lie algebra of simply laced type. This definition arises very naturally from the definition of the Lie bracket and will be the starting point for the construction of our algebra. We use this product in Section 5.2 to give an explicit, albeit impractical, construction of our algebra. In Sections 5.3 and 5.4 we derive an explicit multiplication rule for our algebra. The algebra turns out to be unital, a fact that we prove in Section 5.5.

Then, we use the ideas from Section 2.8 to give the algebra the structure of a decomposition algebra. Their fusion law will be graded and the corresponding Miyamoto group will be a Chevalley group. For the algebra for E_8, our main algebra of interest, we go one step further and prove that it is, in fact, an axial decomposition algebra.

The content of this chapter is based on [DMVC20b]. Shortly after this work was released on the arXiv, Maurice Chayet and Skip Garibaldi also released on explicit construction of these algebras [CG20]. Their approach does not require restricting to the simply laced case and is vastly different from ours. Also the results are complementary.

5.1 A product on the symmetric square

We introduce some terminology and notation that we will use throughout this chapter. The relevant definitions can be found in Sections 1.3, 1.5 and 1.6.

Definition 5.1.1. (i) Let $\mathcal{L}$ be a complex simple Lie algebra of *simply laced type*, i.e., of type A_n, D_n or E_n. To avoid some technicalities that appear when working with low rank, we assume that $n \geq 3$, $n \geq 4$ or $n \in \{6,7,8\}$ when $\mathcal{L}$ is of type A_n, D_n or E_n respectively. Consider a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ and the set of roots $\Phi \subseteq \mathcal{H}^*$ relative to $\mathcal{H}$. For each root $\alpha \in \Phi$, denote its coroot by $h_\alpha \in \mathcal{H}$. Denote the weight lattice by Λ. Let Δ be a base for Φ and denote the set of positive roots with respect to Δ by Φ^+. Let W be the Weyl group of Φ. Recall that this is the group generated by the reflections

$$s_\alpha \colon \mathcal{H} \to \mathcal{H} \colon h \mapsto h - \alpha(h)h_\alpha$$

for the roots $\alpha \in \Phi$.

(ii) Let $\{h_\alpha \mid \alpha \in \Delta\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ be a Chevalley basis for $\mathcal{L}$ with respect to $\mathcal{H}$ and Δ as in Proposition 1.3.30. For $\alpha, \beta \in \Phi$ for which $\alpha + \beta \in \Phi$, define $c_{\alpha,\beta} \in \mathbb{C}$ such that $[e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta}$. Then

$$\begin{aligned}
[h_\alpha, h_\beta] &= 0, && \\
[h_\alpha, e_\beta] &= \beta(h_\alpha) e_\beta, && \\
[e_\alpha, e_{-\alpha}] &= h_\alpha && \\
[e_\alpha, e_\beta] &= c_{\alpha,\beta} e_{\alpha+\beta} && \text{if } \alpha + \beta \in \Phi, \\
[e_\alpha, e_\beta] &= 0 && \text{if } \alpha + \beta \notin \Phi,
\end{aligned}$$

for all $\alpha, \beta \in \Phi$. Also recall the relations on the structure constants $c_{\alpha,\beta}$ from Proposition 1.3.32.

(iii) As usual, we let

$$\mathrm{ad}_\ell : \mathcal{L} \to \mathcal{L} : l \mapsto [\ell, l],$$

for all $\ell \in \mathcal{L}$. Recall from Proposition 1.3.9 that, since $\mathcal{L}$ is simple and of simply laced type, the Weyl group W must act transitively on Φ. Therefore $t \coloneqq \mathrm{tr}(\mathrm{ad}_{h_\alpha} \mathrm{ad}_{h_\alpha})$ does not depend on the choice of $\alpha \in \Phi$. Define

$$\kappa(\ell_1, \ell_2) = 2t^{-1} \, \mathrm{tr}(\mathrm{ad}_{\ell_1} \mathrm{ad}_{\ell_2})$$

for all $\ell_1, \ell_2 \in \mathcal{L}$. Then κ is a rescaling of the Killing form of $\mathcal{L}$, see Definition 1.3.22, such that $\kappa(h_\alpha, h_\alpha) = 2$ for all $\alpha \in \Phi$; in particular, κ is non-degenerate. We will simply refer to κ as the Killing form. This allows us to identify $\mathcal{H}^*$ with $\mathcal{H}$. We have rescaled our Killing form in such a way that $\alpha \in \mathcal{H}^*$ corresponds to h_α under this identification. In particular, $\alpha(h_\beta) = \kappa(h_\beta, \alpha) = \kappa(\beta, \alpha)$ and thus, since Φ is simply laced,

$$\kappa(\alpha,\beta) = \begin{cases} -2 & \text{if } \alpha = -\beta, \\ -1 & \text{if } \alpha + \beta \in \Phi, \\ 1 & \text{if } \alpha - \beta \in \Phi, \\ 2 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Since the Killing form is a Frobenius form for $\mathcal{L}$, we have

$$\kappa(e_\alpha, e_{-\alpha}) = \frac{1}{2}\kappa(e_\alpha, [e_{-\alpha}, h_\alpha]) = \frac{1}{2}\kappa([e_\alpha, e_{-\alpha}], h_\alpha) = \frac{1}{2}\kappa(h_\alpha, h_\alpha) = 1.$$

(iv) Let $S^2(\mathcal{L})$ be the symmetric square of the adjoint representation of $\mathcal{L}$. Write $\ell_1\ell_2$ for the projection of $\ell_1 \otimes \ell_2$ onto $S^2(\mathcal{L})$ for $\ell_1, \ell_2 \in \mathcal{L}$. We denote the action of $\mathcal{L}$ on $S^2(\mathcal{L})$ by $\cdot$:

$$\ell \cdot \ell_1\ell_2 = [\ell, \ell_1]\ell_2 + \ell_1[\ell, \ell_2]$$

for all $\ell, \ell_1, \ell_2 \in \mathcal{L}$.

Recall the definition of $\mathcal{L}$-equivariant maps from Definition 1.5.3. We will define an $\mathcal{L}$-equivariant product and Frobenius form on $S^2(\mathcal{L})$ starting from the Killing form κ.

Definition 5.1.2. The non-degeneracy of the Killing form κ allows us to identify $\mathcal{L}$ with its dual $\mathcal{L}^*$. Hence we can identify $\mathcal{L} \otimes \mathcal{L}$ with $\mathcal{L} \otimes \mathcal{L}^* \cong \mathsf{Hom}(\mathcal{L}, \mathcal{L})$ via the isomorphism defined by

$$\zeta' \colon \mathcal{L} \otimes \mathcal{L} \to \mathsf{Hom}(\mathcal{L}, \mathcal{L}) \colon \ell_1 \otimes \ell_2 \mapsto [\ell' \mapsto \kappa(\ell_2, \ell')\ell_1] \,.$$

Since we are working over the complex numbers, the natural projection $\mathcal{L} \otimes \mathcal{L} \to S^2(\mathcal{L})$ admits an $\mathcal{L}$-equivariant section:

$$\sigma \colon S^2(\mathcal{L}) \to \mathcal{L} \otimes \mathcal{L} \colon \ell_1\ell_2 \mapsto \frac{1}{2}(\ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1).$$

Let $\zeta \coloneqq \zeta' \circ \sigma$. Then ζ is injective and its image consists of the symmetric operators. These are the operators $f \in \mathsf{Hom}(\mathcal{L}, \mathcal{L})$ for which $\kappa(f(a), b) = \kappa(a, f(b))$ for all $a, b \in \mathcal{L}$. Now we can use the construction from Example 1.2.9 (iii) to turn $S^2(\mathcal{L})$ into a Frobenius algebra. Under the above correspondence ζ, the maps $\bullet$ and B are defined by

$$\begin{aligned} \bullet \colon\ & S^2(\mathcal{L}) \times S^2(\mathcal{L}) \to S^2(\mathcal{L}) \\ & (\ell_1\ell_2, \ell_3\ell_4) \mapsto \frac{1}{4}\left(\kappa(\ell_1, \ell_3)\ell_2\ell_4 + \kappa(\ell_1, \ell_4)\ell_2\ell_3 + \kappa(\ell_2, \ell_3)\ell_1\ell_4 + \kappa(\ell_2, \ell_4)\ell_1\ell_3\right), \\ B \colon\ & S^2(\mathcal{L}) \times S^2(\mathcal{L}) \to \mathbb{C} \\ & (\ell_1\ell_2, \ell_3\ell_4) \mapsto \frac{1}{2}\left(\kappa(\ell_1, \ell_3)\kappa(\ell_2, \ell_4) + \kappa(\ell_1, \ell_4)\kappa(\ell_2, \ell_3)\right). \end{aligned}$$

Proposition 5.1.3. *Consider the bilinear maps $\bullet$ and B from Definition* 5.1.2. *Then $(S^2(\mathcal{L}), \bullet, B)$ is a Frobenius algebra for $\mathcal{L}$.*

Proof. The $\mathcal{L}$-equivariance follows from the $\mathcal{L}$-equivariance of κ. Now, this follows immediately from the construction and Example 1.2.9 (iii). □

Also recall Corollary 1.5.9 since these rules will be silently used throughout the following sections.

5.2 Constructing the algebra

We will use the algebra from Proposition 5.1.3 to build a Frobenius algebra of smaller dimension for $\mathcal{L}$. The highest occurring weight in $S^2(\mathcal{L})$, as an $\mathcal{L}$-representation, is the double of a root. Its weight space is one-dimensional. We will explicitly determine a generating set of the subrepresentation $\mathcal{V}$ generated by this weight space in Proposition 5.2.6 below. Next, we define an algebra product on the complement $\mathcal{A}$ of $\mathcal{V}$ in $S^2(\mathcal{L})$ with respect to B. The algebra product on $\mathcal{A}$ will be the composition of the algebra product from Proposition 5.1.3 and the projection onto $\mathcal{A}$. We are grateful to Sergey Shpectorov for providing the central idea of this construction.

Definition 5.2.1. Let $\mathcal{V}$ denote the subrepresentation of $S^2(\mathcal{L})$ generated by $e_\omega e_\omega$, where ω is the highest root with respect to the base Δ.

It will be fairly straightforward to find elements that lie in $\mathcal{V}$. However, in order to determine whether they span $\mathcal{V}$ as a vector space, we will first have to determine the multiplicity of each weight in $\mathcal{V}$, a task requiring some work. We will use the terminology of characters from Definition 1.5.14 to describe these multiplicities.

We introduce some notation to describe the weights and multiplicities of $S^2(\mathcal{L})$ and $\mathcal{V}$.

Definition 5.2.2. (i) For $-2 \leq i \leq 2$ let $\Lambda_i \coloneqq \{\alpha + \beta \mid \alpha, \beta \in \Phi, \kappa(\alpha, \beta) = i\}$. This means that Λ_i contains those weights that can be represented as the sum of two roots $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = i$.

(ii) For $\lambda \in \bigcup_{i=-2}^{2} \Lambda_i$, let $N_\lambda \coloneqq \{\{\alpha, \beta\} \mid \alpha, \beta \in \Phi, \alpha + \beta = \lambda\}$ and $n_\lambda = |N_\lambda|$. Simply put, n_λ is the number of ways to write λ as the sum of two roots.

The elements of Λ_i for $-2 \leq i \leq 2$ are the weights of $S^2(\mathcal{L})$ as an $\mathcal{L}$-representation, cf. Proposition 5.2.4. We prove a few easy statements about these weights.

Lemma 5.2.3. (i) *For each $\lambda \in \Lambda_i$, we have $\kappa(\lambda, \lambda) = 4 + 2i$. In particular, the sets Λ_i are disjoint.*

(ii) *We have* $\Lambda_{-2} = \{0\}$ *and* $\Lambda_{-1} = \Phi$.

(iii) *If* $\alpha + \beta \in \Lambda_i$ *for* $\alpha, \beta \in \Phi$, *then* $\kappa(\alpha, \beta) = i$.

(iv) *Let* $\alpha \in \Phi$ *and* $\lambda \in \Lambda_i$. *Then* $\alpha + \beta = \lambda$ *for some* $\beta \in \Phi$ *if and only if* $\kappa(\alpha, \lambda) = 2 + i$.

Proof. (i) For $\lambda \in \Lambda_i$, we can write $\lambda = \alpha + \beta$ where $\alpha, \beta \in \Phi$ and $\kappa(\alpha, \beta) = i$. Thus $\kappa(\lambda, \lambda) = \kappa(\alpha + \beta, \alpha + \beta) = 4 + 2i$.

(ii) We have $\kappa(\alpha, \beta) = -2$ for $\alpha, \beta \in \Phi$ if and only if $\alpha = -\beta$. Therefore $\Lambda_{-2} = \{0\}$. Also $\Lambda_{-1} = \Phi$ because $\alpha + \beta \in \Phi$ for $\alpha, \beta \in \Phi$ if and only if $\kappa(\alpha, \beta) = -1$.

(iii) By (i) we know that $4 + 2i = \kappa(\alpha + \beta, \alpha + \beta) = 4 + 2\kappa(\alpha, \beta)$ from which the assertion follows.

(iv) This is obvious by (ii) for $i \in \{-2, -1\}$. Suppose that $i \in \{0, 1, 2\}$. If $\lambda = \alpha + \beta$ for some $\beta \in \Phi$, then $\kappa(\alpha, \beta) = i$ and $\kappa(\lambda, \alpha) = 2 + i$ by (i). Conversely, suppose that $\kappa(\lambda, \alpha) = 2 + i$. Write $\lambda = \alpha' + \beta'$ for some $\alpha', \beta' \in \Phi$. Because Φ is simply laced we have either $\alpha \in \{\alpha', \beta'\}$ or $i = 0$ and $\kappa(\alpha', \alpha) = \kappa(\beta', \alpha) = 1$. In the first case the condition is obviously satisfied. In the second case we have that $\alpha' - \alpha$ is a root and $\kappa(\alpha' - \alpha, \beta') = -1$. So $\lambda - \alpha = (\alpha' - \alpha) + \beta'$ is a root. □

Proposition 5.2.4. *The character of* $S^2(\mathcal{L})$ *is given by*

$$\mathrm{ch}_{S^2(\mathcal{L})} = \left(\frac{n(n+1)}{2} + n_0\right) e^0 + \sum_{\lambda \in \Lambda_{-1}} (n_\lambda + n) e^\lambda + \sum_{\lambda \in \Lambda_0} n_\lambda e^\lambda + \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} e^\lambda.$$

Proof. If $\sum_\lambda m_\lambda e^\lambda$ is the character of a representation, then its symmetric square has character $\frac{1}{2} \sum_{\lambda, \mu} m_\lambda m_\mu e^{\lambda+\mu} + \frac{1}{2} \sum_\lambda m_\lambda e^{2\lambda}$; see [FH91, Exercise 23.39]. Since the character of $\mathcal{L}$ as $\mathcal{L}$-representation is given by

$$ne^0 + \sum_{\alpha \in \Phi} e^\alpha,$$

the statement follows from Definition 5.2.2. It is also possible to verify this more explicitly. The Chevalley basis of $\mathcal{L}$ is a basis of weight vectors of $\mathcal{L}$ with respect to $\mathcal{H}$. Now, if $b_1, \ldots, b_n$ is a basis of weight vectors of $\mathcal{L}$ as $\mathcal{L}$-representation, then $b_i b_j$ for $i \le j$ is a basis of weight vectors for $S^2(\mathcal{L})$ from which the character can be computed. □

We are now ready to specify the character of $\mathcal{V}$.

Proposition 5.2.5. *The character of* $\mathcal{V}$ *is given by*

$$\mathrm{ch}_{\mathcal{V}} = n_0 e^0 + \sum_{\lambda \in \Lambda_{-1}} (n_\lambda + 1) e^\lambda + \sum_{\lambda \in \Lambda_0} (n_\lambda - 1) e^\lambda + \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} e^\lambda.$$

Proof. The character can be computed using Freudenthal's formula (Proposition 1.5.15). We refer to Proposition 5.9.2 for the details. □

Next, we compute certain elements of $\mathcal{V}$ and we use the character of $\mathcal{V}$ to verify that these elements, in fact, span $\mathcal{V}$ as a vector space.

Proposition 5.2.6. *Let*

$$\begin{aligned}
\Gamma_0 &\coloneqq \{2e_\alpha e_{-\alpha} - h_\alpha h_\alpha \mid \alpha \in \Phi\},\\
\Gamma_1 &\coloneqq \{e_\alpha h_\alpha \mid \alpha \in \Phi\},\\
\Gamma_2 &\coloneqq \{2e_\alpha e_\beta + c_{\alpha,\beta} e_{\alpha+\beta}(h_\beta - h_\alpha) \mid \alpha, \beta \in \Phi, \kappa(\alpha,\beta) = -1\},\\
\Gamma_3 &\coloneqq \{e_\alpha e_\beta + \frac{c_{\alpha,-\gamma}}{c_{\beta,-\delta}} e_\gamma e_\delta \mid \alpha,\beta,\gamma,\delta \in \Phi, \kappa(\alpha,\beta) = 0, \{\gamma,\delta\} \in N_{\alpha+\beta} \setminus \{\{\alpha,\beta\}\}\},\\
\Gamma_4 &\coloneqq \{e_\alpha e_\beta \mid \alpha,\beta \in \Phi, \kappa(\alpha,\beta) = 1\},\\
\Gamma_5 &\coloneqq \{e_\alpha e_\alpha \mid \alpha \in \Phi\}.
\end{aligned}$$

Then $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ *spans* $\mathcal{V}$ *as a vector space.*

Proof. The Weyl group W acts transitively on Φ and $e_\omega e_\omega \in \mathcal{V}$. Therefore $\Gamma_5 \subseteq \mathcal{V}$. For any $\alpha \in \Phi$, we have $e_{-\alpha} \cdot e_\alpha e_\alpha = -2e_\alpha h_\alpha$, thus $\Gamma_1 \subseteq \mathcal{V}$. Hence $e_\alpha \cdot e_{-\alpha} h_{-\alpha} = 2e_\alpha e_{-\alpha} - h_\alpha h_\alpha \in \mathcal{V}$ for any $\alpha \in \Phi$, which shows that $\Gamma_0 \subseteq \mathcal{V}$. Now let $\alpha, \beta \in \Phi$. Suppose that $\kappa(\alpha,\beta) = 1$. Then $\alpha - \beta \in \Phi$ and $e_{\alpha-\beta} \cdot e_\beta e_\beta = 2c_{\alpha-\beta,\beta} e_\alpha e_\beta \in \mathcal{V}$. Therefore $\Gamma_4 \subseteq \mathcal{V}$. Suppose next that $\kappa(\alpha,\beta) = -1$ such that $\alpha + \beta$ is a root. Then $e_\beta \cdot e_\alpha h_\alpha + e_\alpha \cdot e_\beta h_\beta = 2e_\alpha e_\beta + c_{\alpha,\beta} e_{\alpha+\beta}(h_\beta - h_\alpha) \in \mathcal{V}$. Hence $\Gamma_2 \subseteq \mathcal{V}$.

Finally, let $\alpha,\beta,\gamma,\delta \in \Phi$ such that $\kappa(\alpha,\beta) = 0$ and $\{\gamma,\delta\} \in N_{\alpha+\beta} \setminus \{\{\alpha,\beta\}\}$. Then $\alpha,\beta,\gamma,\delta$ generate a root subsystem of type A_3, $\kappa(\gamma,\delta) = 0$ and $\kappa(\alpha,\gamma) = \kappa(\alpha,\delta) = \kappa(\beta,\gamma) = \kappa(\beta,\delta) = 1$. Now $e_\delta \cdot (2e_{\gamma-\alpha} e_\alpha + c_{\gamma-\alpha,\alpha} e_\gamma (h_\alpha - h_{\gamma-\alpha})) = 2c_{\delta,\gamma-\alpha} e_\alpha e_\beta - 2c_{\gamma-\alpha,\alpha} e_\gamma e_\delta$. Using the identities from Proposition 1.3.32, we see that $c_{\delta,\gamma-\alpha} = -c_{\beta,-\delta}$ and $c_{\gamma-\alpha,\alpha} = c_{\alpha,-\gamma}$. This amounts to $\Gamma_3 \subseteq \mathcal{V}$.

In order to prove that $\Gamma \coloneqq \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ spans $\mathcal{V}$, it suffices to check that the elements of weight λ in Γ span $\mathcal{V}_\lambda$, the weight-λ-space of $\mathcal{V}$. The dimension $\dim(\mathcal{V}_\lambda)$ of $\mathcal{V}_\lambda$ can be derived from Proposition 5.2.5.

The elements of Γ of weight 0 are precisely those contained in Γ_0. Obviously $\dim(\langle \Gamma_0 \rangle) = \frac{|\Phi|}{2} = n_0$ and therefore $\langle \Gamma_0 \rangle = \mathcal{V}_0$.

Let $\alpha \in \Phi$. The elements of Γ of weight α are $e_\alpha h_\alpha$ and the elements $2e_\beta e_\gamma + c_{\beta,\gamma}(h_\gamma - h_\beta)$ where $\{\beta,\gamma\} \in N_\alpha$. Since these elements are linearly independent, they span a subspace of dimension $n_\alpha + 1$, which is the dimension of $\mathcal{V}_\alpha$ by Proposition 5.2.5.

For $\lambda \in \Lambda_0$, the elements of Γ of weight λ are those of the form $e_\alpha e_\beta + \frac{c_{\alpha,-\gamma}}{c_{\beta,-\delta}} e_\gamma e_\delta$ where $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ are two different elements of N_λ. Hence they span a subspace of dimension at least $n_\lambda - 1$ and at most n_λ. Since they are all contained in $\mathcal{V}$, they span $\mathcal{V}_\lambda$, a subspace of dimension $n_\lambda - 1$ by Proposition 5.2.5.

Finally, let $\lambda \in \Lambda_1 \cup \Lambda_2$. Then all elements of $S^2(\mathcal{L})$ of weight λ are contained in $\langle \Gamma_4 \cup \Gamma_5 \rangle$. Therefore $\mathcal{V}_\lambda \leq \langle \Gamma \rangle$. □

As we observed in the previous proof, the elements of Γ_3 of weight $\lambda \in \Lambda_0$ are linearly dependent. We introduce some notation to describe this linear dependence; this will be useful later.

Definition 5.2.7. (i) Recall from Definition 5.2.2 that

$$\Lambda_0 = \{\alpha + \beta \mid \alpha, \beta \in \Phi, \kappa(\alpha, \beta) = 0\}.$$

For each $\lambda \in \Lambda_0$, fix elements $\alpha_\lambda, \beta_\lambda \in \Phi$ such that $\alpha_\lambda + \beta_\lambda = \lambda$. Note that it immediately follows that $\kappa(\alpha_\lambda, \beta_\lambda) = 0$ from Lemma 5.2.3 (iii).

(ii) For all $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = 0$ (and therefore $\alpha + \beta \in \Lambda_0$) we write

$$f_{\alpha,\beta} := \begin{cases} 1 & \text{if } \{\alpha, \beta\} = \{\alpha_\lambda, \beta_\lambda\}, \\ -\frac{c_{\alpha,-\alpha_\lambda}}{c_{\beta,-\beta_\lambda}} & \text{otherwise.} \end{cases}$$

Notice that $f_{\alpha,\beta} \in \{\pm 1\}$.

Proposition 5.2.8. *Let $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = 0$ and let $\lambda := \alpha + \beta \in \Lambda_0$. Then*

(i) $f_{\alpha,\beta} = f_{\beta,\alpha}$ *and,*

(ii) $f_{-\alpha,-\beta} = f_{\alpha,\beta} f_{-\alpha_\lambda,-\beta_\lambda}$.

(iii) *Let Γ_3 be as in Proposition 5.2.6. The subspace of $\mathcal{V}$ spanned by Γ_3 is equal to the subspace of $\mathcal{V}$ spanned by*

$$\{e_{\alpha_\lambda} e_{\beta_\lambda} - f_{\alpha,\beta} e_\alpha e_\beta \mid \lambda \in \Lambda_0; \alpha, \beta \in \Phi; \alpha + \beta = \lambda\}.$$

Proof. The elements of Γ_i for $0 \le i \le 5$ are all weight vectors. Those with weight λ are contained in Γ_3. Thus the weight-λ-space of $\mathcal{V}$ is contained in the span of Γ_3. By Proposition 5.2.5, this weight space has dimension $n_\lambda - 1$. Thus the elements $e_\alpha e_\beta + \frac{c_{\alpha,-\gamma}}{c_{\beta,-\delta}} e_\gamma e_\delta$ for $\alpha, \beta, \gamma, \delta \in \Phi$ with $\alpha + \beta = \gamma + \delta = \lambda$ and $\{\alpha, \beta\} \neq \{\gamma, \delta\}$ must be linearly dependent. In particular $e_\alpha e_\beta \notin \mathcal{V}$ for all $\alpha, \beta \in \Phi$ where $\kappa(\alpha, \beta) = 0$.

(i) This is obvious if $\{\alpha, \beta\} = \{\alpha_\lambda, \beta_\lambda\}$. Assume that $\{\alpha, \beta\} \neq \{\alpha_\lambda, \beta_\lambda\}$. Since $e_\alpha e_\beta = e_\beta e_\alpha$ it follows from the argument above that the elements $e_\alpha e_\beta + \frac{c_{\alpha,-\alpha_\lambda}}{c_{\beta,-\beta_\lambda}} e_{\alpha_\lambda} e_{\beta_\lambda}$ and $e_\beta e_\alpha + \frac{c_{\beta,-\alpha_\lambda}}{c_{\alpha,-\beta_\lambda}} e_{\alpha_\lambda} e_{\beta_\lambda}$ must be linearly dependent. Therefore $\frac{c_{\alpha,-\alpha_\lambda}}{c_{\beta,-\beta_\lambda}} = \frac{c_{\beta,-\alpha_\lambda}}{c_{\alpha,-\beta_\lambda}}$ and thus $f_{\alpha,\beta} = f_{\beta,\alpha}$.

(ii) Since

$$\begin{aligned} &e_{-\alpha} e_{-\beta} + \frac{c_{-\alpha,\alpha_\lambda}}{c_{-\beta,\beta_\lambda}} e_{-\alpha_\lambda} e_{-\beta_\lambda} \in \Gamma_3, \\ &e_{-\alpha_\lambda} e_{-\beta_\lambda} - f_{-\alpha_\lambda,-\beta_\lambda} e_{\alpha_{-\lambda}} e_{\beta_{-\lambda}} \in \Gamma_3, \end{aligned}$$

$$e_{-\alpha}e_{-\beta} - f_{-\alpha,-\beta}e_{\alpha_{-\lambda}}e_{\beta_{-\lambda}} \in \Gamma_3,$$

these elements must be linearly dependent. Thus

$$f_{-\alpha,-\beta} = -\frac{c_{-\alpha,\alpha_\lambda}}{c_{-\beta,\beta_\lambda}} f_{-\alpha_\lambda,-\beta_\lambda}.$$

The assertion follows because $c_{-\alpha,\alpha_\lambda} = -c_{\alpha,-\alpha_\lambda}$ and $c_{-\beta,\beta_\lambda} = -c_{\beta,-\beta_\lambda}$ (see Proposition 1.3.32 (ii)) and therefore

$$-\frac{c_{-\alpha,\alpha_\lambda}}{c_{-\beta,\beta_\lambda}} = -\frac{c_{\alpha,-\alpha_\lambda}}{c_{\beta,-\beta_\lambda}} = f_{\alpha,\beta}.$$

(iii) This follows immediately because the elements of Γ_3 of weight λ span a subspace of dimension $n_\lambda - 1$. □

Next, we want to take a complement of $\mathcal{V}$ in $S^2(\mathcal{L})$ with respect to the bilinear form B. In order for this complement to be well-defined, we need B to be non-degenerate on $\mathcal{V}$.

Proposition 5.2.9. *The restriction of B to $\mathcal{V} \times \mathcal{V}$ is non-degenerate.*

Proof. Since B is $\mathcal{L}$-equivariant, the radical $\{v \in \mathcal{V} \mid B(v, w) \text{ for all } w \in \mathcal{V}\}$ of $B \restriction_{\mathcal{V}\times\mathcal{V}}$ is a subrepresentation of $\mathcal{V}$. However, $\mathcal{V}$ is irreducible as it is a highest weight representation. Since $B \restriction_{\mathcal{V}\times\mathcal{V}}$ is non-zero (e.g., $B(e_\alpha e_\alpha, e_{-\alpha}e_{-\alpha}) = 1$), we conclude that the radical of $B \restriction_{\mathcal{V}\times\mathcal{V}}$ is trivial. □

The previous proposition allows us to define an orthogonal complement of $\mathcal{V}$ with respect to the bilinear form B. This will be the underlying representation of our algebra.

Definition 5.2.10. (i) Let $\mathcal{A}$ be the orthogonal complement of $\mathcal{V}$ in $S^2(\mathcal{L})$ with respect to the $\mathcal{L}$-equivariant bilinear form B:

$$\mathcal{A} \coloneqq \{v \in S^2(\mathcal{L}) \mid B(v, w) = 0 \text{ for all } w \in \mathcal{V}\}.$$

(ii) Denote the orthogonal projection of $S^2(\mathcal{L})$ onto $\mathcal{A}$ by π. For each $v \in S^2(\mathcal{L})$, we will also denote $\pi(v)$ by $\overline{v}$.

The character of $\mathcal{A}$ follows easily from the characters of $S^2(\mathcal{L})$ and $\mathcal{V}$.

Proposition 5.2.11. *The character of $\mathcal{A}$ as a representation for $\mathcal{L}$ is given by*

$$\mathrm{ch}_{\mathcal{A}} = \frac{n(n+1)}{2}e^0 + \sum_{\alpha\in\Phi}(n-1)e^\alpha + \sum_{\lambda\in\Lambda_0} e^\lambda$$

where n is the rank of Φ.

Proof. Since $S^2(\mathcal{L}) = \mathcal{A} \oplus \mathcal{V}$, we have $\mathrm{ch}_{\mathcal{A}} = \mathrm{ch}_{S^2(\mathcal{L})} - \mathrm{ch}_{\mathcal{V}}$. The characters of $S^2(\mathcal{L})$ and $\mathcal{V}$ follow from Propositions 5.2.4 and 5.2.5. $\square$

Using Proposition 5.2.6 we can explicitly describe the weight spaces of $\mathcal{A}$.

Proposition 5.2.12. *The weights of $\mathcal{A}$ are 0, the roots $\alpha \in \Phi = \Lambda_{-1}$ and the sums of orthogonal roots $\lambda \in \Lambda_0$. Any weight vector can be uniquely written as*

(i) $\overline{a}$ *for* $a \in S^2(\mathcal{H}) \leq S^2(\mathcal{L})$ *if the weight vector has weight 0;*

(ii) $\overline{e_\alpha h}$ *for* $h \in \alpha^\perp := \{h \in \mathcal{H} \mid \kappa(\alpha, h) = 0\}$ *if the weight vector has weight* $\alpha \in \Phi$ *(also note that* $\overline{e_\alpha h_\alpha} = 0$*);*

(iii) $c\overline{e_{\alpha_\lambda} e_{\beta_\lambda}}$ *for* $c \in \mathbb{C}$ *when the weight vector has weight* $\lambda \in \Lambda_0$*.*

Proof. Recall that $\mathcal{V}$ is the orthogonal complement of $\mathcal{A}$ in $S^2(\mathcal{L})$ with respect to the bilinear form B. The statement follows from the description of the generating set of $\mathcal{V}$ from Propositions 5.2.6 and 5.2.8. $\square$

The projection π from Definition 5.2.10 can be computed explicitly.

Lemma 5.2.13. *Let $\lambda \in \Lambda_0$. Then*

$$\overline{e_{\alpha_\lambda} e_{\beta_\lambda}} = \frac{1}{n_\lambda} \left(\sum_{\alpha+\beta=\lambda} f_{\alpha,\beta} e_\alpha e_\beta \right), \tag{5.1}$$

where the sum runs over all sets $\{\alpha, \beta\}$ where $\alpha, \beta \in \Phi$ such that $\alpha + \beta = \lambda$, or equivalently, over all elements $\{\alpha, \beta\} \in N_\lambda$. Also

$$\overline{e_{\alpha_\lambda} h_{\beta_\lambda}} = \frac{1}{n_\lambda} \left(e_{\alpha_\lambda} h_{\beta_\lambda} + \sum (c_{\beta,-\alpha_\lambda} e_{\alpha-\beta_\lambda} e_\beta + c_{\alpha,-\alpha_\lambda} e_{\beta-\beta_\lambda} e_\alpha) \right), \tag{5.2}$$

$$\overline{h_{\alpha_\lambda} h_{\beta_\lambda}} = \frac{1}{n_\lambda} \left(h_{\alpha_\lambda} h_{\beta_\lambda} + \sum (e_\alpha e_{-\alpha} + e_\beta e_{-\beta} - e_{\alpha_\lambda-\alpha} e_{\alpha-\alpha_\lambda} - e_{\alpha_\lambda-\beta} e_{\beta-\alpha_\lambda}) \right), \tag{5.3}$$

where each sum runs over all $\{\alpha, \beta\} \in N_\lambda$ with $\{\alpha, \beta\} \neq \{\alpha_\lambda, \beta_\lambda\}$.

Proof. It is immediately verified that

$$B\left(v, \frac{1}{n_\lambda}\left(\sum_{\alpha+\beta=\lambda} f_{\alpha,\beta} e_\alpha e_\beta\right)\right) = 0$$

for all $v \in \Gamma_0 \cup \cdots \cup \Gamma_5$. Since

$$\frac{1}{n_\lambda}\left(\sum_{\alpha+\beta=\lambda} f_{\alpha,\beta} e_\alpha e_\beta\right) - e_{\alpha_\lambda} e_{\beta_\lambda} = -\frac{1}{n_\lambda}\left(\sum_{\alpha+\beta=\lambda} (e_{\alpha_\lambda} e_{\beta_\lambda} - f_{\alpha,\beta} e_\alpha e_\beta)\right) \in \mathcal{V},$$

we have (5.1).

Recall the definition of $f_{\alpha,\beta}$ from Definition 5.2.7 and remember that $f_{\alpha,\beta} = f_{\beta,\alpha}$ by Proposition 5.2.8. Using these, we have, since the projection π is $\mathcal{L}$-equivariant,

$$\begin{aligned}\overline{e_{\alpha_\lambda} h_{\beta_\lambda}} &= e_{-\beta_\lambda} \cdot (-\overline{e_{\alpha_\lambda} e_{\beta_\lambda}}) \\ &= \frac{1}{n_\lambda}\Big(e_{\alpha_\lambda} h_{\beta_\lambda} - \sum (f_{\beta,\alpha} c_{-\beta_\lambda,\alpha} e_{\alpha-\beta_\lambda} e_\beta + f_{\alpha,\beta} c_{-\beta_\lambda,\beta} e_{\beta-\beta_\lambda} e_\alpha)\Big) \\ &= \frac{1}{n_\lambda}\Big(e_{\alpha_\lambda} h_{\beta_\lambda} + \sum (c_{\beta,-\alpha_\lambda} e_{\alpha-\beta_\lambda} e_\beta + c_{\alpha,-\alpha_\lambda} e_{\beta-\beta_\lambda} e_\alpha)\Big)\end{aligned}$$

where each sum runs over the sets $\{\alpha,\beta\}$ with $\alpha,\beta \in \Phi$, $\alpha+\beta=\lambda$ and $\{\alpha,\beta\} \neq \{\alpha_\lambda,\beta_\lambda\}$. Similarly, we have

$$\begin{aligned}\overline{h_{\alpha_\lambda} h_{\beta_\lambda}} &= e_{-\alpha_\lambda} \cdot (-\overline{e_{\alpha_\lambda} h_{\beta_\lambda}}) \\ &= \frac{1}{n_\lambda}\Big(h_{\alpha_\lambda} h_{\beta_\lambda} - \sum (e_\alpha e_{-\alpha} + e_\beta e_{-\beta} - e_{\alpha_\lambda-\alpha} e_{\alpha-\alpha_\lambda} - e_{\alpha_\lambda-\beta} e_{\beta-\alpha_\lambda})\Big)\end{aligned}$$

where we have used that

$$c_{\beta,-\alpha_\lambda} c_{-\alpha_\lambda,\beta} = c_{\alpha,-\alpha_\lambda} c_{-\alpha_\lambda,\alpha} = -1,$$

and

$$c_{\beta,-\alpha_\lambda} c_{-\alpha_\lambda,\alpha-\beta_\lambda} = c_{\alpha,-\alpha_\lambda} c_{-\alpha_\lambda,\beta-\beta_\lambda} = 1,$$

from Proposition 1.3.32. □

We finish this section by defining a suitable product $*$ and bilinear form $\mathcal{B}$ for $\mathcal{A}$ such that $(\mathcal{A}, *, \mathcal{B})$ is a Frobenius algebra for $\mathcal{L}$.

Proposition 5.2.14. *Consider the linear maps*

$$\begin{aligned}* &: \mathcal{A} \times \mathcal{A} \to \mathcal{A} : (v,w) \mapsto \overline{v \bullet w}, \\ \mathcal{B} &: \mathcal{A} \times \mathcal{A} \to \mathbb{C} : (v,w) \mapsto B(v,w).\end{aligned}$$

Then $(\mathcal{A}, *, \mathcal{B})$ is a Frobenius algebra for $\mathcal{L}$.

Proof. The maps $*$ and $\mathcal{B}$ are $\mathcal{L}$-equivariant as a composition of $\mathcal{L}$-equivariant maps. The fact that $\mathcal{B}$ is a Frobenius form follows from Proposition 5.1.3 and because $\mathcal{B}(\overline{v}, w) = B(v,w)$ if $w \in \mathcal{A}$; see also Proposition 6.1.5. □

5.3 The zero weight subalgebra

Consider the zero weight space $\mathcal{A}_0$ of $\mathcal{A}$ with respect to the fixed Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$. Since the product $*$ and bilinear form $\mathcal{B}$ are $\mathcal{L}$-equivariant, $\mathcal{A}_0$ is a Frobenius subalgebra of $\mathcal{A}$. In this section, we describe this subalgebra explicitly. In order to keep a clear distinction with the construction of the previous chapter, we will denote vector spaces occuring in this new construction by gothic letters. First, we will use the monomorphism $\zeta\colon S^2(\mathcal{L}) \to \mathsf{Hom}(\mathcal{L},\mathcal{L})$ from Definition 5.1.2 to describe the zero weight space of $S^2(\mathcal{L})$ as a space of homomorphisms.

Definition 5.3.1. Recall the notation e_α and h_α for $\alpha \in \Phi$ from Definition 5.1.1. Then the zero weight subspace of $S^2(\mathcal{L})$ is spanned by the elements $h_\alpha h_\beta$ and $e_\alpha e_{-\alpha}$ for $\alpha, \beta \in \Phi$.

(i) Let $\mathfrak{J}$ be the subspace of $\mathsf{Hom}(\mathcal{L}, \mathcal{L})$ spanned by the endomorphisms $j_\alpha \coloneqq \zeta(h_\alpha h_\alpha)$ for $\alpha \in \Phi$. Explicitly, the endomorphism j_α is defined by

$$j_\alpha \colon \mathcal{L} \to \mathcal{L} \colon \ell \mapsto \kappa(\ell, h_\alpha) h_\alpha.$$

Since $h_{-\alpha} = -h_\alpha$, we have $j_\alpha = j_{-\alpha}$. Note that $j_\alpha(e_\beta) = 0$ and $j_\alpha(\mathcal{L}) \subseteq \mathcal{H}$ for all $\alpha, \beta \in \Phi$. Therefore we can, and will, view $\mathfrak{J}$ is a subspace of $\mathsf{Hom}(\mathcal{H}, \mathcal{H})$.

(ii) Let $\mathfrak{Z}$ be the subspace of $\mathsf{Hom}(\mathcal{L}, \mathcal{L})$ spanned by the endomorphisms $z_\alpha \coloneqq \zeta(e_\alpha e_{-\alpha})$ for $\alpha \in \Phi$. We have

$$z_\alpha \colon \mathcal{L} \to \mathcal{L} \colon \ell \mapsto \frac{1}{2} \left(\kappa(\ell, e_\alpha) e_{-\alpha} + \kappa(\ell, e_{-\alpha}) e_\alpha \right).$$

Also $z_\alpha = z_{-\alpha}$.

(iii) Define $\mathfrak{S}_0 \coloneqq \mathfrak{J} + \mathfrak{Z}$. Then $\mathfrak{S}_0 = \mathfrak{J} \oplus \mathfrak{Z}$ as vector spaces.

(iv) Consider the product $\bullet$ and bilinear form B on $\mathsf{Hom}(\mathcal{L}, \mathcal{L})$ and on $\mathsf{Hom}(\mathcal{H}, \mathcal{H})$ as defined in Example 1.2.9 (ii). This turns these vector spaces into Frobenius algebras.

We will prove that $\mathfrak{S}_0$ is a Frobenius subalgebra of $\mathsf{Hom}(\mathcal{L}, \mathcal{L})$. In fact, we have $\mathfrak{S}_0 = \mathfrak{J} \oplus \mathfrak{Z}$ as Frobenius algebras. In particular, $\mathfrak{J}$ is a subalgebra of $\mathsf{Hom}(\mathcal{H}, \mathcal{H})$. This subalgebra has already been studied by Tom De Medts and Felix Rehren [DMR17].

Proposition 5.3.2. *The subspace $\mathfrak{J}$ is a Frobenius subalgebra of $\mathsf{Hom}(\mathcal{H}, \mathcal{H})$. More precisely,*

$$j_\alpha \bullet j_\beta = \begin{cases} 2 j_\alpha & \text{if } \alpha = \pm\beta, \\ 0 & \text{if } \kappa(\alpha, \beta) = 0, \\ \frac{1}{2}(j_\alpha + j_\beta - j_{s_\beta(\alpha)}) & \text{if } \kappa(\alpha, \beta) = \pm 1, \end{cases}$$

and

$$B(j_\alpha, j_\beta) = \kappa(\alpha, \beta)^2,$$

for all $\alpha, \beta \in \Phi$. It has dimension $\frac{n(n+1)}{2}$ and hence consists of all endomorphisms $f : \mathcal{H} \to \mathcal{H}$ for which $\kappa(f(a), b) = \kappa(a, f(b))$ for all $a, b \in \mathcal{H}$. The Frobenius algebra $\mathfrak{J}$ is isomorphic to the Frobenius algebra from Example 1.2.9 (iii) *for $V = \mathcal{H}$.*

Proof. This can be calculated using the explicit description of these homomorphisms from Definition 5.3.1. The multiplication also follows from [DMR17, Lemma 3.2]. The dimension follows from [DMR17, Lemma 3.3]. The endomorphisms j_α satisfy the condition that $\kappa(j_\alpha(a), b) = \kappa(a, j_\alpha(b))$ for all $a, b \in \mathcal{H}$. Since the subspace of all such homomorphisms has dimension $n(n+1)/2$, this subspace must be equal to $\mathfrak{J}$. So, in fact, $\mathfrak{J}$ is precisely the Frobenius algebra from Example 1.2.9 (iii) for $V = \mathcal{H}$. $\square$

Also $\mathfrak{Z}$ and $\mathfrak{S}_0$ are Frobenius subalgebras of $\mathsf{Hom}(\mathcal{L}, \mathcal{L})$.

Proposition 5.3.3. *The subspace $\mathfrak{Z}$ is a Frobenius subalgebra of $\mathsf{Hom}(\mathcal{L}, \mathcal{L})$. We have*

$$z_\alpha \bullet z_\beta = \begin{cases} \frac{1}{2} z_\alpha & \text{if } z_\alpha = z_\beta, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$B(z_\alpha, z_\beta) = \begin{cases} \frac{1}{2} & \text{if } z_\alpha = z_\beta, \\ 0 & \text{otherwise}, \end{cases}$$

for all $\alpha, \beta \in \Phi$. The subspace $\mathfrak{Z}$ has dimension $\frac{|\Phi|}{2}$.

Proof. Notice that $\kappa(e_\alpha, e_{-\alpha}) = 1$ and $\kappa(e_\alpha, e_\beta) = 0$ for all $\beta \neq -\alpha$. The assertion now follows from an explicit calculation. $\square$

Proposition 5.3.4. *The subspace $\mathfrak{S}_0$ is a Frobenius subalgebra of $\mathsf{Hom}(\mathcal{L}, \mathcal{L})$. In fact $\mathfrak{S}_0 = \mathfrak{J} \oplus \mathfrak{Z}$ as Frobenius algebras. This means that $\mathfrak{S}_0 = \mathfrak{J} \oplus \mathfrak{Z}$ as vector spaces and*

$$a \bullet b = 0 \quad \text{and} \quad B(a, b) = 0$$

for all $a \in \mathfrak{J}$ and all $b \in \mathfrak{Z}$.

Proof. Since the composition ab of the $\mathcal{L}$-endomorphisms a and b is zero, we also have $a \bullet b = \frac{1}{2}(ab + ba) = 0$ and $B(a, b) = \operatorname{tr}(ab) = 0$. $\square$

Note that the zero weight space $S^2(\mathcal{L})_0$ of $S^2(\mathcal{L})$ is a Frobenius subalgebra of $S^2(\mathcal{L})$ because $\bullet$ and B are $\mathcal{L}$-equivariant. It is isomorphic to $\mathfrak{S}_0$ as a Frobenius algebra.

Proposition 5.3.5. *Let ζ be as in Definition 5.1.2 and let $S^2(\mathcal{L})_0$ be the zero weight subspace of $S^2(\mathcal{L})$ with respect to $\mathcal{H}$. Then ζ induces an isomorphism*

$$\zeta_S \colon S^2(\mathcal{L})_0 \to \mathfrak{S}_0 \colon a \mapsto \zeta(a)$$

of Frobenius algebras.

Proof. Of course, $\mathfrak{S}_0$ must be contained in the image of $S^2(\mathcal{L})_0$ under ζ. However, since both have the same dimension, $\mathfrak{S}_0$ must actually be equal to this image. Notice that this implies that $\zeta(h_\alpha h_\beta) \in \mathfrak{S}_0$ for all $\alpha, \beta \in \Phi$. Now it follows immediately from the construction of the product and bilinear form on $S^2(\mathcal{L})$ (see Definition 5.1.2) that this is an isomorphism of Frobenius algebras. □

Next, we describe the zero weight space $\mathcal{A}_0$ of $\mathcal{A}$. Recall that $\mathcal{A}$ is defined as the orthogonal complement of the $\mathcal{L}$-invariant subspace $\mathcal{V}$ with respect to B. Since B is $\mathcal{L}$-equivariant, the zero weight space $\mathcal{A}_0$ of $\mathcal{A}$ is the orthogonal complement of the zero weight space $\mathcal{V}_0$ of $\mathcal{V}$ in $S^2(\mathcal{L})_0$. By Proposition 5.2.6, we know that the space $\mathcal{V}_0$ is spanned by the elements $2e_\alpha e_{-\alpha} - h_\alpha h_\alpha$. Therefore, we introduce the following definition.

Definition 5.3.6. (i) For each $\alpha \in \Phi$, let $v_\alpha := \zeta(2e_\alpha e_{-\alpha} - h_\alpha h_\alpha) = 2z_\alpha - j_\alpha$ and let $\mathfrak{V}$ be the subspace of $\mathfrak{S}_0$ spanned by these v_α. Then the restriction of B to $\mathfrak{V} \times \mathfrak{V}$ is non-degenerate since $B(v_\alpha, v_\alpha) = 6$. Let $\mathfrak{A}_0$ be the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{S}_0$ with respect to B and let $\pi\colon \mathfrak{S}_0 \to \mathfrak{A}_0$ be the orthogonal projection.

(ii) Define the following product and bilinear form on $\mathfrak{A}_0$:

$$a \diamond b := \pi(a \bullet b) \quad \text{and} \quad B_A(a, b) := B(a, b)$$

for all $a, b \in \mathfrak{A}_0$.

Proposition 5.3.7. (i) *The triple* $(\mathfrak{A}_0, \diamond, B_A)$ *is a Frobenius algebra.*

(ii) *The isomorphism* ζ *from Definition* 5.1.2 *induces an isomorphism*

$$\mathcal{A}_0 \to \mathfrak{A}_0 \colon a \mapsto \zeta(a)$$

of Frobenius algebras.

Proof. This is obvious from Definition 5.3.6 and Proposition 5.3.5 since the subspace $\mathcal{V}_0$ corresponds to $\langle v_\alpha \mid \alpha \in \Phi \rangle$ under ζ. □

Remark 5.3.8. Before we continue, let us give a summary of the notation and the obtained results on the connection between the different algebras. The goal of this section is to get a better understanding of the zero weight space $\mathcal{A}_0$ with respect to $\mathcal{H}$ of the $\mathcal{L}$-module from Definition 5.2.10. Recall that $\mathcal{A}$ is the complement (with respect to B) of the $\mathcal{L}$-module $\mathcal{V}$ inside the symmetric square $S^2(\mathcal{L})$ of the adjoint module of $\mathcal{L}$. By Corollary 1.5.9, the zero weight space $\mathcal{A}_0$ is the complement of the zero weight space $\mathcal{V}_0$ of $\mathcal{V}$ inside the zero weight space $S^2(\mathcal{L})_0$ of $S^2(\mathcal{L})$. So our first step is to obtain a better understanding of $S^2(\mathcal{L})_0$. We can decompose $S^2(\mathcal{L})_0$ as

$$\langle h_\alpha h_\beta \mid \alpha, \beta \in \Phi \rangle \oplus \langle e_\alpha e_{-\alpha} \mid \alpha \in \Phi \rangle.$$

The first component of this decomposition can be identified with the symmetric square $S^2(\mathcal{H})$ of the Cartan subalgebra $\mathcal{H}$. Now we consider the monomorphism $\zeta\colon S^2(\mathcal{L}) \to \mathsf{Hom}(\mathcal{L},\mathcal{L})$. Under this monomorphism the product $\bullet$ and bilinear form B of $S^2(\mathcal{L})$ correspond, by definition, to the Jordan product and trace form on $\mathsf{Hom}(\mathcal{L},\mathcal{L})$. From the results above, we have that ζ induces an isomorphism between the following structures:

$$\begin{aligned}
&\zeta\colon (S^2(\mathcal{H}), \bullet, B) \xrightarrow{\sim} (\mathfrak{J}, \bullet, B),\\
&\zeta\colon (\langle e_\alpha e_{-\alpha} \mid \alpha \in \Phi\rangle, \bullet, B) \xrightarrow{\sim} (\mathfrak{Z}, \bullet, B),\\
&\zeta\colon (S^2(\mathcal{L})_0, \bullet, B) \xrightarrow{\sim} (\mathfrak{S}_0, \bullet, B),\\
&\zeta\colon \mathcal{V}_0 \xrightarrow{\sim} \mathfrak{V},\\
&\zeta\colon (\mathcal{A}_0, *, \mathcal{B}) \xrightarrow{\sim} (\mathfrak{A}_0, \diamond, B_A).
\end{aligned}$$

It turns out that $\mathfrak{J}$ and $\mathfrak{A}_0$ are isomorphic as vector spaces. It will be convenient in the next section to identify both, as the elements of $\mathfrak{J}$ can be viewed as endomorphisms of $\mathcal{H}$.

Proposition 5.3.9. *The restriction π_J of π to $\mathfrak{J}$ is an isomorphism of vector spaces.*

Proof. Note that for each $\alpha \in \Phi$, we have $\pi(j_\alpha) = 2\pi(z_\alpha)$. Thus, since π is surjective, its restriction to $\mathfrak{J}$ is surjective as well. Since $\mathfrak{J}$ and $\mathfrak{A}_0$ have the same dimension, the restriction of π to $\mathfrak{J}$ must be an isomorphism onto $\mathfrak{A}_0$. □

Definition 5.3.10. In the next section we will identify $\mathfrak{A}_0$ with $\mathfrak{J}$ using the isomorphism π_J from Proposition 5.3.9. In particular, we can transfer the product $\diamond$ and the bilinear form B_A to $\mathfrak{J}$:

$$\begin{aligned}
a \diamond b &:= \pi_J^{-1}(\pi_J(a) \diamond \pi_J(b)),\\
B_A(a,b) &:= B_A(\pi_J(a), \pi_J(b)),
\end{aligned}$$

for all $a, b \in \mathfrak{J}$. From Proposition 5.3.7 it follows that $(\mathfrak{J}, \diamond, B_A)$ is isomorphic to $(\mathcal{A}_0, *, \mathcal{B})$ as Frobenius algebras. This will be the starting point of the next section.

Remark 5.3.11. In the spirit of Remark 5.3.8 we now have that ${\pi_J}^{-1} \circ \zeta$ induces an isomorphism between $(\mathcal{A}_0, *, \mathcal{B})$ and $(\mathfrak{J}, \diamond, B_A)$.

The Weyl group of $\mathcal{L}$ acts on the zero weight space of $S^2(\mathcal{L})$; see Definition 1.6.12. Since this zero weight space is isomorphic to $\mathfrak{S}_0$ by Proposition 5.3.5, $\mathfrak{S}_0$ also carries the structure of a representation of the Weyl group of $\mathcal{L}$.

Definition 5.3.12. Consider the natural action of the Weyl group of $\mathcal{L}$ on the zero weight space $S^2(\mathcal{L})_0$ of $S^2(\mathcal{L})$. Due to Proposition 5.3.5, we can transfer this action to $\mathfrak{S}_0$:

$$w \cdot s := \zeta_S^{-1}(w \cdot \zeta_S(s))$$

for all $s \in \mathfrak{S}_0$ and $w \in W$. Notice that the product $\bullet$ and bilinear form B are W-equivariant. Therefore $(\mathfrak{S}_0, \bullet, B)$ is a Frobenius algebra for W. It is readily verified that $w \cdot j_\alpha = j_{w\cdot\alpha}$, $w \cdot z_\alpha = z_{w\cdot\alpha}$ and $w \cdot v_\alpha = v_{w\cdot\alpha}$ for all $\alpha \in \Phi$ and $w \in W$.

Next, we prove that the projection π is W-equivariant. Therefore its image, $\mathfrak{A}_0$, is W-invariant. This fact will be used in Section 5.6 to give $\mathfrak{A}_0$ the structure of a decomposition algebra. This also allows to compute the projection π efficiently, as we will illustrate in the remainder of this section. This is, however, not essential to the rest of our results.

Definition 5.3.13. (i) Consider the transitive action of W on the set $X = \{j_\alpha \mid \alpha \in \Phi^+\}$. Let $O_0, O_1, \ldots, O_d$ be the orbits of W on $X \times X$, where O_0 is the diagonal: $O_0 := \{(x,x) \mid x \in X\}$. Define the following intersection parameters for $0 \le i,j,k \le d$:

$$p_{ij}^k = |\{y \in X \mid (x,y) \in O_i \text{ and } (y,z) \in O_j\}|$$

where (x,z) is any element of O_k. Note that this does not depend on the choice of (x,z). (In fact, $(X, \{O_i\}_{0\le i\le d})$ is an association scheme; see Example 1.7.2.)

(ii) For each $\alpha \in \Phi$, we can write $\pi(j_\alpha)$ uniquely as

$$j_\alpha + \sum_{\beta\in\Phi^+} \mu_{(j_\alpha,j_\beta)} v_\beta$$

for certain constants $\mu_{(j_\alpha,j_\beta)} \in \mathbb{C}$.

Proposition 5.3.14. *The projection $\pi : \mathfrak{S}_0 \to \mathfrak{A}_0$ is W-equivariant. In particular $\mu_x = \mu_y$ for all $x, y \in O_i$, $0 \le i \le d$.*

Proof. Because $w \cdot v_\alpha = v_{w\cdot\alpha}$ for all $\alpha \in \Phi$ and all $w \in W$, the subspace $\mathfrak{V}$ of $\mathfrak{S}_0$ is W-invariant. Since the bilinear form B is W-equivariant, the orthogonal complement $\mathfrak{A}_0$ of $\mathfrak{V}$ as well as the orthogonal projection $\pi\colon \mathfrak{S}_0 \to \mathfrak{A}_0$ with respect to B is W-equivariant. Thus on the one hand we have

$$\pi(w \cdot j_\alpha) = j_{w\cdot\alpha} + \sum_{\beta\in\Phi^+} \mu_{(j_{w\cdot\alpha},j_\beta)} v_\beta$$

while on the other hand

$$\pi(w \cdot j_\alpha) = w \cdot \pi(j_\alpha) = j_{w\cdot\alpha} + \sum_{\beta\in\Phi^+} \mu_{(j_\alpha,j_\beta)} v_{w\cdot\beta}.$$

Since the elements of $\{j_{w\cdot\alpha}\} \cup \{v_\beta \mid \beta \in \Phi^+\}$ are linearly independent, we have $\mu_{(j_\alpha,j_\beta)} = \mu_{(j_{w\cdot\alpha},j_{w\cdot\beta})}$ for all $w \in \Phi$. $\square$

Definition 5.3.15. Let $1 \leq i \leq d$. By Proposition 5.3.14 we can define $\mu_i := \mu_x$ for any $x \in O_i$. Since the bilinear form B is W-equivariant, we can also write $b_i := B(j_\alpha, j_\beta)$ for any $(j_\alpha, j_\beta) \in O_i$.

The following proposition allows us to compute the constants μ_i and hence the projection π by solving a system of $d+1$ linear equations.

Proposition 5.3.16. *For all* $0 \leq k \leq d$ *we have*

$$\sum_{0 \leq i,j \leq d} p_{ij}^k b_j \mu_i = b_k - 2\mu_k.$$

Moreover, these equations uniquely determine the constants μ_i.

Proof. We have that

$$\pi(j_\alpha) = j_\alpha + \sum_{0 \leq i \leq d} \sum_{\substack{\beta \in \Phi^+ \\ (j_\alpha, j_\beta) \in O_i}} \mu_i v_\beta$$

if and only if

$$B(\pi(j_\alpha), v_\gamma) = B(j_\alpha, v_\gamma) + \sum_{0 \leq i \leq d} \sum_{\substack{\beta \in \Phi^+ \\ (j_\alpha, j_\beta) \in O_i}} \mu_i B(v_\beta, v_\gamma) = 0$$

for all $\gamma \in \Phi^+$. If $(j_\alpha, j_\gamma) \in O_k$ then we have, by Propositions 5.3.4 and 5.3.5,

$$\begin{aligned} B(\pi(j_\alpha), v_\gamma) &= -b_k + \sum_{0 \leq i,j \leq d} \sum_{\substack{(j_\alpha, j_\beta) \in O_i \\ (j_\beta, j_\gamma) \in O_j}} \mu_i B(j_\beta, j_\gamma) + 4\mu_k B(z_\gamma, z_\gamma), \\ &= -b_k + \sum_{0 \leq i,j \leq d} p_{ij}^k \mu_i b_j + 2\mu_k. \end{aligned}$$

This proves the statement. $\square$

5.4 Extending the product

The goal of this section is to explicitly describe the algebra $\mathcal{A}$ from Section 5.2. More precisely, we will write the product of any two elements of $\mathcal{A}$ in terms of the product on the zero weight subalgebra $\mathcal{A}_0$ studied in Section 5.3. It suffices to express the product of any two weight vectors of $\mathcal{A}$ with respect to the Cartan subalgebra $\mathcal{H}$. These weight vectors are described in Proposition 5.2.12.

We will use the action of the Lie algebra $\mathcal{L}$ on $\mathcal{A}$ to accomplish this goal. Therefore it will be essential to get a good description of this action. Since $\mathcal{L}$ is generated by the elements e_α for $\alpha \in \Phi$, it suffices to describe the action of e_α

on each of the weight-λ-spaces. This action will of course depend on the W-orbit of (α,λ). Inevitably, we need to distinguish between each of those orbits which makes the following proposition look daunting at first sight. However, in each of the cases, the action is very natural.

Proposition 5.4.1. *Let $\alpha \in \Phi$. Recall Definition 5.1.1 and the linear homomorphism ζ from Definition 5.1.2. The linear action of e_α on $\mathcal{A}$ is uniquely determined as follows.*

$$\begin{aligned}
e_\alpha \cdot \overline{h_1h_2} &= -\overline{e_\alpha(\kappa(\alpha,h_1)h_2+\kappa(\alpha,h_2)h_1)}, && \\
&= -2\overline{e_\alpha(\zeta(h_1h_2)(\alpha))} && \\
e_\alpha \cdot \overline{e_\beta h} &= \overline{h_\alpha h} && \text{if } \beta=-\alpha,\\
e_\alpha \cdot \overline{e_\beta h} &= c_{\alpha,\beta}\overline{e_{\alpha+\beta}(h+\kappa(h,\alpha)h_\beta)} && \text{if } \kappa(\alpha,\beta)=-1,\\
e_\alpha \cdot \overline{e_\beta h} &= -\kappa(\alpha,h)\overline{e_\alpha e_\beta} && \text{if } \kappa(\alpha,\beta)=0,\\
e_\alpha \cdot \overline{e_\beta h} &= 0 && \text{if } \kappa(\alpha,\beta)\geq 1,\\
e_\alpha \cdot \overline{e_{\alpha_\lambda}e_{\beta_\lambda}} &= f_{\lambda+\alpha,-\alpha}\overline{e_{\lambda+\alpha}h_\alpha} && \text{if } \kappa(\alpha,\lambda)=-2, \text{ i.e. } \lambda+\alpha\in\Phi,\\
e_\alpha \cdot \overline{e_{\alpha_\lambda}e_{\beta_\lambda}} &= c_{\alpha,\alpha_\lambda}\overline{e_{\alpha+\alpha_\lambda}e_{\beta_\lambda}} && \text{if } \kappa(\alpha,\alpha_\lambda)=-1 \text{ and } \kappa(\alpha,\beta_\lambda)=0,\\
e_\alpha \cdot \overline{e_{\alpha_\lambda}e_{\beta_\lambda}} &= c_{\alpha,\beta_\lambda}\overline{e_{\alpha_\lambda}e_{\alpha+\beta_\lambda}} && \text{if } \kappa(\alpha,\alpha_\lambda)=0 \text{ and } \kappa(\alpha,\beta_\lambda)=-1,\\
e_\alpha \cdot \overline{e_{\alpha_\lambda}e_{\beta_\lambda}} &= 0 && \text{if } \kappa(\alpha,\lambda)\geq 0,
\end{aligned}$$

where $h_1,h_2\in\mathcal{H}$, $h\in\beta^\perp$, $\alpha,\beta\in\Phi$ and $\lambda\in\Lambda_0$.

Proof. First of all, note that this enumeration exhausts all possible weight vectors of $\mathcal{A}$. Indeed, because the root system Φ is simply laced, we have for any root $\beta\in\Phi$ that $\kappa(\alpha,\beta)\in\{-2,-1,0,1,2\}$ and for any weight $\lambda\in\Lambda_0$ that $\kappa(\alpha,\lambda)\in\{-2,-1,0,1,2\}$. Moreover, if $\kappa(\alpha,\beta)=-2$ then $\alpha=-\beta$ and if $\kappa(\alpha,\lambda)=-2$, then $\lambda+\alpha\in\Phi$ by Lemma 5.2.3 (iv). The form of these weight vectors follows from Proposition 5.2.12.

The statements follow from explicit calculations using the rules from Definition 5.1.1 and the description of the generating set for $\mathcal{V}$ from Proposition 5.2.6. We will do these calculations for the case when $\alpha,\beta\in\Phi$ such that $\kappa(\alpha,\beta)=-1$. The other cases are proven analogously. Let $h\in\mathcal{H}$. By Proposition 5.2.6 it follows that $\overline{e_\beta h_\beta}=0$. Thus $\overline{e_\beta h}=\overline{e_\beta(h+\kappa(h,\alpha)h_\beta)}$. Now we have

$$\begin{aligned}
e_\alpha\cdot\overline{e_\beta h} &= e_\alpha\cdot\overline{e_\beta(h+\kappa(h,\alpha)h_\beta)},\\
&= \overline{[e_\alpha,e_\beta](h+\kappa(h,\alpha)h_\beta)}+\overline{e_\beta[e_\alpha,h+\kappa(h,\alpha)h_\beta]},\\
&= c_{\alpha,\beta}\overline{e_{\alpha+\beta}(h+\kappa(h,\alpha)h_\beta)}+0,
\end{aligned}$$

because the projection $S^2(\mathcal{L})\to\mathcal{A}\colon v\mapsto\overline{v}$ is $\mathcal{L}$-equivariant. □

The next step is to write the product of any zero weight vector and an arbitrary weight vector in terms of products between zero weight vectors. The following lemma will be crucial.

Lemma 5.4.2. *Let τ be an automorphism of the Lie algebra $\mathcal{L}$. Then τ induces an automorphism of the Frobenius algebra $(\mathcal{A}, *, \mathcal{B})$ via*

$$\tau\big(\overline{\ell_1\ell_2}\big) := \overline{\tau(\ell_1)\tau(\ell_2)},$$

for all $\ell_1, \ell_2 \in \mathcal{L}$.

Proof. Since $*$ and $\mathcal{B}$ are $\mathcal{L}$-equivariant, this is of course true if τ is an inner automorphism of the Lie algebra $\mathcal{L}$. By Corollary 1.6.11 we can assume that τ leaves the Cartan subalgebra $\mathcal{H}$ and a fixed Borel subalgebra containing $\mathcal{H}$ invariant, in other words, that it is a graph automorphism. Thus τ acts on the set of highest weights of the irreducible subrepresentations of $S^2(\mathcal{L})$. Since $\mathcal{V}$ is the only subrepresentation of $S^2(\mathcal{L})$ having the double of a root as its highest weight, τ stabilizes the subrepresentation $\mathcal{V}$ globally. Thus τ commutes with the projection $\pi\colon S^2(\mathcal{L}) \to \mathcal{A}$ from Definition 5.2.10. Therefore, by the definition of $*$ and $\mathcal{B}$ (see Proposition 5.2.14), τ must preserve $*$ and $\mathcal{B}$. $\square$

Remark 5.4.3. Note that any automorphism of the root system Φ extends to an automorphism of the Lie algebra $\mathcal{L}$ via the isomorphism theorem; see Proposition 1.6.10.

The $\mathcal{L}$-module contains three different types of weights: the zero weight, the roots $\alpha \in \Phi$ and the sums of two orthogonal roots $\lambda \in \Lambda_0$. In Section 5.3 we described the product of two vectors of weight zero. We determine the product of a zero weight vector and a vector of weight $\alpha \in \Phi$ in Proposition 5.4.4. The computation of the product of a zero weight vector and a vector of weight $\lambda \in \Lambda_0$ is the subject of Proposition 5.4.5.

Proposition 5.4.4. *Let $v, w \in \mathcal{A}$ be weight vectors of respective weights 0 and $\alpha \in \Phi$. Then*

(i) $e_\alpha \cdot e_{-\alpha} \cdot w = 2w$,

(ii) $(e_\alpha \cdot v) * (e_{-\alpha} \cdot w) = 0$ *and therefore*

$$2v * w = e_\alpha \cdot (v * (e_{-\alpha} \cdot w)),$$

(iii) $\mathcal{B}(v, w) = 0$.

Proof. (i) By Proposition 5.2.12, we can write $w = \overline{e_\alpha h}$ for some $h \in \mathcal{H}$ with $\kappa(\alpha, h) = 0$. By Proposition 5.4.1 we have

$$\begin{aligned} e_\alpha \cdot e_{-\alpha} \cdot w &= e_\alpha \cdot e_{-\alpha} \cdot \overline{e_\alpha h}, \\ &= e_\alpha \cdot -\overline{h_\alpha h}, \\ &= 2\overline{e_\alpha h}, \\ &= 2w. \end{aligned}$$

(ii) Recall that s_α is the reflection in the hyperplane orthogonal to α. Now $-s_\alpha\colon \mathcal{H} \to \mathcal{H}\colon h \mapsto -h^{s_\alpha}$ is an automorphism of the root system Φ. By the isomorphism theorem (Proposition 1.6.10) there exists an extension $\tau : \mathcal{L} \to \mathcal{L}$ of $-s_\alpha$ which is an automorphism of the Lie algebra $\mathcal{L}$ and such that $\tau(e_\alpha) = e_\alpha$. By Lemma 5.4.2, the automorphism τ induces an automorphism of the Frobenius algebra $(\mathcal{A}, *, \mathcal{B})$.

Due to Proposition 5.2.12 we can write any weight vector $x \in \mathcal{A}$ of weight α as $x = \overline{e_\alpha h'}$ for some $h' \in \mathcal{H}$ with $\kappa(\alpha, h') = 0$. Thus we have $\tau(x) = \overline{\tau(e_\alpha)\tau(h')} = -\overline{e_\alpha h'} = -x$ for any weight vector $x \in \mathcal{A}$ of weight α. Now $e_\alpha \cdot v$ is a weight vector with weight α and thus $\tau(e_\alpha \cdot v) = -e_\alpha \cdot v$. As we illustrated in part (i) we can write $e_{-\alpha} \cdot w$ as $-\overline{h_\alpha h}$ for some $h \in \mathcal{H}$ with $\kappa(\alpha, h) = 0$. Thus

$$\tau(e_{-\alpha} \cdot w) = \tau\big(\overline{h_\alpha h}\big) = -\overline{h_\alpha h} = -e_{-\alpha} \cdot w.$$

Because $*$ is $\mathcal{L}$-equivariant, the product $(e_\alpha \cdot v) * (e_{-\alpha} \cdot w)$ is a weight vector of weight α. As a result

$$\tau((e_\alpha \cdot v) * (e_{-\alpha} \cdot w)) = -(e_\alpha \cdot v) * (e_{-\alpha} \cdot w).$$

On the other hand, because τ is an automorphism of $(\mathcal{A}, *, \mathcal{B})$, we have

$$\begin{aligned}\tau((e_\alpha \cdot v) * (e_{-\alpha} \cdot w)) &= \tau(e_\alpha \cdot v) * \tau(e_{-\alpha} \cdot w),\\ &= (e_\alpha \cdot v) * (e_{-\alpha} \cdot w).\end{aligned}$$

We conclude that $(e_\alpha \cdot v) * (e_{-\alpha} \cdot w) = 0$. It follows that

$$\begin{aligned}2v * w &= v * (e_\alpha \cdot e_{-\alpha} \cdot w)\\ &= e_\alpha \cdot (v * (e_{-\alpha} \cdot w)) - (e_\alpha \cdot v) * (e_{-\alpha} \cdot w),\\ &= e_\alpha \cdot (v * (e_{-\alpha} \cdot w)),\end{aligned}$$

by (i) and the $\mathcal{L}$-equivariance of $*$.

(iii) This follows from the fact that $\mathcal{B}$ is $\mathcal{L}$-equivariant and Corollary 1.5.9 (ii). □

In a similar fashion, we will now express the product of a zero weight vector and a vector of weight $\lambda \in \Lambda_0$ in terms of products of zero weight vectors. Recall Definition 5.2.7 where we picked $\alpha_\lambda, \beta_\lambda \in \Phi$ such that $\alpha_\lambda + \beta_\lambda = \lambda$ for each $\lambda \in \Lambda_0$.

Proposition 5.4.5. *Let $v, w \in \mathcal{A}$ be weight vectors of respective weights 0 and $\lambda \in \Lambda_0$. Recall from Definition 5.2.2 that n_λ is the number of ways to write λ as the sum of two (orthogonal) roots. Write*

$$\epsilon_\lambda := \frac{1}{2n_\lambda} \sum_{\substack{\alpha \in \Phi\\ \kappa(\alpha,\lambda)=2}} \overline{e_\alpha e_{-\alpha}}.$$

Then

(i) $e_{\alpha_\lambda} \cdot e_{-\alpha_\lambda} \cdot w = 2w$,

(ii) $v * w = \mathcal{B}(v, \epsilon_\lambda)w$,

(iii) $\mathcal{B}(v, w) = 0$.

Proof. (i) By Proposition 5.2.12 we can write $w = c\overline{e_{\alpha_\lambda} e_{\beta_\lambda}}$ for some $c \in \mathbb{C}$. As a result of Proposition 5.4.1 we have

$$\begin{aligned} e_{\alpha_\lambda} \cdot e_{-\alpha_\lambda} \cdot w &= ce_{\alpha_\lambda} \cdot e_{-\alpha_\lambda} \cdot \overline{e_{\alpha_\lambda} e_{\beta_\lambda}}, \\ &= -ce_{\alpha_\lambda} \cdot \overline{h_{\alpha_\lambda} e_{\beta_\lambda}}, \\ &= 2c\overline{e_{\alpha_\lambda} e_{\beta_\lambda}}, \\ &= 2w. \end{aligned}$$

(ii) Note that by Proposition 5.2.12 it suffices to prove this for $w = \overline{e_{\alpha_\lambda} e_{\beta_\lambda}}$. Since $*$ is $\mathcal{L}$-equivariant, the product $v * w$ is a weight vector of weight λ. Because the weight-λ-space of $\mathcal{A}$ is only 1-dimensional, $v*w$ must be a scalar multiple of w. If $a \in \mathcal{A}$ such that $\mathcal{B}(w, a) \neq 0$, then this scalar multiple must be

$$\frac{\mathcal{B}(v * w, a)}{\mathcal{B}(w, a)}.$$

We claim that we can take $a = \overline{e_{\alpha_{-\lambda}} e_{\beta_{-\lambda}}}$. Recall the definition of $\mathcal{B}$ from Proposition 5.2.14 and Definition 5.1.2. We have

$$\begin{aligned} \mathcal{B}(w, a) &= \mathcal{B}(\overline{e_{\alpha_\lambda} e_{\beta_\lambda}}, \overline{e_{\alpha_{-\lambda}} e_{\beta_{-\lambda}}}) \\ &= \frac{1}{n_\lambda^2} \mathcal{B}\left(\sum_{\alpha+\beta=\lambda} f_{\alpha,\beta} e_\alpha e_\beta, \sum_{\alpha+\beta=-\lambda} f_{\alpha,\beta} e_\alpha e_\beta \right) \\ &= \frac{1}{n_\lambda^2} \sum_{\alpha+\beta=\lambda} f_{\alpha,\beta} f_{-\alpha,-\beta} \mathcal{B}(e_\alpha e_\beta, e_{-\alpha} e_{-\beta}) \\ &= \frac{f_{-\alpha_\lambda,-\beta_\lambda}}{2n_\lambda^2} \sum_{\alpha+\beta=\lambda} f_{\alpha,\beta}^2 \\ &= \frac{f_{-\alpha_\lambda,-\beta_\lambda}}{2n_\lambda^2} \sum_{\alpha+\beta=\lambda} 1 \\ &= \frac{f_{-\alpha_\lambda,-\beta_\lambda}}{2n_\lambda} \end{aligned}$$

by Lemma 5.2.13, Proposition 5.2.8 (ii) and because $n_\lambda = n_{-\lambda}$.

The triple $(\mathcal{A}, *, \mathcal{B})$ is a Frobenius algebra, so we have $\mathcal{B}(v*w, a) = \mathcal{B}(v, w*a)$. We compute $w * a$ explicitly using the definition of $*$ and $\bullet$ from Proposition 5.2.14 and Definition 5.1.2. We have

$$w * a = \overline{e_{\alpha_\lambda} e_{\beta_\lambda}} * \overline{e_{\alpha_{-\lambda},\beta_{-\lambda}}},$$

$$
\begin{aligned}
&= \frac{1}{n_\lambda^2}\overline{\left(\sum_{\alpha+\beta=\lambda} f_{\alpha,\beta} e_\alpha e_\beta\right) \bullet \left(\sum_{\alpha+\beta=-\lambda} f_{\alpha,\beta} e_\alpha e_\beta\right)},\\
&= \frac{1}{n_\lambda^2}\sum_{\alpha+\beta=\lambda} f_{\alpha,\beta} f_{-\alpha,-\beta}\overline{(e_\alpha e_\beta \bullet e_{-\alpha}e_{-\beta})},\\
&= \frac{f_{-\alpha_\lambda,-\beta_\lambda}}{4n_\lambda^2}\sum_{\alpha+\beta=\lambda}(\overline{e_\alpha e_{-\alpha}} + \overline{e_\beta e_{-\beta}}),
\end{aligned}
$$

once again by Lemma 5.2.13, Proposition 5.2.8 (ii) and because $n_\lambda = n_{-\lambda}$. By Lemma 5.2.3 (iv) we know that $w * a = \frac{f_{-\alpha_\lambda,-\beta_\lambda}}{n_\lambda}\epsilon_\lambda$. As a result, we have

$$
\begin{aligned}
v * w &= \frac{\mathcal{B}(v, w * a)}{\mathcal{B}(w, a)} w,\\
&= \mathcal{B}(v, \epsilon_\lambda) w.
\end{aligned}
$$

(iii) This follows from Corollary 1.5.9 (ii). □

We are now ready to "build" the Frobenius algebra $(\mathcal{A}, *, \mathcal{B})$. As a first step, we describe its underlying vector space.

Definition 5.4.6. Let $\mathcal{L}$, $\mathcal{H}$ and Φ be as in Definition 5.1.1. Let $\mathfrak{A}$ be the direct sum of the following spaces:

- the space $\mathfrak{J}$ from Definition 5.3.1;
- for each $\alpha \in \Phi$, a copy $\mathfrak{H}_\alpha$ of the subspace $\alpha^\perp := \{h \in \mathcal{H} \mid \kappa(h, \alpha) = 0\}$ of $\mathcal{H}$;
- a vector space with basis $\{x_\lambda \mid \lambda \in \Lambda_0\}$ indexed by the set Λ_0.

For each $h \in \mathcal{H}$, we will denote its orthogonal projection onto $\mathfrak{H}_\alpha$ by $[h]_\alpha$.

Proposition 5.4.7. *Let $\mathcal{A}$ be as in Definition 5.2.10. For each $\lambda \in \Lambda_0$, choose roots $\alpha_\lambda, \beta_\lambda \in \Phi$ such that $\alpha_\lambda + \beta_\lambda = \lambda$. Let θ be the linear map defined by*

$$
\theta : \mathfrak{A} \to \mathcal{A} : \begin{cases} j_\alpha \mapsto \overline{h_\alpha h_\alpha} \\ [h]_\alpha \mapsto \overline{e_\alpha h} \\ x_\lambda \mapsto \overline{e_{\alpha_\lambda} e_{\beta_\lambda}} \end{cases}
$$

for all $\alpha \in \Phi$, $h \in \mathcal{H}$ and $\lambda \in \Lambda_0$. Then θ is an isomorphism of vector spaces.

Proof. The restriction of θ to $\mathfrak{J}$ is the composition of the isomorphism from Proposition 5.3.9 and the inverse of the isomorphism from Proposition 5.3.7 (ii). Since also $\overline{e_\alpha h_\alpha} = 0$ by Proposition 5.2.12, the linear map θ is well-defined. By Proposition 5.2.12 it follows that θ is an isomorphism. □

Now we translate the action of $\mathcal{L}$ on $\mathcal{A}$ to $\mathfrak{A}$ using this isomorphism. We also define bilinear maps $\mathfrak{J} \times \mathfrak{A} \to \mathfrak{A}$ based on Propositions 5.4.4 and 5.4.5.

Definition 5.4.8. (i) Transfer the action of $\mathcal{L}$ on $\mathcal{A}$ to $\mathfrak{A}$ via the isomorphism θ from Proposition 5.4.7:

$$\ell \cdot v := \theta^{-1}(\ell \cdot \theta(v)).$$

Note that it is possible to write this action down explicitly using Proposition 5.4.1.

(ii) For $\lambda \in \Lambda_0$, let $\mathfrak{e}_\lambda := \theta^{-1}(\epsilon_\lambda)$, i.e.

$$\mathfrak{e}_\lambda := \frac{1}{4n_\lambda} \sum_{\substack{\alpha\in\Phi \\ \kappa(\alpha,\lambda)=2}} j_\alpha.$$

(iii) Consider the Frobenius algebra $(\mathfrak{J}, \diamond, B_A)$ from Definition 5.3.10. Note that for $h \in \alpha^\perp$ we have $e_{-\alpha} \cdot [h]_\alpha = -\theta^{-1}\big(\overline{h_\alpha h}\big) \in \mathfrak{J}$ and also $\mathfrak{e}_\lambda \in \mathfrak{J}$. Now define bilinear maps

$$\begin{aligned} *\colon \mathfrak{J} \times \mathfrak{A} &\to \mathfrak{A}, \\ \mathcal{B}\colon \mathfrak{J} \times \mathfrak{A} &\to \mathbb{C}, \end{aligned}$$

such that $\theta(v * w) = \theta(v) * \theta(w)$ for all $v \in \mathfrak{J}$ and $w \in \mathfrak{A}$. More precisely, by Definition 5.3.10 and Propositions 5.4.4 and 5.4.5 let

$$\begin{aligned} v * w &:= v \diamond w && \text{if } w \in \mathfrak{J}, \\ v * w &:= e_\alpha \cdot (v \diamond (e_{-\alpha} \cdot w)) && \text{if } w = [h]_\alpha \text{ for some } h \in \alpha^\perp \text{ and } \alpha \in \Phi, \\ v * w &:= B_A(v, \mathfrak{e}_\lambda) w && \text{if } w = c x_\lambda \text{ for some } c \in \mathbb{C} \text{ and } \lambda \in \Lambda_0. \end{aligned}$$

We prove that we can uniquely extend the maps $*$ and $\mathcal{B}$ to $\mathfrak{A} \times \mathfrak{A}$.

Theorem 5.4.9. *Let $\mathcal{L}$ be a simple complex Lie algebra with root system Φ of type A_n ($n \geq 3$), D_n ($n \geq 4$) or E_n ($n \in \{6,7,8\}$). Let $\mathfrak{A}$ be as in Definition* 5.4.6 *equipped with the $\mathcal{L}$-action from Definition* 5.4.8*. The maps $*$ and $\mathcal{B}$ from Definition* 5.4.8 *uniquely extend to $\mathfrak{A} \times \mathfrak{A}$ such that $(\mathfrak{A}, *, \mathcal{B})$ is a Frobenius algebra for $\mathcal{L}$. Moreover the isomorphism θ from Proposition* 5.4.7 *induces an isomorphism of Frobenius algebras for $\mathcal{L}$ with the Frobenius algebra $(\mathcal{A}, *, \mathcal{B})$ from Proposition* 5.2.14.

Proof. The extensions of $*$ and $\mathcal{B}$ must be $\mathcal{L}$-equivariant. Let $\alpha \in \Phi$, $h \in \alpha^\perp$ and $v \in \mathfrak{A}$. By definition of the action of $\mathcal{L}$ on $\mathfrak{A}$ and Proposition 5.4.4 (i), we have $e_\alpha \cdot e_{-\alpha} \cdot [h]_\alpha = [2h]_\alpha$. Because $*$ and $\mathcal{B}$ must be $\mathcal{L}$-equivariant, we have

$$[h]_\alpha * w = \frac{1}{2} e_\alpha \cdot ((e_{-\alpha} \cdot [h]_\alpha) * w) - \frac{1}{2}(e_{-\alpha} \cdot [h]_\alpha) * (e_\alpha \cdot w)$$

and
$$\mathcal{B}([h]_\alpha, w) = \frac{1}{2}\mathcal{B}(e_{-\alpha} \cdot [h]_\alpha, e_\alpha \cdot w).$$
Since $e_{-\alpha} \cdot [h]_\alpha \in \mathfrak{J}$, this uniquely extends $*$ and $\mathcal{B}$ to $\left(\mathfrak{J} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{H}_\alpha\right) \times \mathfrak{A}$ with $\mathfrak{H}_\alpha$ as in Definition 5.4.6. Analogously, we can extend $*$ and $\mathcal{B}$ to $\mathfrak{A} \times \mathfrak{A}$ by using Proposition 5.4.5 (i) which implies that $e_{\alpha_\lambda} \cdot e_{-\alpha_\lambda} \cdot x_\lambda = 2x_\lambda$.

Now consider the Frobenius algebra $(\mathcal{A}, *, \mathcal{B})$ from Proposition 5.2.14 and the isomorphism $\theta\colon \mathfrak{A} \to \mathcal{A}$ from Proposition 5.4.7. Then also the bilinear maps
$$\begin{aligned}\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}&\colon (v, w) \mapsto \theta^{-1}(\theta(v) * \theta(w))\\ \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}&\colon (v, w) \mapsto \mathcal{B}(\theta(v), \theta(w))\end{aligned}$$
are $\mathcal{L}$-equivariant extensions of $*$ and $\mathcal{B}$. Therefore $(\mathfrak{A}, *, \mathcal{B})$ must be a Frobenius algebra isomorphic via θ with $(\mathcal{A}, *, \mathcal{B})$. □

Remark 5.4.10. (i) The proof of Theorem 5.4.9 is constructive. It allows us to define the product recursively starting from the Frobenius algebra $(\mathfrak{J}, \diamond, B_A)$ and some structure constants, namely the constants $c_{\alpha,\beta}$, of the Lie algebra. This is much more efficient than the construction of Section 5.2 where we start with the symmetric square of the Lie algebra.

(ii) From this explicit construction, it follows that we can pick a basis for $\mathfrak{A}$ in such a way that the structure constants for the algebra $(\mathfrak{A}, *, B)$ are integers. This allows us to define this algebra over an arbitrary field by extension of scalars.

5.5 The story of the unit

Let $(\mathcal{A}, *, \mathcal{B})$ be the Frobenius algebra from Proposition 5.2.14. We prove that this algebra is unital, which means that there exists an element, called a unit, $\mathbf{1} \in \mathcal{A}$ such that $\mathbf{1} * a = a * \mathbf{1} = a$ for all $a \in \mathfrak{A}$.

From Definition 5.1.2 we know that $S^2(\mathcal{L})$ corresponds to the symmetric operators $\mathcal{L} \to \mathcal{L}$ via ζ. From the definition of the product $\bullet$ it will be immediately obvious that the identity operator $\mathsf{id}\colon \mathcal{L} \to \mathcal{L}\colon \ell \mapsto \ell$ corresponds to a unit for the algebra $(S^2(\mathcal{L}), \bullet)$.

Definition 5.5.1. Let ζ be as in Definition 5.1.2. Since the identity operator id is a symmetric operator, it is contained in the image of ζ and we can define
$$C_{\mathcal{L}} \coloneqq \zeta^{-1}(\mathsf{id}).$$
More explicitly, let $\{b_1, b_2, \ldots, b_m\}$ be a basis of $\mathcal{L}$ and let $\{b_1^*, b_2^*, \ldots, b_m^*\}$ be the basis of $\mathcal{L}$ dual to this basis with respect to the Killing form κ. Then
$$C_{\mathcal{L}} \coloneqq \sum_{1\le i\le m} b_i b_i^*.$$

Note that $C_{\mathcal{L}}$ is the Casimir element of $\mathcal{L}$ [Hum72, § 6.2]. Observe that

$$\begin{aligned} B(C_{\mathcal{L}}, \ell_1\ell_2) &= \frac{1}{2}\sum_{1\le i\le m}\kappa(b_i,\ell_1)\kappa(b_i^*,\ell_2) + \frac{1}{2}\sum_{1\le i\le m}\kappa(b_i,\ell_2)\kappa(b_i^*,\ell_1) \\ &= \kappa(\ell_1,\ell_2). \end{aligned}$$

for all $\ell_1, \ell_2 \in \mathcal{L}$ and B as in Definition 5.1.2. Since B is non-degenerate by Proposition 5.2.9, this also uniquely defines $C_{\mathcal{L}}$.

Proposition 5.5.2. *For all $a \in S^2(\mathcal{L})$ we have $C_{\mathcal{L}} \bullet a = a$.*

Proof. For any $\ell_1, \ell_2 \in \mathcal{L}$, we have

$$\begin{aligned} C_{\mathcal{L}} \bullet \ell_1\ell_2 &= \frac{1}{4}\Bigg(\sum_i \kappa(b_i,\ell_1)b_i^*\ell_2 + \sum_i \kappa(b_i^*,\ell_1)b_i\ell_2 \\ &\qquad + \sum_i \kappa(b_i,\ell_2)b_i^*\ell_1 + \sum_i \kappa(b_i^*,\ell_2)b_i\ell_1\Bigg), \\ &= \ell_1\ell_2. \end{aligned}$$

□

Next, we prove that $C_{\mathcal{L}} \in \mathcal{A}$ and that $C_{\mathcal{L}}$ is also a unit for $(\mathcal{A}, *)$.

Proposition 5.5.3. (i) *We have $\ell \cdot C_{\mathcal{L}} = 0$ for all $\ell \in \mathcal{L}$.*

(ii) *We have $C_{\mathcal{L}} \in \mathcal{A}$ and $C_{\mathcal{L}} * a = a$ for all $a \in \mathcal{A}$.*

Proof. (i) For any $\ell, \ell_1, \ell_2 \in \mathcal{L}$ we have

$$\begin{aligned} B(\ell\cdot C_{\mathcal{L}}, \ell_1\ell_2) &= B(C_{\mathcal{L}}, \ell\cdot\ell_1\ell_2) \\ &= B(C_{\mathcal{L}}, [\ell,\ell_1]\ell_2 + \ell_1[\ell,\ell_2]) \\ &= \kappa([\ell,\ell_1],\ell_2) + \kappa(\ell_1,[\ell,\ell_2]) \\ &= 0 \end{aligned}$$

because B and κ are $\mathcal{L}$-equivariant. By Proposition 5.2.9, the bilinear form B is non-degenerate and thus $\ell \cdot C_{\mathcal{L}} = 0$ for all $\ell \in \mathcal{L}$.

(ii) For any $\alpha \in \Phi$, we have $B(C_{\mathcal{L}}, e_\alpha e_\alpha) = \kappa(e_\alpha, e_\alpha) = 0$. Recall that $\mathcal{V}$ is the $\mathcal{L}$-representation generated by the elements $e_\alpha e_\alpha$. By (i), we have $B(C_{\mathcal{L}}, v) = 0$ for all $v \in \mathcal{V}$ and thus $C_{\mathcal{L}} \in \mathcal{A}$.

It follows from the definition of $*$ (see Proposition 5.2.14) and Proposition 5.5.2 that $C_{\mathcal{L}} * a = a$ for all $a \in \mathcal{A}$. □

We transfer the unit $C_{\mathcal{L}}$ from the algebra $(\mathcal{A}, *)$ to the algebra $(\mathfrak{A}, *)$ via the isomorphism θ from Theorem 5.4.9.

Definition 5.5.4. Recall $\mathfrak{A}$ from Definition 5.4.6 and the isomorphism θ from Theorem 5.4.9. Write

$$\mathbf{1} = \theta^{-1}(C_{\mathcal{L}}).$$

Theorem 5.5.5. *Let* $(\mathfrak{A}, *, \mathcal{B})$ *be as in Theorem* 5.4.9 *and* $\mathbf{1}$ *as in Definition* 5.5.4. *Then* $\mathbf{1}$ *is a unit for the algebra* $(\mathfrak{A}, *)$.

Proof. This follows immediately from Definition 5.5.4, Proposition 5.5.3, and Theorem 5.4.9. □

Note that by Proposition 5.5.3 (i) the element $\mathbf{1}$ is contained in the zero weight space of $\mathfrak{A}$, this is $\mathfrak{J}$. We can write down $\mathbf{1}$ explicitly as a linear combination of the generating set $\{j_\alpha \mid \alpha \in \Phi^+\}$ for $\mathfrak{J}$. First, we prove the following lemma.

Lemma 5.5.6. *Let* $\beta \in \Phi$ *and let* r *be the number of positive roots* $\alpha \in \Phi^+$ *such that* $\kappa(\beta, \alpha) = \pm 1$. *Then*

$$\sum_{\alpha \in \Phi^+} j_\alpha = \frac{4+r}{2} \mathsf{id}_{\mathcal{H}}$$

where $\mathsf{id}_{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H} \colon h \mapsto h$.

Proof. Since Φ is irreducible and simply laced, the value of r is independent of the choice of β. It follows from Proposition 5.3.2 that

$$B\Big(\sum_{\alpha \in \Phi^+} j_\alpha, j_\beta\Big) = 4 + r = B\Big(\frac{4+r}{2} \mathsf{id}_{\mathcal{H}}, j_\beta\Big)$$

for all $\beta \in \Phi^+$. Because B is non-degenerate on $\mathfrak{J}$ by Proposition 5.3.2, we have indeed $\sum_{\alpha \in \Phi^+} j_\alpha = \frac{4+r}{2} \mathsf{id}_{\mathcal{H}}$. □

Remark 5.5.7. Note that $r = 2n_\alpha$ for all $\alpha \in \Phi$ where n_α is as in Definition 5.2.2 because Φ is simply laced.

Proposition 5.5.8. *We have*

$$\mathbf{1} = \frac{6+r}{2} \mathsf{id}_{\mathcal{H}}.$$

Proof. Let $v_1, \dots, v_n$ be an orthonormal basis for the Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ with respect to the Killing form κ. Then $\{v_1, \dots, v_n\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ is a basis for $\mathcal{L}$. Note that for the dual basis, we have $v_i^* = v_i$ for all i and $e_\alpha^* = e_{-\alpha}$ for all $\alpha \in \Phi$. Therefore we can write $C_{\mathcal{L}}$ as

$$C_{\mathcal{L}} = \sum_i v_i v_i + \sum_{\alpha \in \Phi} e_\alpha e_{-\alpha}.$$

By Lemma 5.5.6

$$\sum_i v_i v_i = \zeta^{-1}(\mathsf{id}_{\mathcal{H}}) = \frac{2}{4+r} \sum_{\alpha \in \Phi^+} \zeta^{-1}(j_\alpha) = \frac{2}{4+r} \sum_{\alpha \in \Phi^+} h_\alpha h_\alpha.$$

Recall the projection $\pi\colon S^2(\mathcal{L}) \to \mathcal{A}\colon v \mapsto \overline{v}$ from Definition 5.2.10. By Proposition 5.2.6 we have $2\overline{e_\alpha e_{-\alpha}} = \overline{h_\alpha h_\alpha}$. As a result

$$\sum_{\alpha\in\Phi} \overline{e_\alpha e_{-\alpha}} = \sum_{\alpha\in\Phi^+} \overline{h_\alpha h_\alpha}.$$

We have $C_{\mathcal{L}} \in \mathcal{A}$ by Proposition 5.5.3 (ii), so $C_{\mathcal{L}} = \overline{C_{\mathcal{L}}}$. Thus

$$C_{\mathcal{L}} = \left(\frac{2}{4+r} + 1\right) \sum_{\alpha\in\Phi^+} \overline{h_\alpha h_\alpha} = \frac{6+r}{4+r}\theta\Big(\sum_{\alpha\in\Phi^+} j_\alpha\Big).$$

The statement now follows from Lemma 5.5.6. □

5.6 Decompositions of the zero weight subalgebra

The goal of this section is to give the algebra $(\mathfrak{J}, \diamond)$ the structure of a decomposition algebra. We will describe the general procedure and give the explicit decompositions for each of the possible types (A_n, D_n or E_n) afterwards. From now on we will only consider the product $\diamond$ on the space $\mathfrak{J}$ so will simply omit it from our notation.

5.6.1 The general procedure

Note that $\mathfrak{J}$ is the zero weight space of $\mathfrak{A}$ as $\mathcal{L}$-representation. Therefore, the Weyl group W of $\mathcal{L}$ acts by automorphisms on the algebra $\mathfrak{J}$; see Definition 5.3.12. We can use the ideas and terminology from Example 2.8.6 to obtain a decomposition algebra.

Definition 5.6.1. (i) For each $\alpha \in \Phi^+$ let $C_W(s_\alpha)$ be the centralizer in W of the reflection s_α. Since Φ is irreducible and simply laced, these subgroups are conjugate inside W by Proposition 1.3.9.

(ii) Let $\bigoplus_{x\in X_g^0} J_x$ be the global decomposition of $\mathfrak{J}$ with respect to W, cf. Example 2.8.6. Denote its global fusion law by $(X_g^0, \star)$. Let $\bigoplus_{x\in X_l^0} J_x^\alpha$ be the local decompositions of $\mathfrak{J}$ with respect to $(C_W(s_\alpha) \mid \alpha \in \Phi^+)$. Write $(X_l^0, \star)$ for the corresponding fusion law.

(iii) As in Example 2.8.6 (iii) let $J_{x,y}^\alpha := J_x \cap J_y^\alpha$ for $x \in X_g^0$ and $y \in X_l^0$. Let $\mathcal{F}_0$ be the direct product of the fusion laws $(X_g^0, \star)$ and $(X_l^0, \star)$. Write Ω_0 for the Φ^+-tuple of $\mathcal{F}_0$-decompositions $\bigoplus_{x,y} J_{x,y}^\alpha$. Then $(\mathfrak{J}, \Phi^+, \Omega_0)$ is an $\mathcal{F}_0$-decomposition algebra.

We prove that $C_W(s_\alpha)$ is a reflection subgroup of W, this is, a subgroup generated by reflections. This makes it easier to determine the local decompositions and their fusion law.

Proposition 5.6.2. *For each $\alpha \in \Phi^+$, the centralizer $C_W(s_\alpha)$ is a reflection subgroup of W. It is generated by the reflections s_β for which $\kappa(\alpha, \beta) = 0$ or $\beta = \pm\alpha$. Its Dynkin diagram can be obtained by removing the neighbors of the extending node from the extended Dynkin diagram of Φ.*

Proof. Recall from Definition 1.3.10 that the extending node corresponds to the negative of the highest root. Since W acts transitively on Φ by Proposition 1.3.9, it suffices to prove this when α is the highest root of Φ. For $w \in W$, we have ${}^{w}s_\alpha = s_\alpha$ if and only if w fixes the hyperplane orthogonal to α. This means that w must map α to $\pm\alpha$. The statement now follows from Proposition 1.3.15. □

Remark 5.6.3. The following table gives the type of the subsystem

$$\{\pm\alpha\} \cup \{\beta \in \Phi \mid \kappa(\alpha, \beta) = 0\}$$

for each of the possible types of Φ.

W	$C_W(s_\alpha)$
$A_n (n \geq 3)$	$A_1 \times A_{n-2}$
$D_n (n \geq 4)$	$D_2 \times D_{n-2}$
E_6	$A_1 \times A_5$
E_7	$A_1 \times D_6$
E_8	$A_1 \times E_7$

Here we use the convention that $D_2 \cong A_1 \times A_1$ and $D_3 \cong A_3$. This is consistent with our notation for the Weyl groups. Indeed, the Weyl group of type A_n (resp. D_n) is the group S_{n+1} (resp. W'_n) from Example 1.3.12 and we have $W'_2 \cong S_2 \times S_2$ and $W'_3 \cong S_4$.

The fusion law $\mathcal{F}_0$ is $\mathbb{Z}/2\mathbb{Z}$-graded and the corresponding Miyamoto group of the $\mathcal{F}_0$-decomposition algebra from Definition 5.6.1 is isomorphic to W. First we prove that the fusion law $(X_l^0, \star)$ is $\mathbb{Z}/2\mathbb{Z}$-graded.

Lemma 5.6.4. *The fusion law $(X_l^0, \star)$ has a non-trivial $\mathbb{Z}/2\mathbb{Z}$-grading $\xi\colon X_l^0 \to \mathbb{Z}/2\mathbb{Z}$. Let χ be the non-trivial linear $\mathbb{C}$-character of $\mathbb{Z}/2\mathbb{Z}$. The Miyamoto map $\tau_{\alpha,\chi}$ of $(\mathfrak{J}, \Phi^+, \Omega_0)$ is precisely the element s_α in its action on $\mathfrak{J}$.*

Proof. The representation fusion law $(\mathsf{Irr}(C_W(s_\alpha)), \star)$ is graded by the central subgroup $\langle s_\alpha \rangle \leq Z(C_W(s_\alpha))$ by Proposition 3.1.6. This induces a grading on the sublaw $(X_l^0, \star)$. The statement now follows from Theorem 3.3.1. Note that the grading is non-trivial since s_α (and therefore $\tau_{i,\alpha}$) acts non-trivially on $\mathfrak{J}$. □

This $\mathbb{Z}/2\mathbb{Z}$-grading induces a $\mathbb{Z}/2\mathbb{Z}$-grading of the fusion law $\mathcal{F}_0$.

Definition 5.6.5. The $\mathbb{Z}/2\mathbb{Z}$-grading ξ of $(X_l^0, \star)$ from Lemma 5.6.4 induces a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{F}_0$:

$$\xi : \mathcal{F}_0 \to \mathbb{Z}/2\mathbb{Z} : (x, y) \mapsto \xi(y).$$

Proposition 5.6.6. *Let* $(\mathfrak{J}, \Phi^+, \Omega_0)$ *be the* $\mathcal{F}_0$*-decomposition algebra from Definition* 5.6.1. *The Miyamoto group with respect to the grading of* $\mathcal{F}_0$ *from Definition* 5.6.5 *is the Weyl group* W *in its action on* $\mathfrak{J}$.

Proof. This follows from the definitions and Lemma 5.6.4. □

Remark 5.6.7. There are two ways to refine the decomposition and fusion law.

(i) Of course, many of the intersections $J^\alpha_{x,y} = J_x \cap J^\alpha_y$ for $x \in X^0_g$ and $y \in X^0_l$ are trivial. Therefore, we can omit them from our fusion law.

(ii) Instead of considering the global decomposition with respect to W, we can consider the global decomposition with respect to the automorphism group $\mathsf{Aut}(\Phi)$ of the root system Φ. If the Dynkin diagram of Φ admits a non-trivial graph automorphism, then this group is possibly larger than W but still acts by automorphisms on $\mathfrak{J}$. This leads to another global decomposition $\bigoplus_{x\in X_a} J_x$ with fusion law $(X^0_a, \star)$. Let $J^\alpha_{x,y,z} := J_x \cap J_y \cap J^\alpha_z$ for $x \in X^0_a$, $y \in X^0_g$ and $z \in X^0_l$. Then $\bigoplus_{x,y,z} J^\alpha_{x,y,z}$ will be a decomposition of $\mathfrak{J}$ whose fusion law is the direct product of $(X^0_a, \star)$, $(X^0_g, \star)$ and $(X^0_l, \star)$.

It would be cumbersome to give all the computations needed to obtain the explicit decompositions. Instead, we will only give the results, hoping the reader can fill in the details if necessary.

The necessary information about the character theory of Weyl groups can be found in [GP00]. The irreducible characters of Weyl groups are described using compositions and partitions.

Definition 5.6.8. A *composition* of a positive integer n is an ordered sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $|\lambda| := \sum_{i=1}^r \lambda_i = n$. A *partition* of a positive integer n is an unordered sequence $\lambda = [\lambda_1, \dots, \lambda_r]$ such that $|\lambda| := \sum_{i=1}^r \lambda_i = n$. For a composition λ we write $[\lambda]$ for its corresponding partition. For each partition we define a corresponding integer $a(\lambda)$, called the *a-invariant* of λ, by the formula

$$a(\lambda) := \sum_{1\le i<j\le r} \min\{\lambda_i, \lambda_j\}.$$

If we order the sequence λ such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$ then $a(\lambda) = \sum_{i=1}^r (i-1)\lambda_i$.

Let us now describe the characters of the Weyl group of type A_n, which is isomorphic to the symmetric group on $n+1$ elements; see Example 1.3.12 (i).

Definition 5.6.9. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a composition of $n+1$. We denote by S_λ the subgroup of S_{n+1} that permutes amongst themselves the first λ_1 numbers, the next λ_2 numbers and so on. Denote by $\mathbf{1}_\lambda$ the trivial character of S_λ. Then we can index the irreducible characters of S_{n+1} by the partitions of $n+1$. We write

χ_μ for the character corresponding to the partition μ. This can be done in such a way that

$$\mathsf{Ind}_{S_\lambda}^{S_{n+1}}(\mathbf{1}_\lambda) = \chi_{[\lambda]} + \text{ a linear combination of } \chi_\mu \text{ with } a(\mu) < a([\lambda]).$$

See [GP00, Theorem 5.4.7, p. 158] for a proof of this fact.

The characters of the Weyl group of type D_n can be described in a similar manner. Recall the definitions of the groups W_n and W_n' from Example 1.3.12 (ii).

Definition 5.6.10. The irreducible characters of W_n can be indexed by pairs (λ, μ) of partitions such that $|\lambda| + |\mu| = n$. We allow that $|\lambda| = n$ or $|\mu| = n$ and in that case we write $\emptyset$ for the other partition. We write $\chi_{(\lambda,\mu)}$ for the character of W_n corresponding to the partition (λ, μ). If $\lambda \neq \mu$ then the restriction $\chi'_{(\lambda,\mu)} := \mathsf{Res}_{W_n'}^{W_n} \chi_{(\lambda,\mu)}$ is an irreducible character of W_n'. We have $\chi'_{(\lambda,\mu)} = \chi'_{(\mu,\lambda)}$. If $\lambda = \mu$, then $\mathsf{Res}_{W_n'}^{W_n} \chi_{(\lambda,\lambda)}$ is the sum of two distinct irreducible characters for W_n'. We will denote these by $\chi_{(\lambda,+)}$ and $\chi_{(\lambda,-)}$. These characters exhibit all irreducible characters of W_n'.

Next, we illustrate how we can compute the representation fusion law for these groups. We will do this for the Weyl group of type A_n or, equivalently, the symmetric group on $n+1$ elements. In order to determine the representation fusion law of a group G, it suffices, by Proposition 1.4.14, to decide whether $\langle \chi_1\chi_2, \chi_3\rangle = 0$ for $\chi_1, \chi_2, \chi_3 \in \mathsf{Irr}(G)$. To this end, we can use the following lemma.

Lemma 5.6.11. *Let λ be a composition of $n+1$ and $\psi, \psi' \in \mathsf{Irr}(S_{n+1})$. Then*

$$\langle \chi_{[\lambda]}\psi, \psi'\rangle = \langle \mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathsf{Res}_{S_\lambda}^{S_{n+1}} \psi, \psi'\rangle - \langle (\mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathbf{1}_\lambda - \chi_{[\lambda]})\psi, \psi'\rangle.$$

Proof. We have

$$\begin{aligned}\chi_{[\lambda]}\psi &= (\mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathbf{1}_\lambda)\psi - (\mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathbf{1}_\lambda - \chi_{[\lambda]})\psi \\ &= \mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathsf{Res}_{S_\lambda}^{S_{n+1}} \psi - (\mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathbf{1}_\lambda - \chi_{[\lambda]})\psi\end{aligned}$$

by Proposition 1.4.22. This proves the statement. □

We can compute the irreducible constituents of $\mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathsf{Res}_{S_\lambda}^{S_{n+1}} \psi$ and $\mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathbf{1}_\lambda$ using the Littlewood–Richardson rule; see [GP00, § 6.1]. Recall that the partitions corresponding to the irreducible consituents of $\mathsf{Ind}_{S_\lambda}^{S_{n+1}} \mathbf{1}_\lambda - \chi_{[\lambda]}$ have a-invariant less than $a([\lambda])$. Therefore we can use induction on the a-invariant of $[\lambda]$ to compute $\langle \chi_{[\lambda]}\psi, \psi'\rangle$.

Determining the representation fusion law for W_n' is a bit more cumbersome but the same technique applies. The necessary background can be found in [GP00, § 5.5, § 5.6 and § 6.1].

We are now ready to give local and global decompositions of $(\mathfrak{J}, \diamond)$ and their fusion law. Note that this does only depend on the structure of $\mathfrak{J}$ as a representation for the Weyl group W.

For each of the possible types, we will give the following information:

(i) some more notation about the root system that allows a description of the decompositions;

(ii) the global decomposition $\bigoplus_{x\in X_g^0} J_x$ (elements of X_g^0 will be denoted by letters);

(iii) the characters and dimensions of the W-representations J_x for $x \in X_g^0$;

(iv) the global fusion law $(X_g^0, \star)$;

(v) the elements of the full decomposition $\bigoplus_{x\in\mathcal{F}} J_x^\alpha$ with respect to a root α; from this the local decomposition $\bigoplus_{i\in X_l^0} J_i^\alpha$ can be derived (elements of X_l^0 will be denoted by numbers);

(vi) the characters and dimensions of the $C_W(s_\alpha)$-representations J_i^α for $i \in X_l^0$;

(vii) the local fusion law $(X_l^0, \star)$.

5.6.2 Type A_n

(i) We use the description of the root system of type A_n from Example 1.3.12 (i). Denote the orthogonal projection onto $\langle\Phi\rangle = \langle b_i - b_j \mid 0 \le i, j \le n\rangle$ of a basis vector b_i by b_i'. We identify $\mathfrak{J}$ with $S^2(\mathcal{H}) = S^2(\langle\Phi\rangle)$ using Proposition 5.3.5. We will give the full decomposition $\bigoplus_{x\in\mathcal{F}} J_x^\alpha$ with respect to the root $\alpha := b_0 - b_n$. This is the highest root with respect to the base from Example 1.3.12 (i). We use the isomorphisms $W \cong S_{n+1}$ and $C_W(s_\alpha) \cong S_2 \times S_{n-1}$ to describe the characters.

(ii)

$$\begin{aligned}
J_a &:= \langle \sum_{i=0}^{n} b_i' b_i' \rangle \\
J_b &:= \langle b_i' b_i' - b_j' b_j' \mid 0 \le i, j \le n \rangle \\
J_c &:= \langle (b_i - b_j)(b_k - b_l) \mid 0 \le i, j, k, l \le n, \{i, j\} \cap \{k, l\} = \emptyset \rangle
\end{aligned}$$

(iii)

Component	Character	Dimension
J_a	$\chi_{[n+1]}$	1
J_b	$\chi_{[n,1]}$	n
J_c	$\chi_{[n-1,2]}$	$\frac{n^2-n-2}{2}$

Table 5.1: Characters and dimensions for the global decomposition of $\mathfrak{J}$ for type A_n.

(iv)

$\star$	a	b	c
a	a	b	c
b	b	a, b, c	$b, c^\ddagger$
c	c	$b, c^\ddagger$	$a, b^\ddagger, c$

Table 5.2: The fusion law $(X_g^0, \star)$ for type A_n. Entries marked with ‡ should be left out for $n = 3$.

(v)

$$
\begin{aligned}
J^\alpha_{a,1} &\coloneqq \langle \sum_{i=0}^{n} b'_i b'_i \rangle \\
J^\alpha_{b,1} &\coloneqq \langle (n+1)(b'_0 b'_0 + b'_n b'_n) - 2 \sum_{k=0}^{n} b'_k b'_k \rangle \\
J^\alpha_{b,2} &\coloneqq \langle b'_k b'_k - b'_l b'_l \mid 1 \le k, l \le n-1 \rangle \\
J^\alpha_{b,4} &\coloneqq \langle b'_0 b'_0 - b'_n b'_n \rangle \\
J^\alpha_{c,1} &\coloneqq \langle n b'_0 b'_0 + n b'_n b'_n + n(n-1) b'_0 b'_n - \sum_{k=0}^{n} b'_k b'_k \rangle \\
J^\alpha_{c,2} &\coloneqq \langle ((n-1)(b'_0 + b'_n) + 2(b'_k + b'_l))(b'_k - b'_l) \mid 1 \le k, l \le n-1 \rangle \\
J^\alpha_{c,3} &\coloneqq \langle (b'_{k_1} - b'_{l_1})(b'_{k_2} - b'_{l_2}) \mid 1 \le k_1, k_2, l_1, l_2 \le n-1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \{k_1, l_1\} \cap \{k_2, l_2\} = \emptyset \rangle \\
J^\alpha_{c,5} &\coloneqq \langle (b'_0 - b'_n)(b'_k - b'_l) \mid 1 \le k, l \le n-1 \rangle
\end{aligned}
$$

(vi)

Component	Character	Dimension
J^α_1	$3 \cdot \chi_{[2]} \times \chi_{[n-1]}$	$3 \cdot 1$
J^α_2	$2 \cdot \chi_{[2]} \times \chi_{[n-2,1]}$	$2 \cdot (n-2)$
J^α_3	$\chi_{[2]} \times \chi_{[n-3,2]}$	$\frac{n^2-5n-4}{2}$
J^α_4	$\chi_{[1,1]} \times \chi_{[n-1]}$	1
J^α_5	$\chi_{[1,1]} \times \chi_{[n-2,1]}$	$n-2$

Table 5.3: Characters and dimensions for the local decomposition of $\mathfrak{J}$ for type A_n.

(vii)

$\star$	1	2	$3^\dagger$	4	5
1	1	2	$3^\dagger$	4	5
2	2	$1, 2^\ddagger, 3^\dagger$	$2^\dagger, 3^\dagger$	5	$4, 5^\ddagger$
$3^\dagger$	$3^\dagger$	$2^\dagger, 3^\dagger$	$1^\dagger, 2^\dagger, 3^\dagger$	$\emptyset$	$5^\dagger$
4	4	5	$\emptyset$	1	2
5	5	$4, 5^\ddagger$	$5^\dagger$	2	$1, 2^\ddagger, 3^\dagger$

Table 5.4: The fusion law $(X_l^0, \star)$ for type A_n. Entries marked with ‡ should be left out for $n = 3$ and those marked with † should be left out for $n \in \{3, 4\}$.

5.6.3 Type D_n

(i) The root system of type D_n is described in Example 1.3.12 (ii). Once again, we identify $\mathfrak{J}$ with $S^2(\mathcal{H}) = S^2(\langle\Phi\rangle)$ using the isomorphism from Proposition 5.3.5. Local decompositions will be given with respect to the root $\alpha := b_1 + b_2$, the highest root with respect to the base from Example 1.3.12 (ii). We use the isomorphisms $W \cong W_n'$ and $C_W(s_\alpha) \cong W_2' \times W_{n-2}'$ to describe the characters of W and $C_W(s_\alpha)$. For the global decomposition, we make a distinction between $n = 4$ and $n > 4$. For the local decomposition, we restrict to $n > 6$. The given decompositions remain decompositions for $n \in \{4, 5, 6\}$ but the components are not isotypic.

(ii) $n = 4$

$$\begin{aligned}
J_a &:= \langle \sum_{i=1}^{4} b_i b_i \rangle \\
J_b &:= \langle b_i b_i - b_j b_j \mid 1 \le i, j \le 4 \rangle \\
J_c &:= \langle b_i b_j + b_k b_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\} \rangle \\
J_d &:= \langle b_i b_j - b_k b_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\} \rangle
\end{aligned}$$

$n > 4$

$$\begin{aligned}
J_a &:= \langle \sum_{i=1}^{n+1} b_i b_i \rangle \\
J_b &:= \langle b_i b_i - b_j b_j \mid 1 \le i < j \le n \rangle \\
J_c &:= \langle b_i b_j \mid 1 \le i < j \le n \rangle
\end{aligned}$$

(iii) $n = 4$

Component	Character	Dimension
J_a	$\chi'_{([4],\emptyset)}$	1
J_b	$\chi'_{([3,1],\emptyset)}$	3
J_c	$\chi_{([2],+)}$	3
J_d	$\chi_{([2],-)}$	3

Table 5.5: Characters and dimensions for the global decomposition of $\mathfrak{J}$ for type D_4.

$n > 4$

Component	Character	Dimension
J_a	$\chi'_{([n],\emptyset)}$	1
J_b	$\chi'_{([n-1,1],\emptyset)}$	$n-1$
J_c	$\chi'_{([n-2],[2])}$	$\frac{n(n-1)}{2}$

Table 5.6: Characters and dimensions for the global decomposition of $\mathfrak{J}$ for type D_n $(n > 4)$.

(iv) $n = 4$

$\star$	a	b	c	d
a	a	b	c	d
b	b	a, b	d	c
c	c	d	a, c	b
d	d	c	b	a, d

Table 5.7: The fusion law $(X_g^0, \star)$ for type D_4.

$n > 4$

$\star$	a	b	c
a	a	b	c
b	b	a, b	c
c	c	c	a, b, c

Table 5.8: The fusion law $(X_g^0, \star)$ for type D_n $(n > 4)$.

(v) $n > 6$

$$J^{\alpha}_{a,1} \coloneqq \langle \sum_{i=1}^{n+1} b_i b_i \rangle$$
$$J^{\alpha}_{b,1} \coloneqq \langle n(b_1b_1 + b_2b_2) - 2\sum_{k=1}^{n} b_kb_k \rangle$$
$$J^{\alpha}_{b,2} \coloneqq \langle b_kb_k - b_lb_l \mid 3 \leq k, l \leq n \rangle$$
$$J^{\alpha}_{b,6} \coloneqq \langle b_1b_1 - b_2b_2 \rangle$$
$$J^{\alpha}_{c,1} \coloneqq \langle b_1b_2 \rangle$$
$$J^{\alpha}_{c,3} \coloneqq \langle b_kb_l \mid 3 \leq k < l \leq n \rangle$$
$$J^{\alpha}_{c,4} \coloneqq \langle (b_1 + b_2)b_k \mid 3 \leq k \leq n \rangle$$
$$J^{\alpha}_{c,5} \coloneqq \langle (b_1 - b_2)b_k \mid 3 \leq k \leq n \rangle$$

(vi) $n > 6$

Component	Character	Dimension
J_1	$3 \cdot \chi'_{([2],\emptyset)} \times \chi'_{([n-2],\emptyset)}$	$3 \cdot 1$
J_2	$\chi'_{([2],\emptyset)} \times \chi'_{([n-3,1],\emptyset)}$	$n-3$
J_3	$\chi'_{([2],\emptyset)} \times \chi'_{([n-4],[2])}$	$\frac{n^2-5n+6}{2}$
J_4	$\chi_{([1],+)} \times \chi'_{([n-3],[1])}$	$n-2$
J_5	$\chi_{([1],-)} \times \chi'_{([n-3],[1])}$	$n-2$
J_6	$\chi'_{([1,1],\emptyset)} \times \chi'_{([n-2],\emptyset)}$	1

Table 5.9: Characters and dimensions for the local decomposition of $\mathfrak{J}$ for type D_n ($n > 6$).

(vii) $n > 6$

$\star$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1, 2	3	4	5	
3	3	3	1, 2, 3	4	5	
4	4	4	4	1, 2, 3	6	5
5	5	5	5	6	1, 2, 3	4
6	6			5	4	1

Table 5.10: The fusion law $(X^0_l, \star)$ for type D_n ($n > 6$).

Remark 5.6.12. Observe that the local decomposition with respect to $\alpha = b_1 + b_2$ is the same as the one with respect to $b_1 - b_2$ up to the order of the terms. This is due to the fact that the centralizers of their corresponding reflections are equal. As a result, the local fusion law is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-graded.

5.6.4 Type E_n

(i) We use the description and notation for the root systems of type E_n from Example 1.3.12 (iii). As usual, we identify $\mathfrak{J}$ with $S^2(\mathcal{H})$. Local decompositions will be given with respect to the highest root corresponding to the base from Example 1.3.12 (iii). Those are the roots $b_7 + b_8$, $b_7 + b_8$ and $b_7 - b_8$ for $n = 6, 7, 8$ respectively. The characters of the Weyl groups are given in the notation from [GP00, Table C.4 to C.6]. Also recall the Frobenius form B_A for $\mathfrak{J}$ from Definition 5.3.10.

(ii)

$$J_a := \langle \mathsf{id} \in \mathfrak{J} \rangle$$
$$J_b := \langle v \in \mathfrak{J} \mid B_A(v, \mathsf{id}) = 0 \rangle$$

(iii) $n = 6$

Component	Character	Dimension
J_a	1_p	1
J_b	20_p	20

Table 5.11: Characters and dimensions for the global decomposition of $\mathfrak{J}$ for type E_6.

$n = 7$

Component	Character	Dimension
J_a	1_a	1
J_b	27_a	27

Table 5.12: Characters and dimensions for the global decomposition of $\mathfrak{J}$ for type E_7.

$n = 8$

Component	Character	Dimension
J_a	1_x	1
J_b	35_x	35

Table 5.13: Characters and dimensions for the global decomposition of $\mathfrak{J}$ for type E_8.

(iv)

$\star$	a	b
a	a	b
b	b	a, b

Table 5.14: The fusion law $(X_g^0, \star)$ for type E_n.

(v) $n = 6$

$$\begin{aligned}
J^{\alpha}_{a,1} &:= \langle \sum_{i=1}^{8} b_i b_i \rangle \\
J^{\alpha}_{b,1} &:= \langle (b_7 + b_8)(b_7 + b_8) - \frac{1}{3} \sum_{i=1}^{8} b_i b_i \rangle \\
J^{\alpha}_{b,2} &:= \langle b'_i b'_i - b'_j b'_j \mid 1 \le i < j \le 6 \rangle \\
J^{\alpha}_{b,3} &:= \langle (b_i - b_j)(b_k - b_l) \mid 1 \le i, j, k, l \le 6, \{i, j\} \cap \{k, l\} = \emptyset \rangle \\
J^{\alpha}_{b,5} &:= \langle (b_7 + b_8)(b_i - b_j) \mid 1 \le i < j \le 6 \rangle
\end{aligned}$$

$n = 7$

$$\begin{aligned}
J^{\alpha}_{a,1} &:= \langle \mathsf{id} \in \mathfrak{J} \rangle \\
J^{\alpha}_{b,1} &:= \langle (b_7 + b_8)(b_7 + b_8) - \frac{2}{7} \sum_{i=1}^{8} b_i b_i \rangle \\
J^{\alpha}_{b,2} &:= \langle b_i b_i - b_j b_j \mid 1 \le i < j \le 6 \rangle \\
J^{\alpha}_{b,3} &:= \langle b_i b_j \mid 1 \le i < j \le 6 \rangle \\
J^{\alpha}_{b,5} &:= \langle (b_7 + b_8) b_i \mid 1 \le i \le 6 \rangle
\end{aligned}$$

$n = 8$

$$\begin{aligned}
J^{\alpha}_{a,1} &:= \langle \mathsf{id} \in \mathfrak{J} \rangle \\
J^{\alpha}_{b,1} &:= \langle \alpha\alpha - \frac{1}{4} \sum_{i=1}^{8} b_i b_i \rangle \\
J^{\alpha}_{b,3} &:= \langle vw - \frac{\kappa(v, w)}{7} \sum_{i=1}^{8} b_i b_i + \frac{\kappa(v, w)}{14} \alpha\alpha \mid v, w \in \alpha^{\perp} \rangle \\
J^{\alpha}_{b,5} &:= \langle \alpha v \mid v \in \alpha^{\perp} \rangle
\end{aligned}$$

(vi) $n = 6$

Component	Character	Dimension
J_1^α	$2 \cdot \chi_{[2]} \times \chi_{[6]}$	$2 \cdot 1$
J_2^α	$\chi_{[2]} \times \chi_{[5,1]}$	5
J_3^α	$\chi_{[2]} \times \chi_{[4,2]}$	9
J_5^α	$\chi_{[1,1]} \times \chi_{[5,1]}$	5

Table 5.15: Characters and dimensions for the local decomposition of $\mathfrak{J}$ for type E_6.

$n = 7$

Component	Character	Dimension
J_1^α	$2 \cdot \chi_{[2]} \times \chi_{([6],\emptyset)}$	$2 \cdot 1$
J_2^α	$\chi_{[2]} \times \chi_{([5,1],\emptyset)}$	5
J_3^α	$\chi_{[2]} \times \chi_{([4],[2])}$	15
J_5^α	$\chi_{[1,1]} \times \chi_{([5],[1])}$	6

Table 5.16: Characters and dimensions for the local decomposition of $\mathfrak{J}$ for type E_7.

$n = 8$

Component	Character	Dimension
J_1^α	$2 \cdot \chi_{[2]} \times 1_a$	$2 \cdot 1$
J_3^α	$\chi_{[2]} \times 27_a$	27
J_5^α	$\chi_{[1,1]} \times 7'_a$	7

Table 5.17: Characters and dimensions for the local decomposition of $\mathfrak{J}$ for type E_8.

(vii) $n = 6$

$\star$	1	2	3	5
1	1	2	3	5
2	2	$1,2,3$	$2,3$	5
3	3	$2,3$	$1,2,3$	5
5	5	5	5	$1,2,3$

Table 5.18: The fusion law $(X_l^0, \star)$ for type E_6.

$n = 7$

$\star$	1	2	3	5
1	1	2	3	5
2	2	1, 2	3	5
3	3	3	1, 2, 3	5
5	5	5	5	1, 2, 3

Table 5.19: The fusion law $(X_l^0, \star)$ for type E_7.

$n = 8$

$\star$	1	3	5
1	1	3	5
5	3	1, 3	5
5	5	5	1, 3

Table 5.20: The fusion law $(X_l^0, \star)$ for type E_8.

5.7 A decomposition of the algebra

In this section we will give the whole algebra $(\mathfrak{A}, *)$ the structure of a decomposition algebra. Once again, we apply the techniques from Example 2.8.6. This time, we will use the fact that $(\mathfrak{A}, *)$ is an algebra for $\mathcal{L}$. As for the decompositions of the zero weight subalgebra, we will first illustrate the general procedure and give the results for each of the possible types afterwards.

5.7.1 The general procedure

In order to use Example 2.8.6, we look for a class of conjugate subalgebras of $\mathcal{L}$ to obtain local decompositions. Recall the notation for a Chevalley basis from Definition 5.1.1. Since we used the reflection subgroups $C_W(s_\alpha) = N_W(\langle s_\alpha \rangle)$ to obtain local decompositions of $\mathfrak{J}$ in Section 5.6, a natural candidate are the subalgebras of the form $N_{\mathcal{L}}(\langle h_\alpha, e_\alpha, e_{-\alpha}\rangle)$.

Definition 5.7.1. Let $\mathfrak{I}$ be the class of subalgebras of $\mathcal{L}$ conjugate to the subalgebra $\langle h_\alpha, e_\alpha, e_{-\alpha}\rangle$ for some $\alpha \in \Phi$. Note that $\mathfrak{i} \cong \mathfrak{sl}_2(\mathbb{C})$ for each $\mathfrak{i} \in \mathfrak{I}$.

Proposition 5.7.2. *Let $\alpha \in \Phi$. Consider the subalgebra $\mathfrak{i} = \langle\langle e_\alpha, e_{-\alpha}\rangle\rangle$ of $\mathcal{L}$. Then*

$$N_{\mathcal{L}}(\mathfrak{i}) = \mathcal{H} \oplus \langle e_\beta \mid \beta \in \{\pm\alpha\} \cup \{\beta \in \Phi \mid \kappa(\alpha, \beta) = 0\}\rangle.$$

In particular $N_{\mathcal{L}}(\mathfrak{i})$ is reductive for each $\mathfrak{i} \in \mathfrak{I}$.

Proof. Let $\mathcal{H}$ be the Cartan subalgebra from Definition 5.1.1. Clearly we have $\mathcal{H} \leq N_{\mathcal{L}}(\mathfrak{i})$. Thus $\mathcal{H}$ normalizes $N_{\mathcal{L}}(\mathfrak{i})$. As a result, the subalgebra $N_{\mathcal{L}}(\mathfrak{i})$ must be a direct sum of common eigenspaces of the adjoint action of $\mathcal{H}$. This means that $N_{\mathcal{L}}(\mathfrak{i})$ is of the form

$$\mathcal{H} \oplus \langle e_\beta \mid \beta \in S \rangle$$

for some $S \subseteq \Phi$. The first assertion follows because $e_\beta \in N_{\mathcal{L}}(\mathfrak{i})$ for $\beta \in \Phi$ if and only if $\beta \in \{\pm\alpha\} \cup \{\beta \in \Phi \mid \kappa(\alpha, \beta) = 0\}$. Now $N_{\mathcal{L}}(\mathfrak{i})$ is reductive by [Bou05, § VIII.3 Proposition 2]. Since all elements $\mathfrak{i} \in \mathfrak{I}$ are conjugate, the same is true for the subalgebras $N_{\mathcal{L}}(\mathfrak{i})$. □

Definition 5.7.3. Let $\mathfrak{I}$ be as in Definition 5.7.1. For each $\mathfrak{i} \in \mathfrak{I}$, let $\mathcal{L}_{\mathfrak{i}} := [N_{\mathcal{L}}(\mathfrak{i}), N_{\mathcal{L}}(\mathfrak{i})]$. By Proposition 5.7.2 and Proposition 1.3.21 the subalgebra $\mathcal{L}_{\mathfrak{i}}$ is semisimple. For example, we have

$$\mathcal{L}_{\langle\!\langle e_\alpha e_{-\alpha} \rangle\!\rangle} = \langle\!\langle e_\beta \mid \beta \in \{\pm\alpha\} \cup \{\beta \in \Phi \mid \kappa(\alpha, \beta) = 0\} \rangle\!\rangle.$$

Since all elements $\mathfrak{i} \in \mathfrak{I}$ are conjugate, so are all $\mathcal{L}_{\mathfrak{i}}$ for $\mathfrak{i} \in \mathfrak{I}$.

Remark 5.7.4. The type of $\mathcal{L}_{\langle\!\langle e_\alpha, e_{-\alpha} \rangle\!\rangle}$ is given in Remark 5.6.3. Note that the Weyl group of $\mathcal{L}_{\langle\!\langle e_\alpha, e_{-\alpha} \rangle\!\rangle}$ is precisely $C_W(s_\alpha)$ by Proposition 5.6.2.

Definition 5.7.5.
(i) Let $\bigoplus_{x \in X_g} A_x$ be the global decomposition of $\mathfrak{A}$ with respect to $\mathcal{L}$. Denote its fusion law by $(X_g, \star)$. Let $\bigoplus_{x \in X_l} A_x^{\mathfrak{i}}$ be the local decomposition of $\mathfrak{A}$ with respect to $(\mathcal{L}_{\mathfrak{i}} \mid \mathfrak{i} \in \mathfrak{I})$ and $(X_l, \star)$ the local fusion law.

(ii) For each $x \in X_g$ and $y \in X_l$, let $A_{x,y}^{\mathfrak{i}} := A_x \cap A_y^{\mathfrak{i}}$ as in Example 2.8.6 (iii). Let $\mathcal{F}$ be the direct product of the fusion laws $(X_g, \star)$ and $(X_l, \star)$. Then $\bigoplus_{x,y} A_{x,y}^{\mathfrak{i}}$ is an $\mathcal{F}$-decomposition of $\mathfrak{A}$. Let Ω be the $\mathfrak{I}$-tuple of these decompositions. Then $(\mathfrak{A}, \mathfrak{I}, \Omega)$ is an $\mathcal{F}$-decomposition algebra.

As in Section 5.6 we will define a $\mathbb{Z}/2\mathbb{Z}$-grading of the fusion law $\mathcal{F}$ and determine the corresponding Miyamoto group of the $\mathcal{F}$-decomposition algebra $(\mathfrak{A}, \mathfrak{I}, \Omega)$. In Section 5.6 we obtained the $\mathbb{Z}/2\mathbb{Z}$-grading from Lemma 5.6.4 implicitly by restriction to the central subgroup $\langle s_\alpha \rangle \leq C_W(\langle s_\alpha \rangle)$. Similarly, we will obtain a $\mathbb{Z}/2\mathbb{Z}$-grading by restricting to $\mathfrak{i} \leq \mathcal{L}_{\mathfrak{i}}$.

Definition 5.7.6.
(i) Let $\mathfrak{i} \in \mathfrak{I}$. Recall that $\mathfrak{i} \cong \mathfrak{sl}_2(\mathbb{C})$. Let $h \in \mathfrak{i}$ be one of the two coroots with respect to some Cartan subalgebra of $\mathfrak{i}$. Write V for the standard representation of $\mathfrak{i}$. Then the eigenvalues for the action of the element h on V are 1 and -1. Therefore the eigenvalues of the adjoint action of h on $V^{\otimes n}$ are odd (respectively even) integers if n is odd (respectively even). Since any irreducible representation of $\mathfrak{i}$ is some subrepresentation of $V^{\otimes n}$ for some $n \in \mathbb{N}$, this divides the irreducible representations into two parts. The irreducible representations of $\mathfrak{i}$ for which the eigenvalues of the action of h are odd (respectively even), are called odd (respectively even) representations. Also note that the tensor product of two odd (or two

even) representations is a direct sum of even representations and that the tensor product of an odd and an even representation is a direct sum of odd representations. See also [FH91, §11.8 p. 150].

(ii) Obviously, $\mathfrak{i}$ is an ideal of $\mathcal{L}_\mathfrak{i}$. Therefore, the $\mathcal{L}_\mathfrak{i}$-representations $A_x^\mathfrak{i}$ for $x \in X_l$, restricted to $\mathfrak{i}$, are isomorphic to nW_x for some irreducible representation W_x of $\mathfrak{i}$ and some $n \in \mathbb{N}$. Now define

$$\xi : X_l \to \mathbb{Z}/2\mathbb{Z} : x \mapsto \begin{cases} 0 & \text{if } W_x \text{ is even,} \\ 1 & \text{if } W_x \text{ is odd.} \end{cases}$$

Lemma 5.7.7. *The map ξ induces a non-trivial $\mathbb{Z}/2\mathbb{Z}$-grading of the fusion law $(X_l, \star)$.*

Proof. The tensor product of two odd (or two even) representations is a direct sum of even representations and the tensor product of an odd and an even representation is a direct sum of odd representations. Since $(X_l, \star)$ is the representation fusion law on X_l, it follows that ξ defines a grading of $(X_l, \star)$. To prove that this grading is non-trivial it suffices to show that $\mathfrak{A}$ has an odd irreducible $\langle\langle e_\alpha, e_{-\alpha}\rangle\rangle$-subrepresentation. Equivalently, we need to show that one of the coroots of $\langle\langle e_\alpha, e_{-\alpha}\rangle\rangle$, e.g. h_α, has an eigenvector in $\mathfrak{A}$ with an odd eigenvalue. Since e.g. $h_\alpha \cdot [h]_\beta = [h]_\beta$ for any roots $\alpha, \beta \in \Phi$ with $\kappa(\alpha, \beta) = 1$ and any $h \in \mathcal{H}$, it follows that the grading is non-trivial. □

This grading induces a non-trivial grading of $\mathcal{F}$.

Definition 5.7.8. The $\mathbb{Z}/2\mathbb{Z}$-grading ξ of $(X_l, \star)$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{F}$:

$$\xi : X_g \times X_l \to \mathbb{Z}/2\mathbb{Z} : (x, y) \mapsto \xi(y).$$

Now we determine the corresponding Miyamoto group of the $\mathcal{F}$-decomposition algebra $(\mathfrak{A}, \mathfrak{I}, \Omega)$. It turns out that this Miyamoto group is isomorphic to the group of inner automorphism (this is, the adjoint Chevalley group) of $\mathcal{L}$. Recall the terminology from Section 1.6 about Chevalley groups.

The following example explains how the grading coming from odd and even representations of $\mathfrak{sl}_2(\mathbb{C})$ gives rise to involutions.

Example 5.7.9. Consider a Lie algebra $\mathfrak{i} \cong \mathfrak{sl}_2(\mathbb{C})$ together with its standard 2-dimensional representation ρ. Then $\widehat{\mathsf{Int}}\,\mathfrak{i} = \mathsf{Int}(\mathfrak{i}, \rho) \cong \mathsf{SL}_2(\mathbb{C})$ while $\mathsf{Int}\,\mathfrak{i} \cong \mathsf{PSL}_2(\mathbb{C})$. Denote the unique non-trivial element in the center of $\widehat{\mathsf{Int}}\,\mathfrak{i}$ by $\sigma_\mathfrak{i}$. Then $\sigma_\mathfrak{i}$ acts trivially on the representation ρ of $\mathfrak{i}$ (viewed as a representation for $\widehat{\mathsf{Int}\,\mathfrak{i}}$) if and only if the weight lattice $\Lambda(\rho)$ is equal to the root lattice of $\mathfrak{i}$. More precisely, $\sigma_\mathfrak{i}$ acts as 1 (respectively -1) on the even (respectively odd) representations of $\mathfrak{i}$.

Now we are ready to determine the Miyamoto group of $(\mathfrak{A}, \mathfrak{I}, \Omega)$.

Theorem 5.7.10. *Let Φ be an irreducible simply laced root system. Consider the $\mathcal{F}$-decomposition algebra $(\mathfrak{A}, \mathfrak{I}, \Omega)$ from Definition* 5.7.5*. The Miyamoto group of this algebra corresponding to the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{F}$ from Definition* 5.7.8 *is* $\mathsf{Int}(\mathcal{L}, \mathfrak{A})$*, the adjoint Chevalley group of type Φ.*

Proof. From the definition of the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{F}$ (Definitions 5.7.6 and 5.7.8) and Example 5.7.9, it follows that the action of $\tau_{\mathfrak{i},\chi}$ (with χ the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$) corresponds to the action of $\sigma_{\mathfrak{i}}$. This action is non-trivial by Lemma 5.7.7. Since the index set $\mathfrak{I}$ is closed under the action of $\mathsf{Int}\,\mathcal{L}$, the elements $\{\sigma_{\mathfrak{i}} \mid \mathfrak{i} \in \mathfrak{I}\}$ form a conjugacy class of involutions of $\mathsf{Int}(\mathcal{L}, \mathfrak{A})$. Since the weights of $\mathfrak{A}$ are contained in the root lattice $\mathcal{L}$, we have that $\mathsf{Int}(\mathcal{L}, \mathfrak{A})$ is isomorphic to the adjoint Chevalley group of type Φ and therefore simple. So the group generated by the Miyamoto maps must be isomorphic to it. □

Remark 5.7.11. Note that we have never used any information about the algebra product $*$ on $\mathfrak{A}$. Indeed, the technique that we used here is applicable to any algebra on which the Lie algebra $\mathcal{L}$ acts (non-trivially) by derivations, for example the Lie algebra itself. It will be possible to give the algebra the structure of a decomposition algebra with a $\mathbb{Z}/2\mathbb{Z}$-graded fusion law. If this grading is non-trivial then the corresponding Miyamoto group will be a Chevalley group of type Φ (but not necessarily adjoint).

Let us now give an overview of some of the techniques that we used to explicitly obtain the local and global decompositions.

In Section 5.6, we described the decompositions of the zero weight space of $\mathfrak{A}$. We can use the results from [Bro95, Corollary 1] and [Ree98] to extend these decompositions of $\mathfrak{J}$ to decompositions in $\mathfrak{A}$. They introduce the terminology of *small modules* which means that the double of a root is not a weight of the module. More precisely, they prove that, if V is a small module for a semisimple Lie algebra $\mathcal{L}$, then its zero weight space is (almost always) irreducible as a representation for the Weyl group of $\mathcal{L}$. Note that $\mathfrak{A}$ is small as a module for $\mathcal{L}$ or $\mathcal{L}_{\mathfrak{i}}$. If V is now an irreducible subrepresentation of $\mathfrak{J}$ for W (resp. $C_W(s_\alpha)$), then it follows from these results that the $\mathcal{L}$-module (resp. $\mathcal{L}_{\langle\langle e_\alpha, e_{-\alpha}\rangle\rangle}$-module) generated by V is also irreducible. Moreover, we can determine the highest weight of this module from the character of V. This already helps to get a lot of components of the global and local decompositions of $\mathfrak{A}$.

The representation fusion laws for $\mathcal{L}$ and $\mathcal{L}_{\mathfrak{i}}$ can be determined using the results from [FH91, § 25.3].

For each of the types A_n, D_n and E_n, we will continue to use the notation introduced in the corresponding subsection of Section 5.6. In particular, we recall the index sets X_g^0 and X_l^0 for the local and global decomposition. The global decomposition can then be given as follows. For each $x \in X_g^0$ we let A_x be the $\mathcal{L}$-submodule generated by J_x. From the discussion above, it follows that each A_x is an isotypic component of the $\mathcal{L}$-modules. In fact, these are all the isotypic components of $\mathfrak{A}$ as $\mathcal{L}$-module. Therefore we can take $X_g^0 = X_g$. We will give the following additional information about the decompositions of $\mathfrak{A}$.

(i) For each isotypic component A_x for $x \in X_g^0$, we will give its highest weight and dimension. We say that A_x has highest weight $n \cdot w$ if A_x is the isotypic component corresponding to the dominant weight w and the weight w has multiplicity n in A_x. The weight w is given with respect to the basis of fundamental weights. We have ordered this basis with respect to the numbering of the nodes of the Dynkin diagram from Proposition 1.3.11.

(ii) We give the global fusion law $(X_g, \star)$.

(iii) The full decomposition $\bigoplus_{(x,i)\in\mathcal{F}} A^{\mathfrak{i}}_{x,i}$ with respect to $\mathfrak{i} := \langle\langle e_\alpha, e_{-\alpha}\rangle\rangle$ is given, where α is the highest root as in Section 5.6. For $(x,i) \in X_g^0 \times X_l^0$ we let $A^{\mathfrak{i}}_{(x,i)}$ be the $\mathcal{L}_{\mathfrak{i}}$-submodule generated by $J^{\alpha}_{(x,i)}$. We extend X_l^0 to X_l and give $A^{\mathfrak{i}}_{(x,i)}$ for each $x \in X_g$ and $i \in X_l \setminus X_l^0$ for which $A^{\mathfrak{i}}_{(x,i)} \neq 0$.

(iv) We give the highest weight and dimension of each of the components of the local decomposition $\bigoplus_{i\in X_l} A^{\mathfrak{i}}_i$.

(v) Lastly, the local fusion law $(X_l, \star)$ is given.

5.7.2 Type A_n

We restrict to the case where $n > 3$ for the global decomposition and fusion law and to $n > 5$ for the local decomposition and fusion law.

(i)

Component	Highest weight	Dimension
A_a	$(0,0,\dots,0)$	1
A_b	$(1,0,\dots,0,1)$	$n(n+2)$
A_c	$(0,1,0,\dots,0,1,0)$	$\frac{(n+2)(n+1)^2(n-2)}{4}$

Table 5.21: Highest weights and dimensions for the global decomposition of $\mathfrak{A}$ for type A_n $(n > 3)$.

(ii)

$\star$	a	b	c
a	a	b	c
b	b	a,b,c	b,c
c	c	b,c	a,b,c

Table 5.22: The fusion law $(X_g, \star)$ for type A_n $(n > 3)$.

(iii)

$$\begin{aligned}
A^{\mathfrak{i}}_{b,8} &:= \langle [b'_1 + b'_i]_{b_1 - b_i}, [b'_{n+1} + b'_i]_{b_{n+1} - b_i} \mid 1 \le i \le n-1 \rangle \\
A^{\mathfrak{i}}_{b,9} &:= \langle [b'_1 + b'_i]_{b_i - b_1}, [b'_{n+1} + b'_i]_{b_i - b_{n+1}} \mid 1 \le i \le n-1 \rangle \\
A^{\mathfrak{i}}_{c,6} &:= \langle x_{b_1 + b_{n+1} - b_i - b_j} \mid 1 \le i < j \le n-1 \rangle \\
A^{\mathfrak{i}}_{c,7} &:= \langle x_{b_i + b_j - b_1 + b_{n+1}} \mid 1 \le i < j \le n-1 \rangle \\
A^{\mathfrak{i}}_{c,8} &:= \langle [b'_1]_{b_{n+1} - b_i}, [b'_{n+1}]_{b_1 - b_i} \mid 1 \le i \le n-1 \} \rangle \\
A^{\mathfrak{i}}_{c,9} &:= \langle [b'_1]_{b_i - b_{n+1}}, [b'_{n+1}]_{b_i - b_1} \mid 1 \le i \le n-1 \} \rangle \\
A^{\mathfrak{i}}_{c,10} &:= \langle [b'_i]_{b_1 - b_j}, [b'_i]_{b_{n+1} - b_j}, x_{b_1 + b_i - b_j - b_k}, x_{b_{n+1} + b_i - b_j - b_k} \mid \\
&\qquad\qquad 1 \le i, j, k \le n-1, |\{i, j, k\}| = 3 \rangle \\
A^{\mathfrak{i}}_{c,11} &:= \langle [b'_i]_{b_j - b_1}, [b'_i]_{b_j - b_{n+1}}, x_{b_j + b_k - b_1 - b_i}, x_{b_j + b_k - b_{n+1} - b_i} \mid \\
&\qquad\qquad 1 \le i, j, k \le n-1, |\{i, j, k\}| = 3 \rangle
\end{aligned}$$

(iv)

Component	Highest weight	Dimension
$A^{\mathfrak{i}}_1$	$3 \cdot (0; 0, \ldots, 0)$	$3 \cdot 1$
$A^{\mathfrak{i}}_2$	$2 \cdot (0; 1, 0, \ldots, 0, 1)$	$2 \cdot n(n-2)$
$A^{\mathfrak{i}}_3$	$(0; 0, 1, 0, \ldots, 0, 1, 0)$	$\frac{n(n-1)^2(n-4)}{2}$
$A^{\mathfrak{i}}_4$	$(2; 0, \ldots, 0)$	3
$A^{\mathfrak{i}}_5$	$(2; 1, 0, \ldots, 0, 1)$	$3n(n-2)$
$A^{\mathfrak{i}}_6$	$(0; 0, \ldots, 0, 1, 0)$	$\frac{(n-1)(n-2)}{2}$
$A^{\mathfrak{i}}_7$	$(0; 0, 1, 0, \ldots, 0)$	$\frac{(n-1)(n-2)}{2}$
$A^{\mathfrak{i}}_8$	$2 \cdot (1; 0, \ldots, 0, 1)$	$2 \cdot 2(n-1)$
$A^{\mathfrak{i}}_9$	$2 \cdot (1; 1, 0, \ldots, 0)$	$2 \cdot 2(n-1)$
$A^{\mathfrak{i}}_{10}$	$(1; 1, 0, \ldots, 0, 1, 0)$	$n(n-1)(n-3)$
$A^{\mathfrak{i}}_{11}$	$(1; 0, 1, 0, \ldots, 0, 1)$	$n(n-1)(n-3)$

Table 5.23: Highest weights and dimensions for the local decomposition of $\mathfrak{A}$ for type A_n $(n > 5)$.

(v)

$\star$	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11
2	2	1, 2, 3	2, 3	5	4, 5	6	7	8, 10	9, 11	8, 10	9, 11
3	3	2, 3	1, 2, 3		5	6	7	10	11	8, 10	9, 11
4	4	5		1, 4	2, 5			8	9	10	11
5	5	4, 5	5	2, 5	1, 2, 3, 4, 5			8, 10	9, 11	8, 10	9, 11
6	6	6	6			$7^{\ddagger}$	1, 2, 3		8, 10		8, 10
7	7	7	7			1, 2, 3	$6^{\ddagger}$	9, 11		9, 11	
8	8	8, 10	10	8	8, 10		9, 11	6	1, 2, 4, 5	6	2, 3, 5
9	9	9, 11	11	9	9, 11	8, 10		1, 2, 4, 5	7	2, 3, 5	7
10	10	8, 10	8, 10	10	8, 10		9, 11	6	2, 3, 5	6	1, 2, 3, 4, 5
11	11	9, 11	9, 11	11	9, 11	8, 10		2, 3, 5	7	1, 2, 3, 4, 5	7

Table 5.24: The fusion law $(X_l, \star)$ for type A_n $(n > 5)$. Entries marked with $\ddagger$ should be left out for $n \neq 7$.

5.7.3 Type D_n

We restrict to the case where $n > 5$ for the global decomposition and fusion law and to $n > 7$ for the local decomposition and fusion law.

(i)

Component	Highest weight	Dimension
A_a	$(0, 0, \ldots, 0)$	1
A_b	$(2, 0, \ldots, 0, 0)$	$(2n-1)(n+1)$
A_c	$(0, 0, 0, 1, 0, \ldots, 0)$	$\frac{(2n-3)(2n-1)(n-1)n}{6}$

Table 5.25: Highest weights and dimensions for the global decomposition of $\mathfrak{A}$ for type D_n $(n > 5)$.

(ii)

$\star$	a	b	c
a	a	b	c
b	b	a, b	c
c	c	c	a, b, c

Table 5.26: The fusion law $(X_g, \star)$ for type D_n $(n > 5)$.

(iii)

$$
\begin{aligned}
A^{\mathfrak{i}}_{b,7} &:= \langle [b_1]_{\pm b_1 \pm b_i}, [b_2]_{\pm b_2 \pm b_i} \mid 3 \leq i \leq n \rangle \\
A^{\mathfrak{i}}_{c,7} &:= \langle [b_2]_{\pm b_1 \pm b_i}, [b_1]_{\pm b_2 \pm b_i} \mid 3 \leq i \leq n \rangle \\
A^{\mathfrak{i}}_{c,8} &:= \langle [b_k]_{\pm b_1 \pm b_i}, [b_k]_{\pm b_2 \pm b_i}, x_{\pm b_1 \pm b_i \pm b_k \pm b_l}, x_{\pm b_2 \pm b_i \pm b_k \pm b_l} \mid \\
&\qquad\qquad 3 \leq i, j, k \leq n, |\{i, j, k\}| = 3 \rangle
\end{aligned}
$$

(iv)

Component	Highest weight	Dimension
A_1^{i}	$3 \cdot (0; 0; 0, \ldots, 0)$	$3 \cdot 1$
A_2^{i}	$(0; 0; 2, 0, \ldots, 0)$	$(2n-5)(n-1)$
A_3^{i}	$(0; 0; 0, 0, 0, 1, 0, \ldots, 0)$	$\frac{(2n-7)(2n-5)(n-3)(n-2)}{6}$
A_4^{i}	$(2; 0; 0, 1, 0, \ldots, 0)$	$3(2n-5)(n-2)$
A_5^{i}	$(0; 2; 0, 1, 0, \ldots, 0)$	$3(2n-5)(n-2)$
A_6^{i}	$(2; 2; 0, \ldots, 0)$	9
A_7^{i}	$2 \cdot (1; 1; 1, 0, \ldots, 0)$	$2 \cdot 8(n-2)$
A_8^{i}	$(1; 1; 0, 0, 1, 0, \ldots, 0)$	$\frac{8(2n-5)(n-3)(n-2)}{3}$

Table 5.27: Highest weights and dimensions for the local decomposition of $\mathfrak{A}$ for type D_n $(n > 7)$.

(v)

$\star$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1, 2	3	4	5		7	8
3	3	3	1, 2, 3	4	5		8	7, 8
4	4	4	4	1, 2, 3, 4	6	5	7, 8	7, 8
5	5	5	5	6	1, 2, 3, 5	4	7, 8	7, 8
6	6			5	4	1, 6	7	8
7	7	7	8	7, 8	7, 8	7	1, 2, 4, 5, 6	3, 4, 5
8	8	8	7, 8	7, 8	7, 8	8	3, 4, 5	1, 2, 3, 4, 5, 6

Table 5.28: The fusion law $(X_l, \star)$ for type D_n $(n > 7)$.

5.7.4 Type E_n

(i) $n = 6$

Component	Highest weight	Dimension
A_a	$(0, 0, 0, 0, 0, 0)$	1
A_b	$(1, 0, 0, 0, 0, 1)$	650

Table 5.29: Highest weights and dimensions for the global decomposition of $\mathfrak{A}$ for type E_6.

$n = 7$

Component	Highest weight	Dimension
A_a	$(0, 0, 0, 0, 0, 0, 0)$	1
A_b	$(0, 0, 0, 0, 0, 1, 0)$	1539

Table 5.30: Highest weights and dimensions for the global decomposition of $\mathfrak{A}$ for type E_7.

$n = 8$

Component	Highest weight	Dimension
A_a	$(0, 0, 0, 0, 0, 0, 0, 0)$	1
A_b	$(1, 0, 0, 0, 0, 0, 0, 0)$	3875

Table 5.31: Highest weights and dimensions for the global decomposition of $\mathfrak{A}$ for type E_8.

(ii)

$\star$	a	b
a	a	b
b	b	a, b

Table 5.32: The fusion law $(X_g, \star)$ for type E_n.

(iii) $n = 6$
Let $S = \{\beta \in \Phi \mid \kappa(\alpha, \beta) = \pm 1\}$.

$$\begin{aligned} A^{\mathfrak{i}}_{b,12} &\coloneqq \langle [b_7 + b_8]_\beta \mid \beta \in S \rangle \\ A^{\mathfrak{i}}_{b,13} &\coloneqq \langle [b_i - b_j]_\beta, x_{\beta + b_i - b_j} \mid 1 \leq i < j \leq 6, \beta \in S, \\ &\qquad\qquad \kappa(b_i + b_j, \beta) = 1, \kappa(b_i - b_j, \beta) = 0 \rangle \\ A^{\mathfrak{i}}_{b,14} &\coloneqq \langle [b_i - b_j]_\beta, x_{\beta + b_i - b_j} \mid 1 \leq i < j \leq 6, \beta \in S, \\ &\qquad\qquad \kappa(b_i + b_j, \beta) = -1, \kappa(b_i - b_j, \beta) = 0 \rangle \end{aligned}$$

$n = 7$
Also here we let $S = \{\beta \in \Phi \mid \kappa(\alpha, \beta) = \pm 1\}$.

$$\begin{aligned} A^{\mathfrak{i}}_{b,9} &\coloneqq \langle [b_7 + b_8]_\beta \mid \beta \in S \rangle \\ A^{\mathfrak{i}}_{b,10} &\coloneqq \langle [\gamma]_\beta, x_{\beta+\gamma} \mid \gamma = \pm b_i \pm b_j \text{ for } 1 \leq i < j \leq 6, \kappa(\gamma, \beta) = 0 \rangle \end{aligned}$$

$n = 8$

$$\begin{aligned} A^{\mathfrak{i}}_{b,6} &\coloneqq \langle [\alpha]_\beta \mid \beta \in \Phi, \kappa(\alpha, \beta) = \pm 1 \rangle \\ A^{\mathfrak{i}}_{b,7} &\coloneqq \langle [\gamma]_\beta, x_{\beta+\gamma} \mid \gamma, \beta \in \Phi, \kappa(\alpha, \beta) = \pm 1, \kappa(\gamma, \alpha) = \kappa(\gamma, \beta) = 0 \rangle \end{aligned}$$

(iv) $n = 6$

Component	Highest weight	Dimension
$A^{\mathfrak{i}}_1$	$2 \cdot (0;0,0,0,0,0)$	$2 \cdot 1$
$A^{\mathfrak{i}}_2$	$(0;1,0,0,0,1)$	35
$A^{\mathfrak{i}}_3$	$(0;0,1,0,1,0)$	189
$A^{\mathfrak{i}}_5$	$(2;1,0,0,0,1)$	105
$A^{\mathfrak{i}}_{12}$	$(1;0,0,1,0,0)$	40
$A^{\mathfrak{i}}_{13}$	$(1;1,1,0,0,0)$	140
$A^{\mathfrak{i}}_{14}$	$(1;0,0,0,1,1)$	140

Table 5.33: Highest weights and dimensions for the local decomposition of $\mathfrak{A}$ for type E_6.

$n = 7$

Component	Highest weight	Dimension
$A^{\mathfrak{i}}_1$	$2 \cdot (0;0,0,0,0,0,0)$	$2 \cdot 1$
$A^{\mathfrak{i}}_2$	$(0;2,0,0,0,0,0)$	77
$A^{\mathfrak{i}}_3$	$(0;0,0,0,1,0,0)$	495
$A^{\mathfrak{i}}_5$	$(2;0,1,0,0,0,0)$	198
$A^{\mathfrak{i}}_9$	$(1;0,0,0,0,1,0)$	64
$A^{\mathfrak{i}}_{10}$	$(1;1,0,0,0,0,1)$	704

Table 5.34: Highest weights and dimensions for the local decomposition of $\mathfrak{A}$ for type E_7.

$n = 8$

Component	Highest weight	Dimension
$A^{\mathfrak{i}}_1$	$2 \cdot (0;0,0,0,0,0,0,0)$	$2 \cdot 1$
$A^{\mathfrak{i}}_3$	$(0;0,0,0,0,0,1,0)$	1539
$A^{\mathfrak{i}}_5$	$(2;1,0,0,0,0,0,0)$	399
$A^{\mathfrak{i}}_6$	$(1;0,0,0,0,0,0,1)$	112
$A^{\mathfrak{i}}_7$	$(1;0,1,0,0,0,0,0)$	1824

Table 5.35: Highest weights and dimensions for the local decomposition of $\mathfrak{A}$ for type E_8.

(v) $n = 6$

$\star$	1	2	3	5	12	13	14
1	1	2	3	5	12	13	14
2	2	1, 2, 3	2, 3	5	12, 13, 14	12, 13	12, 14
3	3	2, 3	1, 2, 3	5	12, 13, 14	12, 13, 14	12, 13, 14
5	5	5	5	1, 2, 3, 5	12, 13, 14	12, 13	12, 14
12	12	12, 13, 14	12, 13, 14	12, 13, 14	1, 2, 3, 5	2, 3, 5	2, 3, 5
13	13	12, 13	12, 13, 14	12, 13	2, 3, 5	3	1, 2, 3, 5
14	14	12, 14	12, 13, 14	12, 14	2, 3, 5	1, 2, 3, 5	3

Table 5.36: The fusion law $(X_l, \star)$ for type E_6.

$n = 7$

$\star$	1	2	3	5	9	10
1	1	2	3	5	9	10
2	2	1, 2	3	5	10	9, 10
3	3	3	1, 2, 3	5	9, 10	9, 10
5	5	5	5	1, 2, 3, 5	9, 10	9, 10
9	9	10	9, 10	9, 10	1, 3, 5	2, 3, 5
10	10	9, 10	9, 10	9, 10	2, 3, 5	1, 2, 3, 5

Table 5.37: The fusion law $(X_l, \star)$ for type E_7.

$n = 8$

$\star$	1	3	5	6	7
1	1	3	5	6	7
3	3	1, 3	5	6, 7	6, 7
5	5	5	1, 3, 5	6, 7	6, 7
6	6	6, 7	6, 7	1, 3, 5	3, 5
7	7	6, 7	6, 7	3, 5	1, 3, 5

Table 5.38: The fusion law $(X_l, \star)$ for type E_8.

5.8 An algebra for E_8

In this section, we direct some more attention to the case where Φ is of type E_8 since this was the original algebra of interest. We prove that $\mathfrak{A}$ belongs to a one-parameter family of algebras. Each of these can be given the structure of an axial decomposition algebra.

Definition 5.8.1. Let $\mathfrak{A}$, equipped with the $\mathcal{L}$-equivariant bilinear product $*$ and bilinear form $\mathcal{B}$, be as in Section 5.4. Let $\mathbf{1}$ be the unit for $(\mathfrak{A}, *)$ constructed in Section 5.5 and A' the orthogonal complement of $\langle \mathbf{1} \rangle$ with respect to $\mathcal{B}$. Consider a parameter $p \in \mathbb{C}$. Define the following product and bilinear form on $\mathfrak{A}$ that depends on the parameter p:

$$\begin{aligned}(c_1\mathbf{1} + a_1) \odot (c_2\mathbf{1} + a_2) &:= (c_1c_2 + p\mathcal{B}(a_1, a_2))\mathbf{1} + c_1a_2 + c_2a_1 + a_1 * a_2\\ \mathcal{B}_p(c_1\mathbf{1} + a_1, c_2\mathbf{1} + a_2) &:= c_1c_2\mathcal{B}(\mathbf{1}, \mathbf{1}) + \mathcal{B}(a_1, a_2)(1 + p\mathcal{B}(\mathbf{1}, \mathbf{1}))\end{aligned}$$

for all $c_1, c_2 \in \mathbb{C}$ and $a_1, a_2 \in A'$. Note that we retrieve the original product $*$ and bilinear form $\mathcal{B}$ if we put $p = 0$.

The important properties of $*$ and $\mathcal{B}$ from Proposition 5.2.14 still hold for this new product and bilinear form.

Proposition 5.8.2. *The triple* $(\mathfrak{A}, \odot, \mathcal{B}_p)$ *is a Frobenius algebra for* $\mathcal{L}$ *for any choice of the parameter* $p \in \mathbb{C}$ *with* $1 + p\dim(\mathcal{L}) \neq 0$.

Proof. The $\mathcal{L}$-equivariance of $\odot$ follows from the definition of $\odot$ and Proposition 5.2.14. Also from Proposition 5.2.14 we have

$$\begin{aligned}&\mathcal{B}_p((c_1\mathbf{1} + a_1) \odot (c_2\mathbf{1} + a_2), (c_3\mathbf{1} + a_3))\\ &\qquad = c_1c_2c_3\mathcal{B}(\mathbf{1}, \mathbf{1}) + (1 + p\mathcal{B}(1, 1))(c_1\mathcal{B}(a_2, a_3)\\ &\qquad\qquad + c_2\mathcal{B}(a_1, a_3) + c_3\mathcal{B}(a_1, a_2) + \mathcal{B}(a_1 * a_2, a_3))\\ &\qquad = \mathcal{B}_p((c_{\pi(1)}\mathbf{1} + a_{\pi(1)}) \odot (c_{\pi(2)}\mathbf{1} + a_{\pi(2)}), (c_{\pi(3)}\mathbf{1} + a_{\pi(3)})).\end{aligned}$$

for all $c_1, c_2, c_3 \in \mathbb{C}$, $a_1, a_2, a_3 \in A'$ and any permutation π of $\{1, 2, 3\}$. The non-degeneracy of $\mathcal{B}_p$ follows from the non-degeneracy of $\mathcal{B}$ if $1 + p\mathcal{B}(\mathbf{1}, \mathbf{1}) \neq 0$. From the construction of the unit in Section 5.5 it follows that $\mathcal{B}(\mathbf{1}, \mathbf{1}) = \mathrm{tr}(\mathsf{id}_{\mathcal{L}}) = \dim(\mathcal{L})$. □

Next, we show that also the structure as a decomposition algebra from Definition 5.7.5 transfers to this new algebra product.

Proposition 5.8.3. *Let* $\mathfrak{I}$, Ω *and* $\mathcal{F}$ *be as in Definition* 5.7.5. *Then* $(\mathfrak{A}, \mathfrak{I}, \Omega)$ *is an* $\mathcal{F}$*-decomposition algebra for the algebra product* $\odot$ *on* $\mathfrak{A}$.

Proof. Let X_g, X_l, $\mathcal{L}_{\mathsf{i}}$, A_x, A^{i}_y and $A^{\mathsf{i}}_{x,y}$ for $x \in X_g$, $y \in X_l$ and $\mathsf{i} \in \mathfrak{I}$ be as in Definition 5.7.5. Let $e_g \in X_g$ (resp. $e_l \in X_l$) be the element corresponding to the trivial $\mathcal{L}$-module (resp. $\mathcal{L}_{\mathsf{i}}$-module). Then $A_{e_g} = A^{\mathsf{i}}_{e_g,e_l} = \langle \mathbf{1} \rangle$ (as can be seen from Section 5.7.4). Note that (e_g, e_l) is a unit for the fusion law $\mathcal{F}$ and since $\mathbf{1}$ is still a unit for the algebra $(\mathfrak{A}, \odot)$ we have indeed $A^{\mathsf{i}}_{e_g,e_l} \odot A_{x,y} \subseteq A_{x,y}$ for all $(x, y) \in X_g \times X_l$. Also, since $\mathcal{B}$ is $\mathcal{L}$-equivariant, we have $\mathcal{B}(A^{\mathsf{i}}_{e_g,e_l}, A^{\mathsf{i}}_{x,y}) = 0$ for all $(x, y) \in X_g \times X_l \setminus \{(e_g, e_l)\}$. Thus, for all $x_1, x_2 \in X_g$ and $y_1, y_2 \in X_l$ such that $(x_1, y_1) \neq (e_g, e_l) \neq (x_2, y_2)$:

$$\begin{aligned}A^{\mathsf{i}}_{x_1,y_2} \odot A^{\mathsf{i}}_{x_2,y_2} &\subseteq A^{\mathsf{i}}_{x_1,y_1} * A^{\mathsf{i}}_{x_2,y_2} + \langle \mathbf{1} \rangle\\ &\subseteq A^{\mathsf{i}}_{(x_1,y_1)\star(x_2,y_2)} + \mathcal{B}(A^{\mathsf{i}}_{x_1,y_1}, A^{\mathsf{i}}_{x_2,y_2})\langle \mathbf{1} \rangle.\end{aligned}$$

Suppose that $\mathcal{B}(A^{\mathfrak{i}}_{x_1,y_1}, A^{\mathfrak{i}}_{x_2,y_2}) \neq 0$, then there exists an $\mathcal{L}$-equivariant map $A_{x_1} \otimes A_{x_2} \to \langle \mathbf{1} \rangle$ and an $\mathcal{L}_{\mathfrak{i}}$-equivariant map $A^{\mathfrak{i}}_{y_1} \otimes A^{\mathfrak{i}}_{y_2} \to \langle \mathbf{1} \rangle$. By definition of $\mathcal{F}$ this means that $(e_g, e_l) \in (x_1, y_1) \star (x_2, y_2)$. Therefore

$$A^{\mathfrak{i}}_{(i_1,j_1)\star(i_2,j_2)} + \mathcal{B}(A^{\mathfrak{i}}_{i_1,j_1}, A^{\mathfrak{i}}_{i_2,j_2})\langle \mathbf{1} \rangle \subseteq A^{\mathfrak{i}}_{(i_1,j_1)\star(i_2,j_2)}. \qquad \square$$

In the remainder of this section we restrict to the case where Φ is of type E_8. The decomposition $\bigoplus_{x,i} A^{\mathfrak{i}}_{x,i}$ is given in Section 5.7.4. Note that there are only six non-zero components in this decomposition, namely $A^{\mathfrak{i}}_{a,1}$, $A^{\mathfrak{i}}_{b,1}$, $A^{\mathfrak{i}}_{b,3}$, $A^{\mathfrak{i}}_{b,5}$, $A^{\mathfrak{i}}_{b,6}$ and $A^{\mathfrak{i}}_{b,7}$ of respective dimensions 1, 1, 1539, 399, 112 and 1824. Each of these is irreducible as an $\mathcal{L}_{\mathfrak{i}}$-representation. The corresponding sublaw of $\mathcal{F}$ on these components is given in Table 5.39. (To preserve space we have denoted (x, i) by xi.)

$\star$	$a1$	$b1$	$b3$	$b5$	$b6$	$b7$
$a1$	$a1$	$b1$	$b3$	$b5$	$b6$	$b7$
$b1$	$b1$	$a1, b1$	$b3$	$b5$	$b6$	$b7$
$b3$	$b3$	$b3$	$a1, b1, b3$	$b5$	$b6, b7$	$b6, b7$
$b5$	$b5$	$b5$	$b5$	$a1, b1, b3, b5$	$b6, b7$	$b6, b7$
$b6$	$b6$	$b6$	$b6, b7$	$b6, b7$	$a1, b1, b3, b5$	$b3, b5$
$b7$	$b7$	$b7$	$b6, b7$	$b6, b7$	$b3, b5$	$a1, b1, b3, b5$

Table 5.39: The fusion law for type E_8.

We want to look for an axis for this decomposition on which $\mathcal{L}_{\mathfrak{i}}$ acts trivially. Such an axis must be contained in $A^{\mathfrak{i}}_1 = A^{\mathfrak{i}}_{a,1} \oplus A^{\mathfrak{i}}_{b,1}$. Therefore, we need to know the action of $A^{\mathfrak{i}}_1$ on $\mathfrak{A}$ by multiplication. Consider a Chevalley basis of $\mathcal{L}$ as in Definition 5.1.1 and take $\mathfrak{i} := \langle\!\langle e_\alpha, e_{-\alpha} \rangle\!\rangle$ for some root $\alpha \in \Phi^+$ as before. Let a_α be the projection with respect to $\mathcal{B}$ of j_α onto $A_{b,1}$. Then $A^{\mathfrak{i}}_1 = \langle \mathbf{1}, a_\alpha \rangle$ and $\mathcal{B}(\mathbf{1}, a_\alpha) = 0$. Since $\mathbf{1}$ is a unit for $(\mathfrak{A}, \odot)$, it suffices to describe

$$\mathrm{ad}_{a_\alpha} : \mathfrak{A} \to \mathfrak{A} : a \mapsto a_\alpha \odot a.$$

Note that $\mathcal{L}_{\mathfrak{i}}$ fixes a_α and hence ad_{a_α} is an isomorphism of $\mathcal{L}_{\mathfrak{i}}$-representations. Since each $\mathcal{L}_\alpha$-isotypic component $A^{\mathfrak{i}}_i$ is irreducible if $i \neq 1$, the operator ad_{a_α} must act as a scalar on each $A^{\mathfrak{i}}_i$ for $i \neq 1$, by Schur's lemma.

Proposition 5.8.4. *The linear map* ad_{a_α} *is defined by*

$$\begin{aligned}
&\mathbf{1} \mapsto a_\alpha, && \\
&a_\alpha \mapsto \left(\tfrac{1}{496} + \tfrac{1}{2}p\right)\mathbf{1} + \tfrac{9}{98}a_\alpha, && \\
&a \mapsto -\tfrac{3}{196}a && \text{if } a \in A^{\mathfrak{i}}_{b,3}, \\
&a \mapsto \tfrac{9}{196}a && \text{if } a \in A^{\mathfrak{i}}_{b,5}, \\
&a \mapsto \tfrac{9}{196}a && \text{if } a \in A^{\mathfrak{i}}_{b,6}, \\
&a \mapsto 0 && \text{if } a \in A^{\mathfrak{i}}_{b,7}.
\end{aligned}$$

Proof. Since $\mathcal{L}_{\mathfrak{i}}$ acts trivially on $\langle a_\alpha \rangle$, it follows by Schur's lemma that ad_{a_α} must act as a scalar on each $\mathcal{L}_{\mathfrak{i}}$-isotypic component that is irreducible. More precisely, if $A^{\mathfrak{i}}_{x,y}$ is irreducible (this is true for $(x, y) \in \{(b, 3), (b, 5), (b, 6), (b, 7)\}$), then for $a \in A^{\mathfrak{i}}_{x,y}$ we have $a_\alpha \odot a = \lambda a$ for some $\lambda \in \mathbb{C}$ that does not depend on the choice of $a \in A^{\mathfrak{i}}_{x,y}$. So it suffices to compute $a_\alpha \odot a$ for any $a \in A^{\mathfrak{i}}_{x,y} \setminus \{0\}$ to determine λ. We can get such an element a from the explicit description of the decomposition from Section 5.7.4. Thus we only have to compute a few products together with the product $a_\alpha \odot a_\alpha$. We have computed these products using a computer but the computation (although lengthy) can be done by hand. $\square$

If $e_{\mathfrak{i}}$ is an axis, then we must have $e_{\mathfrak{i}} \odot e_{\mathfrak{i}} \in \langle e_{\mathfrak{i}} \rangle$. Therefore, we search for idempotents or nilpotents in $A^{\mathfrak{i}}_1$.

Proposition 5.8.5. (i) *If $p \neq -\frac{614}{74431}$, then the subalgebra $(A^{\mathfrak{i}}_1, \odot)$ of $(\mathfrak{A}, \odot)$ is generated by two primitive, orthogonal idempotents.*

(ii) *If $p = -\frac{614}{74431}$, then the subalgebra $(A^{\mathfrak{i}}_1, \odot)$ of $(\mathfrak{A}, \odot)$ is generated by* $\mathbf{1}$ *and a nilpotent element.*

Proof. An arbitrary element of $A^{\mathfrak{i}}_1$ is of the form $c_1\mathbf{1} + c_2 a_\alpha$ for $c_1, c_2 \in \mathbb{C}$. From Proposition 5.8.4 it follows that

$$(c_1\mathbf{1} + c_2 a_\alpha)^2 = \left(c_1^2 + \left(\frac{1}{496} + \frac{1}{2}p\right) c_2^2\right)\mathbf{1} + \left(2c_1c_2 + \frac{9}{98}c_2^2\right) a_\alpha.$$

Expressing that this element is an idempotent amounts to solving a system of two non-linear equations. A small calculation shows that we have 4 solutions (including the trivial solutions $c_1 = c_2 = 0$ and $c_1 = 1$, $c_2 = 0$) if $p \neq -\frac{614}{74431}$. If not, then we only have the two trivial solutions but then $\mathbf{1} - \frac{196}{9} a_\alpha$ is nilpotent. $\square$

From now on we will always assume that $p \neq -\frac{614}{74431}$.

Definition 5.8.6. Let $e_{\mathfrak{i}}$ be one of the idempotents from Proposition 5.8.5 (i). Then $\mathbf{1} - e_{\mathfrak{i}}$ is the other idempotent. Write $e_{\mathfrak{i}} = c_1\mathbf{1} + c_2 a_\alpha$ where $a_\alpha \in A'$. Explicitly we have

$$c_1 = \frac{1}{2} \pm \frac{9\sqrt{62}}{8\sqrt{74431p + 614}},$$
$$c_2 = \pm \frac{49\sqrt{62}}{2\sqrt{74431p + 614}}.$$

Because we assume $p \neq -\frac{614}{74431}$, we have $c_1 \neq \frac{1}{2}$. Therefore, we can distinguish between the two idempotents by computing $\mathcal{B}(e_{\mathfrak{i}}, \mathbf{1}) = c_1 \mathcal{B}(\mathbf{1}, \mathbf{1})$. Now we pick $e_{\mathfrak{i}}$ for each $\mathfrak{i} \in \mathfrak{I}$ such that $\mathcal{B}(e_{\mathfrak{i}}, \mathbf{1})$ is constant for all $\mathfrak{i} \in \mathfrak{I}$. Also let $A^{\mathfrak{i}}_e := \langle e_{\mathfrak{i}} \rangle$ and $A^{\mathfrak{i}}_{e'} := \langle \mathbf{1} - e_{\mathfrak{i}} \rangle$.

Theorem 5.8.7. *Let Φ be an irreducible root system of type E_8, $(\mathfrak{A}, \odot)$ the algebra parametrized by p from Definition* 5.8.1 *and c_1 as in Definition* 5.8.6. *Let $\mathfrak{I}$ be as in Definition* 5.7.5. *For each $\mathfrak{i} \in \mathfrak{I}$ let $e_\mathfrak{i}$, $A_e^\mathfrak{i}$ and $A_{e'}^\mathfrak{i}$ be as in Definition* 5.8.6 *and $A_3^\mathfrak{i}$, $A_5^\mathfrak{i}$, $A_6^\mathfrak{i}$ and $A_7^\mathfrak{i}$ as in Section* 5.7.4.

(i) *The decomposition $A_e^\mathfrak{i} \oplus A_{e'}^\mathfrak{i} \oplus A_3^\mathfrak{i} \oplus A_5^\mathfrak{i} \oplus A_6^\mathfrak{i} \oplus A_7^\mathfrak{i}$ is an $\mathcal{F}'$-decomposition of $(\mathfrak{A}, \odot)$ where $\mathcal{F}'$ is the fusion law from Table* 5.40. *The element e is a unit for $\mathcal{F}'$.*

(ii) *Let Ω be the tuple of decompositions from* (i) *indexed by $\mathfrak{I}$. Then the quadruple $(\mathfrak{A}, \mathfrak{I}, \Omega, \mathfrak{i} \mapsto e_\mathfrak{i})$ is an axial decomposition algebra with evaluation map*

$$\begin{aligned} e &\mapsto 1 \\ e' &\mapsto 0 \\ 3 &\mapsto \tfrac{4}{3}c_1 - \tfrac{1}{6} \\ 5 &\mapsto \tfrac{1}{2} \\ 6 &\mapsto \tfrac{1}{2} \\ 7 &\mapsto c_1 \end{aligned}$$

(iii) *The algebra $(\mathfrak{A}, \odot)$ is generated by the idempotents $e_\mathfrak{i}$ for $\mathfrak{i} \in \mathfrak{I}$ if $c_1 \neq 0$. If $c_1 = 0$, then these idempotents generate the subalgebra A' with A' as in Definition* 5.8.1.

Proof. Part (i) and (ii) follow immediately from the calculations in Section 5.7.4 and Propositions 5.8.4 and 5.8.5. Since the elements $\mathfrak{i} \in \mathfrak{I}$ are conjugate for the action of $\mathcal{L}$, also the idempotents $e_\mathfrak{i}$ must be conjugate. Hence they span a $\mathcal{L}$-invariant subspace of $\mathfrak{A}$. Assume that $c_1 \neq 0$. From the global decomposition (Table 5.31) we know that $\mathfrak{A}$ only has two proper $\mathcal{L}$-invariant subspaces, namely $\langle \mathbf{1} \rangle$ and its orthogonal complement with respect to $\mathcal{B}$. Since $\mathcal{L}$ acts non-trivial on the idempotents $e_\mathfrak{i}$ and $\mathcal{B}(\mathbf{1}, e_\mathfrak{i}) = c_1\mathcal{B}(\mathbf{1}, \mathbf{1}) \neq 0$ it follows that $\mathfrak{A}$ is spanned by the elements $e_\mathfrak{i}$. In particular the algebra $(\mathfrak{A}, \odot)$ is generated by them. If $c_1 = 0$, then $p = -\frac{1}{248}$ and $\mathcal{B}(\mathbf{1}, e_\mathfrak{i}) = 0$. Moreover, if $a, b \in A'$, then $\mathcal{B}_p(a \odot b, \mathbf{1}) = \mathcal{B}_p(a, b) = 0$ by definition of $\mathcal{B}_p$. Therefore A' is a subalgebra of $(\mathfrak{A}, \odot)$. Since A' is irreducible as $\mathcal{L}$-module, it must be spanned, and therefore generated, by the idempotents $e_\mathfrak{i}$ for $\mathfrak{i} \in \mathfrak{I}$. □

Remark 5.8.8. (i) Because the global decomposition for types A_n and D_n contains three terms (see Section 5.7), it is possible to write down an $\mathcal{L}$-equivariant product, as in Definition 5.8.1, with two degrees of freedom instead of one. If we write $A_a = \langle \mathbf{1} \rangle$, A_b and A_c for the components of the global decomposition, then these subspace are orthogonal with respect to the Frobenius form $\mathcal{B}$. We can define a new product on $\mathfrak{A}$ with two parameters p_1 and p_2 such that

$$a \odot b = a * b + p_1\mathcal{B}(a_b, b_b)\mathbf{1} + p_2\mathcal{B}(a_c, b_c)\mathbf{1}$$

$\star$	e	e'	3	5	6	7
e	e		3	5	6	7
e'		e'	3	5	6	7
3	3	3	$e, e', 3$	5	$6, 7$	$6, 7$
5	5	5	5	$e, e', 3, 5$	$6, 7$	$6, 7$
6	6	6	$6, 7$	$6, 7$	$e, e', 3, 5$	$3, 5$
7	7	7	$6, 7$	$6, 7$	$3, 5$	$e, e', 3, 5$

Table 5.40: The fusion law $\mathcal{F}'$.

where a_x (resp. b_x) is the projection of a (resp. b) onto A_x for $x \in \{b, c\}$.

(ii) If Φ is of type A_n, D_n, E_6 or E_7, we can also try to find idempotents $e_{\mathfrak{i}}$ in the subalgebra $A_1^{\mathfrak{i}}$. However, in Proposition 5.8.4 we used Schur's lemma to derive the adjoint action of the elements of $A_a^{\mathfrak{i}}$. Note that this is no longer possible for the terms of the local decomposition that are not irreducible $\mathcal{L}_{\mathfrak{i}}$-representations. This would lead to further difficulties when trying to establish the diagonalizability of the adjoint action of such an idempotent $e_{\mathfrak{i}}$.

5.9 Computing the character of $\mathcal{V}$

In this section we prove Proposition 5.2.5, which gives the character of $\mathcal{V}$ as an $\mathcal{L}$-representation. We use Freudenthal's formula (Proposition 1.5.15) to compute this character in a combinatorial way. Although this character is essential in Proposition 5.2.6, the computation is quite technical. We use the notation from Definition 5.1.1. Recall the definition of Λ_i for $-2 \leq i \leq 2$ and n_λ for $\lambda \in \Lambda_i$ from Definition 5.2.2. In addition to Lemma 5.2.3, we prove the following combinatorial properties about the weights $\lambda \in \Lambda_i$.

Lemma 5.9.1. (i) *$n_0 = \frac{|\Phi|}{2}$ and $n_\lambda = 1$ for $\lambda \in \Lambda_1 \cup \Lambda_2$.*

(ii) *Suppose $\omega \in \Phi$ is the highest root of Φ and $\lambda \in \bigcup_{0 \leq i \leq 2} \Lambda_i$ is dominant. Then $\lambda = \omega + \psi$ for some $\psi \in \Phi^+$.*

(iii) *If $\lambda \in \Lambda_k$ and $\lambda + i\alpha \in \Lambda_j$ for $i \geq 1$, $\alpha \in \Phi$ and $-2 \leq j, k \leq 2$, then $j = i^2 + i\kappa(\lambda, \alpha) + k$.*

(iv) *Let $\lambda \in \Lambda$ be dominant and $\omega \in \Phi$ the highest root. Suppose $f \in W$ such that $f(\lambda) = \lambda$ and $f(2\omega - \lambda) = \lambda - 2\omega$. Then $\sum_{\alpha \in \Phi^+} \kappa(2\omega - \lambda, \alpha) = \sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\lambda,\alpha)=0}} \kappa(2\omega, \alpha)$.*

(v) *Suppose $\alpha, \beta \in \Phi$ such that $\kappa(\alpha, \beta) = 0$. The number of roots $\gamma \in \Phi^+$ such that $\kappa(\alpha, \gamma) = -\kappa(\beta, \gamma) = \pm 1$ is $2(n_{\alpha+\beta} - 1)$.*

(vi) *For each $\lambda \in \Lambda_0$, we have $n_\lambda > 1$.*

Proof. (i) Of course $n_0 = \frac{|\Phi|}{2}$ because $\alpha + \beta = 0$ for $\alpha, \beta \in \Phi$ if and only if $\alpha = -\beta$. Let $\lambda \in \Lambda_1 \cup \Lambda_2$. Suppose $\lambda = \alpha + \beta = \alpha' + \beta'$ for some $\alpha, \beta, \alpha', \beta' \in \Phi$ for which $\kappa(\alpha, \beta) > 0$. Then $\kappa(\alpha', \alpha + \beta) \geq 3$. Since Φ is simply laced, $\kappa(\alpha', \alpha) = 2$ or $\kappa(\alpha', \beta) = 2$ and thus $\alpha' = \alpha$ or $\beta' = \beta$. Therefore $n_\lambda = 1$.

(ii) Let $\lambda \in \bigcup_{0 \leq i \leq 2} \Lambda_i$ be dominant and write $\lambda = \alpha + \beta$ for some $\alpha, \beta \in \Phi$. Suppose that α is maximal with respect to the partial order $\preccurlyeq$ induced by the base Δ. If $\alpha \neq \omega$, then there exists a root $\gamma \in \Phi^+$ for which $\kappa(\alpha, \gamma) = -1$. Since λ is dominant and $\gamma \in \Phi^+$, we have $\kappa(\beta, \gamma) = 1$. Therefore, both $\alpha + \gamma$ and $\beta - \gamma$ are roots and $\lambda = (\alpha + \gamma) + (\beta - \gamma)$. This contradicts the fact that α was maximal. Thus $\lambda = \omega + \psi$ for some $\psi \in \Phi$. Because $\kappa(\lambda, \psi) > 1$, the root ψ must be positive.

(iii) Suppose $\lambda + i\alpha \in \Lambda_j$ for some $i \geq 1$, $\alpha \in \Phi^+$ and $-2 \leq j \leq 2$. Then, by Lemma 5.2.3 (i), $4 + 2j = \kappa(\lambda + i\alpha, \lambda + i\alpha) = 4 + 2k + 2i\kappa(\lambda, \alpha) + 2i^2$. Thus $j = i^2 + i\kappa(\lambda, \alpha) + k$.

(iv) If $\alpha \in \Phi^+$ such that $\kappa(\lambda, \alpha) > 0$ then also $\kappa(\lambda, f(\alpha)) > 0$. Because λ is dominant, $f(\alpha)$ must be positive. Now $\kappa(2\omega - \lambda, \alpha + f(\alpha)) = 0$.

(v) If $\gamma \in \Phi^+$ such that $\kappa(\alpha, \gamma) = -\kappa(\beta, \gamma) = \pm 1$ then $\{s_\gamma(\alpha), s_\gamma(\beta)\} \in N_{\alpha+\beta}$. Conversely, if $\{\gamma, \delta\} \in N_{\alpha+\beta} \setminus \{\{\alpha, \beta\}\}$, then precisely two of the four roots $\pm(\gamma - \alpha)$ and $\pm(\gamma - \beta)$ are positive and they satisfy the necessary requirement.

(vi) Suppose that $\lambda = \omega + \psi$ for some $\omega, \psi \in \Phi$. We claim that there exists a root $\beta \in \Phi$ such that $\kappa(\omega, \beta) = \pm 1$ and $\kappa(\psi, \beta) = \pm 1$. If not, then every root would be orthogonal to either ω or ψ, which contradicts the irreducibility of Φ. Now ω, ψ and β form a root subsystem of Φ of type A_3 and inside this subsystem, we can find another way to write λ as the sum of two orthogonal roots. □

We are ready to prove Proposition 5.2.5.

Proposition 5.9.2. *The character of $\mathcal{V}$ is given by*

$$\mathrm{ch}_{\mathcal{V}} = n_0 e^0 + \sum_{\lambda \in \Lambda_{-1}} (n_\lambda + 1) e^\lambda + \sum_{\lambda \in \Lambda_0} (n_\lambda - 1) e^\lambda + \sum_{\lambda \in \Lambda_1 \cup \Lambda_2} e^\lambda.$$

Proof. Let m_λ be the dimension of the weight-λ-space of $\mathcal{V}$. Write ρ for the half-sum of all positive roots, this is, $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and ω for the highest root of Φ. According to Freudenthal's formula (Proposition 1.5.15):

$$(\kappa(2\omega + \rho, 2\omega + \rho) - \kappa(\lambda + \rho, \lambda + \rho)) m_\lambda = 2 \sum_{\alpha \in \Phi^+} \sum_{i \geq 1} \kappa(\lambda + i\alpha, \alpha) m_{\lambda + i\alpha}. \quad (5.4)$$

We can rewrite the left hand side as

$$\left(\kappa(2\omega, 2\omega) - \kappa(\lambda, \lambda) + \sum_{\alpha \in \Phi^+} \kappa(2\omega - \lambda, \alpha)\right) m_\lambda.$$

We compute the values of m_λ inductively.

Claim 1: $m_\lambda = 1$ for all $\lambda \in \Lambda_2$. Since $\mathcal{V}$ is isomorphic to the highest weight representation of $\mathcal{L}$ of highest weight 2ω, we have $m_{2\omega} = 1$. Because Φ is irreducible and simply laced, W acts transitively on Φ. Therefore $m_{2\alpha} = 1$ for all $\alpha \in \Phi$, or $m_\lambda = 1$ for all $\lambda \in \Lambda_2$.

Claim 2: $m_\lambda = 1$ for all $\lambda \in \Lambda_1$. Now, let $\lambda \in \Lambda_1$ be dominant. Suppose $\lambda + i\alpha \in \Lambda_j$ for some $\alpha \in \Phi^+$ and $i \geq 1$. Because λ is dominant $\kappa(\lambda, \alpha) \geq 0$ and, by Lemma 5.9.1 (iii), this is only possible if $j = 2$, $i = 1$ and $\kappa(\lambda, \alpha) = 0$. Thus $\lambda + \alpha = 2\beta$ for some $\beta \in \Phi$ and $\kappa(\beta, \alpha) = 1$. Thus $\beta - \alpha$ is a root. Because $n_\lambda = 1$, $\beta + (\beta - \alpha)$ is the unique way to write λ as the sum of two roots. By Lemma 5.9.1 (ii) $\beta = \omega$ and $\alpha = 2\omega - \lambda \in \Phi^+$. Thus there exists precisely one $\alpha \in \Phi^+$ for which $\lambda + \alpha \in \Lambda_2$. From equation (5.4) we have

$$\left(\kappa(2\omega, 2\omega) - \kappa(\lambda, \lambda) + \sum_{\alpha \in \Phi^+} \kappa(2\omega - \lambda, \alpha)\right) m_\lambda = 2\kappa(2\omega, \lambda - 2\omega) = 4$$

In order to calculate the left hand side we write $\lambda = \omega + \psi$ for some $\psi \in \Phi^+$. Then $2\omega - \lambda = \omega - \psi$ is a positive root. Apply Lemma 5.9.1 (iv) with $f = s_{\omega - \psi}$. If $\alpha \in \Phi^+$ and $\kappa(\lambda, \alpha) = 0$ then $\{s_\alpha(\omega), s_\alpha(\psi)\} \in N_\lambda$. Because $n_\lambda = 1$ either $\kappa(\omega, \alpha) = 1$ or $\alpha = \omega - \psi$. Thus we have

$$\begin{aligned}\left(\kappa(2\omega, 2\omega) - \kappa(\lambda, \lambda) + \sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\lambda, \alpha) = 0}} \kappa(2\omega, \alpha)\right) m_\lambda &= 4, \\ (8 - 6 + \kappa(2\omega, \omega - \psi))\, m_\lambda &= 4.\end{aligned}$$

Hence $m_\lambda = 1$ for all $\lambda \in \Lambda_1$.

Claim 3: $m_\lambda = n_\lambda - 1$ for all $\lambda \in \Lambda_0$. Consider a dominant weight $\lambda \in \Lambda_0$. If $\lambda + i\alpha \in \Lambda_j$ for some $i \geq 1$, $\alpha \in \Phi^+$ and $-2 \leq j \leq 2$ then, by Lemma 5.9.1 (iii), we have $i = 1$, $j \geq 1$ and $\kappa(\lambda, \alpha) = j - 1$. If $j = 2$ then $\kappa(\lambda, \alpha) = 1$ and $\kappa(\alpha, \lambda + \alpha) = 3$ which is impossible since $\lambda + \alpha$ has to be the double of a root. The only remaining case is where $j = 1$ and $\kappa(\lambda, \alpha) = 0$. Since $\lambda + \alpha \in \Lambda_1$, $\lambda + \alpha$ can be written uniquely as the sum of two roots β and γ. Of course $\alpha \neq \beta$ and $\alpha \neq \gamma$ because otherwise λ would be a root. Thus $\kappa(\alpha, \beta) = \kappa(\alpha, \gamma) = 1$ and $(\beta - \alpha) + \gamma$ and $\beta + (\gamma - \alpha)$ are two ways to write λ as the sum of two roots. Conversely, if $\lambda = \delta + \varepsilon$ with $\delta, \varepsilon \in \Phi$ and $\alpha \in \Phi^+$ such that $\kappa(\delta, \alpha) = -\kappa(\varepsilon, \alpha) = \pm 1$, then

$\lambda+\alpha \in \Lambda_1$. Lemma 5.9.1 (v) and a double counting argument gives us the number of $\alpha \in \Phi^+$ for which $\lambda+\alpha \in \Lambda_1$: $n_\lambda(n_\lambda - 1)$. The right hand side of (5.4) becomes $4n_\lambda(n_\lambda - 1)$.

As far as the left hand side goes, we write $\lambda = \omega + \psi$ for some $\psi \in \Phi^+$ using Lemma 5.9.1 (iii). By Lemma 5.9.1 (vi) we can find $\{\omega', \psi'\} \in N_{\omega+\psi} \setminus \{\{\omega, \psi\}\}$. Let $\beta_1 := \omega - \omega'$ and $\beta_2 := \omega - \psi'$. Then $\beta_1, \beta_2 \in \Phi$ and $\kappa(\omega, \beta_1) = \kappa(\omega, \beta_2) = 1$ and $\kappa(\psi, \beta_1) = \kappa(\psi, \beta_2) = -1$. Apply Lemma 5.9.1 (iv) with $f = s_{\beta_1} s_{\beta_2}$. The left hand side of (5.4) reduces to

$$\left(4 + \sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\lambda,\alpha)=0}} \kappa(2\omega, \alpha) \right) m_\lambda.$$

The number of $\alpha \in \Phi^+$ for which $\kappa(\omega, \alpha) = 1$ and $\kappa(\lambda, \alpha) = 0$ is, because ω is dominant, equal to $2(n_\lambda - 1)$ by Lemma 5.9.1 (v). Also $\kappa(\omega, \alpha) > 0$ since ω is dominant and $\kappa(\omega, \alpha) = 2$ if and only if $\alpha = \omega$ (but then $\kappa(\lambda, \alpha) = 2 \neq 0$). Thus

$$\sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\lambda,\alpha)=0}} \kappa(2\omega, \alpha) = 4(n_\lambda - 1).$$

We conclude, by (5.4), that $4n_\lambda m_\lambda = 4n_\lambda(n_\lambda - 1)$ and thus $m_\lambda = n_\lambda - 1$ for all $\lambda \in \Lambda_0$.

Claim 4: $m_\lambda = n_\lambda + 1$ for all $\lambda \in \Lambda_{-1}$. The only dominant weight in $\Lambda_{-1} = \Phi$ is ω. Now, by Lemma 5.9.1 (iii), if $\omega + i\alpha \in \Lambda_j$ for $i \geq 1$ and $\alpha \in \Phi^+$ then $i = 1$ and $\kappa(\omega, \alpha) = j$. Obviously, the converse is also true. Hence, for the right hand side of (5.4):

$$2 \sum_{\alpha \in \Phi^+} \kappa(\omega + \alpha, \alpha) m_{\omega+\alpha} = 2 \sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\omega,\alpha)=0}} 2 \cdot (n_{\omega+\alpha} - 1) + 2 \sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\omega,\alpha)=1}} 3 \cdot 1 + 2 \sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\omega,\alpha)=2}} 4 \cdot 1$$

Let $\alpha \in \Phi^+$ such that $\kappa(\omega, \alpha) = 0$. Let $\beta \in \Phi^+$ such that $\kappa(\beta, \omega) = 1$ and $\kappa(\beta, \alpha) = -1$. Since ω is dominant, there are precisely $2(n_{\alpha+\omega} - 1)$ choices for β by Lemma 5.9.1 (v). Then $\{\beta, \omega - \beta\}$ and $\{\alpha + \beta, \omega - \alpha + \beta\}$ are two different elements of N_ω. Conversely, let $\{\gamma, \delta\}$ and $\{\varepsilon, \zeta\}$ be two different elements of N_ω. Then $\kappa(\gamma, \varepsilon) = 1$ or $\kappa(\gamma, \zeta) = 1$. Without loss of generality, assume that $\kappa(\gamma, \varepsilon) = 1$. Also assume that $\gamma - \varepsilon$ is positive (otherwise take $\varepsilon - \gamma$). Since $\kappa(\omega, \varepsilon) = 1$ and ω is dominant, the root ε must be positive. Thus $\alpha := \gamma - \varepsilon \in \Phi^+$ and $\beta := \varepsilon \in \Phi^+$ are positive roots for which $\kappa(\omega, \alpha) = 0$, $\kappa(\omega, \beta) = 1$ and $\kappa(\alpha, \beta) = -1$. The same reasoning applies when γ is replaced by δ but leads to the same α and β. This double counting argument gives us

$$\sum_{\substack{\alpha \in \Phi^+ \\ \kappa(\omega,\alpha)=0}} 2 \cdot (n_{\omega+\alpha} - 1) = n_\omega(n_\omega - 1).$$

Now consider $\alpha \in \Phi^+$ such that $\kappa(\omega, \alpha) = 1$. Then $\{\alpha, \omega - \alpha\} \in N_\omega$. Conversely, if $\{\beta, \gamma\} \in N_\omega$ then $\kappa(\omega, \beta) = \kappa(\omega, \gamma) = 1$. Thus

$$\sum_{\substack{\alpha\in\Phi^+\\ \kappa(\omega,\alpha)=1}} 1 = n_\omega \cdot 2. \tag{5.5}$$

The only root $\alpha \in \Phi^+$ such that $\kappa(\omega, \alpha) = 2$, is ω itself.

So the right hand side of (5.4) equals

$$2n_\omega(n_\omega - 1) + 12n_\omega + 8 = 2n_\omega^2 + 10n_\omega + 8 = 2(n_\omega + 1)(n_\omega + 4).$$

As far as the left hand side goes, we have

$$\left(\kappa(2\omega, 2\omega) - \kappa(\omega, \omega) + \sum_{\alpha\in\Phi^+} \kappa(\omega, \alpha)\right) m_\omega = \left(8 - 2 + \sum_{\substack{\alpha\in\Phi^+\\ \kappa(\omega,\alpha)=1}} 1 + 2\right) m_\omega.$$

Using (5.5), we conclude

$$(2n_\omega + 8)m_\omega = 2(n_\omega + 1)(n_\omega + 4).$$

Hence $m_\lambda = n_\lambda + 1$ for all $\lambda \in \Lambda_{-1}$.

Claim 5: $m_0 = n_0$. Finally, we compute m_0, once again using (5.4). By (5.5), the left hand side equals $4(n_\omega + 3)m_0$. Obviously $0 + i\alpha \in \Lambda_j$ for $i \geq 0$ and $\alpha \in \Phi^+$ if and only if $i = 1$ and $j = -1$ or $i = 2$ and $j = 2$. Because W acts transitively on Φ, we have $n_\alpha = n_\omega$ for all $\alpha \in \Phi$. The right hand side becomes

$$\begin{aligned} 2\sum_{\alpha\in\Phi^+} 2 \cdot (n_\alpha + 1) + 2\sum_{\alpha\in\Phi^+} 4 \cdot 1 &= 4\sum_{\alpha\in\Phi^+} n_\alpha + 12\sum_{\alpha\in\Phi^+} 1, \\ &= \frac{|\Phi|}{2} \cdot (4n_\omega + 12) \end{aligned}$$

Hence $m_0 = \frac{|\Phi|}{2} = n_0$. $\square$

Remark 5.9.3. The dimension of $\mathcal{V}$ can also be computed using the formulas from [LM06].

6

Further developments

Although we have come a long way since the definition of an axial algebra was given in 2015, we are far from a complete understanding of their structure. In this chapter, we give an overview of some unfinished projects where we have reached some partial results.

6.1 Expansions

A typical way to construct new associative algebras is to take extensions of other ones. Trying to classify all possible extensions of one algebra by another, is therefore an important problem in the theory of associative algebras. Extensions also make sense for non-associative algebras since we have a notion of ideals and quotients. However, we typically use other techniques to construct non-associative algebras. In this section we introduce expansions, a concept generalizing split extensions.

6.1.1 Introduction to expansions

As we have seen a few times, e.g. in Definition 2.9.1 and Proposition 5.2.14, we often construct non-associative algebras in the following way. We start from an algebra B and take an R-submodule A of B together with a projection $\pi\colon B \to A$ and define a product on A by

$$a * b \coloneqq \pi(ab)$$

for all $a, b \in A$. The definition of an expansion tries to formalize such a construction.

Definition 6.1.1. Let R be a commutative ring with identity and let A and B be R-algebras. Then we call B an *expansion* of A if there exist morphisms $\sigma\colon A \to B$ and $\pi\colon B \to A$ of R-modules such that $\pi \circ \sigma = \mathsf{id}_A$ and

$$ab = \pi(\sigma(a)\sigma(b))$$

for all $a, b \in A$. If both σ and π are morphisms of *algebras* then we call A a *split extension* of B.

Remark 6.1.2. If B is an expansion of A then the corresponding morphism σ has to be injective. Then $\sigma(A) \cong A$ and we can view A is a submodule of B.

Recall the adjoint action of an algebra A on itself by left multiplication:

$$\mathrm{ad}\colon A \to \mathsf{End}(A)\colon a \mapsto \mathrm{ad}_a$$

where

$$\mathrm{ad}_a\colon A \to A\colon b \mapsto ab.$$

Denote by $\langle\!\langle \mathrm{ad}(A) \rangle\!\rangle$ the associative subalgebra of $\mathsf{End}(A)$ generated by $\mathrm{ad}(A)$. This associative algebra is sometimes called the *(left) multiplication algebra* of A; see e.g. [Sch66, § II.2].

Proposition 6.1.3. *If A is a unital algebra then $\langle\!\langle \mathrm{ad}(A) \rangle\!\rangle$ is an associative expansion of A.*

Proof. Consider the linear maps

$$\pi\colon \langle\!\langle \mathrm{ad}(A) \rangle\!\rangle \to A\colon f \mapsto f(\mathbf{1})$$

and

$$\sigma\colon A \to \langle\!\langle \mathrm{ad}(A) \rangle\!\rangle\colon a \mapsto \mathrm{ad}_a\,.$$

Then $\pi(\sigma(a)) = \mathrm{ad}_a(\mathbf{1}) = a$ and

$$\pi(\sigma(a)\sigma(b)) = \mathrm{ad}_a(\mathrm{ad}_b(\mathbf{1})) = ab$$

for all $a, b \in A$. □

A unital commutative algebra also has an expansion which is a Jordan algebra. Recall that $\mathsf{End}(A)^+$ is the Jordan algebra with the Jordan product defined by

$$a \bullet b = \tfrac{1}{2}(ab + ba);$$

see Example 1.2.3 (i).

Proposition 6.1.4. *Let A be a unital commutative algebra and J the Jordan subalgebra of $\mathsf{End}(A)^+$ generated by $\mathrm{ad}(A)$. Then J is an expansion of A.*

Proof. Consider the linear maps

$$\pi\colon J \to A\colon f \mapsto f(\mathbf{1})$$

and

$$\sigma\colon A \to J\colon a \mapsto \mathrm{ad}_a\,.$$

Then indeed $\pi(\sigma(a)) = \mathrm{ad}_a(\mathbf{1}) = a$ and

$$\pi(\sigma(a) \bullet \sigma(b)) = \frac{1}{2}(\mathrm{ad}_a(\mathrm{ad}_b(\mathbf{1})) + \mathrm{ad}_b(\mathrm{ad}_a(\mathbf{1}))) = ab$$

for all $a, b \in A$ because A is commutative. □

If an expansion of an algebra is commutative, then the algebra itself must be commutative as well. This raises the question which properties, and under what conditions, can be transferred between an algebra and its expansions. An example of such a property is the existence of a Frobenius form.

Proposition 6.1.5. *Let B be a Frobenius algebra. Suppose that B is an expansion of an algebra A with linear maps $\pi\colon B \to A$ and $\sigma\colon A \to B$ such that $\sigma \circ \pi$ is an orthogonal projection with respect to the Frobenius form of B, this is*

$$\langle \sigma(\pi(a)), b\rangle = \langle a, \sigma(\pi(b))\rangle$$

for all $a, b \in B$. Then

$$A \times A \to R\colon (a, b) \mapsto \langle \sigma(a), \sigma(b)\rangle$$

defines a Frobenius form for A.

Proof. By assumption, we have $\langle \sigma(\pi(a)), b\rangle = \langle a, \sigma(\pi(b))\rangle$ for all $a, b \in B$ and $\pi \circ \sigma = \mathsf{id}_A$. Thus, for each $a, b \in B$, we have $\langle \sigma(\pi(a)), \sigma(b)\rangle = \langle a, \sigma(\pi(\sigma(b)))\rangle = \langle a, \sigma(b)\rangle$. Now we have

$$\begin{aligned} \langle \sigma(ab), \sigma(c)\rangle &= \langle \sigma(\pi(\sigma(a)\sigma(b))), \sigma(c)\rangle \\ &= \langle \sigma(a)\sigma(b), \sigma(c)\rangle \\ &= \langle \sigma(a), \sigma(b)\sigma(c)\rangle \\ &= \langle \sigma(a), \sigma(\pi(\sigma(b)\sigma(c)))\rangle \\ &= \langle \sigma(a), \sigma(bc)\rangle \end{aligned}$$

for all $a, b, c \in A$.

It remains to prove that this bilinear form is non-degenerate. We need to show that

$$\varphi\colon A \to \mathsf{Hom}(A, R)\colon a \mapsto [b \mapsto \langle \sigma(a), \sigma(b)\rangle]$$

is bijective. Analogously, it follows that

$$\varphi\colon A \to \mathsf{Hom}(A, R)\colon a \mapsto [b \mapsto \langle \sigma(b), \sigma(a)\rangle]$$

is bijective. We know that

$$\psi\colon B \to \mathsf{Hom}(B, R)\colon a \mapsto [b \mapsto \langle a, b\rangle]$$

is bijective.

Suppose that $a \in A$ such that $\langle \sigma(a), \sigma(b)\rangle = 0$ for all $b \in A$. Then, for all $c \in B$, we have

$$\langle \sigma(a), c\rangle = \langle \sigma(a), \sigma(\pi(c))\rangle = 0.$$

Because ψ is injective, it follows that $\sigma(a) = 0$. Now $a = 0$ because σ is injective.

Let $f \in \mathsf{Hom}(A, R)$ be arbitrary. Because ψ is surjective, there exists an element $b \in B$ such that $\langle b, c\rangle = f(\pi(c))$ for all $c \in B$. Therefore we have

$$\begin{aligned}\langle \sigma(\pi(b)), \sigma(a)\rangle &= \langle b, \sigma(a)\rangle \\ &= f(\pi(\sigma(a))) \\ &= f(a)\end{aligned}$$

for all $a \in A$. This shows that $\varphi(\pi(b)) = f$. Since $f \in \mathsf{Hom}(A, R)$ was arbitrary, the morphism φ is surjective. □

Next, we consider expansions of axial decomposition algebras. We want to be able to compare the decompositions of the algebra with those of its expansion. The following definition introduces extra conditions on expansions of axial decomposition algebras.

Definition 6.1.6. Let $\mathcal{A} = (A, I, \Omega, \alpha)$ be an axial $\mathcal{F}_A$-decomposition algebra. An *axial expansion* of $\mathcal{A}$ is an axial $\mathcal{F}_B$-decomposition algebra $\mathcal{B} = (B, I, \Sigma, \beta)$ for which there exist morphisms $\pi\colon B \to A$ and $\sigma\colon A \to B$ of R-modules, a map $\psi\colon J \to I$ and a morphism $\zeta\colon \mathcal{F}_B \to \mathcal{F}_A$ (preserving the distinguished unit) such that the following conditions hold:

(E1) $\pi \circ \sigma = \mathsf{id}_A$,

(E2) $\pi(\sigma(a)\sigma(b)) = ab$ for all $a, b \in A$,

(E3) $\pi(B_x^j) \subseteq A_{\zeta(x)}^{\psi(j)}$ for all $x \in \mathcal{F}_B$ and $j \in J$.

We will discuss two special types of axial expansions where $\ker(\pi)$ has a special form.

6.1.2 One-point expansions

The first type of expansions we are going to consider are those where $\ker(\pi)$ is a subalgebra generated by an element e such that $e^2 \in \langle e\rangle$. The discussion of these expansions is based on joint work with Simon F. Peacock and Justin McInroy. For convenience, we will, from now on, only consider commutative algebras.

Definition 6.1.7. Let $\mathcal{B} = (B, I, \Sigma, \beta)$ be an axial expansion of $\mathcal{A} = (A, I, \Omega, \alpha)$ with A and B commutative. Let $\pi\colon B \to A$, $\sigma\colon A \to B$ and $\psi\colon I \to I$ be as in Definition 6.1.6. Suppose that $\psi = \mathsf{id}_I$. Assume that $\ker(\pi) = \langle e\rangle$ for some $e \in B$ such that $e^2 \in \langle e\rangle$. Let $\mathcal{F}_A = (X_A, \star)$ be the fusion law of $\mathcal{A}$ and suppose that there exists a map $\mu\colon X_A \to R$ such that $e\sigma(v) = \mu(x)\sigma(v)$ for all $v \in A_x^i$, $x \in X_A$ and $i \in I$. Then we call $\mathcal{B}$ a *one-point expansion* of $\mathcal{A}$.

Remark 6.1.8. Suppose that $\mathcal{B}$ is a one-point expansion of $\mathcal{A}$ as in Definition 6.1.7. Then there exists a bilinear form $\varphi\colon A \times A \to R$ such that $\sigma(v)\sigma(w) = \sigma(vw) + \varphi(v, w)e$ for all $v, w \in A$.

A typical application of one-point expansions is the discussion of units for axial decomposition algebras. We illustrate this for the Norton-Sakuma algebra of type $2A$.

Example 6.1.9. Denote by B the Norton-Sakuma algebra of type $2A$. This is the unique commutative 3-dimensional $\mathbb{R}$-algebra with basis $\{b_0, b_1, b_2\}$ of idempotents such every permutation of these idempotents defines an automorphism of the algebra and such that

$$b_0 b_1 = \frac{1}{2^3}(b_0 + b_1 - b_2);$$

see [IPSS10, Table 3, p. 2449]. It is an axial decomposition algebra

$$\mathcal{B} = (B, \{0,1,2\}, \Omega, i \mapsto b_i)$$

with b_0, b_1 and b_2 as its axes, the Jordan fusion law (see Example 2.4.5) as its fusion law and evaluation map λ defined by $\lambda(e) = 1$, $\lambda(z) = 0$ and $\lambda(h) = \frac{1}{4}$. This algebra is endowed with a Frobenius form defined by

$$\begin{aligned} \langle b_0, b_0 \rangle = \langle b_1, b_1 \rangle = \langle b_2, b_2 \rangle &= 1, \\ \langle b_0, b_1 \rangle = \langle b_0, b_\rho \rangle = \langle b_1, b_2 \rangle &= \frac{1}{2^3}. \end{aligned}$$

The element $\mathbf{1} := \frac{4}{5}(b_0 + b_1 + b_2)$ is a unit for this algebra.

Now, let A be the 2-dimensional $\mathbb{R}$-algebra generated by two idempotents e and f such that $ef = -e - f$. We can give this algebra the structure of an axial decomposition algebra

$$\mathcal{A} = (A, \{0,1,2\}, \Omega_A, i \mapsto a_i)$$

with $a_0 := e$, $a_1 := f$ and $a_2 := -e - f$ as its axes, the group fusion law $(\mathbb{Z}/2\mathbb{Z}, \star)$ as its fusion law and evaluation map λ defined by $\lambda(0) = 1$ and $\lambda(1) = -1$.

Then $\mathcal{B}$ is a one-point expansion of $\mathcal{A}$. The maps π, σ and ζ from Definition 6.1.7 are defined as follows. We have $\pi(b_i) = \frac{1}{6}a_i$ and $\sigma(a_i) = 6b_i - \frac{5}{12}\mathbf{1}$ for $0 \leq i \leq 2$ while ζ is the non-trivial $\mathbb{Z}/2\mathbb{Z}$-grading of the Jordan fusion law. Note that $\sigma \circ \pi$ is the orthogonal projection onto the orthogonal complement of $\langle \mathbf{1} \rangle$ with respect to the Frobenius form. We have $\ker(\pi) = \langle \mathbf{1} \rangle$ and so we can take $\mu(0) = \mu(1) = 1$.

Also the Norton-Sakuma algebra of type $3A$ can be described in terms of one-point expansions.

Example 6.1.10. Consider the Norton-Sakuma algebra of type $3A$. This is a 4-dimensional algebra C with a basis $\{c_0, c_1, c_2, u_\rho\}$ of idempotents. Every permutation of the idempotents c_0, c_1 and c_2 induces an automorphism of the algebra and

$$c_0 c_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_2) - \frac{3^3 \cdot 5}{2^{11}} u_\rho,$$

$$c_0 u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_1) + \frac{5}{2^5}u_\rho.$$

We can view this algebra as an axial decomposition algebra

$$\mathcal{C} = (C, \{0,1,2\}, \Omega_C, i \mapsto c_i)$$

for the Ising fusion law $\mathcal{F}_C$ from Example 2.4.6. Its axes are c_0, c_1 and c_2. The evaluation map λ is the same as for the Griess algebra, i.e. $\lambda(e) = 1$, $\lambda(z) = 0$, $\lambda(q) = \frac{1}{4}$, $\lambda(t) = \frac{1}{32}$. The element

$$\mathbf{1} := \frac{16}{21}(c_0 + c_1 + c_2) + \frac{9}{14}u_\rho$$

is a unit for this algebra.

Similarly to Example 6.1.9, we can view this algebra as a one-point expansion by projecting onto the orthogonal complement of $\langle \mathbf{1} \rangle$ with respect to a Frobenius form for C. We obtain that $\mathcal{C}$ is a one-point expansion of an axial decomposition algebra

$$\mathcal{B} = (B, \{0,1,2\}, \Omega_B, i \mapsto b_i).$$

The axes b_0, b_1 and b_2 are idempotents and form a basis for B. Every permutation of these idempotents defines an automorphism of B and we have

$$b_0 b_1 = \frac{1}{2^5}\left(-13b_0 - 13b_1 + \frac{7 \cdot 19}{23}b_2\right).$$

The fusion law $\mathcal{F}_B$ of $\mathcal{B}$ is given by the following fusion table.

$\star$	e	q	t
e	e	q	t
q	q	e	t
t	t	t	e, q

The maps π and σ from Definition 6.1.6 are defined by

$$\begin{aligned} \pi &\colon C \to B \colon c_i \mapsto \frac{23}{58}b_i \\ \sigma &\colon B \to C \colon b_i \mapsto \frac{58}{23}c_i - \frac{35}{46}\mathbf{1} \end{aligned}$$

for all $0 \leq i \leq 2$. The morphism $\zeta\colon \mathcal{F}_C \to \mathcal{F}_B$ is defined by $\zeta(e) = \zeta(z) = e$, $\zeta(q) = q$ and $\zeta(t) = t$.

The axial decomposition algebra $\mathcal{B}$ is in turn a one-point expansion of the algebra $\mathcal{A}$ from Example 6.1.9. Indeed, let

$$\pi\colon B \to A\colon b_i \mapsto \frac{5 \cdot 29}{2^3 \cdot 23}a_i$$

$$\sigma\colon A \to B\colon a_i \mapsto \frac{2^3 \cdot 23}{5 \cdot 29} b_i + \frac{9}{2 \cdot 5 \cdot 29} v$$

where $v := -\frac{2^4 \cdot 23}{27}(b_0 + b_1 + b_2)$. Then $\pi \circ \sigma = \mathsf{id}_A$. The kernel of π is a subalgebra of B generated by the idempotent v. For all $a \in A$ we have $v\sigma(a) = \frac{-13}{3}\sigma(a)$. If we define $\zeta\colon \mathcal{F}_B \to \mathbb{Z}/2\mathbb{Z}$ by $\zeta(e) = \zeta(q) = 0$ and $\zeta(t) = 1$ then we have $\pi(B_x^i) \subseteq A^i_{\zeta(x)}$ for all $x \in \mathcal{F}_B$ and $0 \le i \le 2$. This proves that $\mathcal{B}$ is indeed a one-point expansion of $\mathcal{A}$.

The joint work with Simon F. Peacock and Justin McInroy tries to classify all one-point expansions of axial decomposition algebras. Although this project is not yet finished, the partial results we obtained look promising.

6.1.3 $\mathbb{Z}/2\mathbb{Z}$-expansions

The next type of expansions we are going to consider are $\mathbb{Z}/2\mathbb{Z}$-expansions. The results and ideas presented in this subsection are based on joint work with Tom De Medts and Madeleine Whybrow.

Definition 6.1.11. Let A and B be commutative algebras, $\mathcal{B} = (B, J, \Omega_B, \beta)$ an axial expansion of $\mathcal{A} = (A, I, \Omega_A, \alpha)$ and π, σ, ψ and ζ as in Definition 6.1.6. Denote the fusion law of $\mathcal{B}$ and $\mathcal{A}$ by $\mathcal{F}_B$ and $\mathcal{F}_A$ respectively. Let $M := \ker(\pi)$. Then $\pi_M := \mathsf{id}_B - \sigma \circ \pi$ is the projection onto M with kernel $\sigma(A)$. For each $i \in I$ and $x \in \mathcal{F}_A$, define

$$M_x^i := \pi_M\Big(\sum_{j \in \psi^{-1}(i)} B^j_{\zeta^{-1}(x)}\Big).$$

Then we call $\mathcal{B}$ a $\mathbb{Z}/2\mathbb{Z}$-*expansion* of $\mathcal{A}$ if the following hold:

(i) $\sigma(A)^2 \subseteq \sigma(A) \supseteq M^2$ and $\sigma(A)M \subseteq M$,

(ii) $\Omega_M[i] := (M_x^i)_{x \in \mathcal{F}_A}$ defines a decomposition of M for each $i \in I$,

(iii) the map
$$A \times M \to M\colon (a, m) \mapsto \sigma(a)m$$
gives (M, Ω_M) the structure of an $\mathcal{A}$-module.

We present a typical way to construct such a $\mathbb{Z}/2\mathbb{Z}$-expansion.

Example 6.1.12. Let $\mathcal{A} = (A, I, \Omega_A, \alpha)$ be a commutative axial decomposition algebra with fusion law $\mathcal{F}_A = (X_A, \star)$ and (M, Ω_M) an $\mathcal{A}$-module. Consider a symmetric bilinear map $\pi\colon M \times M \to A$ such that $\pi(M_x^i, M_y^i) \subseteq A^i_{x \star y}$ for all $i \in I$ and $x, y \in \mathcal{F}_A$. Write $B := A \oplus M$ and consider the product on B defined by

$$(a, m)(b, n) := (ab + \pi(m, n), a \cdot n + b \cdot m)$$

for all $a, b \in A$ and $n, m \in M$. Let $e \in X_A$ be the unit of the fusion law $\mathcal{F}_A$. Note that $A^i_e \oplus M^i_e$ is a subalgebra of B. We can think of $A^i_e \oplus M^i_e$ as the subalgebra where we will allow for distortion of the decompositions. Since we do not have much control over this distortion, we will need to impose restrictions on this algebra and its action on $\bigoplus_{x\in X_a\setminus\{e\}}(A^i_x \oplus M^i_x)$.

Suppose that $\mathcal{C} = (C, K, \Omega_C, \gamma)$ is an axial decomposition algebra such that the following conditions hold.

- For every $i \in I$ there exists an isomorphism $\theta_i \colon C \to A^i_e \oplus M^i_e$ of algebras.
- For every $x \in X_A \setminus \{e\}$ there exists a scalar $\lambda_x \in R$ such that

$$\theta_i(\gamma(k))v = \lambda_x v$$

  for all $v \in A^i_x \oplus M^i_x$.

Let $\mathcal{F}_C = (X_C, \star)$ be the fusion law of $\mathcal{C}$ with distinguished unit $e' \in X_C$.

Define the symmetric fusion law $\mathcal{F}_B = (X_B, \circledast)$ with $X_B = X_C \cup (X_A \setminus \{e\})$ and

$$x \circledast y \coloneqq \begin{cases} x \star y & \text{if } x, y \in X_C, \\ \{y\} & \text{if } x \in X_C \text{ and } y \in X_A \setminus \{e\}, \\ x \star y & \text{if } x, y \in X_A \setminus \{e\} \text{ and } e \notin x \star y, \\ (x \star y) \cup X_C & \text{if } x, y \in X_A \setminus \{e\} \text{ and } e \in x \star y. \end{cases}$$

The element e' is clearly a unit for this fusion law. We have constructed this fusion law such that the sublaw on X_C is isomorphic to $\mathcal{F}_C$ and such that

$$\zeta \colon \mathcal{F}_B \to \mathcal{F}_A \colon x \mapsto \begin{cases} e & \text{if } x \in X_C, \\ x & \text{if } x \in X_A \setminus \{e\}, \end{cases}$$

defines a morphism of fusion laws.

For every $i \in I$, $k \in K$ and $x \in \mathcal{F}_B$ let

$$B^{(i,k)}_x \coloneqq \begin{cases} A^i_x \oplus M^i_x & \text{if } x \in X_A \setminus \{e\}, \\ \theta_i(C^k_x) & \text{if } x \in X_C. \end{cases}$$

Since

$$A^i_e \oplus M^i_e = \bigoplus_{x\in X_C} \theta_i(C^k_x)$$

for every $i \in I$ and $k \in K$ and also

$$B = (A^i_e \oplus M^i_e) \oplus \bigoplus_{x\in X_A\setminus\{e\}} (A^i_x \oplus M^i_x),$$

it follows that

$$\Omega_B[(i, k)] = (B^{(i,k)}_x)_{x\in X_B}$$

defines a decomposition of B as R-module.

Our restrictions and definition of $\mathcal{F}_B$ imply that

$$(B, I \times K, \Omega_B, (i,k) \mapsto \theta_i(\gamma(k)))$$

is an $\mathcal{F}_B$-decomposition algebra. If $\mu\colon X_C \to R$ is the evaluation map of $\mathcal{C}$, then $\mathcal{B}$ has evaluation map

$$\lambda\colon X_B \to R\colon x \mapsto \begin{cases} \mu(x) & \text{if } x \in X_C, \\ \lambda_x & \text{if } x \in X_A \setminus \{e\}. \end{cases}$$

By construction, $\mathcal{B}$ is a $\mathbb{Z}/2\mathbb{Z}$-expansion of $\mathcal{A}$.

Although this construction is very technical and seemingly restrictive, it is widely applicable. Let us illustrate this by giving an explicit example.

Example 6.1.13. We begin with the 3-dimensional Jordan algebra A over $\mathbb{R}$ with generating idempotents a_0 and a_1 that has the following algebra product:

$$\begin{aligned} a_0(a_0a_1) &= \frac{1}{32}a_0 + \frac{1}{2}a_0a_1, \\ a_1(a_0a_1) &= \frac{1}{32}a_0 + \frac{1}{2}a_0a_1, \\ (a_0a_1)(a_0a_1) &= \frac{1}{32}(a_0 + a_1 + a_0a_1). \end{aligned}$$

Consider the axial decomposition algebra $\mathcal{A} = (A, \{0,1\}, \Omega_A, i \mapsto a_i)$ for the Ising fusion law from Example 2.4.6 and evaluation map λ' defined by $\lambda'(e) = 1$, $\lambda'(z) = 0$, $\lambda'(q) = \frac{1}{2}$ and $\lambda'(t) = \frac{1}{16}$. Note that $A_t^0 = A_t^1 = 0$.

We try to construct a $\mathbb{Z}/2\mathbb{Z}$-expansion of $\mathcal{A}$. Therefore, we consider the $\mathcal{A}$-module (M, Ω_M) with basis $\{m_0, m_1\}$ and action of A defined by

$$\begin{aligned} a_i \cdot m_i &= m_i && \text{for all } i \in \{0,1\}, \\ a_0 \cdot m_1 &= \frac{1}{16}m_1 \\ a_1 \cdot m_0 &= \frac{1}{16}m_0 \\ (a_0a_1) \cdot m_i &= \frac{1}{16}m_i && \text{for all } i \in \{0,1\} \end{aligned}$$

This defines indeed a module for $\mathcal{A}$ with $M_e^i = \langle m_i \rangle$ and $M_z^i = M_q^i = 0$ for all $i \in \{0,1\}$ and $M_t^0 = \langle m_1 \rangle$ and $M_t^1 = \langle m_0 \rangle$.

We also define a symmetric bilinear map $\pi\colon M \times M \to A$ by

$$\begin{aligned} \pi(m_i, m_i) &= a_i && \text{for all } i \in \{0,1\}, \\ \pi(m_0, m_1) &= 0. \end{aligned}$$

It is readily verified that the map π satisfies the conditions from Example 6.1.12.

Now consider B as in Example 6.1.12. Then, in fact, the algebra B is isomorphic to the Norton-Sakuma algebra of type $4A$

For each $i \in \{0,1\}$ the subalgebra $A_e^i \oplus M_e^i$ is an associative algebra generated by two idempotents $b_i^+ := \frac{1}{2}(a_i, m_i)$ and $b_i^- := \frac{1}{2}(a_i, m_i)$. It can be given the structure of an axial decomposition algebra with axes b_i^+ and b_i^- for the following fusion law and evaluation map defined by $e' \mapsto 1$ and $z' \mapsto 0$.

$\star$	e'	z'
e'	e'	
z'		z'

Using the construction from Example 6.1.12, we obtain an axial decomposition algebra $\mathcal{B}$ with the following fusion law.

$\star$	e'	z'	z	q	t
e'	e'			q	t
z'		z'		q	t
z			z	q	t
q	q	q	q	e', z', z	t
t	t	t	t	t	e', z', z, q

The evaluation map λ is defined by $\lambda(e') = 1$, $\lambda(z') = \lambda(z) = 0$, $\lambda(q) = \frac{1}{4}$ and $\lambda(t) = \frac{1}{32}$. Note how we obtain more information about the structure of this algebra when viewed as axial *decomposition* algebra. As an axial algebra, we cannot make the distinction between z and z'.

6.2 Axial decomposition algebras for the pariahs

The 26 sporadic simple groups can be further subdivided into two classes: the ones that can be retrieved as a subquotient of the monster and the ones that do not. The first class is sometimes referred to as the *happy family* and consists out of 20 of the 26 sporadic simple groups. We can often realize them as Miyamoto groups of axial decomposition algebras by considering a subalgebra of the Griess algebra. This raises the question whether we can construct an axial decomposition algebra for the 6 other sporadic groups who are called the *pariahs*.

Together with Simon F. Peacock and Justin McInroy, we try to construct such algebras for two of the pariahs: the Lyons group Ly and the third Janko group J_3. We use the techniques from Section 2.8.

Example 6.2.1. Let us start with the pariah which has the largest order: the Lyons group $G = Ly$. This group has order $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$ and a unique irreducible representation A of degree 45694 [CCN+85]. A character computation shows that $\dim(\mathsf{Hom}_G(A \otimes A, A)) = 2$, i.e. A admits a 2-dimensional space of G-invariant products. Now let C be the conjugacy class of G denoted by $3A$ in [CCN+85]. The centralizer $C_G(g)$ of an element $g \in C$ is an extension of the McLaughlin group McL, another sporadic simple group, by a cyclic group of order 3. The restriction of A to $C_G(g)$ is a multiplicity-free representation and contains a unique trivial subrepresentation. From Theorem 2.8.7 it follows that we can give A the structure of an axial decomposition algebra (A, C, Ω, α). Its fusion law is the representation fusion law on the irreducible constituents of the character of $C_G(g)$ afforded by A. Since $Z(C_G(g)) = \langle g \rangle$ has order three, this fusion law is $\mathbb{Z}/3\mathbb{Z}$-graded by Proposition 3.1.6. By Theorem 3.3.1, we know that the corresponding Miyamoto group is the Lyons group in its action on A.

Example 6.2.2. Next, we consider the third Janko group J_3. This group has order $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$. Daniel Frohardt has shown that there exists a 85-dimensional representation M for J_3 that admits a J_3-invariant symmetric trilinear form [Fro83]. He shows that this form is unique up to scalar multiplication and can be used to define a product on $A = M \oplus M^*$ that turns A into a (non-associative) Frobenius algebra. Once again, we look for a family of conjugate subgroups to apply Theorem 2.8.7 and give this algebra the structure of an axial decomposition algebra. A suitable choice seems to be the family of Sylow-3-subgroups of J_3. Such a Sylow-3-subgroup H has order 3^5 and a center isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. The representation M (and therefore also M^*) has a unique trivial H-subrepresentation. This allows us to define axes for this algebra. Unfortunately, the H-representation A is not multiplicity-free and therefore Theorem 2.8.7 does not apply. However, we hope to take advantage of the specific nature of this algebra and its Frobenius form to establish the diagonalizability of the adjoint action of these axes.

A

Overview of definitions and results

In this chapter, we give an overview of the most important definitions and results of this dissertation.

A.1 Axial and decomposition algebras

Before we introduce decomposition algebras, let us fix our terminology concerning algebras.

Definition A.1.1. Let R be a commutative ring with identity. An *R-algebra* is an R-module A equipped with a bilinear map

$$A \times A \to A\colon (a, b) \mapsto ab.$$

We do not assume any extra conditions on this product. In particular, our algebras need not be associative.

The study of a special type of bilinear forms is crucial in the study of non-associative algebras. Associative algebras equipped with such a form are known as Frobenius algebras. We generalize this terminology to non-associative algebras.

Definition A.1.2. A *Frobenius algebra* is an R-algebra A endowed with a bilinear form, called a *Frobenius form*,

$$\langle\ ,\ \rangle\colon A \times A \to R\colon (a, b) \mapsto \langle a, b\rangle$$

such that $\langle a, bc\rangle = \langle ab, c\rangle$ for all $a, b, c \in A$. Moreover, this form must be non-degenerate, i.e.

$$A \to \mathsf{Hom}(A, R)\colon a \mapsto [b \mapsto \langle a, b\rangle]$$

and

$$A \to \mathsf{Hom}(A, R)\colon a \mapsto [b \mapsto \langle b, a\rangle]$$

are isomorphisms of R-modules.

An axial algebra A is generated by idempotents $e \in A$ for which the operator $\mathrm{ad}_e\colon A \to A\colon a \mapsto ea$ is diagonalizable. The multiplication of the corresponding eigenvectors has to obey a fusion law. The notion of a decomposition algebra is based on the observation that these decompositions are the most crucial. We start by defining (general) fusion laws which no longer depend on an underlying ring. This is in contrast to their original definition for axial algebras where the elements of fusion laws needed to be eigenvalues.

Definition A.1.3. A *fusion law* is a pair $(X, \star)$ where X is an arbitrary set and $\star$ is a map $X \times X \to 2^X$ where 2^X denotes the power set of X. We call $x \in X$ a *unit* if $x \star y \subseteq \{y\} \supseteq y \star x$ for all $y \in X$.

Example A.1.4. Let G be a finite group.

(i) We define the *group fusion law* $(G, \star)$ by

$$g \star h := \{gh\}$$

for all $g, h \in G$.

(ii) Write X for the set of conjugacy classes of G. Then we define a fusion law on X by setting

$$E \in C \star D \iff E \cap CD \neq \emptyset.$$

We refer to $(X, \star)$ as the *class fusion law* of G.

(iii) Let $\mathsf{Irr}(G)$ be the set of irreducible complex characters of G. Then we can define a fusion law on $\mathsf{Irr}(G)$ by defining

$$\chi \in \chi_1 \star \chi_2 \iff \chi \text{ is a constituent of } \chi_1\chi_2.$$

This fusion law is called the *representation fusion law* of G. We can also define the representation fusion law of a complex semisimple Lie algebra in a similar manner.

Let us introduce (axial) decomposition algebras.

Definition A.1.5. Let $\mathcal{F} = (X, \star)$ be a fusion law.

(i) An *$\mathcal{F}$-decomposition algebra* is a triple (A, I, Ω) where A is an algebra, I is an index set and Ω is an I-tuple of decompositions $A = \bigoplus_{x\in X} A^i_x$ of A as R-module. Moreover, these decompositions have to obey the fusion law $(X, \star)$:

$$A^i_x A^i_y \subseteq A^i_{x\star y} := \bigoplus_{z\in x\star y} A^i_z.$$

for all $x, y \in X$ and $i \in I$.

(ii) Let $\lambda\colon X \to R$ be a map and $e \in X$ a unit for $\mathcal{F}$. An *axial $\mathcal{F}$-decomposition algebra with evaluation map* λ is a quadruple (A, I, Ω, α) where (A, I, Ω) is an $\mathcal{F}$-decomposition algebra and $\alpha\colon I \to A$ is a map such that $\alpha(i) \in A_e^i \setminus \{0\}$ and

$$\alpha(i)a = \lambda(x)a$$

for all $a \in A_x^i$, $x \in X$ and $i \in I$. The elements $\alpha(i) \in A$ are called the *axes* of the axial decomposition algebra.

In Sections 2.8 and 2.9 we introduce two generic examples.

Example A.1.6. (i) Supose that a finite group or complex semisimple Lie algebra G acts by automorphisms (resp. derivations) on an algebra A. Then the decomposition of A into G-isotypic components obeys the representation fusion law of G (Theorem 2.8.1). This observation can be used to give A the structure of a decomposition algebra (Corollary 2.8.4).

(ii) Norton algebras are non-associative algebras related to association schemes. We prove that these Norton algebras are generated by idempotents. For each of these idempotents, its adjoint action on the Norton algebra is diagonalizable (Lemma 2.9.5). This is in particular true for association schemes constructed from a generously transitive action of a group G. In that case, the group G acts by automorphisms on the Norton algebra and the Norton algebra is an axial decomposition algebra (Theorem 2.9.7).

Both fusion laws and decomposition algebras allow for a natural definition of morphisms.

Definition A.1.7. (i) A morphism $(X, \star) \to (Y, \star)$ of fusion laws is a map $\xi\colon X \to Y$ such that

$$\xi(x \star y) \subseteq \xi(x) \star \xi(y)$$

for all $x, y \in X$. Such a morphism induces a functor from the category of $(X, \star)$-decomposition algebras to the category of $(Y, \star)$-decomposition algebras, cf. Proposition 2.5.9.

(ii) A morphism $(A, I, \Omega) \to (B, J, \Sigma)$ of $(X, \star)$-decomposition algebras is a pair (φ, ψ) where $\varphi\colon A \to B$ is an algebra morphism and $\psi\colon I \to J$ is a map such that $\varphi(A_x^i) \subseteq B_x^{\psi(i)}$ for all $x \in X$ and $i \in I$.

This allows us to define the category Fus of fusion laws and the category $\mathcal{F}\text{-}\mathsf{Dec}_R$ of $\mathcal{F}$-decomposition algebras. Both these categories are complete (Propositions 2.10.4 and 2.11.3). The category of fusion laws is also cocomplete (Proposition 2.10.5). We also introduce *decomposition ideals* and prove that they can be seen as kernels in the category $\mathcal{F}\text{-}\mathsf{Dec}_R$ (Definition 2.11.4 and Proposition 2.11.5).

A.2 Miyamoto groups

Gradings of fusion laws lie at the root of the important connection between decomposition algebras and groups.

Definition A.2.1. A *grading* of a fusion law $\mathcal{F}$ is a morphism $\xi\colon \mathcal{F} \to \Gamma$ for a group fusion law Γ.

Every fusion law has a unique *finest grading* and we can give an explicit presentation for the corresponding grading group Γ (Proposition 3.1.2). We can determine the finest grading of the class fusion law and representation fusion law of a group G explicitly.

Proposition A.2.2. *Let G be finite group.*

(i) *The finest grading of the class fusion law $(X, \star)$ of G is given by the group $\Gamma \coloneqq G/[G,G]$ with grading map $X \to \Gamma\colon {}^G g \mapsto g[G,G]$.*

(ii) *The finest grading of the representation fusion law $(\mathsf{Irr}(G), \star)$ is given by the group $\mathsf{Irr}(Z(G))$ with grading map $\mathsf{Irr}(G) \to \mathsf{Irr}(Z(G))\colon \chi \mapsto \frac{\chi_{Z(G)}}{\chi(1)}$.*

If the fusion law of a decomposition algebra is graded, then we can associate a group of automorphisms to the decomposition algebra.

Definition A.2.3. Let (A, I, Ω) be a decomposition algebra for a fusion law $\mathcal{F}$ that is graded by $\xi\colon \mathcal{F} \to \Gamma$. Let $\chi\colon \Gamma \to R^\times$ be a group homomorphism. Define the *Miyamoto map* $\tau_{i,\chi}\colon A \to A$ by

$$\tau_{i,\chi}(a) = \chi(g)a$$

for all $i \in I$, $a \in A^i_x$, $x \in \xi^{-1}(g)$ and $g \in \Gamma$. Then $\tau_{i,\chi}$ is an automorphism of A. Let $\mathcal{Y}$ be a set of group homomorphisms $\Gamma \to R^\times$. Then we define the *Miyamoto group* $\mathsf{Miy}_{\mathcal{Y}}(A, I, \Omega) \coloneqq \langle \tau_{i,\chi} \mid i \in I, \chi \in \mathcal{Y} \rangle \leq \mathsf{Aut}(A)$.

Example A.2.4. In Section 3.3 we determine the Miyamoto groups for the examples from Example A.1.6.

Next, we study the functorial properties of the Miyamoto group. It turns out that the Miyamoto group is hard to control with respect to morphisms of decomposition algebras. Therefore, we introduce a more universal concept, the *universal Miyamoto group*, in Definition 3.6.1. This group is a central extension of the Miyamoto group, cf. Proposition 3.6.4. The generators $\tau_{i,\chi}$ for the Miyamoto group lift to generators $t_{i,\chi}$ for the universal Miyamoto group.

Example A.2.5. The Miyamoto group of a Matsuo algebra is a 3-transposition group by Proposition 3.5.13. Conversely, every 3-transposition group gives rise to a Matsuo algebra (Proposition 3.5.14). This correspondence between Matsuo algebras and 3-transposition groups is one-to-one up to the center of the 3-transposition group. The universal Miyamoto group of a Matsuo algebra A is the 3-transposition group corresponding to A with the largest possible center by Theorem 3.6.5.

Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a morphism of decomposition algebras for a graded fusion law. We wonder whether φ induces a morphism $\mathsf{Miy}(\mathcal{A}) \to \mathsf{Miy}(\mathcal{B})$ of the corresponding (universal) Miyamoto groups. This is true for both the Miyamoto group as its universal version if φ is an epimorphism (Proposition 3.7.2). In general, this is not true for the Miyamoto group itself; see Example 3.7.11.

However, we can impose some minor restrictions such that there exists a morphism between the universal Miyamoto groups. We need to assume that φ is *Miyamoto-admissable*, a natural condition that can be formulated in terms of a commutative diagram in the category of decomposition algebras; see Definition 3.7.4.

Theorem A.2.6 (See Theorem 3.7.8)**.** *Let $(\varphi, \psi)\colon \mathcal{A} \to \mathcal{B}$ be a Miyamoto-admissable morphism of "nice" decomposition algebras for a graded fusion law. Then there exists a corresponding morphism $\widehat{\mathsf{Miy}}(\varphi)\colon\ \widehat{\mathsf{Miy}}(\mathcal{A}) \to \widehat{\mathsf{Miy}}(\mathcal{B})\colon t_{i,\chi} \mapsto t_{\psi(i),\chi}$ between the universal Miyamoto groups.*

A morphism of "nice" axial decomposition algebras is automatically Miyamoto-admissable.

Theorem A.2.7 (See Theorem 3.7.10)**.** *Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a morphism of "nice" axial decomposition algebras for a graded fusion law. Then there exists a corresponding morphism $\widehat{\mathsf{Miy}}(\varphi)\colon\ \widehat{\mathsf{Miy}}(\mathcal{A}) \to \widehat{\mathsf{Miy}}(\mathcal{B})\colon t_{i,\chi} \mapsto t_{\psi(i),\chi}$ between the universal Miyamoto groups.*

A.3 Modules over (axial) decomposition algebras

In Chapter 4, we introduce modules for (axial) decomposition algebras.

Definition A.3.1. Let $\mathcal{F} = (X, \star)$ be a fusion law.

(i) Let $\mathcal{A} = (A, I, \Omega)$ be an $\mathcal{F}$-decomposition algebra. An *$\mathcal{A}$-module* is a pair (M, Σ) where M is an R-module equipped with a (left) R-bilinear action of A:
$$A \times M \to M\colon (a, m) \mapsto a \cdot m$$
Moreover, Σ is an I-tuple of decompositions $M = \bigoplus_{x\in X} M_x^i$ of M as R-module such that
$$A_x^i \cdot M_y^i \subseteq M_{x\star y}^i := \bigoplus_{z \in x\star y} M_z^i,$$
for all $x, y \in X$ and $i \in I$.

(ii) Let $\mathcal{A} = (A, I, \Omega, \alpha)$ be an axial $\mathcal{F}$-decomposition algebra with evaluation map $\lambda\colon X \to R$. An *$\mathcal{A}$-module* is a module (M, Σ) for (A, I, Ω) such that

$$\alpha(i) \cdot m = \lambda(x)m$$

for all $m \in M_x^i$, $x \in X$ and $i \in I$.

If $\xi\colon \mathcal{F} \to \Gamma$ is a grading of the fusion law, then we can also define Miyamoto maps of modules.

Definition A.3.2. Let (M, Σ) be a module for a decomposition algebra $\mathcal{A} = (A, I, \Omega)$. For any group homomorphism $\chi\colon \Gamma \to R^\times$ we can define a *Miyamoto map* $\mu_{i,\chi}$ such that

$$\mu_{i,\chi}(m) = \chi(g)m$$

for all $i \in I$, $a \in A_x^i$, $x \in \xi^{-1}(g)$ and $g \in \Gamma$. The elements $\mu_{i,\chi}$ are invertible, i.e. contained in $\mathsf{GL}(M)$.

In general, we don't have a morphism $\mathsf{Miy}(\mathcal{A}) \to \mathsf{GL}(M)\colon \tau_{i,\chi} \to \mu_{i,\chi}$. However, for the universal Miyamoto group, we have more control over the situation. We are interested in whether $\widehat{\mathsf{Miy}}(\mathcal{A}) \to \mathsf{GL}(M)\colon t_{i,\chi} \to \mu_{i,\chi}$ defines a morphism of groups, i.e. a linear representation of $\widehat{\mathsf{Miy}}(\mathcal{A})$. Again, we can formulate a sufficient condition on (M, Σ); see Definition 4.1.9. A module that satisfies this condition is called *Miyamoto-admissable.*

Theorem A.3.3 (See Theorem 4.1.11)**.** *If (M, Σ) is a Miyamoto-admissable module for a "nice" decomposition algebra $\mathcal{A} = (A, I, \Omega)$ with a graded fusion law, then*

$$\widehat{\mathsf{Miy}}(\mathcal{A}) \to \mathsf{GL}(M)\colon t_{i,\chi} \mapsto \mu_{i,\chi}$$

defines a group homomorphism.

A module (M, Σ) for an axial decomposition algebra (A, I, Ω, α) with fusion law $(X, \star)$ is said to be *of axial type* if $M_x^i = \{m \in M \mid \alpha(i) \cdot m = \lambda(x)m\}$ for all $x \in X$ and $i \in I$. A module of axial type is always Miyamoto-admissable.

Theorem A.3.4 (See Theorem 4.2.7)**.** *If (M, Σ) is a module of axial type for a "nice" axial decomposition algebra $\mathcal{A} = (A, I, \Omega, \alpha)$ with a graded fusion law, then*

$$\widehat{\mathsf{Miy}}(\mathcal{A}) \to \mathsf{GL}(M)\colon t_{i,\chi} \mapsto \mu_{i,\chi}$$

defines a group homomorphism.

For Matsuo algebras, the situation is even nicer and we can also define a module for any representation of the universal Miyamoto group (Proposition 4.3.3). For each of these modules (M, Σ) we have $M_1^i = 0$ for all $i \in I$ and 1 the unit of the fusion law. In fact, the category of representations of the universal Miyamoto group is equivalent to the full subcategory of such modules (Corollary 4.3.5). We also study those modules for which $M_1^i \neq 0$. Under some conditions, these are direct sums of adjoint modules of the Matsuo algebra; see Theorems 4.3.10 and 4.3.13.

A.4 Frobenius algebras for simply laced Chevalley groups

Skip Garibaldi and Robert M. Guralnick observed in 2015 that there exists a 3875-dimensional non-associative algebra on which the simple algebraic group of type E_8 acts by automorphisms [GG15]. We give two explicit constructions for this algebra. Moreover, we show that a similar algebra can be constructed for all Chevalley groups of simply laced type.

Let $\mathcal{L}$ be a complex simple Lie algebra of type A_n ($n \geq 3$), D_n ($n \geq 4$) or E_n ($n \in \{6, 7, 8\}$). We will construct an algebra on which $\mathcal{L}$ acts by derivations. The underlying module for this algebra will be a submodule $\mathcal{A}$ of the symmetric square $S^2(\mathcal{L})$ of the adjoint module. We can identify $S^2(\mathcal{L})$ with the space of symmetric operators $\mathcal{L} \to \mathcal{L}$ with respect to the Killing form of $\mathcal{L}$; see Definition 5.1.2. The Jordan product and trace form turn this space, and therefore $S^2(\mathcal{L})$, into a Frobenius algebra; see Example 1.2.9 and Definition 5.1.2. We can define a product on $\mathcal{A}$ by multiplying in $S^2(\mathcal{L})$ and projecting onto $\mathcal{A}$. Together with the restriction of the trace form, this gives $\mathcal{A}$ the structure of a Frobenius algebra (Proposition 5.2.14).

We also give an independent, more efficient, construction of this Frobenius algebra that does not rely on the (much larger) symmetric square $S^2(\mathcal{L})$. This construction starts by describing the product and Frobenius form on the zero weight space $\mathcal{A}_0$ of $\mathcal{A}$. Next, we extend this product to the whole of $\mathcal{A}$, relying on the fact that $\mathcal{L}$ acts by derivations on $\mathcal{A}$ and the specific nature of $\mathcal{A}$ (Theorem 5.4.9). To this end, we also prove an important lemma that shows that any automorphism of $\mathcal{L}$ naturally extends to an automorphism of $\mathcal{A}$ (Lemma 5.4.2).

We use the techniques developed in Sections 2.8 and 3.3 to give both $\mathcal{A}$ and its zero weight subalgebra $\mathcal{A}_0$ the structure of a decomposition algebra. We explicitly describe the decompositions for each of the possible types, i.e. A_n, D_n and E_n. The fusion law of these decomposition algebras turns out to be $\mathbb{Z}/2\mathbb{Z}$-graded. Therefore, we can consider the corresponding Miyamoto group of $\mathcal{A}$. This is the complex adjoint Chevalley group of $\mathcal{L}$ in its action on $\mathcal{A}$; see Theorem 5.7.10.

The original motivation for constructing this algebra is the case where $\mathcal{L}$ is of type E_8. In that case, we go one step further. We show that the constructed Frobenius algebra belongs to a one-parameter family of Frobenius algebras on $\mathcal{A}$ (Proposition 5.8.2). Each of these is an axial decomposition algebra with a $\mathbb{Z}/2\mathbb{Z}$-graded fusion law and Miyamoto group the adjoint Chevalley group of type E_8. More precisely we show the following theorem, cf. Theorem 5.8.7.

Theorem A.4.1. *There exists a one-parameter family of non-associative commutative 3876-dimensional Frobenius algebras on which the complex Chevalley group of type E_8 acts by automorphisms. Each of these contains a set Ω of idempotents. For each idempotent $e \in \Omega$, there exists a decomposition $A = \bigoplus_{1 \leq i \leq 6} A_i^e$ of the algebra A as a complex vector space. Moreover $ea = \lambda_i a$ when $a \in A_i^e$ for $\lambda_1 = 1$,*

$\lambda_2 = 0$, $\lambda_3 = \frac{4}{3}c_1 - \frac{1}{6}$, $\lambda_4 = \lambda_5 = \frac{1}{2}$ and $\lambda_6 = c_1$ where $c_1 \in \mathbb{C}$ depends on the parameter. The linear map defined by

$$\tau_e(a) := \begin{cases} a & \text{if } a \in \bigoplus_{1\le i\le 4} A_i^e, \\ -a & \text{if } a \in A_5^e \oplus A_6^e, \end{cases}$$

defines an automorphism of the Frobenius algebra. These automorphisms τ_e for $e \in \Omega$ generate the complex Chevalley group of type E_8.

A.5 Further developments

In the final chapter, we provide some ideas for further investigation and present some unfinished projects.

First, expansions are introduced as natural generalizations of split extensions.

Definition A.5.1. Let R be a commutative ring with identity and let A and B be R-algebras. Then we call B an *expansion* of A if there exist morphisms $\sigma\colon A \to B$ and $\pi\colon B \to A$ of R-modules such that $\pi \circ \sigma = \mathsf{id}_A$ and

$$ab = \pi(\sigma(a)\sigma(b))$$

for all $a, b \in A$. If both σ and π are morphisms of *algebras* then we call A a *split extension* of B.

We illustrate how many constructions of non-associative algebras can be formulated in terms of expansions. We show how some properties, e.g. the existence of a Frobenius form (Proposition 6.1.5), can be transferred between an algebra and its expansions. This is particularly useful if the expansion is better understood, e.g. if it is associative or a Jordan algebra. We show that such expansions exist for unital algebras in Propositions 6.1.3 and 6.1.4.

For axial decomposition algebras we introduce axial expansions.

Definition A.5.2. Let $\mathcal{A} = (A, I, \Omega, \alpha)$ be an axial $\mathcal{F}_A$-decomposition algebra. An *axial expansion* of $\mathcal{A}$ is an axial $\mathcal{F}_B$-decomposition algebra $\mathcal{B} = (B, I, \Sigma, \beta)$ for which there exist morphisms $\pi\colon B \to A$ and $\sigma\colon A \to B$ of R-modules, a map $\psi\colon J \to I$ and a morphism $\zeta\colon \mathcal{F}_B \to \mathcal{F}_A$ (preserving the distinguished unit) such that the following conditions hold:

(E1) $\pi \circ \sigma = \mathsf{id}_A$,

(E2) $\pi(\sigma(a)\sigma(b)) = ab$ for all $a, b \in A$,

(E3) $\pi(B_x^j) \subseteq A_{\zeta(x)}^{\psi(j)}$ for all $x \in \mathcal{F}_B$ and $j \in J$.

We discuss two special types of axial expansions. *One-point expansions* try to formalize the idea of adjoining a unit to a non-associative algebra while $\mathbb{Z}/2\mathbb{Z}$-*expansions* expand an axial decomposition algebra by a module (Definitions 6.1.7 and 6.1.11). These concepts help us to gain a better understanding of some of the Norton-Sakuma algebras as illustrated in Examples 6.1.9, 6.1.10 and 6.1.13.

Another idea that is worth persuing is the attempt to construct axial decomposition algebras whose Miyamoto group is a pariah, this is a sporadic group that cannot be realized as a subquotient of the monster group. In Section 6.2, we suggest how to construct such algebras for the Lyons group and the third Janko group using the techniques from Section 2.8.

B

Nederlandstalige samenvatting

B.1 Historische context

Niet-associatieve algebra's spelen een belangrijke rol in verschillende deelgebieden van de wiskunde. De studie van Lie algebra's is wellicht het meest gekende voorbeeld. Lie algebra's werden geïntroduceerd om infinitesimale transformaties te bestuderen maar zijn belangrijk in bijna elk domein binnen de wiskunde en theoretische fysica. Ook andere types van niet-associatieve algebra's hebben belangrijke toepassingen. Jordan algebra's, bijvoorbeeld, formaliseren observabelen in kwantummechanica maar spelen ook een cruciale rol in Zel'manov's oplossing van het beperkte Burnside probleem. Hoewel we Lie algebra's en Jordan algebra's zullen aantreffen in dit proefschrift, is ons voornaamste doel om axiale algebra's te bestuderen. Zij vormen een nieuw type van niet-associatieve algebra's.

Wellicht één van de meest spectaculaire verbanden tussen groepentheorie en niet-associatieve algebra's is de constructie van de monstergroep als automorfismengroep van de Griess algebra. Deze algebra heeft dimensie 196884 hetgeen, zoals John McKay ontdekte, ook de eerste niet-triviale coëfficiënt is in de Fourier ontwikkeling van de j-functie, een modulaire vorm. Dit opmerkelijke verband tussen de monstergroep en modulaire functies leidde tot een aantal vermoedens die de 'monstrous moonshine' vermoedens worden genoemd. Ze werden gepostuleerd door John H. Conway en Simon P. Norton [CN79]. Richard E. Borcherds bewees deze vermoedens [Bor92], hetgeen hem in 1998 de Fields medaille opleverde. Hij maakte hierbij gebruik van de theorie van vertex operator algebra's (VOA's). Deze algebra's werden oorspronkelijk, op minder formele wijze, bestudeerd door fysici binnen het gebied van de conforme veldentheorie. De Griess algebra kan worden geïnterpreteerd in een dergelijke vertex operator algebra, de 'moonshine module' VOA [FLM84, FLM88].

In 2009 introduceerde Alexander A. Ivanov Majorana algebra's om de Griess algebra te kunnen bestuderen zonder de bijhorende VOA [Iva09]. Zijn aanpak is gebaseerd op de studie van idempotenten in de Griess algebra en gemotiveerd door de volgende observatie van John H. Conway [Con85]. Een bepaalde toevoegingsklasse van involuties in de monstergroep is in bijectief verband met een klasse van idempotenten van de Griess algebra A. De toegevoegde actie $\mathrm{ad}_e\colon A \to A\colon a \mapsto ea$ van een dergelijke idempotent $e \in A$ is diagonaliseerbaar met eigenwaarden 1, 0, $\frac{1}{4}$ en $\frac{1}{32}$. Bovendien gelden er bijzondere restricties, zogenaamde fusieregels, op de vermenigvuldiging van de bijhorende eigenvectoren. Majorana algebra's worden, bij definitie, voortgebracht door dergelijke idempotenten en deze worden de assen van de algebra genoemd.

Door gebruik te maken van Majorana algebra's herbewezen Alexander A. Ivanov, Dmitrii V. Pasechnik, Àkos Seress en Sergey Shpectorov een resultaat van Shinya Sakuma dat oorspronkelijk werd geformuleerd in de taal van vertex operator algebra's [Sak07, IPSS10]. Dit resultaat zegt dat de Majorana algebra's voortgebracht door twee assen behoren tot één van negen isomorfismenklassen, de Norton-Sakuma algebra's. Elk van deze is isomorf met een deelalgebra van de Griess algebra. Majorana theorie wordt tevens gebruikt om andere deelalgebra's van de Griess algebra te beschrijven [CRI14, Dec14, FIM16a, FIM16b, IPSS10, IS12b, IS12a, Iva11a, Iva11b].

Het is erg gebruikelijk dat, wanneer een belangrijke stelling wordt bewezen, men nadien de voorwaarden probeert te verfijnen. Dit leidt vaak tot nieuwe definities en interessante inzichten. Axiale algebra's vinden zo hun oorsprong in een veralgemening van de stelling van Sakuma. Ze werden geïntroduceerd door Jonathan I. Hall, Felix Rehren en Sergey Shpectorov [HRS15a, HRS15b]. Enerzijds zijn het veralgemeningen van Majorana algebra's gedefinieerd over willekeurige velden. Anderzijds omvatten ze ook commutatieve, associatieve algebra's en Jordan algebra's. Hun definiërende eigenschap is dat ze worden voorgebracht door idempotenten die tot decomposities in eigenruimten leiden. De vermenigvuldiging van eigenvectoren wordt beperkt door een fusiewet. De Peirce decomposities van associatieve en Jordan algebra's zijn belangrijke voorbeelden van dergelijke decomposities.

Sindsdien werd de theorie van axiale algebra's verder ontwikkeld en nieuwe voorbeelden werden geconstrueerd. Matsuo algebra's, een klasse van algebra's geassocieerd aan 3-transpositiegroepen, vormen een typisch voorbeeld van een axiale algebra. Hun fusiewet lijkt op die van Jordan algebra's en axiale algebra's met een dergelijke fusiewet werden uitvoerig bestudeerd [HRS15a, HSS18, DMR17]. Computeralgoritmen werden ontwikkeld om nieuwe voorbeelden te construeren en vermoedens te testen [Ser12, PW18, MS20].

De studie van axiale algebra's is echter nog maar pas begonnen en er ontbreekt een theoretisch framework. Dit proefschrift heeft als één van de voornaamste doelen om een dergelijk framework te voorzien. We zullen (axiale) decompositie-algebra's invoeren die alle algebra's beschrijven met gelijkaardige eigenschappen als axiale algebra's. Decompositie-algebra's vormen een interessante categorie hetgeen hun bruikbaarheid nog versterkt.

Het verband met groepentheorie is steeds een belangrijke motivatie geweest voor het onderzoeken van axiale algebra's. Op het niveau van VOA's merkte Masahiko Miyamoto reeds op dat er involuties bestaan horende bij conforme vectoren in vertex operator algebra's [Miy96]. Voor de 'moonshine module' VOA brengen deze involuties de monstergroep voort. Deze belangrijke connectie bestaat ook voor Majorana algebra's, axiale algebra's en decompositie-algebra's. We maken gebruik van het feit dat decompositie-algebra's een categorie vormen om dit verband verder te versterken.

Daarnaast introduceren we modulen over (axiale) decompositie-algebra's. Ook hier is er een duidelijke link met groepen. In het bijzonder voor Matsuo algebra's is dit een heel sterk verband.

Een belangrijke onderzoeksvraag bestaat er in om te bepalen welke groepen aanleiding geven tot interessante axiale algebra's. Bovendien kunnen nieuwe voorbeelden ons nieuwe inzichten verschaffen. We geven een aantal manieren om dergelijke voorbeelden te construeren. Daarnaast construeren we, expliciet, decompositie-algebra's voor de complexe Chevalley groepen van ADE-type. In het bijzonder beschrijven we een 3876-dimensionale algebra waarop de complexe Chevalley groep van type E_8 werkt door automorfismen. Het bestaan van een dergelijke algebra werd reeds bewezen door Skip Garibaldi en Robert M. Guralnick in 2015 [GG15], onafhankelijk van de studie van axiale algebra's. Hun bewijs is echter niet constructief en, voor zover we weten, was er tot nog toe geen expliciete constructie voor deze algebra bekend.

B.2 Overzicht van de resultaten

In Hoofdstuk 2 geven we meer achtergrond omtrent Peirce decomposities, de Griess algebra en axiale algebra's. Nadien voeren we (algemene) fusiewetten en decompositie-algebra's in. We tonen aan hoe axiale algebra's passen binnen het framework van decompositie-algebra's door axiale decompositie-algebra's te introduceren.

We geven twee generieke methoden om decompositie-algebra's te construeren (Secties 2.8 en 2.9). Een eerste methode gaat uit van een groep (of complexe Lie algebra) die werkt via automorfismen (respectievelijk derivaties) op een algebra. Door de isotypische decompositie van de algebra te beschouwen kunnen we deze algebra de structuur geven van een decompositie-algebra. De tweede methode is gebaseerd op de theorie van Norton algebra's. Dit is een klasse niet-associatieve algebra's gerelateerd aan associatie-schema's.

Daarnaast tonen we aan dat zowel de definitie van fusiewetten als decompositie-algebra's aanleiding geeft tot een natuurlijke notie van morfismen. We bestuderen een aantal eigenschappen van de bijhorende categorieën. We bewijzen onder andere dat beide categorieën compleet zijn (Proposities 2.10.4 en 2.11.3). De categorie van fusiewetten is tevens co-compleet (Propositie 2.10.5). Daarnaast introduceren we decompositie-idealen en tonen we aan dat deze de rol van kernen vervullen binnen de categorie van decompositie-algebra's (Definitie 2.11.4 en Propositie 2.11.5).

In Hoofdstuk 3 gaan we dieper in op het verband tussen axiale algebra's en groepen. Graderingen van fusiewetten liggen hier aan de basis. We bestuderen de graderingen van enkele belangrijke fusiewetten: de klasse-fusiewet (cf. Propositie 3.1.5) en de representatie-fusiewet (cf. Propositie 3.1.6).

Indien de fusiewet van een decompositie-algebra gegradeerd is, geeft dit aanleiding tot een belangrijke groep van automorfismen van de algebra. Deze groep noemen we de Miyamoto groep. We bepalen die Miyamoto groep voor de generieke voorbeelden uit Hoofdstuk 2; zie Sectie 3.3.

Vervolgens bestuderen we of het nemen van de Miyamoto groep een functor induceert tussen de categorie van decompositie-algebra's en die van groepen. Dit blijkt vaak niet het geval (zie Voorbeeld 3.7.11) en daarom introduceren we de universele Miyamoto groep. Deze groep is een centrale extensie van de Miyamoto groep en gedraagt zich beter met betrekking tot morfismen van decompositie-algebra's. We tonen aan dat een epimorfisme van decompositie-algebra's aanleiding geeft tot een bijhorend morfisme van zowel de Miyamoto groep als zijn universele variant (Propositie 3.7.2). Voor de universele Miyamoto groep kunnen we nog een stap verder gaan. We kunnen een natuurlijke voorwaarde leggen op een morfisme van decompositie-algebra's, uitgedrukt in termen van een commutatief diagram. Deze voorwaarde impliceert dat er een bijhorende morfisme van de universele Miyamoto groep bestaat zoals we bewijzen in Stelling 3.7.8. Voor morfismen van axiale decompositie-algebra's is deze extra conditie automatisch voldaan; zie Lemma 3.7.9 en Stelling 3.7.10.

Hoofdstuk 4 behandelt de theorie van modulen over (axiale) decompositie-algebra's. Ook hier is er een notie van Miyamoto groepen indien de fusiewet gegradeerd is. Onder bepaalde voorwaarden correspondeert ieder moduul van een decompositie-algebra met een representatie van de universele Miyamoto groep (Stelling 4.1.11). Voor axiale decompositie-algebra's kunnen deze voorwaarden opnieuw vereenvoudigd worden (Stelling 4.2.7). Voor Matsuo algebra's over Fischer ruimten kunnen we deze resultaten nog verder verbeteren. In dit geval kunnen we de universele Miyamoto groep interpreteren als de 'universele' 3-transpositiegroep horende bij de Matsuo algebra (Stelling 3.6.5). Een representatie van deze groep geeft nu ook aanleiding tot een moduul van de algebra (Propositie 4.3.3) hetgeen resulteert in een equivalentie van categorieën (Gevolg 4.3.5). In Stellingen 4.3.10 en 4.3.13 bespreken we de modulen die niet op deze manier verkregen kunnen worden.

In Hoofdstuk 5 construeren we decompositie-algebra's met als Miyamoto-groepen de complexe Chevalley groepen van ADE-type. Dit omvat een expliciete beschrijving van de algebra voor E_8 uit [GG15]. In dit geval kunnen we deze algebra zelfs de structuur geven van een axiale decompositie-algebra. Meer bepaald bewijzen we de volgende stelling, cf. Stelling 5.8.7.

Stelling B.2.1. *Er bestaat een één-parameter familie van niet-associatieve com-*

mutatieve 3876-dimensionale algebra's uitgerust met een associatieve bilineaire vorm. Elk van deze algebra's bevat een verzameling Ω *van idempotenten. Voor elk element* $e \in \Omega$*, bestaat er een decompositie* $A = \bigoplus_{1 \le i \le 6} A_i^e$ *van de algebra* A *als complexe vectorruimte. Bovendien geldt er dat* $ea = \lambda_i a$ *voor alle* $a \in A_i^e$ *met* $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = \frac{4}{3}c_1 - \frac{1}{6}$, $\lambda_4 = \lambda_5 = \frac{1}{2}$ *en* $\lambda_6 = c_1$ *en* $c_1 \in \mathbb{C}$ *afhankelijk van de parameter. De lineaire afbeelding gedefinieerd door*

$$\tau_e(a) := \begin{cases} a & \textit{indien } a \in \bigoplus_{1 \le i \le 4} A_i^e, \\ -a & \textit{indien } a \in A_5^e \oplus A_6^e, \end{cases}$$

bepaalt een automorfisme van deze algebra. De automorfismen τ_e *voor* $e \in \Omega$ *brengen de complexe Chevalley groep van type* E_8 *voort.*

Tot slot suggereren we een aantal ideeën voor verder onderzoek. Een eerste idee is gebaseerd op het concept van *expansies*, een begrip dat we introduceren in Definitie 6.1.1. Expansies zijn cruciaal in vele constructies van niet-associatieve algebra's en ze kunnen ons bijgevolg meer informatie geven over hun structuur. Ook voor axiale decompositie-algebra's is dit het geval zoals we illustreren voor twee bijzondere types van expansies. We tonen aan hoe ze ons helpen om (sommige van) de Norton-Sakuma algebra's beter te begrijpen. Een tweede idee heeft betrekking tot de constructie van axiale decompositie-algebra's voor twee andere sporadische groepen: de Lyons groep en derde Janko groep. We geven suggesties voor geschikte algebra's waarop deze groepen door automorfismen werken en illustreren hoe de ideeën uit Sectie 2.8 kunnen helpen om ze de structuur te geven van axiale decompositie-algebra's.

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