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Graphs and finite geometries

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Chapter 1

Introduction

During my doctoral studies, I was working on questions which arise both from finite geometry and graph theory. Apart from the first problem which comes from spectral graph theory, the other three investigated topics are in the intersection of these two huge parts of mathematics. Interestingly, the latter three areas have something in common, namely in all of them we would like to understand the extremal objects, their structure and whether a stability phenomenon occurs around these extremal examples.

One of the investigations was motivated by Tamás Héger and Marcella Takáts [12], who gave a sharp result concerning resolving sets in the incidence graph of projective planes. While there are several papers in the literature about resolving sets and metric dimension, there is far less known about another concept, the *resolving partitions* and *partition dimension* which can be defined very similarly, although it turned out that they behave quite differently.

We also considered a natural question regarding finite projective planes. For a fixed prime power q, we would like to determine the number of collinearity constraints, reckon them as 3-tuples of points of the projective plane which must lie on a line, such that no matter how they are given we can embed them into a finite projective plane of order q. Quite surprisingly, this investigation unfolded some deep connection between our motivation and connectivity properties of hypergraphs, in particular linear 3-uniform hypergraphs. These connections also helped us to observe some interesting properties of Steiner triple systems which are well-known and immensely studied by top researchers.

There are a large number of different type of hypergraph coloring problems, most of them are broadly studied. In contrast to the usual chromatic number, in the mid-nineties V. Voloshin in his papers [7, 8, 9] introduced a new notion, *the upper* *chromatic number*, which requires quite the opposite from a coloring. Namely, here we would like to use as many colors as we can to color the points such that there are no rainbow hyperedges, which means that every hyperedge must have points with the same color. Since projective planes can be considered as hypergraphs where the vertices correspond to the points, and the hyperedges correspond to the lines of the projective plane, therefore these structures provide us natural examples for which determining the upper chromatic number is rather intriguing. In 2007 Gábor Bacsó and Zsolt Tuza [13] gave bounds on the upper chromatic number of an arbitrary projective plane of order q and then later in 2013 Gábor Bacsó, Tamás Héger and Tamás Szőnyi [14] determined the upper chromatic number of $\mathrm{PG}(2, q)$ which depends only on $\tau_2(\mathrm{PG}(2, q))$ which is the minimum number of points in a 2-fold blocking set in $\mathrm{PG}(2, q)$. All of these results motivated us to investigate this question in higher dimensional finite projective spaces, too.

The thesis is built up as follows. In Chapter 2 we recall the basic definitions and important notions which are indispensable in the sequel. Furthermore we also try to emphasize some known results that we will often refer to later on. In Chapter 3 we will prove a conjecture of Willem H. Haemers by constructing two b-regular graphs which have the same adjacency spectrum, but exactly one of them has a perfect matching. This construction came from a joint work [A] with Willem H. Haemers and Jay Cummings.

Together with Zoltán Lóránt Nagy in [B] we managed to determine the partition dimension of the incidence graph of a finite projective plane up to a constant factor of 2 and in Chapter 4 we will discuss the details. Chapter 5 is devoted to the investigation of collinearity constraints and their embeddability properties which interestingly have connection with connectivity notions in hypergraph theory. Moreover this led to some nice observations about Steiner triple systems too. The results are joint with Zoltán Lóránt Nagy (see [C]).

In Chapter 6, together with Tamás Héger and Tamás Szőnyi (see [D]) we generalize the results of Gábor Bacsó, Tamás Héger and Tamás Szőnyi regarding the upper chromatic number to higher dimensional finite projective spaces. The method is quite similar to their former argument but we need to evolve it in some sense, because in our case we often have to deal with weighted blocking sets, too (for example after suitable projections). I would like to highlight that there will be a series of observations if the order of the projective plane is a prime number which led to Theorem 6.1.9 that is exciting in its own right. Nevertheless, as a corollary to this theorem, we can achieve a considerably greater interval of stability if the order of the projective space is a prime number.

For the sake of clarity we will use the word *Result* when referring to other authors' work, and use *Theorem*, *Lemma*, etc. if it is our original result. The thesis is based on those articles which are denoted by roman numerals in the bibliography. Let me mention that there are two papers of ours [E, F] which have also been published, but I chose not to include them. Although they deal with graph theoretic questions I think they do not fit in smoothly in the current frame of the thesis.

Chapter 2

Preliminaries

2.1 Graphs

If we don't say it otherwise we consider finite, simple, undirected graphs throughout the thesis.

Definition 2.1.1. *$G = (V; E)$ denotes a graph with vertex-set V and edge-set E. Two vertices are called* adjacent *if there is an edge connecting them. The set of* neighbors *of a vertex v is denoted by $N(v)$ that is the collection of those vertices which are adjacent to v. The number of neighbors of a vertex v is called the* degree *of v, and we denote it by $d(v)$. A graph is d-*regular *if the degree of every vertex is exactly d.*

Definition 2.1.2. *P_n and C_n denote the path and the cycle on n vertices. The* length *of a path or a cycle is the number of edges contained in it. For distinct vertices x and y, their* distance *is denoted by $d(x, y)$, that is the length of the shortest path between x and y. A graph is said to be* connected *if there exists a path between any two distinct vertices. If there is no path between two vertices then let their distance be ∞ by definition. The* diameter *of a graph $\mathrm{diam}(G)$ is equal to the largest distance among the pair of vertices in G. The diameter is finite if and only if G is connected (and finite).*

Definition 2.1.3. *The* adjacency matrix *of a graph G is a $|V| \times |V|$ matrix denoted by $A(G)$ (or shortly A) such that for vertices x and y the corresponding element of the matrix $A(x, y) = 1$ if and only if x and y are adjacent, otherwise $A(x, y) = 0$. The multiset of the eigenvalues of the adjacency matrix is called the* spectrum *of a graph with respect to the adjacency matrix. We call two graphs* cospectral *if and only if their spectra is the same.*

There is a widely investigated set of problems concerning a graph's spectrum. Namely, whether a property of a graph is determined by its' adjacency spectrum. In other words, what are those graph properties which happen to be the same for any two cospectral graph. An even simpler, but very natural question is the following. How can we find two graphs that are cospectral in an easy way? There exist some methods to create a new graph from the original graph which are cospectral. Nevertheless, most of the time the methods are useful too, because the two graphs will not be isomorphic. We recall one of such methods here which we will use later in Chapter 3. It is due to Godsil and McKay [29] (see also van Dam, Haemers [27]).

Result 2.1.4. *Let G be a graph and let $\{X, Y\}$ be a partition of the vertex set. Suppose that X induces a regular subgraph, and that each vertex $y \in Y$ has $0, \frac{|X|}{2}$, or $|X|$ neighbors in X. Make a new graph G' from G as follows. For each $y \in Y$ with $\frac{|X|}{2}$ neighbors in X, delete the $\frac{|X|}{2}$ edges between y and X, and join y to the $\frac{|X|}{2}$ other vertices of X. Then G and G' are cospectral.*

The set X is called a *switching set*. The operation that changes G to G' is called *Godsil-McKay switching*.

2.2 Incidence structures

Definition 2.2.1. *An* incidence structure *is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ such that $\mathcal{P}$ and $\mathcal{L}$ are disjoint, nonempty sets and $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ is a binary relation between $\mathcal{P}$ and $\mathcal{L}$ which is called* incidence.

In this thesis we will consider only such incidence structures where both $\mathcal{P}$ and $\mathcal{L}$ have finitely many elements. The elements of $\mathcal{P}$ and $\mathcal{L}$ usually be called *points* and *lines* or *blocks* depending on the context. Instead of $(P, \ell) \in \mathcal{I}$ we often use the geometric language such as P and ℓ are incident or ℓ passes through P.

Definition 2.2.2. *An incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is* isomorphic *to another incidence structure $(\mathcal{P}', \mathcal{L}', \mathcal{I}')$ if and only if there is a bijection $\varphi : \mathcal{P} \cup \mathcal{L} \to \mathcal{P}' \cup \mathcal{L}'$ for which $\varphi(\mathcal{P}) = \mathcal{P}'$, $\varphi(\mathcal{L}) = \mathcal{L}'$ and $(P, \ell) \in \mathcal{I} \iff (\varphi(P), \varphi(\ell)) \in \mathcal{I}'$ holds.*

Definition 2.2.3. *For an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ the* dual structure *is $(\mathcal{L}, \mathcal{P}, \mathcal{I}^*)$ for which $(\ell, P) \in \mathcal{I}^*$ if and only if $(P, \ell) \in \mathcal{I}$. An isomorphism between an incidence structure and its dual is called a* correlation*. A correlation of order 2 is called a* polarity*. An incidence structure is said to be* self-dual *if there* *exist a correlation of the incidence structure. A class of incidence structures is called* self-dual *if all the dual structures are in the class.*

For instance, the class of all finite projective planes is self-dual. Note that for a self-dual class of incidence structures we can apply the *principle of duality* which states the following. Given a theorem on all structures in the class we can obtain another theorem valid for all of them by interchanging the words „points" and „lines".

Definition 2.2.4. *The* incidence graph *of an incidence structure* $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ *is a bipartite graph with vertex classes* $\mathcal{P}$ *and* $\mathcal{L}$ *and* (P, ℓ) *is an edge of this graph if and only if* $(P, \ell) \in \mathcal{I}$*.*

We refer the reader to the books of Beth, Jungnickel and Lenz [17, 18] for further details.

2.3 Projective and affine spaces

Let us start by giving the combinatorial definition of affine and projective planes.

Definition 2.3.1. *Let* $\mathcal{P}$ *be a non-empty set (whose elements are called* points*) and* $\mathcal{L}$ *is a collection of some subsets of* $\mathcal{P}$ *(whose elements are called* lines*). The incidence structure* $(\mathcal{P}, \mathcal{L})$ *is an* affine plane *if and only if the following axioms hold:*

A1) *For every two different points there is exactly one line incident with both;*

A2) *For any line* ℓ *and any point* $P \notin \ell$ *there exists a unique line* ℓ' *such that* $P \in \ell'$ *and* $\ell \cap \ell' = \emptyset$*;*

A3) *There exist four points, no three of which belong to the same line.*

Definition 2.3.2. *Let* $\mathcal{P}$ *be a non-empty set (whose elements are called* points*) and* $\mathcal{L}$ *is a collection of some subsets of* $\mathcal{P}$ *(whose elements are called* lines*). The incidence structure* $(\mathcal{P}, \mathcal{L})$ *is a* projective plane *if and only if the following axioms hold:*

P1) *For every two different points there is exactly one line incident with both;*

P2) *For every two different lines there is exactly one point incident with both;*

P3) *There exist four points, no three of which belong to the same line.*

If two lines of an affine plane do not intersect each other, we call them *parallel*. One can construct an affine plane from a projective plane by removing a line from it together with its points. It is well-known that any affine plane has q^2 points and q^2+q lines and every projective plane has q^2+q+1 points and q^2+q+1 lines for some integer $q \geq 2$, which is called the *order* of the affine plane or the projective plane, respectively. Moreover, every affine plane of order q can be embedded into a unique projective plane of order q. The n-dimensional projective space over the finite field $\mathrm{GF}(q)$, denoted by $\mathrm{PG}(n,q)$, can be defined as follows.

Definition 2.3.3. *Consider an $(n+1)$-dimensional vector space denoted by V over $\mathrm{GF}(q)$. Let the point-set $\mathcal{P} = \mathcal{F}_0$ of $\mathrm{PG}(n,q)$ be the set of 1-dimensional subspaces of V, let the set of lines $\mathcal{L} = \mathcal{F}_1$ of $\mathrm{PG}(n,q)$ be the set of 2-dimensional subspaces of V; and in general, the set $\mathcal{F}_k$ of $(k+1)$-dimensional subspaces of V, be the k-dimensional subspaces of $\mathrm{PG}(n,q)$ for $0 \leq k \leq n$. Incidences are given by containment.*

The elements of $\mathcal{F}_k$ are called the k-dimensional subspaces of $\mathrm{PG}(n,q)$ in the sequel. Thus $\mathcal{F}_2$ will correspond to the 2-dimensional subspaces of $\mathrm{PG}(n,q)$ which are called *planes*. The *codimension* of a k-dimensional subspace of $\mathrm{PG}(n,q)$ is defined as $n-k$. For example we will refer to subspaces of $\mathrm{PG}(n,q)$ with codimension 1 as *hyperplanes*. Let us introduce the notation

$$\theta_{q,n} = \theta_n = q^n + q^{n-1} + \ldots + q + 1 = \frac{q^{n+1}-1}{q-1} \tag{2.3.1}$$

for the number of points in an n-dimensional projective space of order q. We recall that a projective plane of order q has $\theta_2 = q^2+q+1$ points. Finally, we recall that the number of $(k+1)$-spaces containing a fixed k-space in $\mathrm{PG}(n,q)$ is θ_{n-k-1}. This can be seen easily by taking an $(n-k-1)$-space disjoint from the fixed k-space and observing that each appropriate $(k+1)$-space intersects it in a unique point.

It is well-known that for $n \geq 3$ any n-dimensional finite projective space must be isomorphic to $\mathrm{PG}(n,q)$ by Wedderburn's theorem. Notice that for $n=2$ there could be other finite projective planes of order q which are not isomorphic to $\mathrm{PG}(2,q)$ for some appropriate order. These other finite projective planes are called *non-Desarguesian*. For more information and details about the basics of finite geometry we refer the reader to the book of Hirschfeld [16].

2.4 Blocking sets

Definition 2.4.1. *An* m-space *is a subspace of* $\mathrm{PG}(n,q)$ *of dimension* m*. A point-set* $\mathcal{B} \subseteq \mathrm{PG}(n,q)$ *is called a* t-fold k-blocking set *if every* $(n-k)$*-space intersects* $\mathcal{B}$ *in at least* t *points. A point* $P \in \mathcal{B}$ *is* essential *if* $\mathcal{B}\setminus\{P\}$ *is not a* t*-fold* k*-blocking set, in other words, if there is an* $(n-k)$*-space through* P *that intersects* $\mathcal{B}$ *in precisely* t *points.* $\mathcal{B}$ *is called* minimal*, if all of its points are essential which means that* $\mathcal{B}$ *does not contain a smaller* t*-fold* k*-blocking set. A blocking set is also commonly defined as a point-set which intersects every line, but does not contain a line. In the language of hypergraphs,* t*-fold blocking sets are called* t-transversals*.*

In $\mathrm{PG}(n,q)$, every k-space intersects every $(n-k)$-space non-trivially. If $k < \frac{n}{2}$, it is easy to find two (or more, say, t) disjoint k-spaces, whose union is clearly a 2-fold (or t-fold) k-blocking set of size $2\theta_k$ (or $t\theta_k$). If $k \geq \frac{n}{2}$, this does not work and, in fact, not much is known even about the size of a smallest double k-blocking set, let alone its structure. Even for the particular case $n = 2k$, no general constructions had been known for small double k-blocking sets until 2016, when a construction appeared in [15] by De Beule, Héger, Szőnyi, Van de Voorde. Note that, however, *weighted* t-fold blocking sets can be obtained easily in this way.

Definition 2.4.2. *A weighted point set of* $\mathrm{PG}(n,q)$ *is a multiset* $\mathcal{B}$ *of the points of* $\mathrm{PG}(n,q)$*. We may refer to the multiplicities of the points of* $\mathcal{B}$ *via a function* $w = w_{\mathcal{B}}$ *mapping the point set of* $\mathrm{PG}(n,q)$ *to the set of non-negative integers, where* w *is also called a* weight function*. Points not contained in* $\mathcal{B}$ *have weight zero by* w *and, vice versa, zero weight points are considered to be not in* $\mathcal{B}$*. We call a weighted point set* $\mathcal{B}$ *of* $\mathrm{PG}(n,q)$ *a* weighted t-fold $(n-k)$-blocking set *if for every* k*-space* U*,* $\sum_{P\in U} w(P) \geq t$*, and* $\mathcal{B}$ *is called* minimal *if decreasing the weight of any point results in a* k*-space violating the previous property; in other words, if* $\mathcal{B}$ *does not contain a strictly smaller* t*-fold* $(n-k)$*-blocking set, where the size of a weighted point set is defined as the sum of weights in it. Also, for any point set* S*,* $|S \cap \mathcal{B}|$ *is defined as* $\sum_{P\in S} w(P)$*, and in general, any quantity referring to a number of points of* $\mathcal{B}$ *is usually considered with multiplicities. For example an* i*-secant line* ℓ *(with respect to* $\mathcal{B}$*) is a line such that* $|\ell \cap \mathcal{B}| = i$*.*

Weighted multiple blocking sets were studied recently by Ferret, Storme, Sziklai, Weiner [10] and Harrach [11]. We recall some important results which we will use in the rest of the thesis concerning weighted multiple blocking sets.

Result 2.4.3 (Harrach [11])**.** *Suppose that a weighted t-fold k-blocking set $\mathcal{B}$ in* $\mathrm{PG}(n,q)$ *has less than* $(t+1)q^k+\theta_{k-1}$ *points. Then* $\mathcal{B}$ *contains a unique minimal weighted t-fold k-blocking set* $\mathcal{B}'$*.*

The next theorem and its' variants are often referred as the $t \pmod p$ results.

Result 2.4.4 (Ferret, Storme, Sziklai, Weiner [10] Theorem 4.2 and Corollary 5.2)**.** *Let* $\mathcal{B}$ *be a minimal weighted t-fold* $(n-k)$*-blocking set of* $\mathrm{PG}(n,q)$, $q=p^h$, *p prime,* $h \geq 1$, *of size* $|\mathcal{B}| = tq^{n-k}+t+k'$, *with* $t+k' \leq \frac{q^{n-k}-1}{2}$*. Then* $\mathcal{B}$ *intersects every k-space in* $t \pmod p$ *points. Moreover if* $e \geq 1$ *denotes the largest integer for which each k-space intersects* $\mathcal{B}$ *in* $t \pmod{p^e}$ *points, then* $|\mathcal{B}| > tq^{n-k} + \frac{q^{n-k}}{p^e+1} - 1$*.*

Chapter 3

Cospectral regular graphs with and without a perfect matching

3.1 Introduction

In the last decades there were quite a lot of interest in the following type of questions. Is the graph G *determined by its spectrum* (or shortly *DS*)? It would mean that every graph which is cospectral to G with respect to the adjacency matrix (see Definition 2.1.3) needs to be isomorphic with G, too. Another heavily investigated variant is whether a property $\mathcal{P}$ of a graph G is determined by its spectrum. In other words, any graph cospectral to G (even the non-isomorphic ones) must also have property $\mathcal{P}$.

These type of questions originated from chemistry about half a century ago. In 1956, Günthard and Primas [19] raised a very similar question that relates spectral graph theory to chemistry. For about a year it was believed that every graph is DS, until Collatz and Sinogowitz [20] found two cospectral, non-isomorphic trees. After 1967 many examples of cospectral graphs were found. One result standing out is due to Schwenk [21], who stated that almost all trees are not determined by their spectrum.

On the other hand, van Dam and Haemers recently conjectured that almost all graphs are DS. The fraction of known non-DS graphs on n vertices is much larger than the fraction of DS graphs, but both fractions tend to zero as $n \to \infty$. The conjecture is false not even for trees but for strongly regular graphs, too. Since it is hard to prove that a graph is DS, only a very small number of graphs are known to be DS. However, Wang and Xu [22] developed a method and ran some experiments which showed that a very large part of the tested graphs are determined by their generalized spectrum, that is the spectrum of G together with the spectrum of the complement of G. For further details about these type of questions we refer the reader to the book of Brouwer and Haemers and a survey paper from van Dam and Haemers [24, 28].

In the rest of this chapter we focus on a problem posed by Haemers (believing that the answer should be negative) at the 22nd British Combinatorial Conference (see [25], Problem 22.8). The problem was also mentioned by Sebastian Cioabă at the REGS program in Combinatorics (Problem 48, 2011).

Question 3.1.1. *Does there exist a pair of regular, cospectral graphs where one has a perfect matching and the other has none?*

By Kőnig's theorem, regular bipartite graphs of positive degree have a perfect matching. For regular graphs which are not bipartite, there exists a powerful sufficient condition for existence of a perfect matching in terms of the spectrum of the adjacency matrix; see the papers of Haemers [23, 37, 30]. Bipartiteness as well as regularity can be deduced from the spectrum (see [27] by van Dam, Haemers). So it seems natural to ask whether for a regular graph the existence of a perfect matching can be seen from the spectrum.

For non-regular graphs there exist easy examples. The disjoint union of the 4-cycle C_4 and the path P_{n-4} has a perfect matching when n is even, and is cospectral with a graph consisting of the path P_{n-4} with two pendant vertices attached to each endpoint, which obviously has no perfect matching. More interesting connected examples were also found.

Proving that a property $\mathcal{P}$ of a graph G is not determined by the spectrum can be done by using one of the known methods of creating a new cospectral graph from G. Our main tool was the so-called Godsil-McKay switching (or shortly GM switching) which was introduced earlier in Result 2.1.4.

3.2 Construction

Theorem 3.2.1. *For each $b \geq 5$ there exists a pair of cospectral connected b-regular graphs, where one has a perfect matching and the other one does not.*

Proof. We will prove the theorem by constructing a b-regular graph with a Godsil-McKay switching set X and no perfect matching, for which switching will introduce many perfect matchings.

First assume b is odd. Define the graph H_b to be the complement of the disjoint union of $\frac{b-1}{2}$ paths P_2 and the path P_3. Then H_b has $b+2$ vertices, and each vertex has degree b except for one vertex u of degree $b-1$. To u we attach a pendant edge $\{u,v\}$, which increases its degree to b. Call the graph obtained this way with $\widetilde{H_b}$. Notice that H_b has an odd number of vertices, and therefore no perfect matching, while $\widetilde{H_b}$ has many perfect matchings, each of which contains the edge $\{u,v\}$. Consequently, by attaching any other graph F to $\widetilde{H_b}$ by identifying v with some vertex in F, the result has the property that no edge in F which is incident with v can be in a perfect matching. Figure 3.1 shows these gadgets for $b=5,6$.

We define the graph on the switching set X to be $K_3 + C_{2b-5}$, the disjoint union of a triangle and a cycle with $2b-5$ vertices. The construction of Y starts with $b-2$ disjoint copies of $\widetilde{H_b}$. We define W to be the set of vertices consisting of the $b-2$ copies of v. Each $w \in W$ will be joined to $b-1$ vertices of X, such that w is adjacent to every vertex of the triangle and no two vertex degrees of the larger cycle differ by more than one. Notice that our graph is now connected, every vertex except those in the larger cycle has degree b, and every vertex in Y is adjacent to 0, $\frac{|X|}{2}$, or $|X|$ vertices in X, so X is a switching set.

We will enlarge Y and add $(b-2)(b-1)$ edges between Y and X such that X remains a switching set, and each vertex gets degree b, as desired. To this end, first add one more copy of $\widetilde{H_b}$ and insert $b-1$ edges between the copy of v and the vertices in X belonging to the larger cycle, such that the degrees of these vertices still differ by at most one. Next we add $\frac{b-3}{2}$ disjoint unions of P_2, and join both vertices of each P_2 to $b-1$ vertices of the larger cycle in X, such that the degree of the vertices in X become equal to b. The result is shown in Figure 3.2. The

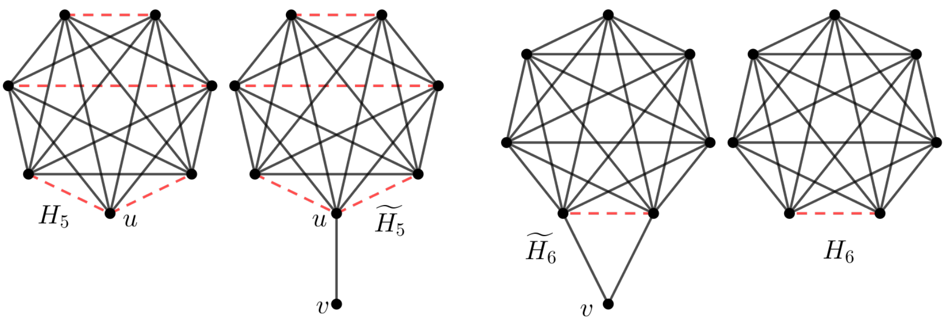


Figure 3.1: H_b and $\widetilde{H_b}$ for $b=5,6$

obtained graph is b-regular and connected, and X is a Godsil-McKay switching set. Furthermore, by deleting the $b-2$ vertices of W, the corresponding $b-2$ copies of H_b,

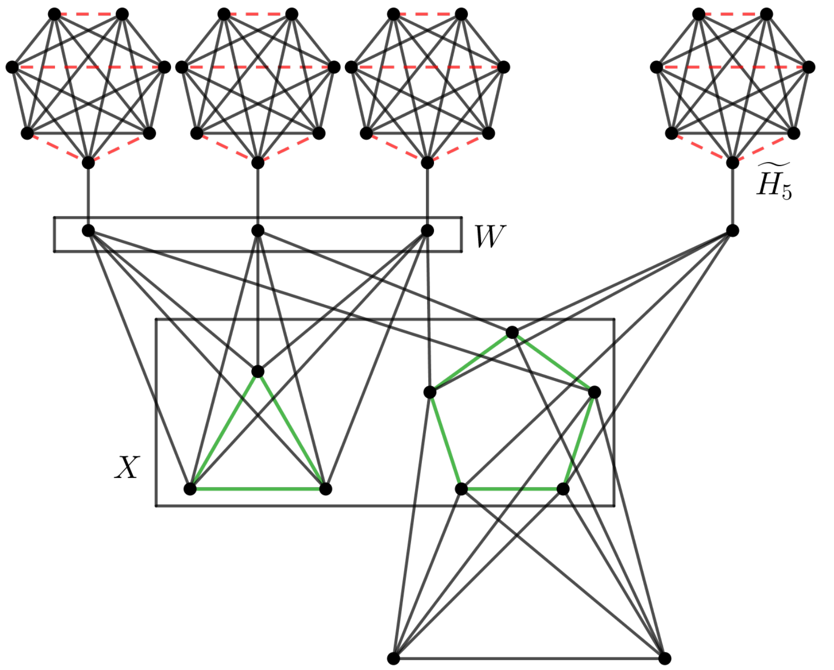


Figure 3.2: The 5-regular graph with no perfect matching

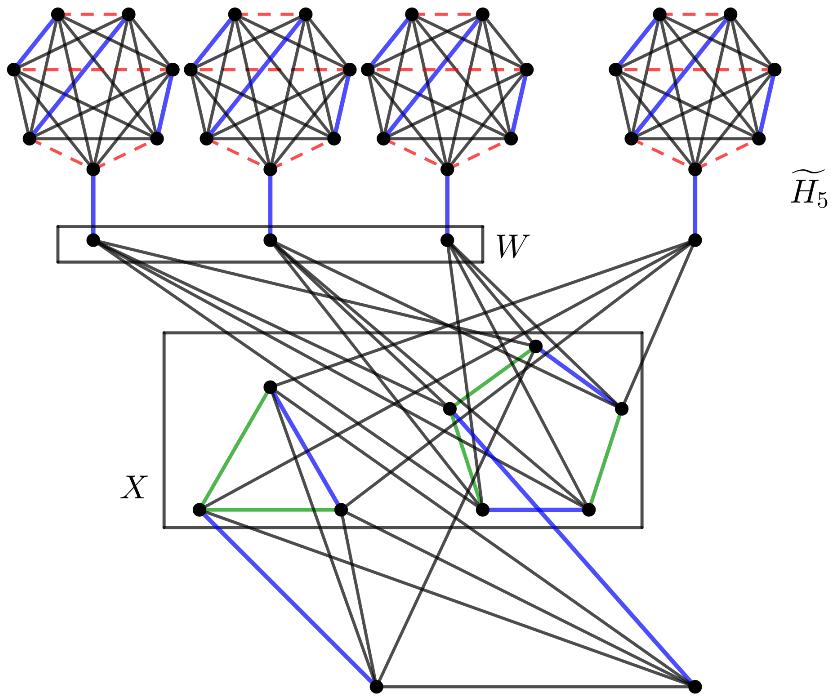


Figure 3.3: After the Godsil-McKay switching there is a perfect matching

the triangle and the remainder become b components, each having an odd number of vertices. Consequently this graph does not have a perfect matching. However, after performing a Godsil-McKay switch, one easily finds (many) perfect matchings and one of them is shown in Figure 3.3 with blue edges. This concludes the proof for the odd case.

When b is even, we make a few small changes. First, the graphs $\widetilde{H}_b$ should be replaced with a graph obtained as follows. Delete one edge from the complete graph K_{b+1}, add an additional vertex v and connect it to the two vertices of degree $b-1$. Then, since $b+1$ is odd, any perfect matching must contain one of these two new edges, which precludes any additional edge incident with v from being in a matching, just as before. The switching set X now has $2b-4$ vertices and induces $K_3 + C_{2b-7}$.

Lastly, we make a small alteration to the final step, where we increased the degrees of the vertices in the larger cycle to b. Because b is even, we can do this without adding an additional $\widetilde{H}_b$. Instead, we add a cycle C_{b-2} to Y and join each vertex in the added cycle to $b-2$ vertices of the larger cycle in X, such that all degrees become b. With the mentioned modifications we complete the proof of the theorem by imitating the above steps for the even case. □

3.3 Remarks

In the previous section, we have constructed a pair of connected b-regular graphs where one has a perfect matching, and the other one not, for every $b \geq 5$. The smallest example, which was shown in Figures 3.1 and 3.3, is a pair of 5-regular graphs on 42 vertices. The 6-regular example contains 44 vertices. In general, if b is odd the example contains $b^2 + 5b - 8$ vertices, and if b is even the example contains $b^2 + 3b - 10$ vertices.

We remark that for some $b \geq 5$ we can create smaller examples than the above explained one with a slightly different switching set. However, we do not know how to modify the construction to make it work for $b \leq 4$. In fact, if $b \leq 2$ there exist no non-isomorphic cospectral graphs (see [27] by van Dam, Haemers). Moreover, we observed that this method cannot work for $b = 3$.

Remark 3.3.1. *If $b = 3$ it can be seen that there cannot exist a Godsil-McKay switching between a graph with a perfect matching and one with none.*

Furthermore, Stephen Hartke checked by computer all 3-regular pairs of cospectral graphs on at most 20 vertices and found no example. So it is not unlikely that among 3-regular graphs a cospectral pair does not exist. Hence the following problem remains open.

Problem 3.3.2. *Is there any pair of b-regular, cospectral graphs for $b \in \{3, 4\}$ such that exactly one of them has a perfect matching?*

Let me mention another similar question which is also intriguing and, as far as I know, it is still open.

Problem 3.3.3. *Is there any pair of b-regular, cospectral graphs such that their chromatic index are different? Or equivalently is the chromatic index of a regular graph determined by its spectrum?*

That problem asks for two cospectral, regular graphs G_1 and G_2 such that the edge set of G_1 can be partitioned into b perfect matchings, but this cannot be done with the edge set of G_2. Notice that in our construction one of the graphs did not have any perfect matchings at all. Although the other one has a perfect matching but after deleting an arbitrary perfect matching from it the remaining graph does not have a perfect matching. Thus in our construction both of the graphs have chromatic index $b + 1$ by Vizing's theorem.

Chapter 4

Partition dimension of projective planes

4.1 Introduction

In graph theory, a large number of different concepts were introduced to distinguish or identify every vertex in a given graph. Notably the vertices are usually distinguished via adjacency to a certain set of vertices — like in case of identifying codes in graphs and distinguishing sets or locating-dominating sets [31, 40, 42, 43] — or via distance from a certain set of vertices — like in case of resolving sets, metric and partition dimension (see Bailey, Cameron and Chartrand, Salehi, Zhang [34, 36]). In this chapter we will study the latter concept on a highly symmetric graph family which appears naturally in many branches of combinatorics, namely on the incidence graphs of projective planes.

As in Definition 2.1.2, we denote the distance of two vertices u and v in a connected graph G with $d(u, v)$. For an ordered set $W = \{w_1, w_2, \dots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the k-vector (ordered k-tuple)

$$r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$$

is referred to as the *(metric) representation* of v with respect to W. The set W is called a *resolving set* for G if the vertices of G have distinct representations. A resolving set containing a minimum number of vertices is called a minimum resolving set of G. The number of vertices of a minimum resolving set is the so-called *metric dimension* and denoted by $\mu(G)$.

The distance concept was naturally generalized to subsets of points due to Chartrand et al. [36]. For an ordered k-partition $\mathbf{S} = \{S_1, S_2, \dots, S_k\}$ of $V(G)$ and a vertex v of G, the representation of v with respect to $\mathbf{S}$ is defined as the k-vector

$$r(v|\mathbf{S}) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$$

where $d(v, S_i) := \min_{s \in S_i} d(v, s)$ for $i = 1, 2, \ldots, k$. The partition $\mathbf{S}$ is called a *resolving partition* if the k-vectors $r(v|\mathbf{S})$, $v \in V(G)$, are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is the *partition dimension* $\mathrm{pd}(G)$ of G. We note that the notation also make sense if $\mathbf{S}$ is any family of sets. Generally, we say that a set system *separates* a vertex set if no two vertices have equal distances from every set of the set system.

Both dimension concepts has been widely investigated, see [34, 37, 45] for surveys. Although they are analogously defined and there are connections between the two parameters, in general, they are not similar in nature. To illustrate this phenomenon, we recall some results concerning $\mu(G)$ and $\mathrm{pd}(G)$.

Result 4.1.1 (Chartrand et al. [36]). $\mathrm{pd}(G) \leq \mu(G) + 1$ *for all graphs* G*.*

Result 4.1.2 (Chappell et al. [37]). *For any given natural numbers* α *and* β *with* $3 \leq \alpha \leq \beta + 1$*, there exists a graph* G *where* $\mu(G) = \beta$ *and* $\mathrm{pd}(G) = \alpha$*.*

The study of dimension parameters concerning incidence graphs of designs or geometries has been initiated only recently by Bailey and others in [32, 33, 37, 12]. Note that a similar concept of identifying codes in special graphs is also studied recently [38, 42].

Let Π_q be an arbitrary finite projective plane of order q with point set $\mathcal{P}$ and line set $\mathcal{L}$. As in Definition 2.3.2 we denote the plane by $\mathrm{PG}(2, q)$ if we assume that the plane is built on a finite field $\mathbb{F}_q$. The incidence graph of a plane Π_q is denoted by $G(\Pi_q)$ (see Definition 2.2.4). We will denote the classes of the bipartite graph $G(\Pi_q)$ by $\mathcal{P}[G(\Pi_q)]$ and $\mathcal{L}[G(\Pi_q)]$, corresponding to points and lines, respectively. Similarly, we introduce this notation for any subset Z of the vertex set of $G(\Pi_q)$ in general, namely $\mathcal{P}[Z]$ will denote those vertices of Z which correspond to the points of Π_q and $\mathcal{L}[Z]$ will denote those vertices of Z which correspond to the lines of Π_q. The metric dimension of this incidence graph was determined by Héger and Takáts in [12].

Result 4.1.3 (Héger,Takáts [12]). *If* q *is large enough, then* $\mu(G(\Pi_q)) = 4q - 4$*.*

Chappell, Gimbel and Hartman [37] gave bounds on the partition dimension $\mathrm{pd}(G)$ in terms of the *diameter* $\mathrm{diam}(G)$ of the graph G and investigated the case $\mathrm{diam}(G) = 2$. They mentioned that investigating the order of a graph with given partition dimension and diameter appears to be more difficult when the diameter exceeds two. The incidence graph of projective planes provides an infinite family for well structured graphs of diameter three, so this can also be considered as a partial motivation for the following problem, besides Result 4.1.3 by Héger and Takáts.

Problem 4.1.4. *Determine the partition dimension of the incidence graph of a finite projective plane.*

We determine the partition dimension of the incidence graph $G(\Pi_q)$ of the projective plane Π_q up to a constant factor 2. Our main results are as follows.

Theorem 4.1.5. *The partition dimension of the incidence graph of a projective plane of order q is at least $(2+o(1))\log_2 q$.*

Theorem 4.1.6. *The partition dimension of the incidence graph of a projective plane of order q is at most $(4+o(1))\log_2 q$.*

Note that in view of a general bound of Theorem 3.1 in Chappell [37] concerning the maximal degree of the graph, $\mathrm{pd}(G(\Pi_q)) \geq \log_3(q+2)$.

In the next section we prove Theorem 4.1.5, and the section after that is devoted to derive Theorem 4.1.6 using probabilistic and graph theoretic tools. A survey on applications of the probabilistic method in finite geometry can be found in a paper of Gács and Szőnyi [39]. Finally, we discuss open problems in the last section of this chapter.

4.2 Proof of the lower bound

We are going to show that $\mathrm{pd}(G(\Pi_q))$ is at least of size $(2+o(1))\log_2 q$. To this end, let us consider a resolving partition $\mathbf{S}$ with sets $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s, \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_t\}$ such that $\mathcal{P}_i \subseteq \mathcal{P}[G(\Pi_q)]$, $L_j \subseteq \mathcal{L}[G(\Pi_q)]$ and $\mathcal{M}_k$ is a mixed subset containing vertices from both $\mathcal{P}[G(\Pi_q)]$ and $\mathcal{L}[G(\Pi_q)]$. Since $\mathbf{S}$ is a resolving partition we know that for all vertices of the incidence graph the corresponding vectors are pairwise different. Let us examine the possible values of the coordinates of these vectors depending on the type (corresponding to point or line) of the vertex v.

1. If $v \in \mathcal{P}[G(\Pi_q)]$, then

   - $d(v, \mathcal{P}_i) = \begin{cases} 0 & v \in \mathcal{P}_i \\ 2 & \text{otherwise.} \end{cases}$

   - $d(v, \mathcal{L}_j) = \begin{cases} 1 & \text{there is a line in } \mathcal{L}_j \text{ which is incident with } v \\ 3 & \text{otherwise.} \end{cases}$

   - $d(v, \mathcal{M}_k) = \begin{cases} 0 & v \in \mathcal{M}_k \\ 1 & \text{there is a line in } M_k \text{ which is incident with } v \\ 2 & \text{otherwise.} \end{cases}$

2. If $v \in \mathcal{L}[G(\Pi_q)]$, then

   - $d(v, \mathcal{L}_j) = \begin{cases} 0 & v \in \mathcal{L}_j \\ 2 & \text{otherwise.} \end{cases}$

   - $d(v, \mathcal{P}_i) = \begin{cases} 1 & \text{there is a point in } \mathcal{P}_i \text{ which is incident with } v \\ 3 & \text{otherwise.} \end{cases}$

   - $d(v, \mathcal{M}_k) = \begin{cases} 0 & v \in \mathcal{M}_k \\ 1 & \text{there is a point in } M_k \text{ which is incident with } v \\ 2 & \text{otherwise.} \end{cases}$

Note that if there is a partition class which contains only one type of vertices, then by this last observation one can distinguish all the vertices of $\mathcal{P}[G(\Pi_q)]$ from the vertices of $\mathcal{L}[G(\Pi_q)]$. Moreover, if we consider two vertices which are of the same type then their vector is different if they are not in the same partition class. But we know that there is a partition class which contains at least $\frac{q^2+q+1}{s+t}$ many lines and also there is a class which contains at least $\frac{q^2+q+1}{r+t}$ many points.

For these lines, their representation vectors are the same in the coordinates corresponding to the subsets $\mathcal{L}_j$. The number of coordinates remaining to distinguish two such lines depends on the type of their partition class. Namely, if this class is a class with just lines then it's $r+t$ but when this class is a mixed class then it's just only $r+t-1$. Since the values of these remaining coordinates could only be 1 or 3 for the coordinates corresponding to a subset $\mathcal{P}_i$ and could only be 1 or 2 for the mixed classes because they do not contain these. Hence the following inequality has to hold:

$$2^{r+t-1} \geq \frac{q^2+q+1}{s+t}.$$

By the principle of duality (see Definition 2.2.3 and the next paragraph) a similar argument works for the points, too. It gives that

$$2^{s+t-1} \geq \frac{q^2+q+1}{r+t}.$$

We would like to minimize $r+s+t$ under the above conditions. One can easily see that the minimum could be reached by taking $s=r=0$ (Note: in this setup one should be careful because the points and the lines are not automatically distinguished). In that case the two conditions happen to be the same:

$$t \cdot 2^{t-1} \geq q^2+q+1.$$

Hence the theoretical lower bound on the partition dimension number follows.

4.3 Proof of the upper bound

Here we prove that $\mathrm{pd}(G(\Pi_q)) \leq (4+o(1))\log_2 q$. First, we will outline the construction that provides the desired bound and introduce some key tools. Next, we prove two lemmas concerning the main ingredients of our constructions. Finally, we show that the construction is indeed a resolving partition.

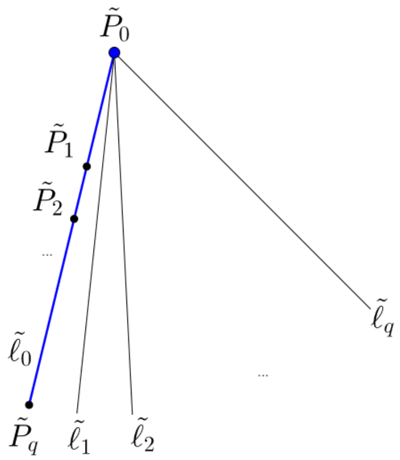
Notation 4.3.1. Choose an incident pair of point $\widetilde{P_0}$ and line $\widetilde{\ell_0}$ and call them the *support* of the construction (see Figure 4.1a). Denote the points incident to $\widetilde{\ell_0}$ by $\widetilde{P_i}$ ($i \in [0,q]$) and the lines incident to $\widetilde{P_0}$ by $\widetilde{\ell_i}$ ($i \in [0,q]$). We call the point set $\{\widetilde{P_i} : 0 < i \leq q\}$ and line set $\{\widetilde{\ell_i} : 0 < i \leq q\}$ *major points* and *major lines* in the construction, respectively. The set $\mathcal{C} := \{\widetilde{P_i} : 0 < i \leq q\} \cup \{\widetilde{\ell_i} : 0 < i \leq q\}$ is the *core* of the construction. Let us call the points and lines which are not in the core *common points* and *common lines* (altogether the *common vertices*).

Construction 4.3.2. *Our partition set system $\mathcal{H}$ consists of 4 subsystems:*

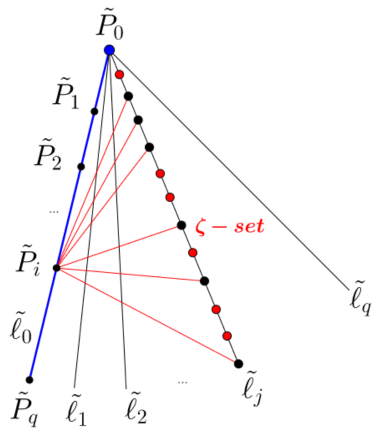
$$\mathcal{H} = \{H_0\} \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \{H_{-1}\},$$

where $\mathcal{H}_1 = \bigcup_{i=1}^{k} H_i$, $\mathcal{H}_2 = \bigcup_{i=k+1}^{k+l} H_i$. H_0 is defined as $H_0 := \{\widetilde{P_i} \mid i \in [1\ldots q]\}$, furthermore H_{-1} is defined so that it completes the system, that is, $\bigcup_{i=-1}^{k+l} H_i = \mathcal{P}[G(\Pi_q)] \cup \mathcal{L}[G(\Pi_q)]$ with H_{-1} being disjoint from any other H_i.

Let us define the set system $\mathcal{H}_1$. Any $H \in \mathcal{H}_1$ is built up as follows: choose a major point $\widetilde{P_i}$ and a major line $\widetilde{\ell_j}$ which will be the base of the set H. Divide the point set $\widetilde{\ell_j} \setminus \widetilde{P_0}$ into two equal parts (if q is odd then divide it into two almost equal



(a) The support of the construction



(b) Example of a ζ-set

Figure 4.1:

parts, i.e. of size $\lceil \frac{q}{2} \rceil$ *and of size* $\lfloor \frac{q}{2} \rfloor$*). Put the points of the first part into* H *and put also every line determined by* $\widetilde{P_i}$ *and points of the second part. This way* $|H| = q$ *will hold for every element of* $\mathcal{H}_1$*. Any subset of* $\mathcal{P} \cup \mathcal{L}$ *that can be created this way is called a* ζ*-set (on the base point* $\widetilde{P_i}$ *and base line* $\widetilde{\ell_j}$*). We include Figure 4.1b here to help the reader imagine a* ζ*-set.*

Approximately $k \approx 3 \log_2 q$ ζ*-sets will be chosen randomly in such a way that almost all points and lines are uniquely determined by the distances from the sets of* $\mathcal{H}_1$*, if we restrict ourselves to the common vertices.*

Finally, $\mathcal{H}_2$ *consists of* $l \approx \log_2 q$ *sets which will distinguish all of the remaining non-separated pairs of vertices.*

Through the following lemmas, we show the existence of such set systems.

Lemma 4.3.3. *One can choose* k *suitable* ζ*-sets in such a way that they separate almost all pair of elements from* $(\mathcal{P} \setminus \{\widetilde{P_i}\})$ *and from* $(\mathcal{L} \setminus \{\widetilde{\ell_i}\})$*. There can be at most* $m(k)$ *pairs in total which remained unseparated with*

$$m(k) = \frac{2\binom{q^2}{2}}{2^k}.$$

Proof. First, we choose k points and k lines from the major point and line set uniformly at random, and index them by $\widetilde{P_i}$ and $\widetilde{\ell_i}$, $i \in [1 \dots k]$. Next, we choose k ζ-sets (on the base points $\widetilde{P_i}$ and base line $\widetilde{\ell_i}$, by taking $\lfloor \frac{q}{2} \rfloor$ points from each line $\widetilde{\ell_i}$ uniformly at random, leaving the support point intact. This enables us to calculate the expected value of not-separated pairs.

We distinguish the cases when the pair is a pair of point or pair of lines, but note that the two cases are similar due to the symmetry of the ζ-sets and the duality of the structure.

In the calculation below, we omit the integer part for transparency (that is, we consider the case q is even) but it is straightforward to see that the reasoning works for the q odd case as well.

Let Q and Q' be two random points in $\mathcal{P}\setminus\{\widetilde{P_i}\}$. The probability of the separation essentially depends on two factors: whether or not QQ' intersects $\widetilde{\ell_0}$ in a point $\widetilde{P_t}$, $t \in [1\dots k]$; and whether or not one of Q and Q' is incident to $\widetilde{\ell_t}$. These subcases provide the following for a random point pair:

$$\mathbb{P}(Q, Q' \text{ not separated by } k \text{ random } \zeta\text{-set}) =$$
$$\mathbb{P}(Q, Q' \text{ not separated by } k \text{ random } \zeta\text{-set } \& \ QQ' \cap \widetilde{\ell_0} \notin \{\widetilde{P_t}, t \in [1\dots k]\})+$$
$$\mathbb{P}(Q, Q' \text{ not separated by } k \text{ random } \zeta\text{-set } \& \ QQ' \cap \widetilde{\ell_0} \in \{\widetilde{P_t}, t \in [1\dots k]\})$$
$$\leq \frac{q-k+1}{q+1}\left(\frac{q-2}{2q-2}\right)^k + \frac{k}{q+1}\left(\frac{q-1}{q}\left(\frac{q-2}{2q-2}\right)^{k-1}\right).$$

Indeed, suppose first that QQ' intersects $\widetilde{\ell_0}$ in a point outside $\widetilde{P_t}$, $t \in [1\dots k]$. That case, for every ζ-set H on the base $\widetilde{P_i}$, $\widetilde{\ell_i}$, $d(Q,H) \neq d(Q',H)$ holds if exactly one of the lines $\widetilde{P_i}Q$, $\widetilde{P_i}Q'$ belongs to H, hence the probability of separation by H is at least $\frac{\frac{q}{2}}{q-1}$. (Note that equality does not hold here as Q or Q' might be a point of the ζ-set.)

On the other hand, if QQ' intersects $\widetilde{\ell_0}$ in a point $\widetilde{P_t}$, $t \in [1\dots k]$, then the above argument works for all but one ζ-set, H (on the base $\widetilde{P_t}$, $\widetilde{\ell_t}$). However, if $Q \notin \widetilde{\ell_t}, Q' \notin \widetilde{\ell_t}$, then $d(Q,H) = d(Q',H)$ surely holds, while if $Q \in \widetilde{\ell_t}$, or $Q' \in \widetilde{\ell_t}$, then $d(Q,H) \neq d(Q',H)$ only if Q or Q' is a point in H.

An easy calculation shows that

$$\frac{q-k+1}{q+1}\left(\frac{q-2}{2q-2}\right)^k + \frac{k}{q+1}\left(\frac{q-1}{q}\left(\frac{q-2}{2q-2}\right)^{k-1}\right) < \left(\frac{1}{2}\right)^k.$$

Taking into consideration the number of point pairs, and the dual case for the number of line pairs, we obtain a bound on the expected value of the non-separated point pairs and line pairs:

$$\mathbb{E}(\text{not separated pairs by } k \text{ random } \zeta\text{-set}) =$$
$$2\binom{q^2}{2}\mathbb{P}(Q, Q' \text{ not separated by } k \text{ random } \zeta\text{-set}) < \frac{2\binom{q^2}{2}}{2^k}.$$

The statement thus follows. $\square$

Lemma 4.3.4. *One can choose a system $\mathcal{H}_2$ of $\lceil \log_2 q \rceil$ disjoint sets consisting of both points and lines which are disjoint to the ζ-sets and to the major points as well such that with these new sets the corresponding representations of the vertices will be pairwise different.*

Proof. Let us look at a table of the representations of the vertices so far, with H_{ij} denoting a chosen ζ-set on base point $\widetilde{P_i}$ and base line $\widetilde{\ell_j}$.

	$\widetilde{P_0}$	$\widetilde{P_1}$	$\dots$		$\dots$	$\widetilde{P_q}$	$P_1 \dots P_{q^2}$	$\widetilde{\ell_0}$	$\widetilde{\ell_1}$	$\dots$	$\widetilde{\ell_j}$	$\dots$	$\widetilde{\ell_q}$	$\ell_1 \dots \ell_{q^2}$
H_0	2	0	$\dots$	0	$\dots$	0	2	1	3	$\dots$	3	$\dots$	3	1
$H_{ij} \in \mathcal{H}_1$	2	2	$\dots 2$	1	$2 \dots$	2	0/1/2	2	2	$\dots 2$	1	$2 \dots$	2	0/1/2

Considering this table, one can determine which pairs of vertices could have the same representation:

1. pairs of common vertices not being separated after Lemma 4.3.3,

2. pairs of major points and also pairs of major lines (which are not basis of ζ-sets in $\mathcal{H}_1$),

3. the support line and some common lines and dually the support point and some common points may form some non-separated pairs.

These are the only possibilities which we need to take care of with the suitable choice of $\mathcal{H}_2$. We will build up an element H of $\mathcal{H}_2$ in the following way. By using the known results in the theory of *searching sets* we can define $\mathcal{L}[H]$, and then $\mathcal{P}[H]$ will be chosen in the same way dually. Before that we need some structural observations.

We make an auxiliary graph X with vertex set consisting of those common vertices which are in the remaining non-separated pairs and two such vertices are joined with an edge if they have the same representation so far. Clearly X is just the disjoint union of some cliques. Moreover, by the choice of H_0 in every clique either every vertex is a point of Π_q or every vertex is a line of Π_q. Just for convenience let us assume that we choose $k = \lceil 3 \log_2 q \rceil + 3$ ζ-sets in the first part of the construction. It means by Lemma 4.3.3 that we have at most $\frac{q}{8}$ pairs of vertices which have not been separated yet. Hence the number of edges in X is at most $\frac{q}{8}$. Together with the observation above we get that X has at most $\frac{q}{4}$ vertices.

Modify X a little bit by adding the support point and line to it if needed (let us call the graph we get this way X'), namely whenever there is a common point or line which is non-separated from the support point or line respectively. There are three options for $\widetilde{P_0}$ (and similarly for $\widetilde{\ell_0}$):

- $\widetilde{P_0}$ has already been separated from every other vertex in $G(\Pi_q)$.
- $\widetilde{P_0}$ has the same representation as some of the points of X, hence joins that clique in X'.
- $\widetilde{P_0}$ has the same representation as a common point not in X (hence it's exactly one vertex), therefore they both inserted into X' with an edge between them.

Either way the number of vertices in X' is at most $\frac{q}{4} + 4$. Denote the vertices in $X' \cap \mathcal{P}[G(\Pi_q)]$ with $\mathcal{P}[X']$ and similarly $\mathcal{L}[X']$ will denote the vertices in X' corresponding to a line of Π_q. By the note above we know that there is no edge in X' between $\mathcal{P}[X']$ and $\mathcal{L}[X']$.

Now let us recall the notion of a *searching set* for a search problem. Roughly speaking we would like to distinguish all elements of a set by pointing out some subsets and create a 0-1 vector for every element of the set where the value of the j^{th} coordinate is 1 if and only if this element is inside the j^{th} chosen subset. These distinguishing subsets are also called searching sets. It is known that for an n element set we need $\lceil \log_2 n \rceil$ such searching sets to reach our goal by distinguishing all of the elements of the set.

The main idea behind the selection of $\mathcal{L}[H]$ is based on these searching sets. Namely, we will consider a family of searching sets on the q major points and also on $\mathcal{P}[X']$ and on $\mathcal{L}[X']$ too, with sizes half of their corresponding domain set. For an $H \in \mathcal{H}_2$ we will choose one of the searching sets for the major points, for $\mathcal{P}[X']$ and also for $\mathcal{L}[X']$. Let us denote these searching sets with $T(H)$, $Q(H)$ and $R(H)$ respectively. We can assume that $\widetilde{P_0}$ and $\widetilde{\ell_0}$ are always outside of every chosen searching set without loss of generality.

We are going to prove that we can find $\frac{q}{2}$ such common lines (they will form $\mathcal{L}[H]$) which satisfy the following properties:

- through every point in $T(H)$ there is exactly one line from $\mathcal{L}[H]$,
- through every point in $Q(H)$ there is exactly one line from $\mathcal{L}[H]$,
- they are not going through any point of $\mathcal{P}[X'] \setminus Q(H) =: Q(H)^C$,
- they are not in $\mathcal{L}[X'] \setminus R(H) =: R(H)^C$,
- they have not been assigned to any set in our construction yet.

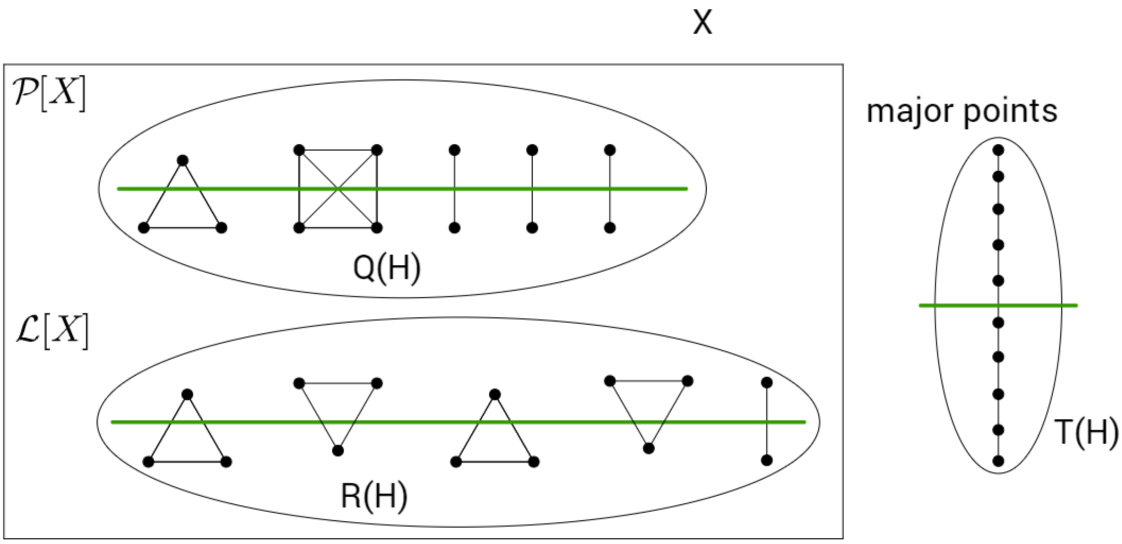


Figure 4.2: An appropriate searching set

Let us call those common lines which satisfy the last three requirements *free* lines. Let us make another auxiliary graph denoted by Y which is a bipartite graph where the first class consists of the vertices of $T(H)$ and the other class is just $\mathcal{P}[G(\Pi_q)] \setminus \left(\cup_{i=0}^{q+1}\widetilde{\mathcal{P}_i}\right) \setminus Q(H)^C$ and there is an edge between two vertices (obviously from different classes) iff the line defined by these two vertices is a free line. We need to give a lower bound on the number of free lines through an arbitrary point from $T(H)$ and also from Q because of the first two requirements.

Consider a point $v \in T(H)$. Through this point there are q common lines but it may happen that this point was chosen as a base to a ζ-set therefore it is possible that $\frac{q}{2}$ of these common lines were used before. Since we are not going to use lines through the points of $Q(H)^C$ and lines from $R(H)^C$ it could rule out another $|Q(H)^C| + |R(H)^C| = \frac{1}{2}\left(\frac{q}{4} + 4\right) = \frac{q}{8} + 2$ lines through v. Furthermore, if v was in some searching sets on the major points before this phase then in every such case there was one line which was inserted into that particular set from $\mathcal{H}_2$. This gives us a lower bound on the number of free lines through an arbitrary point of $T(H)$:

$$|\{\text{free lines through } v \in T(H)\}| \geq q - \frac{q}{2} - \left(\frac{q}{8} + 2\right) - \log_2 q = \frac{3q}{8} - \log_2 q - 2$$

Note that the degree of v in Y is at least $q \cdot \left(\frac{3q}{8} - \log_2 q - 2\right)$ because every free line through v has q points on it which correspond to q edges in the graph. Consider now an arbitrary point $u \in Q(H)$. Through u there are $\frac{q}{2}$ common lines which meets the support line in $T(H)$. Again we do not want to use lines through the points of $Q(H)^C$ and lines from $R(H)^C$ which could rule out $\frac{q}{8} + 2$ lines just as above. If u was in some searching sets on $\mathcal{P}[X']$ before this phase then in every such case there was one line which was inserted into that particular set from $\mathcal{H}_2$. Hence:

$$|\{\text{free lines through } u \in Q(H)\}| \geq \frac{q}{2} - \left(\frac{q}{8} + 2\right) - \log_2 q = \frac{3q}{8} - \log_2 q - 2$$

Now we can see that if q is large enough then there are many free lines through these points in Y. Moreover, for an arbitrary point $u \in Q(H)$ if we ignore those lines which contains another point of $Q(H)$ then there is at least $\frac{3q}{8} - \log_2 q - 2 - \left(\frac{q}{8} + 2\right) = \frac{q}{4} - \log_2 q - 4$ such free lines through u. Let us consider the points of $Q(H)$ one by one then we can choose an edge (thus a free line) for the first one which fulfills the requirements and does not contain any other points from $Q(H)$. Drop the meeting point of this line and the support line for any other points of $Q(H)$ because of the first requirement. Then we can continue this in a greedy way because of the counting above (for the last member of $Q(H)$ we drop another at most $\frac{q}{8} + 2$ meeting points but the number of free lines through that point is still at least $\frac{q}{4} - \log_2 q - 4 - \left(\frac{q}{8} + 2\right) = \frac{q}{8} - \log_2 q - 6$).

Now we just need to choose one line through every uncovered points of $T(H)$ carefully. Note that if we drop those lines through these points which meet $Q(H)$ (and in parallel delete the q edges for each of them from Y) then again there remains at least $\frac{3q}{8} - \log_2 q - 2 - \left(\frac{q}{8} + 2\right) = \frac{q}{4} - \log_2 q - 4$ free lines through them. By choosing from the free lines greedily works again because of the calculations above. Dually one can construct $\mathcal{P}[H]$ in a similar way which completes the set $H \in \mathcal{H}_2$. By repeating this argument we can construct $\lceil \log_2 q \rceil$ such sets in $\mathcal{H}_2$ (we included the decreasing of the degrees above therefore this greedy approach will work).

In the preceding paragraphs we just showed a way of choosing $\log_2 q$ sets all of which fulfills the requirements for its lines and dually for its points, too. The only thing remaining to verify is that these sets in $\mathcal{H}_2$ really take care of every pair of non-separated vertices after choosing H_0 and $\mathcal{H}_1$.

The major points (and dually the major lines) are indeed separated because for a particular $H \in \mathcal{H}_2$ those major points will have coordinate 1 which are inside $T(H)$ since we chose a line going through them and the other major points have coordinate 2. Since we chose a family of searching sets on these major points then eventually they will be distinguished at the end. If we examine all of the other remaining pairs we can notice that the coordinates for $\widetilde{P_0}$ and $\widetilde{\ell_0}$ will be 2 for every set in $\mathcal{H}_2$ and for a particular $H \in \mathcal{H}_2$ the points of $Q(H)$ will have coordinate at most 1 (we chose a line through them and maybe we put them into $\mathcal{P}[H]$) but for the points of $Q(H)^C$ the coordinate is surely 2 (did not choose a line through them and we exclude them from being in $\mathcal{P}[H]$). Similar arguments hold for the lines of $R(H)$ and $R(H)^C$. Again, since we chose a family of searching sets on the $\mathcal{P}[X']$ and on $\mathcal{L}[X']$ all of the remaining pairs will be separated, too. $\square$

Observe that by adding all of the non-used vertices of $G(\Pi_q)$ to H_{-1} we obtain a resolving partition indeed, with $1 + (3\lceil \log_2 q\rceil + 3) + \lceil \log_2 q\rceil + 1 \leq 4\lceil \log_2 q\rceil + 5$ classes which completes the proof of Theorem 4.1.6.

Remark 4.3.5. *One can easily see that in these Lemmas we do not rely heavily on the parity of q, if q is odd everything still works with slight modifications. Indeed, the probability that a point pair Q, Q' is not separated by k random ζ-sets is still at most $\frac{1}{2^k}$ in Lemma 4.3.3 if q is odd, and likewise in Lemma 4.3.4, the inequalities were not sharp in the conditions of the greedy algorithm and hold in the corresponding case as well.*

4.4 Further related problems and remarks

Although the lower and upper bounds we proved do not match, we strongly believe that the construction given in the upper bound is optimal in some sense. The reason for this is the following: one has to create a set system where the majority of the sets contain neither more points nor more lines than cq for a small constant c. Indeed, the result of Blokhuis [35] implies that cq lines are incident with at least roughly $\frac{c}{c+1}q^2$ points, hence a set containing this many lines assign the same distance for the majority of the points.

This observation in fact improves our lower bound via the result of Katona [41] on separating systems of given size, but only in the remainder term. Thus several questions remain open concerning the optimal construction.

Problem 4.4.1. *Can the above bounds be improved if the plane is coordinatized, that is, G is the incidence graph of the plane* $\mathrm{PG}(2,q)$*?*

Problem 4.4.2. *Prove that there exists a constant c for which* $\mathrm{pd}(G(\Pi_q)) = (c + o(1))\log_2 q$.

Our construction is mainly based on sets of collinear points and lines forming a pencil, in order to separate approximately half of the lines (incident to the points) from the other lines and do the same for the points for every set in the resolving partition. Note that the points of a maximal arc and the dual configuration has the same property. This motivates the following natural question.

Problem 4.4.3. *Does there exist a set of disjoint ovals in* Π_q *of cardinality* $c\log_2 q$ *($c \leq 4$) which separates the lines of the projective plane* Π_q*?*

A related problem only requires a set of ovals to cover (intersect) every line.

Problem 4.4.4. *What is the minimal cardinality of a set of (disjoint) ovals in* Π_q *for which no line is skew to all of them?*

It is believed, see Illés, Szőnyi, Wettl and Ughi [44, 46], that the order of magnitude is $O(\log q)$ for q odd, which provides $O(q\log q)$ points on the plane, even the size of small minimal blocking (point)sets is much less. Note that the q even case is completely different, where 3 ovals can cover every line in the Galois plane $\mathrm{PG}(2,q)$ due to Illés, Szőnyi and Wettl [44].

Our result can also be considered as a first step to the determination of the partition dimension of incidence graphs of symmetric structures in general, analogously to the metric dimension case.

Chapter 5

Spreading linear triple systems and expander triple systems

5.1 Introduction

A Steiner triple system $\mathcal{S}$ of order n, briefly STS(n), consists of an n-element set V and a collection of triples (or blocks) of V, such that every pair of distinct points in V is contained in a unique block. It is well known due to Kirkman [63] that there exists an STS(n) if and only if $n \equiv 1, 3 \pmod 6$, these values are called *admissible*. Steiner triple systems correspond to triangle decompositions of the complete graph $G = K_n$. In the context of triangle decompositions of a graph G, an edge will always refer to a pair of vertices which is contained in one triple of a certain triple system, $E(G)$ denotes the edge set of G, while $|\mathcal{S}|$ is the number of triples in the system, which obviously equals $\frac{1}{3}|E(G)|$ in the case of triple systems obtained from triangle decompositions of a graph G.

A nontrivial Steiner subsystem of $\mathcal{S}$ is a STS(n') induced by a proper subset of V, with $n' > 3$. Speaking about a subsystem, we always suppose that it is of order greater than 3. Similarly, we call a subset $V' \subset V$ of the underlying set of a triple system $\mathcal{F}$ *nontrivial* if it has size at least 3 and it is not an element of the triple system. Our aim is to generalize and strengthen the results concerning the subsystem-free property of Steiner triple systems, and in general, linear triple systems, also called linear 3-graphs.

This chapter is devoted to the study of two main features of linear triples systems in an extremal hypergraph theory aspect. The first property is the *expander property* while the second is the so-called *spreading property*. In 1973, Erdős formulated the following conjecture.

Conjecture 5.1.1 (Erdős, [57]). *For every $k \geq 2$ there exists a threshold n_k such that for all admissible $n > n_k$, there exists a Steiner triple system of order n with the following property: every $t+2$ vertices induce less than t triples of $\mathcal{S}$ for $2 \leq t \leq k$.*

This conjecture is still open, although recently Glock, Kühn, Lo and Osthus [61] and independently Bohman and Warnke [50] proved its asymptotic version. In other words, this conjecture asserts the existence of arbitrarily sparse Steiner triple systems.

One should note here that it is also a natural question whether typical Steiner triple systems are sparse in a very robust sense, namely that they do not contain Steiner subsystems. Indeed, this is equivalent to avoid a set of $t < n$ vertices inducing quadratically many, $\frac{1}{3}\binom{t}{2}$ triples. The first result in this direction was due to Doyen [55], who proved the existence of *at least one* subsystem-free STS(n) for every admissible order n. In the language of triangle decompositions of the edge set, a subsystem-free STS may be seen as a decomposition where every subset $V' \subset V(G)$ contains at least one edge which belongs to a triangle not induced by V'. In order to capture this phenomenon and its generalisation, we require some notation and definitions.

Definition 5.1.2. *Given a 3-uniform linear hypergraph $\mathcal{F}$ (i.e. linear triple system), let $E(\mathcal{F})$ be the collection of vertex pairs (x, y) for which there exists a triple (x, y, z) from the system $\mathcal{F}$, containing x and y. The corresponding graph $G(\mathcal{F})$ is referred to as the* shadow of the system.

Definition 5.1.3. *Consider a graph $G = G(V, E)$ that admits a triangle decomposition. This decomposition corresponds to a linear triple system $\mathcal{F}$. For an arbitrary set $V' \subset V$, $N(V')$ denotes the set of its* neighbours*:*

$$z \in N(V') \Leftrightarrow z \in V \setminus V' \quad \textit{and} \quad \exists xy \in E(G[V']) : \{x, y, z\} \in \mathcal{F}.$$

The closure $\mathrm{cl}(V')$ *of a subset V' w.r.t. a (linear) triple system $\mathcal{F}$ is the smallest set $W \supseteq V'$ for which $|N(W)| = 0$ holds. Note that the closure uniquely exists for each set V'. We call a (linear) triple system $\mathcal{F}$* spreading *if $\mathrm{cl}(V') = V$ for every nontrivial subset $V' \subset V$.*

Consequently, a STS(n) is subsystem-free if and only if $|N(V')| > 0$ holds for all nontrivial subsets V' of the underlying set V of the system. Note that Doyen used the term *non-degenerate plane* for STSs with the spreading property [55, 56].

Two natural extremal questions arise here. The first one concerns the lower bound on $|N(V')|$ in terms of $|V'|$ in the case of Steiner triple systems, while the second one seeks for edge-density conditions on triangle decompositions of general graphs $G = G(V, E)$, i.e. linear triple systems, where the condition $|N(V')| > 0$ must hold for all nontrivial subsets of V.

Problem 5.1.4 (Expander STSs)**.** *Does there exist an infinite family of Steiner triple systems* $\mathrm{STS}(n)$ *such that for some* $\varepsilon > 0$, $\frac{|N(V')|}{|V'|} \geq \varepsilon$ *for every nontrivial* $V' \subset V(G)$ *provided that* $|V'| \leq \frac{|V|}{2}$*? How large* $\varepsilon > 0$ *can be?*

This can be interpreted as the analogue of the expander property of graphs and the vertex isoperimetric number [47]. Similar generalized concepts for expanding triple systems were introduced very recently by Conlon and his coauthors [52, 53], see also the related paper [59] by Fox and Pach. Observe however that their definition is slightly different for a triple system to be expander.

Problem 5.1.5 (Sparse spreading linear triple systems)**.** *What is the minimum size* $\xi_{sp}(n)$ *of a linear spreading triple system* $\mathcal{F}$ *on* n *vertices?*

For these triple systems, the closure of any nontrivial subset with respect to the underlying graph of the triple system is the whole system.

Note that one might require only a weaker condition, namely that the closure of any nontrivial subset of the triple system $\mathcal{F}$ (i.e. consisting of at least two triples) should be the whole system. In applications this condition is equally important, since it models whether every set of hyperedges has a direct influence on the whole system. For this concept, we introduce the following notation.

Notation 5.1.6. *A triple system* $\mathcal{F}$ *is* weakly spreading *if* $\mathrm{cl}(V') = V$ *holds for every*

$$V' = V(\mathcal{F}') \ : \ \mathcal{F}' \subseteq \mathcal{F}, \ |\mathcal{F}'| > 1.$$

Problem 5.1.7 (Sparse weakly spreading linear triple systems)**.** *What is the minimum size* $\xi_{wsp}(n)$ *of a linear weakly spreading triple system* $\mathcal{F}$ *on* n *vertices?*

Our main results are as follows.

Theorem 5.1.8. *For odd prime number* p*, there exists a Steiner triple system* $\mathrm{STS}(3p)$ *of order* $3p$*, for which* $|N(V')| \geq |V'| - 3$ *holds for every* $V' \subset V(G)$ *of size* $|V'| \leq \frac{|V|}{2}$.

The result is clearly sharp.

Corollary 5.1.9. *For every sufficiently large n, there exists a Steiner triple system* $\mathrm{STS}(\overline{n})$ *of order $\overline{n}$, for which*

$$|N(V')| \geq |V'| - 3$$

for every $V' \subset V(G)$ of size $|V'| \leq \frac{|V|}{2}$, where $\overline{n} \in [n - n^{0.525}, n]$. Consequently, for every n one can find a Steiner triple system $\mathcal{S}$ of size $|\mathcal{S}| = (1 + o(1))\frac{n^2}{6}$ which is almost 1-expander.

As we will see, much smaller edge density compared to that of STSs' still enables us to construct spreading linear triple systems.

Theorem 5.1.10. *For the minimum size of a spreading linear triple system, we have*

$$0.1103n^2 < \xi_{sp}(n) < \left(\frac{5}{36} + o(1)\right) n^2 \approx 0.139n^2$$

Surprisingly, the weak spreading property does not require a dense structure at all.

Theorem 5.1.11. *For the minimum size of a weakly spreading linear triple system, we have*

$$n - 3 \leq \xi_{wsp}(n) < \frac{8}{3}n - 6\sqrt{n}.$$

5.2 Expander property of Steiner triple systems

In order to prove Theorem 5.1.8, we recall first the STS construction of Bose and Skolem for $n = 6k + 3$ where $2k + 1$ is a prime number, and the well-known Cauchy-Davenport theorem with its closely related variant, the result of Dias da Silva and Hamidoune about the conjecture of Erdős and Heilbronn. We refer to the book of Tao and Vu [67] on the subject.

Result 5.2.1 (Cauchy-Davenport)**.** *For any prime p and nonempty subsets A and B of the prime order cyclic group $\mathbb{Z}_p$, the size of the sumset $A + B = \{a_i + b_j \mid a_i \in A, b_j \in B\}$ can be bounded as $|A + B| \geq \min\{p,\ |A| + |B| - 1\}$.*

Result 5.2.2 (Erdős-Heilbronn conjecture, Dias da Silva and Hamidoune '94)**.** *For any prime p and any subset A of the prime order cyclic group $\mathbb{Z}_p$, the size of the restricted sumset $A \dot{+} A = \{a_i + a_j \mid a_i \neq a_j \in A\}$ can be bounded as $|A \dot{+} A| \geq \min\{p,\ 2|A| - 3\}$.*

Construction 5.2.3 (Bose and Skolem, case $n = 6k+3$)**.** *Let the triple system $\mathcal{S}$ be defined in the following way. The underlying set is partitioned into three sets of equal sizes, $V(\mathcal{S}) = A \cup B \cup C$, where $|A| = |B| = |C| = 2k+1$. Elements of each partition class are indexed by the elements of the additive group $\mathbb{Z}_{2k+1}$. The system $\mathcal{S}$ contains the triple T, if*

- $T = \{a_i, b_i, c_i\}$, $i \in \mathbb{Z}_{2k+1}$, *or*
- $T = \{a_i, a_j, b_k\}$, $i \neq j \in \mathbb{Z}_{2k+1}$, $k = \frac{1}{2}(i+j)$, *or*
- $T = \{b_i, b_j, c_k\}$, $i \neq j \in \mathbb{Z}_{2k+1}$, $k = \frac{1}{2}(i+j)$, *or*
- $T = \{c_i, c_j, a_k\}$, $i \neq j \in \mathbb{Z}_{2k+1}$, $k = \frac{1}{2}(i+j)$.

See Stinson [66] for further details and generalisations.

Proof of Theorem 5.1.8. Consider a subset $V_0 = A_0 \cup B_0 \cup C_0$ of the underlying set $V(\mathcal{S}) = A \cup B \cup C$, where $|V_0| \leq \frac{n}{2}$. In order to bound $N(V_0)$, we prove a lower bound on $V_0 \cup N(V_0)$. Observe that a vertex v belongs to $V_0 \cup N(V_0)$ if and only if there exist two elements of V_0 together which they form a triple of $\mathcal{S}$. Let us denote by A^*, B^*, C^* the restrictions of $V_0 \cup N(V_0)$ to the partition classes A, B, C. The structure of Construction 5.2.3 and Results 5.2.1 and 5.2.2 in turn implies the following sets of inequalities of type Cauchy–Davenport and Erdős–Heilbronn, respectively.

$$
\begin{aligned}
|A^*| &\geq \min\{p,\ |-A_0| + |B_0| - 1\} \ \text{ if } \ |A_0|, |B_0| > 0,\\
|B^*| &\geq \min\{p,\ |-B_0| + |C_0| - 1\} \ \text{ if } \ |B_0|, |C_0| > 0,\\
|C^*| &\geq \min\{p,\ |-C_0| + |A_0| - 1\} \ \text{ if } \ |C_0|, |A_0| > 0.
\end{aligned}
\tag{5.2.1}
$$

$$
\begin{aligned}
|A^*| &\geq \min\{p,\ 2|C_0| - 3\},\\
|B^*| &\geq \min\{p,\ 2|A_0| - 3\},\\
|C^*| &\geq \min\{p,\ 2|B_0| - 3\}.
\end{aligned}
\tag{5.2.2}
$$

Note that in the Erdős-Heilbronn-type inequalities (5.2.2), the lower bound can be improved by one if the set consists of a single element. We distinguish several cases according to the sizes of the sets A_0, B_0, and C_0.

First suppose that two of these partition sets are empty. In this case, one Erdős-Heilbronn-type inequality (5.2.2) in turn provides the desired bound.

Next suppose that exactly one of these sets, say C_0, is empty. Thus we may apply two Erdős-Heilbronn type and one Cauchy-Davenport type inequality to obtain

$$|(A^* \setminus A_0) \cup (B^* \setminus B_0) \cup (C^* \setminus C_0)| \geq$$
$$\min\{p,\ |A_0| + |B_0| - 1\} - |A_0| + \min\{p,\ 2|A_0| - 3\} - |B_0| + \min\{p,\ 2|B_0| - 3\}.$$

Hence it is enough to show that

$$\min\{p,\ |A_0|+|B_0|-1\}+\min\{p,\ 2|A_0|-3\}+\min\{p,\ 2|B_0|-3\} \geq 2(|A_0|+|B_0|)-3$$

holds when both sets consist of at least two elements, otherwise the proof is straightforward. Then, depending on the relation between p, $|A_0|$ and $|B_0|$, we may apply either $3p \geq 2(|A_0| + |B_0|)$ or $p \geq \{|A_0|, |B_0|\} \geq 2$ to get the desired bound.

Finally, suppose that none of A_0, B_0, C_0 are empty, i.e., we can apply all the inequalities of (5.2.1) and (5.2.2). In order the finish the proof, consider the following proposition, the proof of which is straightforward.

Proposition 5.2.4. *Suppose that* $z \geq \min\{p,\ q_1\}$ *and* $z \geq \min\{p,\ q_2\}$ *holds for* $z, q_1, q_2 \in \mathbb{Z}$. *Then*

$$z \geq \min\{p,\ \lceil \lambda q_1 + (1-\lambda)q_2 \rceil\}$$

also holds for $\lambda \in [0, 1]$.

We apply Proposition 5.2.4 where $|A_0|, |B_0|$ and $|C_0|$ takes the role of z with the corresponding lower bounds of (5.2.1) and (5.2.2) and $\lambda = \frac{1}{3}$, which provides

$$\begin{aligned}
|A^*| &\geq \min\left\{p,\ \frac{1}{3}(2|C_0| - 3) + \frac{2}{3}(|A_0| + |B_0| - 1)\right\} \\
|B^*| &\geq \min\left\{p,\ \frac{1}{3}(2|A_0| - 3) + \frac{2}{3}(|B_0| + |C_0| - 1)\right\} \\
|C^*| &\geq \min\left\{p,\ \frac{1}{3}(2|B_0| - 3) + \frac{2}{3}(|C_0| + |A_0| - 1)\right\}
\end{aligned} \tag{5.2.3}$$

By summing them up, this would imply a slightly weaker bound

$$|A^* \cup B^* \cup C^*| \geq 2(|A_0| + |B_0| + |C_0|) - 5.$$

However, it is impossible to have equality in all the inequalities of (5.2.3). Indeed, suppose that C_0 has the least size among the three sets A_0, B_0, C_0. Then we could have use a better lower bound $(|A_0| + |B_0| - 1)$ for $|A^*|$ in the first line of (5.2.3), which would yield an improvement of at least $\frac{4}{3}$ except when $|C_0| \geq |A_0| - 1$ and $|C_0| \geq |B_0| - 1$ moreover one of these inequalities is strict, say the one corresponding to B_0. But in the latter case, we still get an improvement of $\frac{2}{3}$ corresponding to $|A^*| \geq \min\{p,\ \frac{1}{3}(2|C_0|-3)+\frac{2}{3}(|A_0|+|B_0|-1)\}$, and we get another improvement of $\frac{2}{3}$ corresponding to $|C^*| \geq \min\{p,\ \frac{1}{3}(2|B_0|-3)+\frac{2}{3}(|C_0|+|A_0|-1)\}$, as B_0 is a set of least size among the three sets A_0, B_0, C_0 as well. Thus by taking the ceiling, we get the desired bound. □

5.3 Spreading linear triple system

5.3.1 Proofs – lower bounds

Doyen [55] proved the existence of spreading Steiner triple systems for every admissible order n, and applied the name *non-degenerate plane* for such systems. In this section, we investigate how much sparser a linear triple system can be to keep its spreading property. It follows immediately that such a system $\mathcal{F}$ should be dense enough compared to a STS(n). Indeed, the complement of the shadow $G(\mathcal{F})$ must be triangle-free, which in turn implies $\frac{1}{12}n^2 < |\mathcal{F}|$ according to the theorem of Mantel and Turán.

Proof of Theorem 5.1.10, lower bound. Our aim is to obtain an upper bound on $E(\overline{G})$, the number of edges not covered by the triples of a linear spreading system that is denoted by $\mathcal{F}$. We start with three simple observations.

(1) $\overline{G}$ does not contain K_3.

(2) For every claw $K_{1,3}$ in $\overline{G}$, the leaves cannot determine a triple of $\mathcal{F}$.

(3) For every pair of triples of $\mathcal{F}$ which share a vertex, the corresponding 5-vertex graph in $\overline{G}$ cannot contain more than 3 edges.

Let F denotes a 4-vertex subgraph of the shadow G obtained from a triple T of $\mathcal{F}$ and a vertex adjacent to exactly one vertex of the triple in G. Such a vertex is called the *private neighbour* of T. Counting the pairs of edges of $\overline{G}$, we get that the number of F subgraphs of G is

$$\sum_v \binom{\overline{d}(v)}{2}.$$

Indeed, every such pair adjacent non-edge vu, vu' spans an edge hence determines the triple $\{u, u', u''\}$ by observation (1), and vu'' must be an edge in G in view of observation (2).

On the other hand, every subgraph F can be determined by a triple T and one of its private neighbours. Let the *value of the triple* T, $\mathrm{Val}(T)$ denote the number of private neighbours of the triple T, i.e., the number of F subgraphs corresponding to the triple. We thus obtain

$$\sum_v \binom{\overline{d}(v)}{2} = \sum_{T \in \mathcal{F}} \mathrm{Val}(T). \tag{5.3.1}$$

Observe that $|\mathcal{F}| = \frac{1}{6}(2\binom{n}{2} - \sum_v \overline{d}(v))$, moreover $\mathrm{Val}(T) \leq n-3$ clearly holds for every triple T. By the application of the bound $\mathrm{Val}(T) \leq n-3$, one would directly derive $E(\overline{G}) \leq \frac{\sqrt{13}-1}{12}n^2 + O(n) \approx 0.21n^2$ from Equation 5.3.1. However, this upper bound on $\mathrm{Val}(T)$ cannot be sharp for every triple: if the value of a triple is much larger than $\frac{n}{2}$, then many triples have value less than $\frac{n}{2}$. To understand better this situation, take a triple $T = \{v_1, v_2, v_3\}$, and denote by N_i^* the vertices which are connected only to v_i from the triple $\{v_1, v_2, v_3\}$, for $i \in \{1, 2, 3\}$.

Observation 5.3.1. *$G[N_1^* \cup N_2^* \cup N_3^*]$ is a complete graph.*

Proof. Indeed, since every pair of vertices from this class has a common non-neighbour, thus they must be joined in G to avoid a K_3 in $\overline{G}$. □

Now we define a new graph $\mathcal{G} = \mathcal{G}(\mathcal{F})$ as follows: we assign a vertex to every triple $T \in \mathcal{F}$, and we join T and T' if a pair from each span a C_4 in $\overline{G}$.

Proposition 5.3.2. *Suppose that $T \sim T'$ in $\mathcal{G}$. Then* $\mathrm{Val}(T) + \mathrm{Val}(T') \leq n$.

Proof. Without loss of generality, we may suppose that $T = \{v_1, v_2, v_3\}$, $T' \supset \{u, w\}$, and $\{u, w\} \subset N_1^*$. Observe that $\mathrm{Val}(T) = |N_1^* \cup N_2^* \cup N_3^*|$. On the other hand, Observation 5.3.1 implies that each vertex of the private neighbourhood set $N_1^* \cup N_2^* \cup N_3^*$ is connected to at least 2 vertices of T', hence $\mathrm{Val}(T') \leq n - \mathrm{Val}(T)$. □

We partition the vertex set of $\mathcal{G}$ to vertices with $\mathrm{Val}(T) \geq \frac{n}{2}$ (class A) and with $\mathrm{Val}(T) < \frac{n}{2}$ (class B). Consider now the bipartite graph $\mathcal{G}[A, B]$. We obtain lower and upper bound in this bipartite graph as follows.

Proposition 5.3.3.

$$\deg(T) \geq \frac{1}{3}\binom{\mathrm{Val}(T)}{2} \quad if\ T \in A,$$

$$\deg(T') \leq \binom{n - \mathrm{Val}(T)}{2} \quad if\ T' \in B,\ T \sim T'.$$

Proof. To prove the first bound, observe that every neighbour of $T = \{v_1, v_2, v_3\}$ in $\mathcal{G}[A, B]$ corresponds to a pair of vertices in one of the sets N_i^* $(i = 1, 2, 3)$ that supports a C_4 in $\overline{G}$, so

$$\deg(T) = \sum_{i \in \{1,2,3\}} \binom{|N_i^*|}{2} \geq \frac{1}{3}\binom{\mathrm{Val}(T)}{2}$$

by Jensen's inequality.

To prove the second bound, observe that if T' and T'' span a C_4 in $\overline{G}$ and T' and T also span a C_4 in $\overline{G}$ by T' having two vertices in N_i^*, then the pair from T'' supporting the C_4 must be in $V \setminus \bigcup_i N_i^*$. This in turn implies the assertion by the formula $\mathrm{Val}(T) = |N_1^* \cup N_2^* \cup N_3^*|$. ☐

Proposition 5.3.2 and 5.3.3 enables us to improve the upper bound on the average value of the triples $\mathrm{Val}(T) \leq n - 3$, and is carried out in the following Lemma.

Lemma 5.3.4. *Suppose that a weighted bipartite graph $\mathcal{G}(A, B)$ is given under the set of conditions*

- $\mathrm{Val} : A \to [\frac{n}{2}, n]$ *and* $\mathrm{Val} : B \to [0, \frac{n}{2})$ *holds for the weight function;*
- $\mathrm{Val}(v) + \mathrm{Val}(v') \leq n\ \forall vv' \in E(\mathcal{G})$ *;*
- $\deg(v) \geq \frac{1}{3}\binom{\mathrm{Val}(v)}{2}$ *if* $v \in A$*;*
- $\deg(v') \leq \binom{n-\mathrm{Val}(v)}{2}$ *if* $v' \in B,\ vv' \in E(\mathcal{G})$.

Then

$$\sum_{v \in V(\mathcal{G})} \mathrm{Val}(v) \leq \tau n \cdot |V(\mathcal{G})|, \tag{5.3.2}$$

where $\tau \approx 0.51829$ *is the unique local extremum of the rational function* $\frac{z(1-z)(3-2z)}{4z^2-6z+3}$ *in the interval* $z \in [\frac{1}{2}, 1]$.

We finish the proof by applying Lemma 5.3.4, and then return to the proof of Lemma 5.3.4. Equality (5.3.1) and the bound (5.3.2) together gives

$$\sum_{v\in V(G)} \binom{\overline{d}(v)}{2} = \sum_{T\in\mathcal{F}} \mathrm{Val}(T) < 0.5183\cdot n|\mathcal{F}| \tag{5.3.3}$$

On the other hand, since $|\mathcal{F}| = \frac{1}{6}(2\binom{n}{2} - \sum_{v\in V(G)} \overline{d}(v))$, this provides

$$n\left(\frac{\sum_{v\in V(G)} \overline{d}(v)}{n}\right)^2 + \left(\frac{0.5183n}{3} - 1\right)\sum_{v\in V(G)} \overline{d}(v) \le \frac{0.5183}{3}(n^3 - n^2)$$

by the AMQM inequality. Introducing $E(\overline{G}) = \frac{1}{2}\sum_{v\in V(G)} \overline{d}(v)$, we get a quadratic inequality for $E(\overline{G})$ in terms of n, which gives the desired bound $E(\overline{G}) < 0.169n^2 + O(n)$. □

Proof of Lemma 5.3.4. Instead of considering it as an involved convex optimisation problem, the general idea is to obtain a biregular bipartite graph in which the vertices has larger average value and optimise the average in the class of biregular bipartite graphs. The proof is carried out in three main steps.

First take a vertex v_0 of maximal value. We claim that for all of its neighbours $v' \in B$, the inequalities corresponding to them in Lemma 5.3.4 would hold with equalities:

(i) $\mathrm{Val}(v') = n - \mathrm{Val}(v_0)$,

(ii) $\deg(v') = \binom{n-\mathrm{Val}(v_0)}{2}$,

or else the average value could be increased. The claim for (i) is straightforward, while for (ii) suppose that $v' \in N(v_0)$ has smaller degree. Then one could take $\frac{1}{3}\binom{\mathrm{Val}(v_0)}{2}$ disjoint copies of $\mathcal{G}$, add a new vertex v_0^* (of value $\mathrm{Val}(v_0)$) and join to every copy of v'. Hence the conditions were fulfilled, while the average value would be increased.

Similar argument shows that for each $u \in A$ for which $N(v_0) \cap N(u) \neq \emptyset$, $\deg(u) = \frac{1}{3}\binom{\mathrm{Val}(u)}{2}$. Suppose it is not the case. Then for any $v' \in N(v_0) \cap N(u)$ one could delete the edge uv' in $\mathcal{G}$, then take $\frac{1}{3}\binom{\mathrm{Val}(v_0)}{2}$ disjoint copies of the derived graph and finally add a new vertex v_0^* (of value $\mathrm{Val}(v_0)$) and join to every copy of v'.

Without loss of generality we can assume that for each $u \in A$ for which $|N(v_0) \cap N(u)| = \lambda_u > 0$ with a maximum value vertex v_0, every neighbour v'

of u is connected to a vertex of maximum value. Consider the following construction. We take $m \cdot \deg(u)$ disjoint copies of $\mathcal{G}$ for an arbitrarily chosen $m \in \mathbb{Z}^+$ and redistribute the neighbours of the copies of u in such a way that $m \cdot \lambda_u$ copies are each joined to $\deg(u)$ distinct vertices from the copies of $N(v_0) \cap N(u)$, and the rest of the copies of u are each joined to $\deg(u)$ distinct vertices from the copies of $N(u) \setminus N(v_0)$. Since m can be chosen arbitrarily, this step can be performed at the same time for each such vertex u (as m can be chosen as the least common multiple of all of the corresponding degrees).

In order to maximize the average value of the vertices, we can clearly delete all but one connected components of the graph, and hence we assume that every vertex $v' \in B$ is connected to a vertex of maximum value. Now let us rewrite the average value as

$$\frac{1}{|V(\mathcal{G})|} \sum_{v \in V(\mathcal{G})} \mathrm{Val}(v) = \frac{1}{|V(\mathcal{G})|} \sum_{v \in A} \left(\mathrm{Val}(v) + \sum_{v' \in N(v)} \frac{\mathrm{Val}(v')}{d(v')} \right).$$

Observe that the contribution of each vertex $v \in A$ to the average is the weighted sum

$$\frac{\mathrm{Val}(v) + \sum_{v' \in N(v)} \frac{\mathrm{Val}(v')}{d(v')}}{1 + \sum_{v' \in N(v)} \frac{1}{d(v')}}.$$

According to our previous considerations, we may assume that for all $v' \in B$, we have $\mathrm{Val}(v') = n - \mathrm{Val}(v_0)$, moreover $d(v') = \binom{n-\mathrm{Val}(v_0)}{2}$. In order to show that all the vertices of A have the same degree we may compare the corresponding contributions of a vertex v_0 of maximum value and some other vertex $u \in A$ which has the second largest value.

Clearly either

$$\frac{\mathrm{Val}(v_0) + (n - \mathrm{Val}(v_0))\frac{d(v_0)}{d(v')}}{1 + \frac{d(v_0)}{d(v')}} \geq \frac{\mathrm{Val}(u) + (n - \mathrm{Val}(v_0))\frac{d(u)}{d(v')}}{1 + \frac{d(u)}{d(v')}},$$

or

$$\frac{\mathrm{Val}(v_0) + (n - \mathrm{Val}(v_0))\frac{d(v_0)}{d(v')}}{1 + \frac{d(v_0)}{d(v')}} < \frac{\mathrm{Val}(u) + (n - \mathrm{Val}(v_0))\frac{d(u)}{d(v')}}{1 + \frac{d(u)}{d(v')}}.$$

In both cases once again we can apply the above argument of copying the graph, eliminating a vertices of a certain degree and redistributing its neighbourhood for among new vertices of another fixed degree. This way we eliminate either the vertices of maximum degree or of second maximum degree, while the average value is monotonically increasing. Doing so repeatedly, after a suitable number of steps we end up with a bipartite graph where all vertices $v \in A$ have the same degree.

The argument implies that in order to determine the maximum of the average value under the constraints of Lemma 5.3.4, it is enough to determine the maximum average value in the class of biregular subgraphs as the third step to finish the proof.

To this end, consider the maximum of the function

$$w \to \frac{w\binom{n-w}{2} + \frac{1}{3}(n-w)\binom{w}{2}}{\binom{n-w}{2} + \frac{1}{3}\binom{w}{2}}$$

in the interval $w \in [\frac{n}{2}, n]$, which is an equivalent reformulation of the problem. Introducing $z = \frac{w}{n}$, we obtain the function

$$z \to n\frac{z(1-z)(3-2z-\frac{4}{n})}{4z^2-6z+3-\frac{3-2z}{n}}$$

on the domain $z \in [\frac{1}{2}, 1]$. One can verify that

$$n\frac{z(1-z)(3-2z-\frac{4}{n})}{4z^2-6z+3-\frac{3-2z}{n}} \leq n\frac{z(1-z)(3-2z)}{4z^2-6z+3},$$

which in turn implies the statement of the theorem. □

Proof of Theorem 5.1.11, lower bound. Take an arbitrary triple T_1 of the weakly spreading system $\mathcal{F}$. Observe that there must exist a triple T_2 sharing a common vertex with T_1, otherwise their union would violate the weakly spreading property. From now on, the weakly spreading condition guarantees the existence of an ordering of the triples $T_1, T_2, \dots T_m$ of $\mathcal{F}$, such that

$$|T_k \cap \bigcup_{i=1}^{k-1} T_i| \geq 2 \quad (\forall k \leq m).$$

This in turn implies the lower bound. Notice that it is sharp for $5 \leq n \leq 10$. □

5.3.2 Upper bounds – construction for sparse spreading systems

We will construct a spreading triple system $\mathcal{F}$ on $n = 6p + 3$ vertices for every p such that p is an odd prime number, with $|E(G(\mathcal{F}))| \approx \frac{5}{12}n^2$.

Construction 5.3.5. *The vertex set of $\mathcal{F}$ is the disjoint union of 6 smaller subsets (we refer to them as classes), namely $V = A \cup B \cup C \cup A' \cup B' \cup C'$, where $|A| = |B| = |C| = p+1$ and $|A'| = |B'| = |C'| = p$. Denote the elements of A with $a_0, a_1, \dots, a_{p-1}$ and a special vertex a. Similarly $B = \{b_0, b_1, \dots, b_{p-1}, b\}$ and*

$C = \{c_0, c_1, \ldots, c_{p-1}, c\}$. *For* A', B', C' *we note the corresponding vertices by* α, β, γ *respectively, and index their elements again from* 0 *up to* $p-1$*. The set of triples in* $\mathcal{F}$ *are defined as follows:*

- *black triples:*
  - *between* A *and* B'*:* $\{a, a_j, \beta_j\}$ *(for* $0 \leq j \leq p-1$*); and* $\{a_i, a_{2j-i \pmod p}, \beta_j\}$ *(for* $0 \leq i \neq j \leq p-1$*)*
  - *between* B *and* C'*:* $\{b, b_j, \gamma_j\}$ *(for* $0 \leq j \leq p-1$*); and* $\{b_i, b_{2j-i \pmod p}, \gamma_j\}$ *(for* $0 \leq i \neq j \leq p-1$*)*
  - *between* C *and* A'*:* $\{c, c_j, \alpha_j\}$ *(for* $0 \leq j \leq p-1$*); and* $\{c_i, c_{2j-i \pmod p}, \alpha_j\}$ *(for* $0 \leq i \neq j \leq p-1$*)*
- *brown triples:*
  - *between* A' *and* B*:* $\{\alpha_i, \alpha_{2j-i \pmod p}, b_j\}$ *(for* $0 \leq i \neq j \leq p-1$*)*
  - *between* B' *and* C*:* $\{\beta_i, \beta_{2j-i \pmod p}, c_j\}$ *(for* $0 \leq i \neq j \leq p-1$*)*
  - *between* C' *and* A*:* $\{\gamma_i, \gamma_{2j-i \pmod p}, a_j\}$ *(for* $0 \leq i \neq j \leq p-1$*)*
- *orange triples:*
  - *between* $A \setminus \{a\}$*,* $B \setminus \{b\}$ *and* $C \setminus \{c\}$*:* $\{a_i, b_j, c_{i+j \pmod p}\}$ *(for* $0 \leq i, j \leq p-1$*)*
  - *between* A'*,* B' *and* C'*:* $\{\alpha_i, \beta_j, \gamma_{i+j+1 \pmod p}\}$ *(for* $0 \leq i, j \leq p-1$*)*
  - $\{a, b, c\}$
- *red triples:*
  $\{a, \alpha_j, b_j\}$*,* $\{b, \beta_j, c_j\}$ *and* $\{c, \gamma_j, a_j\}$ *(for* $0 \leq j \leq p-1$*)*
- *blue triples:*
  $\{a, \gamma_j, c_j\}$*,* $\{b, \alpha_j, a_j\}$ *and* $\{c, \beta_j, b_j\}$ *(for* $0 \leq j \leq p-1$*)*

Proposition 5.3.6. *The triple system* $\mathcal{F}$ *defined in Construction 5.3.5 has the spreading property.*

Proof. The first step is to verify the statement for those sets V' which have a large enough intersection with either $A \cup B \cup C$ or $A' \cup B' \cup C'$, by the application of the Cauchy–Davenport theorem.

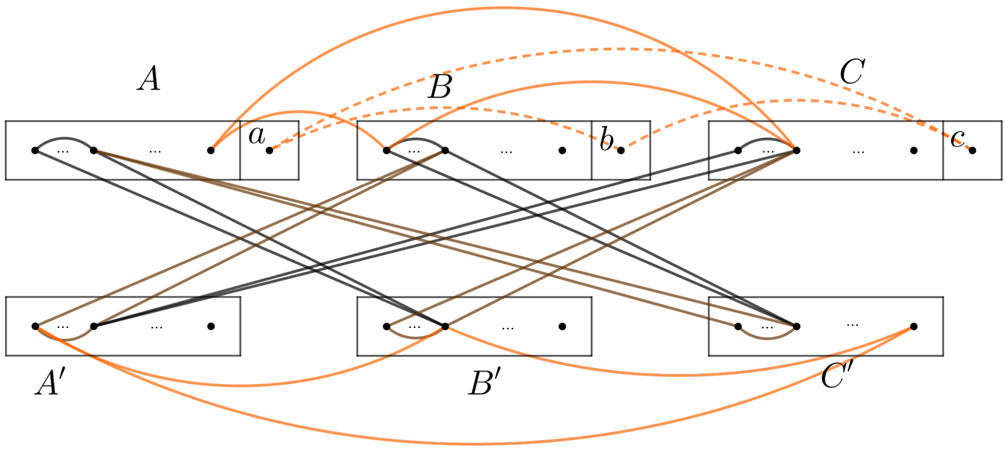


Figure 5.1: Black, brown and orange triples and $\{a, b, c\}$

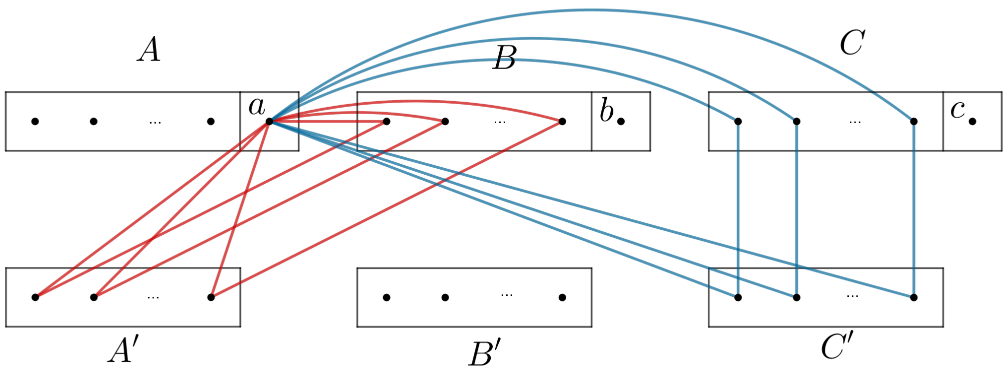


Figure 5.2: Red and blue triples through a

Observation 5.3.7. *Let us denote the set* $(A \setminus \{a\}) \cup (B \setminus \{b\}) \cup (C \setminus \{c\})$ *by* U*. If* $|V' \cap U| > 3$ *or* $|V' \cap (A' \cup B' \cup C')| > 3$*, and* V' *intersects with at least two different classes then* $\mathrm{cl}(V') = V$*.*

Indeed, without loss of generality, suppose to the contrary that there exists a set $|V' \cap (A' \cup B' \cup C')| > 3$ with $A_0 = V' \cap A'$, $B_0 = V' \cap B'$, $C_0 = V' \cap C'$ from which C_0 has the least size (smaller than p), such that $\mathrm{cl}(V') = V'$. Apply the Cauchy–Davenport theorem (Result 5.2.1) for A_0 and B_0 to obtain $|A_0 + B_0| \geq \min\{|A_0| + |B_0| - 1, p\} > |C_0|$. Thus the orange triples with their additive structure ensure that $|\mathrm{cl}(V') \cap C'| > |C_0|$, a contradiction.

In the rest of the proof, we point out that no matter how we choose a nontrivial set V' of 3 elements $\{x, y, z\}$, its closure contains at least 4 elements form either U or $A' \cup B' \cup C'$, coming from more than one classes, thus the application of Observation 5.3.7 in turn completes the proof.

1. $\{x, y, z\} \subset A \cup B \cup C$:

   a) $|\{x, y, z\} \cap \{a, b, c\}| = 0$:

      a1) If the starting elements are not in the same class (A, B or C) then two of them from different classes determine a new element (moreover it cannot be the special vertex) from the third class via an orange triple and now we have 4 elements of the closure in U not from the same class.

      a2) Without loss of generality we can assume that $x = a_i, y = a_j, z = a_k$ from $A \setminus \{a\}$. By using black and brown triples a_i, a_j determine some β_l; a_i, a_k determine some β_m $(m \neq l)$ and β_l, β_m determine some $c_s \in C \setminus \{c\}$.

   b) $|\{x, y, z\} \cap \{a, b, c\}| = 1$, without loss of generality let us assume that $z = a$:

      b1) If x, y are in different classes then they determine a new element from the third class via an orange triple thus we have got now 4 elements: a_i, a, b_j, c_k. From a and c_k we get γ_k due to a blue triple. If $j \neq k$ then b_j and γ_k determine some b_m through a black triple, and the closure meets U in more than 3 elements. If $j = k$ then $i = 0$ must hold, therefore b is in the closure from b_k and γ_k, moreover β_0 also in the closure from a and a_0. Now b and β_0 determine c_0 and we are done unless $i = j = k = 0$. In that case one can verify that the closure contains $\{a_0, b_0, c_0, \alpha_0, \beta_0, \gamma_0, a, b, c\}$ and by α_0 and β_0 we get that γ_1 is in the closure via an orange triple hence γ_0 and γ_1 determine $a_{\frac{p+1}{2}}$ via a brown triple that is the fourth element from U.

      b2) If x, y are in the same class then a, x and a, y determine different elements of the same class from $A' \cup B' \cup C'$ therefore these two elements determine a new element of U hence we trace back to case a).

c) $|\{x, y, z\} \cap \{a, b, c\}| = 2$, without loss of generality let us assume that $z = a$ and $y = b$:

c1) if $x = c_i \in C \setminus \{c\}$ then a and c_i determine γ_i via a blue triple, then b and γ_i determine $b_i \in B \setminus \{b\}$ due to a black triple therefore we trace back to case b1).

c2) if $x = b_i \in B \setminus \{b\}$ then b and b_i determine γ_i via a black triple, then a and γ_i determine $c_i \in C \setminus \{c\}$ therefore we trace back to case b1).

c3) if $x = a_i \in A \setminus \{a\}$ then b and a_i determine α_i via a blue triple, then a and α_i determine $b_i \in B \setminus \{b\}$ due to a red triple therefore we trace back to case b1).

2. $\{x, y, z\} \subset A' \cup B' \cup C'$:

One can deduce that this case can be discussed precisely the same way as case 1.a).

3. $|\{x, y, z\} \cap (A \cup B \cup C)| = 2$:

Assume that $\{y, z\} \subset A \cup B \cup C$ and $x \in A' \cup B' \cup C'$.

a) $|\{y, z\} \cap \{a, b, c\}| = 0$:

a1) If y and z are not in the same class then they determine a new non-special element from the third class. Together with the element from $A' \cup B' \cup C'$ one of these elements will form a triple which gives another new element from $A \cup B \cup C$. Either the closure meets U in more than 3 elements or trace back to case 1.b).

a2) Without loss of generality we can assume that $z = a_i$ and $y = a_j$. These two elements determine some β_k due to a black triple. Now if $x = \beta_l$ then from β_k and β_l we can get a c_m and then apply 1.a1). If $x = \gamma_l$ then at least one of the pairs γ_l, a_i or γ_l, a_j can determine a new element γ_m and we get a situation like in case 2. If $x = \alpha_l$ then α_l, β_k determine some γ_m and we get back the previous case.

b) $|\{y, z\} \cap \{a, b, c\}| \neq 0$:

Without loss of generality suppose that $z = a$. Now a together with the element from $A' \cup B' \cup C'$ will determine a new element from U hence we trace back to case 1.b) or 1.c).

4. $|\{x, y, z\} \cap (A \cup B \cup C)| = 1$:

   a) If the two elements from $A' \cup B' \cup C'$ are in different classes then via an orange triple they determine a new element from the third class and together with the element from $A \cup B \cup C$ they can determine at least one new element which is either in $A' \cup B' \cup C'$ and we are done or from $A \cup B \cup C$ thus trace back to case 3.

   b) If the two elements from $A' \cup B' \cup C'$ are in the same class then they determine a new element from U and we trace back to case 3.

□

We continue with a construction which shows a linear upper bound on the minimum size of weakly spreading systems. This will be derived from the upper bound of Theorem 5.1.10 and completes the proof of the upper bound of Theorem 5.1.11.

Construction 5.3.8 (Crowning construction)**.** *Consider a linear spreading system $\mathcal{F}$ on n vertices and $\xi_{sp}(n) = \frac{1}{3}\left(\binom{n}{2} - Cn^2\right)$ triples, with the appropriate constant C. Assign a new vertex $v(xy)$ to every not-covered edge xy of the underlying graph $G = G(\mathcal{F})$, and add newly formed triples by taking $\{\{x, y, v(xy)\} : xy \notin G\}$.*

Proposition 5.3.9. *Construction 5.3.8 provides a weakly spreading system on $n + Cn^2$ vertices with $\frac{1}{3}\left(\binom{n}{2} + 2Cn^2\right)$ triples, hence we obtain*

$$\xi_{wsp}(N) \leq \frac{2}{3}N + \frac{1}{6C}N.$$

Proof. It is easy to verify that any two triples, whose underlying set is denoted by V', determine at least three vertices which are not newly added such that they do not form a triple in $\mathcal{F}$. By the spreading property of $\mathcal{F}$, we get that $\mathrm{cl}(V')$ contains all points besides the new ones. Through the newly formed triples we get that actually every vertex is contained in $\mathrm{cl}(V')$. □

Proof of Theorem 5.1.11, upper bound. Applying Proposition 5.3.9 with $C = \frac{1}{12}$ in turn provides the upper bound. □

5.4 Related results and open problems

In this section we point out several related areas. First we discuss the connection to the topic of Latin squares, a message of which is that similar structures often provide constructions for the problem in view.

5.4.1 Latin squares

A Latin square of order n is an $n \times n$ matrix in which each one of n symbols appears exactly once in every row and in every column. A subsquare of a Latin square is a submatrix of the Latin square which is itself a Latin square. Note that Latin squares of order n and 1-factorizations of complete bipartite graphs $K_{n,n}$ are corresponding objects. We will apply the following theorem due to Maenhaut, Wanless and Webb [64], who were building on the work of Andersen and Mendelsohn [48].

Result 5.4.1 (Maenhaut, Wanless and Webb, [64])**.** *Subsquare-free Latin squares exists for every odd order.*

Note that for prime order the statement follows from the Cauchy–Davenport theorem. The construction presented below not only gives a simple weakly spreading construction, but it may provide an ingredient to a possible extension of Construction 5.3.5, where the triangle decomposition of the balanced complete tripartite graph, denoted by the set of orange triples, were obtained by a Cauchy–Davenport argument in the prime order case.

Construction 5.4.2. *Take a subsquare-free Latin square of odd order n with row set U, column set V and symbol set W. We assign a triple system $\mathcal{T}$ on $U \cup V \cup W$ to the Latin square as follows. Let $T = \{u_i, v_j, w_k\} \in \mathcal{T}$ if and only if w_k is the symbol in position (i, j) in the Latin square.*

Proposition 5.4.3. *Construction 5.4.2 yields that the minimum size of a weakly spreading triple system is at most $\zeta_{wsp}(n) \leq \frac{n^2}{9}$ for $n \equiv 3 \pmod 6$.*

Proof. Observe first that every pair of elements from different classes is contained exactly once in the system $\mathcal{T}$. Thus we have to show that there does not exist a subsystem spanned by $U' \subseteq U$, $V' \subseteq V$ and $W' \subseteq W$ for which every pair or elements from different classes is contained exactly once in triple of the subsystem. Clearly the existence would only be possible if $1 < |U'| = |V'| = |W'| < n$ but such a system would be equivalent to a Latin subsquare, a contradiction. □

5.4.2 Influence maximization

A social network represented by the graph of relationships and interactions in a group of individuals plays a fundamental role as a medium for the spread of information, ideas, and influence among its members. Models for the processes where some sort of influence or information propagate through a social network have been studied in a number of domains, including sociology, psychology, economy and computer science. The influence of a set of nodes is the (expected) number of active nodes at the end of the propagation process in the model and the influence maximization asks for the set of given size which has the largest influence. In one of the models, called the threshold model (see Chen [51]) there exists a threshold value $t(v)$ for every vertex $v \in V$ and in each discrete step a vertex is activated if it has at least $t(v)$ active neighbours. For more details we refer to the recent surveys [49, 58] and to the pioneer papers of Domingos and Richardson [54] and Kempe et. al. [62].

Mostly in models of social networks one only considers the graph of relationships, however in applications the propagation may depend more on whether an individual is influenced by the majority of the group members of social groups he or she belongs to. In that context, one has to describe the groups as hyperedges of a hypergraph, and in case of linear 3-graphs, the propagation of a vertex set V would clearly influence its closure $\mathrm{cl}(V)$. Hence our results determine bounds on the number of 3-sets needed so that every set of 3 vertices besides the triples themselves, or every pair of triples has maximum influence.

5.4.3 Connectivity, backward and forward 3-graphs

First we recall the concept of k-vertex-connectivity of hypergraphs, which is strongly related to the properties in view, and introduce a new edge-connectivity concept for triple systems.

Definition 5.4.4. *A hypergraph $\mathcal{F}$ is* k-vertex connected *if the removal of any $k-1$ vertices and adjacent edges results a connected hypergraph. A 3-uniform hypergraph $\mathcal{F}$ is* strongly connected *if every vertex partition $U \cup (V \setminus U)$ induces a triple T with $|T \cap U| = 2$, provided $|U| \geq 4$.*

The latter definition implies that if the partition classes U and $(V \setminus U)$ are large enough, then triples of type $|T \cap U| = 2$ and $|T \cap U| = 1$ both should appear. The condition $|U| \geq 4$ enables us to apply this concept for linear 3-graphs. We note that the spreading property is stronger than the strong connectivity, while the weakly spreading property is weaker.

Observation 5.4.5. *A Steiner triple system is subsystem-free, that is, spreading if and only if it is strongly connected. Every spreading linear triple system is strongly connected. Every strongly connected 3-graph is weakly spreading.*

Notice that the converse is not true for the latter statements.

Proposition 5.4.6. *If a linear 3-graph $\mathcal{F}$ is not 2-vertex connected then it is not weakly spreading.*

Proof. If the 3-graph $\mathcal{F}$ is not connected, then the assertion is obvious. Suppose now that there is a vertex v whose removal makes the 3-graph disconnected. This means one can find two triples T, $T' \in \mathcal{F}$ sharing v as a common vertex, with $T \setminus v$ and $T' \setminus v$ being in distinct connected components after the removal of v. Hence the $\mathrm{cl}(\{T, T'\}) = \{T, T'\}$. □

Finally, we underline that the weakly spreading property is not a local one, as the condition $\mathrm{cl}(V') \supset V'$ restricted to every pair of triples, $V' = V(\mathcal{F}')$ with $|\mathcal{F}'| = 2$ by no means imply weakly spreading. This follows from the construction below.

Construction 5.4.7. *Consider the complete graph K_n on n vertices $n > 3$, and add a vertex v_{ij} to every graph edge v_iv_j. The obtained triple system $\mathcal{F}_{(n)} = \{\{v_i, v_j, v_{ij}\} | i \neq j \leq n\}$ on $\binom{n}{2} + n$ vertices with $\binom{n}{2}$ hyperedges has the property that every pair of triples generate at least one further triple, but their closure will correspond to either $\mathcal{F}_{(3)}$ or $\mathcal{F}_{(4)}$.*

We finish this subsection by mentioning a connection to directed hypergraphs. A directed hyperedge is an ordered pair, $E = (X, Y)$, of disjoint subsets of vertices where X is the tail while Y is the head of the hyperedge. *Backward, resp. forward* 3-*graphs* are defined as directed 3-uniform hypergraphs with hyperedges having a singleton head, resp. tail, see Gallo et al. [60]. These objects have many applications in computer science, operations research, bioinformatics and transport networks.
It is easy to see that if one directs each triple of a linear 3-graph in all possible three ways to obtain a backward edge, then the connectivity, described above, of the triple system and the connectivity of the resulting directed hypergraph are equivalent.

5.4.4 Further results and open problems

We also mention the recent related work of Nenadov, Sudakov and Wagner [65] on embedding partial Steiner triple system to a small complete STS, and in general, embedding certain partial substructures to complete structures. In the spreading problem of linear 3-graphs, one may consider the triples of the hypergraph as collinearity prescription for triples of points, and under this condition the aim would be to embed the partial linear space to an affine of projective plane of small order. Here if two triples T, T' is incident to the same line, then the points of $\mathrm{cl}(\{T, T'\})$ would also be incident.

While our Theorem 5.1.8 on the expander property was sharp, our results Theorem 5.1.10 and 5.1.11 concerning spreading and weakly spreading determined the corresponding parameter only up to a small constant factor. The authors believe that if n is large enough, then neither of the bounds are sharp; however it seems a hard problem to asymptotically determine the exact values, similarly to many other extremal problems in hypergraph theory. Let us finish with several open problems.

Problem 5.4.8. *Is the asymptotically best upper bound on the minimum size $\xi_{wsp}(n)$ of a linear weakly spreading triple system obtained by the Crowning Construction 5.3.8 from an optimal construction for $\xi_{sp}(n)$?*

Although the lower bound $\xi_{wsp}(n)$ is tight for $n \leq 10$, we conjecture that this might be the case, meaning that $(C + o(1))n \leq \xi_{wsp}(n)$ for some $C > 1$.

Problem 5.4.9. *Generalize the results to r-uniform (linear) hypergraphs $\mathcal{F}$.*

In order to do this, one should define the neighbourhood and closure accordingly: a vertex z in the neighbourhood of V', if and only if there exist a hyperedge $F \in \mathcal{F}$ containing z such that

- either $|F \cap V'| \geq \frac{r}{2}$ (majority rule)
- or $|F \cap V'| \geq t$, $t < r$ fixed (large intersection).

Problem 5.4.10. *Prove the existence of Steiner triple system* $\mathrm{STS}(n)$ *of arbitrary admissible order n, for which*

$$|N(V')| \geq |V'| - 3$$

for every $V' \subset V(G)$ of size $|V'| \leq \frac{|V|}{2}$.

Chapter 6

Upper chromatic number of $\mathrm{PG}(n,q)$ and blocking sets

6.1 Introduction

Throughout the chapter, let $\mathcal{H}$ denote a hypergraph with point set V and edge set E. A strict N-coloring $\mathcal{C}$ of $\mathcal{H}$ is a coloring of the elements of V using exactly N colors; in other words, $\mathcal{C} = \{C_1, \ldots, C_N\}$ is a partition of V where each C_i is nonempty $(1 \le i \le N)$. Given a coloring $\mathcal{C}$, we define the mapping $\varphi_{\mathcal{C}} \colon V \to \{1, 2, \ldots, N\}$ by $\varphi_{\mathcal{C}}(P) = i$ if and only if $P \in C_i$. We call the numbers $1, \ldots, N$ colors and the sets $C_1, \ldots C_N$ color classes. We call a hyperedge $H \in E$ *rainbow (with respect to $\mathcal{C}$)* if no two points of H have the same color; that is, $|H \cap C_i| \le 1$ for all $1 \le i \le N$. The upper chromatic number (or shortly UCN) of the hypergraph $\mathcal{H}$, denoted by $\bar{\chi}(\mathcal{H})$, is the maximum number N for which $\mathcal{H}$ admits a strict N-coloring without rainbow hyperedges. Let us call such a coloring *proper* or *rainbow-free*. It is easy to see that for an ordinary graph G (that is, a 2-uniform hypergraph), $\bar{\chi}(G)$ is just the number of connected components of G.

As one can see, the above defined hypergraph coloring problem is a counterpart of the traditional one, where we seek the least number of colors with which we can color the vertices of a hypergraph while forbidding hyperedges to contain two vertices of the same color. The general mixed hypergraph model, introduced by Voloshin [7, 8], combines the above two concepts. This mixed model is better known but here we do not discuss it; the interested reader is referred to [9].

It is clear that if we find a vertex set $T \subset V$ in $\mathcal{H}$ which intersects every hyperedge in at least two points, then by coloring the points of T with one color and all the other points of V by mutually distinct colors, we obtain a proper, strict $(|V| - |T| + 1)$-coloring.

Definition 6.1.1. *Let $\mathcal{H} = (V; E)$ be a hypergraph, t a nonnegative integer. A vertex set $T \subset V$ is called a* t-transversal *of $\mathcal{H}$ if $|T \cap H| \geq t$ for all $H \in E$. The size of the smallest t-transversal of $\mathcal{H}$ is denoted by $\tau_t(\mathcal{H})$.*

Definition 6.1.2. *We say that a coloring of $\mathcal{H}$ is* trivial *if it contains a monochromatic 2-transversal.*

As seen above, the best trivial colorings immediately yield a lower bound for $\bar{\chi}(\mathcal{H})$.

Proposition 6.1.3.

$$\bar{\chi}(\mathcal{H}) \geq |V| - \tau_2(\mathcal{H}) + 1.$$

Two general problems are to determine whether this bound is sharp (for a particular class of hypergraphs), and to describe the colorings attaining the upper chromatic number. In this chapter, the hypergraphs we consider consist of the points of the n-dimensional projective space PG(n, q) over the finite field GF(q) of q elements with its k-dimensional subspaces as hyperedges, $n \geq 2$, $1 \leq k \leq n - 1$. We denote this hypergraph by $\mathcal{H}(n, k, q)$. The study of this particular case was started in the mid-nineties by Bacsó and Tuza [13], who established general bounds for the upper chromatic number of arbitrary finite projective planes (considered as a hypergraph whose points and hyperedges are the points and lines of the plane). We will use a notation which was introduced in Chapter 2, namely Equation (2.3.1), for the number of points in a n-dimensional projective space of order q.

Result 6.1.4 (Bacsó, Tuza [13])**.** *Let Π_q be an arbitrary finite projective plane of order q, and let $\tau_2(\Pi_q) = 2(q + 1) + c(\Pi_q)$. Then*

$$\bar{\chi}(\Pi_q) \leq q^2 - q - \frac{c(\Pi_q)}{2} + o(\sqrt{q}).$$

Note that Proposition 6.1.3 claims $\bar{\chi}(\Pi_q) \geq q^2 - q - c(\Pi_q)$. Recently, Bacsó, Héger, and Szőnyi have obtained exact results for the Desarguesian projective plane PG$(2, q)$.

Result 6.1.5 (Bacsó, Héger, Szőnyi [14])**.** *Let $q = p^h$, p prime. Suppose that either $q > 256$ is a square, or $p \geq 29$ and $h \geq 3$ odd. Then $\bar{\chi}(\mathrm{PG}(2, q)) = \theta_2 - \tau_2(\mathrm{PG}(2, q)) + 1$, and equality is reached only by trivial colorings.*

In this work, we determine $\bar{\chi}(\mathcal{H}(n,k,q))$ and aim not only to characterise trivial colorings as the only ones achieving the upper chromatic number of the hypergraph $\mathcal{H}(n,k,q)$, but to obtain results showing that proper colorings of $\mathcal{H}(n,k,q)$ using a little less number of colors than $\bar{\chi}(\mathcal{H}(n,k,q))$ are trivial; in other words, to prove that trivial colorings are stable regarding the number of colors. For the sake of convenience, we will formulate our results in three theorems for the hypergraph $\mathcal{H}(n,n-k,q)$. We note that if $k<\frac{n}{2}$, then $\tau_2(\mathcal{H}(n,n-k,q))=2\theta_k$, where equality can be reached by the union of two disjoint k-spaces, but not much is known if $k\geq\frac{n}{2}$.

Theorem 6.1.6. *Let $n\geq 3$, $1\leq k<\frac{n}{2}$, and assume that $q\geq 17$ if $k=1$ and $q\geq 13$ if $k\geq 2$. Then*

$$\bar{\chi}(\mathcal{H}(n,n-k,q))=\theta_n-\tau_2(\mathcal{H}(n,n-k,q))+1=\theta_n-2\theta_k+1.$$

Theorem 6.1.7. *Let $n\geq 2$, $q=p^h$, p prime, $1\leq k\leq n-1$. Suppose that*

- $\delta=\frac{1}{2}\left((\sqrt{2}-1)q^k-3\theta_{k-1}-8\right)\geq 0$ *and* $q\geq 11$ *if* $h=1$,
- $\delta=\frac{1}{2}\left(q^{k-1}-\theta_{k-2}-3\right)$, $k\geq 2$ *and* $q\geq 25$ *if* $h\geq 2$.

Under these assumptions the following hold:

a) If $k<\frac{n}{2}$, then any rainbow-free coloring of $\mathcal{H}(n,n-k,q)$ using

$$N\geq\theta_n-\tau_2(\mathcal{H}(n,n-k,q))+1-\delta=\theta_n-2\theta_k+1-\delta$$

colors contains a monochromatic pair of disjoint k-spaces, and hence is trivial.

b) If $k\geq\frac{n}{2}$, then

$$\bar{\chi}(\mathcal{H}(n,n-k,q))<\theta_n-2\theta_k+1-\delta.$$

Note that the stability gap δ in the above result is far much weaker in the non-prime case (in particular, the case $k=1$ is missing). The next theorem gives a much better result at the expense of requiring much stronger assumptions on the order and the characteristic of the field.

Theorem 6.1.8. *Let $n\geq 2$, $q=p^h$, p prime, $1\leq k\leq n-1$. Suppose that $p\geq 11$, $q\geq 239$ and $\delta=\frac{q^k}{200}-\theta_{k-1}$. Then any rainbow-free coloring of $\mathcal{H}(n,n-k,q)$ using*

$$N\geq\theta_n-2\theta_k+1-\delta$$

colors contains a monochromatic 2-fold k-blocking set, and hence is trivial.

The requirements on q and N in the above theorem could be chosen differently, see Remark 6.3.19 for the details. Note that Theorem 6.1.8 is not phrased in terms of $\tau_2(\mathcal{H}(n, n-k, q))$, the parameter found in the trivial lower bound Proposition 6.1.3. If $k < \frac{n}{2}$, $\tau_2(\mathcal{H}(n, n-k, q)) = 2\theta_k$. If $k = \frac{n}{2}$, then [15][Corollary 4.13] asserts the existence of a double k-blocking set in PG$(2k, q)$ of size $2q^k + 2\frac{q^k-1}{p-1}$, where $q = p^h$, $p > 5$ prime, $h \geq 2$. Thus, if $p \geq 409$, then $\tau_2(\mathcal{H}(2k, k, q)) \leq 2\theta_k + \delta$, whence Theorem 6.1.8 yields that the trivial bound is again sharp for $\mathcal{H}(2k, k, q)$, regardless the exact value of $\tau_2(\mathcal{H}(2k, k, q))$.

In finite geometrical language, t-transversals are called t-fold blocking sets. In the proof of the above theorem, we rely on weighted 2-fold blocking sets as well, so we devote the next section to this topic, and we obtain the following new result. The precise definitions are given in the next section.

Theorem 6.1.9. *Let $\mathcal{B}$ be a minimal weighted t-fold k-blocking set of* PG(n, p)*, p prime. Assume that $|\mathcal{B}| \leq (t + \frac{1}{2})p^k - \frac{1}{2}$ and $t \leq \frac{3}{8}p + 1$. Then $\mathcal{B}$ is the (weighted) union of t not necessarily distinct k-dimensional subspaces.*

This result, in fact, follows from the similar Theorem 6.2.5 about $t \pmod p$ sets.

6.2 Small, weighted multiple $(n-k)$-blocking sets

In the sequel, we will use all of the notions which were introduced in Chapter 2, particularly in Section 2.4. Moreover, we will use Results 2.4.3 and 2.4.4 several times going forward. For the sake of convenience, we will consider $(n-k)$-blocking sets throughout this section.

We refer to 1-fold and 2-fold blocking sets as blocking sets and double blocking sets, respectively; the term multiple blocking set refers to a t-fold blocking set with $t \geq 2$. We call a point of weight one simple. It is easy to see that a weighted t-fold k-blocking set must contain at least $t\theta_k$ points unless $t \geq q+1$. We include this supposedly folklore result with proof for the sake of completeness.

Proposition 6.2.1. *Let $\mathcal{B}$ be a t-fold $(n-k)$-blocking set in* PG(n, q)*. If $t \leq q$, then $|\mathcal{B}| \geq t\theta_{n-k}$.*

Proof. We prove by induction on k. If $k = 1$, we may take a point $P \notin \mathcal{B}$ (otherwise $|\mathcal{B}| \geq \theta_n > q\theta_{n-1}$ and there is nothing to prove). There are θ_{n-1} lines through P, each containing at least t points of $\mathcal{B}$, whence $|\mathcal{B}| \geq t\theta_{n-1}$. Suppose now $k \geq 2$. If $\mathcal{B}$ is an $(n-k+1)$-blocking set then, by induction, $|\mathcal{B}| \geq \theta_{n-k+1} = q\theta_{n-k} + 1 > t\theta_{n-k}$ and we are done. If there is a $(k-1)$-space Π disjoint from $\mathcal{B}$, then each of the θ_{n-k} distinct k-spaces containing Π intersects $\mathcal{B}$ in at least t points, whence $|\mathcal{B}| \geq t\theta_{n-k}$. $\square$

Note that $t \leq q$ is necessary here, as if $\mathcal{B}$ contains each point of an $(n-k+1)$-space with weight one, then $\mathcal{B}$ is a $(q+1)$-fold $(n-k)$-blocking set of size $\theta_{n-k+1} = q\theta_{n-k} + 1 < (q+1)\theta_{n-k}$; moreover, adding s further $(n-k)$-spaces to $\mathcal{B}$ we obtain a weighted $(q+1+s)$-fold $(n-k)$-blocking set of size less than $(q+1+s)\theta_{n-k}$ for any $s \geq 0$.

A stability result for weighted t-fold $(n-k)$-blocking sets of size close to this lower bound was proven by Klein and Metsch [70, Theorem 11].

Result 6.2.2 (Klein, Metsch [70]). *Let $\mathcal{B}$ be a weighted t-fold $(n-k)$-blocking set in* $\mathrm{PG}(n,q)$. *Suppose that $|\mathcal{B}| \leq t\theta_{n-k} + r\theta_{n-k-2}$, where t and r satisfy the following:*

a) $1 \leq t \leq \frac{q+1}{2}$;

b) $t + r \leq q$, $r \geq 0$ *is an integer;*

c) any blocking set of $\mathrm{PG}(2,q)$ *of size at most $q+t$ contains a line.*

Then $\mathcal{B}$ contains the (weighted) union of t not necessarily distinct $(n-k)$-spaces.

Let us remark that for $k = 1$ (that is, when $\mathcal{B}$ is a t-fold weighted blocking set with respect to lines), [70, Theorem 7] shows that condition *c)* can be omitted in the above result. However, a blocking set of $\mathrm{PG}(2,q)$ not containing a line must contain at least $q + \sqrt{q} + 1$ points in general (see [69] by Bruen), and, according to the following result of Blokhuis, at least $\frac{3}{2}(q+1)$ if q is prime, hence condition *c)* holds accordingly.

Result 6.2.3 (Blokhuis [68]). *Suppose that $\mathcal{B}$ is a blocking set in* $\mathrm{PG}(2,p)$*, p prime, not containing a line. Then $|\mathcal{B}| \geq \frac{3}{2}(p+1)$.*

Let us recall, what was already explained in Chapter 2, that the number of $(k+1)$-spaces containing a fixed k-space in $\mathrm{PG}(n,q)$ is θ_{n-k-1}. This can be seen easily by taking an $(n-k-1)$-space disjoint from the fixed k-space and observing that each appropriate $(k+1)$-space intersects it in a unique point.

6.2.1 Proof of Theorem 6.1.9

We prove a theorem closely related to Theorem 6.1.9 by considering an analogous problem in a slightly more general setting.

Definition 6.2.4. *Let us call a weighted point set $\mathcal{B}$ in $\mathrm{PG}(n,q)$ a $t \pmod p$ set with respect to the k-dimensional subspaces if $\mathcal{B}$ intersects every k-space in $t \pmod p$ points (counted with weights), for $1 \le k \le n-1$.*

Clearly, $t \pmod p$ sets are t-fold blocking sets if $t < p$ and, by Result 2.4.4, small minimal t-fold blocking sets are $t \pmod p$ sets.

Theorem 6.2.5. *Let $\mathcal{B}$ be a $t \pmod p$ set with respect to the k-dimensional subspaces in $\mathrm{PG}(n,p)$, p prime. Suppose that $t \le \frac{3}{8}p+1$ and $|\mathcal{B}| \le (t+1)\theta_{n-k}+p-2$. Then $|\mathcal{B}| = t\theta_{n-k}$ and $\mathcal{B}$ consists of the weighted union of t not necessarily disctinct $(n-k)$-spaces.*

Proof. The proof will use induction on k. Clearly, $\mathcal{B}$ is a t-fold $(n-k)$-blocking set. We will need the existence of a point not in $\mathcal{B}$. This follows if $|\mathcal{B}| < \theta_n$. If $p-1 \ge t+1$, then our assumption gives $|\mathcal{B}| \le (p-1)\theta_{n-1}+p-2 = \theta_n - 1 - \theta_{n-1}+p-2 < \theta_n$. If $p \le t+1$, then from $t \le \frac{3}{8}p+1$ it follows that $p \le 3$ must hold. If $p=2$, then $t=1$ and $|\mathcal{B}| \le (t+1)\theta_{n-1}+p-2 = \theta_n - 1$. If $p=3$, the problematic case is $t=2$, when $|\mathcal{B}| \le 3\theta_{n-1}+1 = \theta_n$. If $|\mathcal{B}| = \theta_n$ and $\mathcal{B}$ contains every point of the space, then it is clearly a $1 \pmod p$ set for every subspace, in contradiction with $t=2$. Hence we always find a point not contained in $\mathcal{B}$.
Case 1: $k=1$ (and $n \ge 2$). Notice first that every point of $\mathcal{B}$ has weight at most t. Indeed, by taking the weights of the points modulo p, we may assume that no point has weight at least p; and if $t+1 \le w(P) \le p-1$ for a point P, then all the θ_{n-1} lines through P must contain at least $p+t-w(P)$ more weights, whence $|\mathcal{B}| \ge w(P) + (p+t-w(P))\theta_{n-1} \ge p-1+(t+1)\theta_{n-1}$, a contradiction.

It follows from Results 6.2.2 and 6.2.3 that the assertion holds if $|\mathcal{B}| = t\theta_{n-1}$, hence we may assume that $|\mathcal{B}| > t\theta_{n-1}$ and prove by contradiction.

We will call lines that are neither t-secants (to $\mathcal{B}$), nor contained fully in $\mathcal{B}$ *long* lines; lines contained in $\mathcal{B}$ will be referred to as *full* lines. Non-t-secant lines are, therefore, either full or long. Long lines exist as on any point not in $\mathcal{B}$ (an *outer* point) we find a line intersecting $\mathcal{B}$ in more than t points, since $|\mathcal{B}| = t\theta_{n-1}$ would follow otherwise. Suppose that the minimum weight of a long line is $sp+t$. Clearly, $1 \le s \le t-1$ (the weight of a long line is at most tp). Let ℓ be a long line of weight $sp+t$, and let $P \in \ell \setminus \mathcal{B}$. We want to show that for any 2-space Π containing ℓ, there is a long line through P in Π different from ℓ. Fix such a plane Π (if $n=2$, then this is unique) and suppose to the contrary. Let $\mathcal{B}' = \mathcal{B} \cap \Pi$. Then, looking around from P in Π, $|\mathcal{B}'| = (p+1)t + sp$. Similarly as before, there must be a non-t-secant line on any point $R \in \Pi$; in other words, long and full lines form a blocking set in the dual plane of Π. It follows that long lines cover each outer point of Π exactly once. Moreover, the number of non-t-secant lines must be at least $\frac{3}{2}(p+1)$ for the following reason. By Blokhuis' Result 6.2.3, a blocking set of $\mathrm{PG}(2,p)$ of size less than $\frac{3}{2}(p+1)$ contains a line. In our setting this situation would result in a point Q through which all lines are either long or full. But then $(2t-1)p+t \geq tp+t+sp = |\mathcal{B}'| \geq w(Q)+(p+1)(p+t-w(Q)) \geq t+(p+1)p$, a contradiction even under $t < \frac{1}{2}p+1$.

Let e be a t-secant to $\mathcal{B}'$ (such a line exists as seen above). Let $P_1,\dots,P_r$ be the mutually distinct points of $e \cap \mathcal{B}'$, $1 \leq r \leq t$. Let $h_1(P_i)$ and $h_2(P_i)$ denote the number of full and long lines on P_i, respectively, and let h_1 and h_2 be the total number of full and long lines, respectively; then $h_1+h_2 \geq \frac{3}{2}(p+1)$. Looking around from P_i we see that

$$(p+1)t+sp = |\mathcal{B}'| \geq w(P_i)+(p+1)(t-w(P_i))+h_1(P_i)p+h_2(P_i)sp,$$

whence $w(P_i)+s \geq h_1(P_i)+sh_2(P_i)$. Let $h_2' := h_2-(p+1-r)$ be the number of long lines intersecting e in a point of $\mathcal{B}'$. Then $h_1+h_2' \geq \frac{3}{2}(p+1)-(p+1-r) \geq \frac{1}{2}(p+1)+r$, and we obtain that

$$t+rs = \sum_{i=1}^{r}(w(P_i)+s) \geq \sum_{i=1}^{r}(h_1(P_i)+sh_2(P_i)) = h_1+sh_2' = h_1+s(h_2-(p+1-r)),$$

whence

$$\begin{aligned} t &\geq h_1+s(h_2-(p+1)) > h_1+s\left(\left(\frac{3(p+1)}{2}-h_1\right)-(p+1)\right) = h_1+s\left(\frac{p+1}{2}-h_1\right) \\ &= (s-1)\left(\frac{p+1}{2}-h_1\right)+\frac{p+1}{2}. \end{aligned}$$

As $t < \frac{1}{2}(p+1)$, it follows that $s \geq 2$ and $h_1 > \frac{1}{2}(p+1)$.

It is clear that there are at least $p+1-r \geq 2$ long lines. Take now two long lines and the $h_1 \geq \frac{1}{2}p+1$ full lines one by one. The first line contains at least $sp+t$ weights of $\mathcal{B}'$. The second line may intersect it in a point of weight at most t, hence we see at least sp more weights on it. Turning to the full lines, the ith full line contains at least $p+1-2-(i-1) = p-i$ points of $\mathcal{B}'$ not contained by any of the previous lines. Altogether we obtain

$$\begin{aligned} (p+1)t+sp = |\mathcal{B}'| &\geq 2sp+t+\sum_{i=1}^{\frac{p}{2}+1}(p-i) = 2sp+t+\left(\frac{p}{2}+1\right)p-\binom{\frac{p}{2}+2}{2} \\ &= 2sp+t+\frac{p^2}{2}+p-\frac{p^2}{8}-\frac{3p}{4}-1, \end{aligned}$$

whence

$$t \geq \frac{3}{8}p + s + \frac{1}{4} - \frac{1}{p} > \frac{3}{8}p + 1,$$

a contradiction. Thus we see that all planes containing ℓ indeed contain at least one other long line through P, so we find at least $1 + \theta_{n-2}$ long lines through P, hence on all the θ_{n-1} lines through P we find that $|\mathcal{B}| \geq t\theta_{n-1} + (\theta_{n-2} + 1)sp = t\theta_{n-1} + s(\theta_{n-1} - 1) + sp \geq (t+1)\theta_{n-1} + p - 1$, a contradiction.

Case 2: $2 \leq k \leq n-1$ (and $n \geq 3$). Take a point $P \notin \mathcal{B}$ in PG(n,p). Project the points of $\mathcal{B}$ from P into an arbitrary hyperplane H. We get a weighted point set $\tilde{\mathcal{B}} \subseteq \Pi$ for which $|\tilde{\mathcal{B}}| = |\mathcal{B}|$. Let W be a $(k-1)$-space in H, and let $U = \langle P, W\rangle$ be the k-space spanned by P and W. Then $|W \cap \tilde{\mathcal{B}}| = |U \cap \mathcal{B}|$, hence $\tilde{\mathcal{B}}$ is a $t \pmod p$ set with respect to $(k-1)$-spaces in the $(n-1)$-space H thus, by induction on k, $|\mathcal{B}| = |\tilde{\mathcal{B}}| = t\theta_{n-1-(k-1)} = t\theta_{n-k}$. Results 6.2.2 and 6.2.3 finish the proof. □

Theorem 6.1.9 now follows from Theorem 6.2.5 and the $t \mod p$ Result 2.4.4.

6.3 On the upper chromatic number of $\mathcal{H}(n, n-k, q)$

6.3.1 Proof of Theorems 6.1.6 and 6.1.7

The steps of the proof have a lot in common with those in [14]. We recall that we want to color the points of PG(n,q) with as many colors as possible so that each $(n-k)$-space contains two equicolored points. For two points P and Q, PQ denotes the line joining them.

Definition 6.3.1. *Let* $[m]_q = \frac{q^m-1}{q-1} = q^{m-1} + q^{m-2} + \ldots + q + 1$. *Let* $[m]_q! = \prod_{i=1}^{m}[i]_q$, *where* $[0]_q! = 1$, *and let* $\left[{n \atop m}\right]_q = \frac{[n]_q!}{[m]_q![n-m]_q!} = \frac{(q^n-1)(q^{n-1}-1)\ldots(q^{n-m+1}-1)}{(q^m-1)(q^{m-1}-1)\ldots(q-1)}$ *denote the number of m dimensional subspaces of an n dimensional* vector space; *thus the number of m-spaces in* PG(n,q) *is* $\left[{n+1 \atop m+1}\right]_q$.

Note that $\left[{k+1 \atop 1}\right]_q = \theta_k$. Let us collect some facts regarding the above defined q-binomial coefficients.

Lemma 6.3.2. *Let $q \geq 2$, $n \geq 1$, $s \geq 0$. Then*

a) $\left[{n \atop 1}\right]_q = q^{n-1} + \left[{n-1 \atop 1}\right]_q$, *that is,* $\theta_n = q^{n-1} + \theta_{n-1}$;

b) the number of m-spaces containing a given k-space in GF$(q)^n$ *is* $\left[{n-k \atop m-k}\right]_q$;

c) $\theta_s = \left[{s+1 \atop 1}\right]_q < \left(1 + \frac{1}{q-1}\right) q^s \leq \left(1 + \frac{2}{q}\right) q^s$;

d) if $s \leq n-1$, *then* $\dfrac{\left[{n+1 \atop n-s+1}\right]_q}{\left[{n-1 \atop s}\right]_q} \geq q^{2s}$.

Proof. The first statement is trivial. As for the second one, let U be the given k-space. The quantity in question is just the number of $(m-k)$-spaces in the $(n-k)$-dimensional quotient space $\mathrm{GF}(q)^n/U$. The third assertion is trivial for $s=0$; otherwise $\theta_s = q^s + q^{s-1} + \frac{q^{s-1}-1}{q-1} < q^s + (1 + \frac{1}{q-1})q^{s-1}$. Finally, regarding the fourth: it is trivial if $s = 0$; for $s \geq 1$,

$$\frac{\left[{n+1 \atop n-s+1}\right]_q}{\left[{n-1 \atop s}\right]_q} = \frac{[n+1]_q![s]_q![n-1-s]_q!}{[n-s+1]_q![s]_q![n-1]_q!} = \frac{(q^{n+1}-1)(q^n-1)}{(q^{n-s+1}-1)(q^{n-s}-1)}$$

$$> \frac{q^{2n+1} - 2q^{n+1}}{q^{2n-2s+1} - q^{n-s+1}} = \frac{q^{n+s} - 2q^{-s}}{q^{n-s} - 1} > \frac{q^{n+s} - 2q^{-s} + 1}{q^{n-s}}$$
$$= q^{2s} - 2q^{-n} + q^s q^{-n} \geq q^{2s},$$

as $s \geq 1$ and $q \geq 2$. □

General notation and assumptions. Suppose that a strict proper coloring $\mathcal{C}$ of $\mathcal{H}(n, n-k, q)$ using N colors is given. We denote the color classes of $\mathcal{C}$ by $C_1, \ldots, C_N$. For the sake of simplicity, we will compare the N with $\theta_n - 2\theta_k + 1$ (note that $\tau_2(\mathcal{H}(n, n-k, q)) = 2\theta_k$ iff $k < \frac{n}{2}$, and compare with the trivial lower bound $\theta_n - \tau_2 + 1$). We define the *deficit* $d = d(\mathcal{C})$ of $\mathcal{C}$ by $N = \theta_n - 2\theta_k + 1 - d$, which, in principle, may be negative as well. Without loss of generality we may assume that $C_1, \ldots, C_m$ are precisely the color classes of size at least two for some $m \geq 1$. Let $\mathcal{B} = \mathcal{B}(\mathcal{C}) = \cup_{i=1}^m C_i$.

Definition 6.3.3. *We say that a color class* C colors the $(n-k)$-space U *if* $|C \cap U| \geq 2$.

As every $(n-k)$-space must be colored by at least one of the color classes among $C_1, \ldots, C_m$, we clearly see that $\mathcal{B}$ is a 2-fold k-blocking set.

Proposition 6.3.4.

a) $m = |\mathcal{B}| - 2\theta_k - d + 1$

b) $m \leq 2\theta_k + d - 1$

c) $|\mathcal{B}| \leq 4\theta_k + 2(d-1)$

Proof. The first assertion follows from $\theta_n - |\mathcal{B}| + m = N = \theta_n - 2\theta_k + 1 - d$. As $|C_i| \geq 2$ for all $1 \leq i \leq m$, $|\mathcal{B}| \geq 2m$. This and the previous equality imply $m \leq 2\theta_k + d - 1$ and $|\mathcal{B}| \leq 4\theta_k + 2(d-1)$. $\square$

Lemma 6.3.5. *A color class C colors at most $\binom{|C|}{2}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ distinct $(n-k)$-spaces.*

Proof. If C colors an $(n-k)$-space U, then U contains a line spanned by the points of C. The number of such lines is at most $\binom{|C|}{2}$. By Lemma 6.3.2, the number of $(n-k)$ spaces containing a given line is $\begin{bmatrix} n-1 \\ n-k-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$. $\square$

The next proposition says that $\mathcal{B}$ cannot be too large; roughly speaking, $|\mathcal{B}| \leq (4-\sqrt{2})q^k + 2d + o(q^k)$.

Proposition 6.3.6. *Suppose that $d \leq \alpha q^k$, and $q > \left(\frac{5}{\sqrt{2}} - 2 - \alpha - \frac{4}{q}\right)^{-1} > 0$. Then $|\mathcal{B}| < (4-\sqrt{2})q^k + 4\theta_{k-1} + 2d + 2$.*

Proof. As every $(n-k)$-space must be colored, by Lemma 6.3.5 and convexity we have

$$\begin{bmatrix} n+1 \\ n-k+1 \end{bmatrix}_q \leq \sum_{i=1}^{m} \binom{|C_i|}{2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \leq \left(\binom{|\mathcal{B}| - 2(m-1)}{2} + (m-1)\right) \begin{bmatrix} n-1 \\ k \end{bmatrix}_q .$$

By Lemma 6.3.2 *d*) and Proposition 6.3.4 *a*), *b*)

$$q^{2k} \leq \binom{|\mathcal{B}| - 2(m-1)}{2} + (m-1) \leq \binom{4\theta_k - |\mathcal{B}| + 2d}{2} + 2\theta_k + d - 2.$$

Suppose to the contrary that $|\mathcal{B}| \geq 4\theta_k - \sqrt{2}q^k + 2d + 2 = (4-\sqrt{2})q^k + 4\theta_{k-1} + 2d + 2$ (here we use Lemma 6.3.2 *a*)). Then by the assumption and Lemma 6.3.2 *c*), the right-hand-side of the above expression is at most

$$\binom{\sqrt{2}q^k - 2}{2} + 2\theta_k + d - 2 < q^{2k} - \left(\frac{5}{\sqrt{2}} - 2\left(1 + \frac{2}{q}\right) - \alpha\right) q^k + 1 < q^{2k},$$

a contradiction. $\square$

The following lemma will be very useful as it provides us large color classes if $\mathcal{B}$ is not large. The proof is based on Result 2.4.3. Right now, we do not need the following stronger version of this lemma since our blocking set has no weights, but respecting its future use we will state it in a more general setting. This version can deal with colorings which come from weighted blocking sets.

Lemma 6.3.7. *Suppose that a color class C contains a simple essential point P of $\mathcal{B}$. Then there exists a set of simple points $S \subset C \setminus \{P\}$ such that $|S| \geq 3q^k + \theta_{k-1} - |\mathcal{B}|$, and for any point $Q \in S$ there exists an $(n-k)$ space U such that $U \cap \mathcal{B} = \{P; Q\}$ (so these points are essential for $\mathcal{B}$). In particular, $|C| \geq 3q^k + \theta_{k-1} + 1 - |\mathcal{B}|$.*

Proof. For $Q \in \mathcal{B}$, $Q \neq P$, let $P \sim Q$ iff Q is also a simple point and there exists an $(n-k)$-space U such that $U \cap \mathcal{B} = \{P, Q\}$. As P is simple and essential, we find at least one such point. Let $\{Q_1, \ldots, Q_r\} = \{Q \in \mathcal{B} \colon P \sim Q\}$. For all $1 \leq i \leq r$, take a point R_i from $PQ_i \setminus \{P, Q_i\}$, and let $R = \{R_1, \ldots, R_r\}$. Then the set $(\mathcal{B} \cup R) \setminus \{P\}$ is also a 2-fold k-blocking set. Thus $\mathcal{B} \cup R$ contains two different minimal 2-fold k-blocking sets, so by Harrach's Result 2.4.3 we have $|\mathcal{B}| + r \geq 3q^k + \theta_{k-1}$. As $Q_1, \ldots, Q_r$ must have the same color as P, the proof is finished. □

Now we are ready to show that if $\mathcal{B}$ is not large, then it is, in fact, quite small. Roughly speaking, if $|\mathcal{B}| < 3q^k$, then $|\mathcal{B}| < 2q^k + 2d + o(q^k)$. We will use that $3q^k + \theta_{k-1} - 2\theta_k = 3q^k + \theta_{k-1} - 2\left(q^k + \theta_{k-1}\right) = q^k - \theta_{k-1}$.

Proposition 6.3.8. *Let $\beta \geq 2$. Assume $|\mathcal{B}| \leq 3q^k + \theta_{k-1} + 1 - \beta$ and $(\beta - 4)q^k > (\beta + 4)\theta_{k-1} + \beta(2d + \beta - 3)$. Then $|\mathcal{B}| < 2\theta_k + 2d - 2 + \beta$.*

Proof. By Result 2.4.3 there is a unique minimal 2-fold k-blocking set $\mathcal{B}'$ contained in $\mathcal{B}$. By Lemma 6.3.7 we know that if a color class contains a point of $\mathcal{B}'$, then it contains at least $3q^k + \theta_{k-1} + 1 - |\mathcal{B}|$ points of it, while all other color classes in $\mathcal{B}$ have at least two points. This and Proposition 6.3.4 *a*) imply that

$$|\mathcal{B}| - 2\theta_k + 1 - d = m \leq \frac{|\mathcal{B}'|}{3q^k + \theta_{k-1} + 1 - |\mathcal{B}|} + \frac{|\mathcal{B}| - |\mathcal{B}'|}{2} \leq$$

$$\leq \frac{2\theta_k}{3q^k + \theta_{k-1} + 1 - |\mathcal{B}|} + \frac{|\mathcal{B}| - 2\theta_k}{2},$$

so

$$\left(|\mathcal{B}| - (2\theta_k + 2d - 2)\right)\left(3q^k + \theta_{k-1} + 1 - |\mathcal{B}|\right) \leq 4\theta_k.$$

The left-hand side expression is concave in $|\mathcal{B}|$. Substituting either $|\mathcal{B}| = 2\theta_k + 2d - 2 + \beta = 2q^k + 2\theta_{k-1} + 2d - 2 + \beta$ or $|\mathcal{B}| = 3q^k + \theta_{k-1} + 1 - \beta$ we obtain

$$\beta\left(3q^k + \theta_{k-1} - 2\theta_k - 2d + 3 - \beta\right) \leq 4\theta_k,$$

which, due to simple calculations and rearrangement, leads to $(\beta - 4)q^k \leq (\beta + 4)\theta_{k-1} + \beta(2d + \beta - 3)$, a contradiction. As $|\mathcal{B}| \leq 3q^k + \theta_{k-1} + 1 - \beta$, we conclude that $|\mathcal{B}| < 2\theta_k + 2d - 2 + \beta$ must hold. □

Using these results, we may assume that $\mathcal{B}$ is quite small. As shown by the next proposition, this immediately gives the desired result on the upper chromatic number provided that $\mathcal{B}$ contains the union of two disjoint one-fold blocking sets, which property can be deduced from a stability type result on multiple blocking sets like Theorem 6.1.9 or Result 6.2.2; however, the strength of the result obtained in this way will be utterly dependent on the strength of the stability result.

Proposition 6.3.9. *Suppose that $\mathcal{B}$ contains two disjoint k-blocking sets, U_1 and U_2. If the coloring is nontrivial, then $|U_1|+|U_2| \geq 4(3q^k - |\mathcal{B}| + \theta_{k-1})$; in particular, $|\mathcal{B}| \geq 2.4q^k + 0.8\theta_{k-1}$ and, if U_1 and U_2 are k-spaces, then $|\mathcal{B}| \geq 2.5q^k + \frac{1}{2}\theta_{k-1}$.*

Proof. We may assume that $|\mathcal{B}| < 3q^k + \theta_{k-1}$, otherwise the assertions are trivial. Then, by Result 2.4.3, $\mathcal{B}' = U_1 \cup U_2$ is precisely the set of essential points of $\mathcal{B}$. If the coloring is not trivial, then there are at least two colors used in $\mathcal{B}'$, say, red and green. Without loss of generality we may take a red point $P \in U_1$. By Lemma 6.3.7, we find a set S of essential points of $\mathcal{B}$ such that $|S| = 3q^k - |\mathcal{B}| + \theta_{k-1}$, and for each point $Q \in S$ there is an $(n-k)$-space U_Q such that $\mathcal{B} \cap U_Q = \{P, Q\}$. Thus all points of S are red. As U_2 is a k-blocking set, $\forall\, Q \in S\colon U_Q \cap U_2 = \{Q\}$, so $S \subset U_2$. By interchanging the role of U_1 and U_2, we see that U_1 and U_2 both contain at least $3q^k - |\mathcal{B}| + \theta_{k-1}$ red points. As the same holds for green points as well, we find that $4(3q^k - |\mathcal{B}| + \theta_{k-1}) \leq |U_1| + |U_2| \leq |\mathcal{B}|$, thus $|\mathcal{B}| \geq 2.4q^k + 0.8\theta_{k-1}$ in general; if U_1 and U_2 are k-spaces, substituting $|U_1| = |U_2| = \theta_k = q^k + \theta_{k-1}$ gives the assertion. $\square$

The next lemma shows under what conditions does Proposition 6.3.6 provide a good enough bound on $|\mathcal{B}|$ to make Proposition 6.3.8 work with $\beta = 5$, the value we will tipically use.

Lemma 6.3.10. *Assume $d \leq \alpha q^k$ for some $0 \leq \alpha \leq \frac{1}{2}$. Suppose that either*

1. *$k = 1$, $q \geq 5$ and $d \leq \min\left\{\frac{q}{10} - 2, \frac{q(\sqrt{2}-1)}{2} - \frac{9}{2}\right\}$, or*
2. *$k \geq 2$, $q \geq 13$ and $d \leq \frac{q^k}{10} - \frac{9q^{k-1}}{10} - \frac{28q^{k-2}}{10}$.*

Then $|\mathcal{B}| \leq 2\theta_k + 2d + 2$.

Proof. It is easy to see that the requirement $q > \left(\frac{5}{\sqrt{2}} - 2 - \alpha - \frac{4}{q}\right)^{-1} > 0$ of Proposition 6.3.6 holds under $\alpha \leq \frac{1}{2}$ and $q \geq 5$, hence we can conclude that $|\mathcal{B}| < (4-\sqrt{2})q^k + 4\theta_{k-1} + 2d + 2$. Thus to meet the assumptions of Proposition

6.3.8 with $\beta = 5$, it is enough to have

$$\begin{aligned} q^k &> 9\theta_{k-1} + 5(2d+2) \text{ and} && (6.3.1)\\ (4-\sqrt{2})q^k + 4\theta_{k-1} + 2d + 2 &\leq 3q^k + \theta_{k-1} - 4 \text{ or, equivalently,} && (6.3.2)\\ (\sqrt{2}-1)q^k &\geq 3\theta_{k-1} + 2d + 6. && (6.3.3)\end{aligned}$$

For $k = 1$, (6.3.1) and (6.3.3) demand $d < \frac{q-19}{10}$ and $d \leq \frac{q(\sqrt{2}-1)}{2} - \frac{9}{2}$.

For $k \geq 2$, using Lemma 6.3.2 *a*) and *c*) we see that to satisfy (6.3.1) it is enough to have

$$9\left(1 + \frac{2}{q}\right) q^{k-1} + 10\alpha q^k + 10 \leq q^k,$$

hence, as $k \geq 2$, it is sufficient to require

$$\alpha \leq \frac{1}{10} - \frac{9}{10q} - \frac{28}{10q^2}.$$

Regarding (6.3.3), we can similarly deduce that

$$(\sqrt{2}-1)q^k \geq 3\left(1 + \frac{2}{q}\right) q^{k-1} + 2\alpha q^k + 6$$

is enough, hence so is

$$\alpha \leq \frac{\sqrt{2}-1}{2} - \frac{3}{2q} - \frac{6}{q^2}.$$

It is easy to see that the latter requirement is weaker for $q \geq 9$, so the former one is enough, which is positive if $q \geq 13$. Thus under these conditions Proposition 6.3.8 yields $|\mathcal{B}| < 2\theta_k + 2d + 3$. As the quantities on both sides are integers, the proof is finished. □

Remark 6.3.11. *If $q \geq 25$, all conditions of Lemma 6.3.10 are satisfied under $d \leq \frac{q^k}{10} - 2q^{k-1}$.*

The considerations so far are enough to prove Theorems 6.1.6 and 6.1.7.

Proof of Theorem 6.1.6. We recall the assumptions $n \geq 3$, $1 \leq k < \frac{n}{2}$, $q \geq 17$ if $k = 1$ and $q \geq 13$ if $k \geq 2$. Under these, the requirements of Lemma 6.3.10 are met for $d \leq -1$, $\alpha = 0$, so we conclude that $|\mathcal{B}| \leq 2\theta_k$. Result 6.2.2 asserts that $\mathcal{B}$ contains the union of two k-spaces (which are disjoint as $\mathcal{B}$ is not weighted). Proposition 6.3.9 yields that either $|\mathcal{B}| \geq 2.5q^k + 0.5\theta_{k-1}$, a contradiction due to q being large enough, or the coloring is trivial, in which case $d \geq 0$, a contradiction. Thus there is no coloring with $d \leq -1$, in other words, $\bar{\chi}(\mathcal{H}(n, n-1, q)) \geq \theta_n - 2\theta_k + 1$. Equality can be reached by trivial colorings since, as $k < \frac{n}{2}$, we can always find two disjoint k-spaces, whose union is clearly a 2-fold k-blocking set. □

Proof of Theorem 6.1.7. Suppose $d \leq \frac{1}{2}\left((\sqrt{2}-1)q^k - 3\theta_{k-1} - 8\right)$ and $q \geq 11$, q prime. As $11 > \left(\frac{5}{\sqrt{2}} - 2 - \frac{1}{4} - \frac{4}{11}\right)^{-1} \approx 1.08$, we can apply Proposition 6.3.6 with $\alpha = \frac{1}{4}$ to obtain $|\mathcal{B}| < (4-\sqrt{2})q^k + 4\theta_{k-1} + 2d + 2 = 3q^k + \theta_{k-1} - 6$. Set $\beta = 8$. Then $|\mathcal{B}| \leq 3q^k + \theta_{k-1} + 1 - \beta$ and, using $d < \frac{q^k}{4} - \theta_{k-1} - 4$, we also have $(\beta - 4)q^k > (\beta+4)\theta_{k-1} + \beta(2d + \beta - 3)$, so Proposition 6.3.8 applies and yields $|\mathcal{B}| < 2\theta_k + 2d + 6 < 2.5q^k$. Hence, by Theorem 6.1.9, $\mathcal{B}$ contains two disjoint k-spaces; call them U_1 and U_2. (Note that this is possible only if $k < \frac{n}{2}$, hence we obtain a contradiction for $k \geq \frac{n}{2}$ showing that no proper coloring satisfies the condition on d.) As $|\mathcal{B}| < 2.5q^k$, Proposition 6.3.9 claims that our coloring is trivial.

Suppose now that q is not a prime, and recall that our assumptions in this case are $q \geq 25$ and $d \leq \frac{1}{2}\left(q^{k-1} - {k-1 \brack 1}_q - 3\right)$. To apply Lemma 6.3.10 we need $d \leq \frac{q^k}{10} - 2q^{k-1}$, which follows from $d < \frac{q^{k-1}}{2}$ and $q \geq 25$; hence we obtain $|\mathcal{B}| \leq 2\theta_k + 2d + 2$. The assumed upper bound for d is equivalent to $2d + 2 \leq (q-2)\frac{q^{k-1}-1}{q-1}$, so we may apply Result 6.2.2 with $t = 2$ and $r = q-2$ to see that $\mathcal{B}$ contains the union of two disjoint k-spaces (again, $k \geq \frac{n}{2}$ gives a contradiction). As $|\mathcal{B}| < 2\theta_k + q^{k-1} < 2.5q^k$ clearly holds, Proposition 6.3.9 claims that the coloring is trivial. $\square$

Remark 6.3.12. *We do not believe that the upper bound $d \lesssim 0.2q^k$ for the q prime case in the above result is close to be sharp. We think that the limit should be roughly $d \lesssim 0.5q^k$ but to achieve this, one needs to improve Propositions 6.3.6 and 6.3.8 significantly, or to use a different approach. Improving only Proposition 6.3.6 would allow us to prove the same assertion under $d \lesssim 0.25q^k$ (this is the best allowed by Proposition 6.3.8).*

6.3.2 Improvements when q is not a prime

We recall that $\mathcal{B} = \mathcal{B}(\mathcal{C})$ denotes the union of color classes in the proper coloring $\mathcal{C}$ with at least two elements, so $\mathcal{B}$ is a 2-fold k-blocking set in PG(n,q) colored in a way that each $(n-k)$-space contains at least two points of $\mathcal{B}$ of the same color.

General assumptions. In the sequel, we will always assume $q \geq 25$ and $d \leq \frac{q^k}{10} - 2q^{k-1}$, thus by Remark 6.3.11 we have that $|\mathcal{B}| \leq 2\theta_k + 2d + 2 \leq 2\theta_k + 2(\frac{q^k}{10} - 2q^{k-1}) + 2$; this is the bound on the size of $\mathcal{B}$ we will use. Let $\mathcal{B}'$ denote the unique minimal 2-fold k-blocking set contained in $\mathcal{B}$ (which is the set of essential points for $\mathcal{B}$, cf. Result 2.4.3). We want to prove that $\mathcal{B}'$ is monochromatic; to this end, let us suppose to the contrary that $\mathcal{B}$ contains a red and a green essential point as well. As $|\mathcal{B}| \leq 2\theta_k + 2d + 2 \leq 2.2q^k - 2q^{k-1} + 2\theta_{k-2} + 2 \leq 2.2q^k < \frac{5}{2}q^k - \frac{1}{2}$, the t mod p property (Result 2.4.4) holds for $\mathcal{B}'$.

We consider three cases depending on the relation between n and $2k$. Our main case is when $n = 2k$, in which situation the famous André–Bruck–Bose representation of projective planes shall be used to enable us using planar tools; the cases $n > 2k$ and $n < 2k$ will be traced back to this one in the following way.

$n < 2k$**:**

If $n < 2k$ then we simply embed this projective space into $\mathrm{PG}(2k, q)$. Color the new points with new and pairwise different colors. After the embedding we get a proper coloring of $\mathrm{PG}(2k, q)$ (a k-space of $\mathrm{PG}(2k, q)$ intersects the embedded n-space containing $\mathcal{B}$ in a $k + n - 2k = n - k$ dimensional subspace) without any change in the color classes of size at least two, therefore we get to the situation of our main case.

$n > 2k$**:**

In this case we will assume $d \leq \frac{q^k}{10} - 2q^{k-1} - 1$, which is one less than the general assumption; hence, this will be the one we will have to meet to satisfy the assumptions for all cases.

Let us embed $\mathrm{PG}(n, q)$ into $\mathrm{PG}(2n - 2k, q)$ and let us take an $(n - 2k - 1)$-space $\mathcal{V} \subset \mathrm{PG}(2n - 2k, q)$ which is disjoint from $\mathrm{PG}(n, q)$ (considered now as a given n-space of $\mathrm{PG}(2n - 2k, q)$); thus $\mathrm{PG}(2n - 2k, q)$ is generated by the original $\mathrm{PG}(n, q)$ and $\mathcal{V}$. We build a cone $\mathcal{K}$ upon the base $\mathcal{B}$ with vertex $\mathcal{V}$; that is, the cone $\mathcal{K}$ consists of the points of the lines joining a point $X \in \mathcal{B}$ with a point $Y \in \mathcal{V}$.

Lemma 6.3.13. *For an arbitrary point $P \in \mathrm{PG}(2n - 2k, q) \setminus (\mathrm{PG}(n, q) \cup \mathcal{V})$ there exist a unique pair of points $X \in \mathrm{PG}(n, q)$ and $Y \in \mathcal{V}$ such that the line defined by X and Y contains P.*

Proof. If a good pair X, Y exists then, clearly, the line XY is contained in $\langle P, \mathcal{V}\rangle \cap \langle P, \mathrm{PG}(n, q)\rangle$, which is a subspace of dimension $(n-2k-1+1)+(n+1)-(2n-2k) = 1$. Hence a line of this type is unique, and it defines the points X and Y in a unique way. $\square$

The points of $\mathrm{PG}(2n - 2k)$ not in $\mathcal{K}$ get pairwise distinct new colors, and let us color the points of $\mathcal{K}$ in the following way. The points of $\mathcal{V}$ will get the color of an arbitrarily chosen point of $\mathcal{B}$, and the points of $\mathcal{K} \setminus (\mathcal{B} \cup \mathcal{V})$ get the color of their well-defined ancestor (the unique point X in Lemma 6.3.13) in $\mathcal{B}$. Finally, let us give weight two to the points of $\mathcal{V}$. In this way, the coloring of $\mathrm{PG}(2n - 2k)$ is proper, since if an $(n-k)$-space U meets $\mathcal{V}$ then it is blocked by $\mathcal{K}$ trivially, and if it is skew to $\mathcal{V}$ then $\langle \mathcal{V}, U\rangle$ will be an $(2n-3k)$-space such that it meets PG(n, q) in an $(n-k)$-space W, thus W contains two points of $\mathcal{B}$ of the same color and, by the cone structure, U contains two points of $\mathcal{K}$ of the same color. Also, the red and the green essential points for $\mathcal{B}$ in PG(n, q) remain essential for $\mathcal{K}$, hence $\mathcal{K}$ contains a red and a green essential point of weight one.

Note that except for the points of the $(n-2k-1)$-space $\mathcal{V}$, each point of $\mathcal{K}$ has weight one; $n-2k-1 = \dim(\mathcal{V}) \leq \frac{2n-2k}{2} - 2 = \frac{\dim(\mathrm{PG}(2n-2k))}{2} - 2$; furthermore, the number of points of $\mathcal{K}$ (with weights) is $|\mathcal{B}| + 2|\mathcal{V}| + |\mathcal{B}||\mathcal{V}|(q-1) = |\mathcal{B}| + 2\theta_{n-2k-1} + |\mathcal{B}|(q^{n-2k}-1) = |\mathcal{B}|q^{n-2k} + 2\theta_{n-2k-1}$, and as $|\mathcal{B}| = 2\theta_k + 2d + 2 \leq 2\theta_k + 2(\frac{q^k}{10} - 2q^{k-1})$, we obtain $|\mathcal{K}| \leq 2(\theta_{n-k} - \theta_{n-2k-1}) + 2(\frac{q^{n-k}}{10} - 2q^{n-k-1}) + 2\theta_{n-2k-1} = 2\theta_{n-k} + 2(\frac{q^{n-k}}{10} - 2q^{n-k-1})$, which is exactly our assumption for the main case.

The main case, $n = 2k$**:**

In both of the above cases, we ended up in a projective space of order n, n even, whose points admit a proper coloring, and the union of the color classes of size at least two form a 2-fold $\frac{n}{2}$-blocking set $\mathcal{B}$ of size at most $2\theta_{\frac{n}{2}} + 2(\frac{q^{\frac{n}{2}}}{10} - 2q^{\frac{n}{2}-1}) + 2$, which set is either non-weighted, or the set of points with weight more than one is a subplane of dimension at most $\frac{n}{2} - 2$, and all points in this subplane are of weight two. In both cases, our indirect assumption assures that there exist red and green essential points of weight one. From now on we fix this notation for the appropriate dimensions and set $n = 2k$.

For future purposes, we need to find a hyperplane H that intersects $\mathcal{B}$ in at most $2.2\theta_{k-1}$ points and contains all points of weight two (if there is any). If $k = 1$, we are done (otherwise $\mathcal{B}$ blocks every line of PG$(2, q)$ at least three times, so $|\mathcal{B}| \geq 3(q+1)$, a contradiction). Suppose now $k \geq 2$. Then $|\mathcal{B}| \leq 2.2\theta_k - 4q^{k-1}$. Let U_{-2} be the $(k-2)$-space consisting of the points of weight two or, if there are no such points, an arbitrary $(k-2)$-space. Among the θ_{k+1} distinct $(k-1)$-spaces containing U_{-2}, there must be one, say, U_{-1}, that contains no point of $\mathcal{B} \setminus U_{-2}$, otherwise $|\mathcal{B}| \geq \theta_{k+1} > 2.2\theta_k$, a contradiction. Among the θ_k distinct k-spaces containing U_{-1} there must be one, say, U_0, that contains at most two points of $\mathcal{B} \setminus U_{-1}$, otherwise $|\mathcal{B}| \geq 3\theta_k > 2.2\theta_k$. Suppose now that the $(k+i)$-space U_i contains at most $2.2q^i$ points of $\mathcal{B} \setminus U_{i-1}$ $(0 \leq i \leq k-3)$. Then among the θ_{k-i-1} distinct $(k+i+1)$-spaces containing U_i, there must be one, say, U_{i+1}, that contains at most $2.2q^{i+1}$ points of $\mathcal{B} \setminus U_i$, otherwise $|\mathcal{B}| > 2.2q^{i+1}\theta_{k-i-1} = 2.2\theta_k - 2.2\theta_i > 2.2\theta_k - 2.2\theta_{k-2} > 2.2\theta_k - 4q^{k-1} \geq |\mathcal{B}|$, a contradiction. To find an appropriate hyperplane U_{k-1}, we claim that among the $\theta_1 = q+1$ distinct $(2k-1)$-spaces containing U_{k-2} there is one that contains at most $2.2q^{k-1} - 2\theta_{k-2}$ points of $\mathcal{B} \setminus U_{k-2}$, otherwise $|\mathcal{B}| > \left(2.2q^{k-1} - 2\theta_{k-2}\right)(q+1) = 2.2q^k + 2.2q^{k-1} - 2(\theta_{k-1} - 1) - 2\theta_{k-2} = 2.2\theta_k - 2\theta^{k-1} - 4.2\theta_{k-2} + 2 = 2.2\theta_k - 2q^{k-1} - 6.2\frac{q^{k-1}-1}{q-1} + 2 > 2.2\theta_k - 4q^{k-1} \geq |\mathcal{B}|$, a contradiction (where we use $q \geq 25$). Thus we find an $(n-1)$-space U_{k-1} such that

$$|\mathcal{B} \cap U_{k-1}| = |\mathcal{B} \cap U_{-2}| + \sum_{i=0}^{k-1} |(\mathcal{B} \setminus U_i) \cap U_{i+1}| \leq |\mathcal{B} \cap U_{-2}| + 2.2\theta_{k-2} +$$

$$+2.2q^{k-1} - 2\theta_{k-2} \leq 2\theta_{k-2} + 2.2\theta_{k-1} - 2\theta_{k-2} = 2.2\theta_{k-1}.$$

We set $H = U_{k-1}$ to be the hyperplane (a $(2k-1)$-space) admitting the properties claimed. André [71] and independently Bruck and Bose [72, 73] developed a method, the well-known *André-Bruck-Bose representation*, for representing translation planes of order q^h with kernel containing $\mathrm{GF}(q)$ in the projective space $\mathrm{PG}(2h, q)$. It arises from a suitable $(h-1)$-spread of the hyperplane at infinity in $\mathrm{PG}(2h, q)$. The affine lines of the plane are h-dimensional subspaces containing the $(h-1)$-spaces of the $(h-1)$-spread. The ideal points correspond to the elements of the spread. Thus a point set intersecting every h-space yields a blocking set in the plane $\mathrm{PG}(2, q^h)$.

It is well-known that an arbitrary $(k-1)$-space can be mapped to any other $(k-1)$-space with a suitable linear transformation. By the previous observations we can take a $(k-1)$-spread $\mathcal{S}$ of H (i.e., a set of $(k-1)$-spaces that partition H) in such a way that if $\mathcal{V}$ exist it will be contained in one of the spread elements. Moreover, this transitivity property allows us to choose such a Desarguesian (also called regular) spread, too.

Remember that we have already assumed on the contrary that $\mathcal{B}'$, the minimal part of $\mathcal{B}$, contains red and green essential points of weight one. By using Lemma 6.3.7 and the choice of H one can see that $\mathcal{B}'$ must have both red and green affine points. In the following we will show that the minimal part of $\mathcal{B}$ must be monochromatic which will give us a contradiction.

Let us define a point-line incidence structure $\Pi = \Pi(H, \mathcal{S})$ in the following way:

- the points of Π are the points of $\mathrm{PG}(2k, q) \setminus H$ and the elements of $\mathcal{S}$;
- for each k-dimensional subspace U of $\mathrm{PG}(2k, q)$ such that $U \cap H \in \mathcal{S}$, the set $(U \setminus H) \cup \{U \cap H\}$ is considered to be a line of Π, as well as $\mathcal{S}$;
- a point is incident with a line if it is an element of it.

Then Π is well-known to be a projective plane of order $\tilde{q} := q^k$ by the André-Bruck-Bose representation, and since $\mathcal{S}$ is a Desarguesian spread, then $\Pi \simeq \mathrm{PG}(2, \tilde{q})$.

We will consider $\mathcal{S}$ as the line at infinity in Π, and a point of Π is called ideal or affine according to whether it is on the ideal line or not.

Definition 6.3.14. *From the coloring $\mathcal{C}$ of* $\mathrm{PG}(2k, q)$*, we define a coloring $\widetilde{\mathcal{C}}$ of the points of Π in the following way.*

- *For an affine point P of Π, let P inherit its color naturally from the coloring $\mathcal{C}$.*
- *For an ideal point $S \in \mathcal{S}$, we distinguish two cases. On the one hand, if each point of S forms a singleton color class of $\mathcal{C}$ (i.e., $\mathcal{B} \cap S = \emptyset$), then let the color of S be the color of an arbitrarily chosen point of S. On the other hand, if there is a color class of $\mathcal{C}$ of size at least two containing a point of S (i.e., $\mathcal{B} \cap S \neq \emptyset$), then color S with a color i such that $C_i \cap S \neq \emptyset$, $|C_i| \geq 2$, and for all $j \in \{1, 2, \ldots, N\}$ we have $|C_i \cap S| \geq |C_j \cap S|$.*

Note that $\widetilde{\mathcal{C}}$ is an N-coloring of Π that might not be strict.

Definition 6.3.15. *From a weighted point set Z of* $\mathrm{PG}(2k, q)$ *with weight function w_Z, we define a weight function $\widetilde{w}_Z$ on the points of Π in the following way.*

- *For an affine point P of Π, let $\widetilde{w}_Z(P) = w_Z(P)$ if $P \in Z$ and $\widetilde{w}_Z(P) = 0$ otherwise.*
- *For an ideal point $S \in \mathcal{S}$, let $\widetilde{w}_Z(S) = |S \cap Z|$ (counted with weights; that is, $\widetilde{w}_Z(S) = \sum_{P \in S} w_Z(P)$).*

For the point set Z, $\widetilde{Z}$ denotes the weighted point set of Π corresponding to the weight function $\widetilde{w}_Z$ (zero weight points are not considered as elements of $\widetilde{Z}$).

Consider now $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}(\mathcal{C})}$ (recall that $\mathcal{B}$ may be weighted). If $S \notin \widetilde{\mathcal{B}}$, then the points of S have pairwise distinct colors in $\mathcal{C}$ (and all are singletons). If $S \in \widetilde{\mathcal{B}}$ is of weight one, then the color of S at $\widetilde{\mathcal{C}}$ is the same as the color of the unique point in $S \cap \mathcal{B}$ at $\mathcal{C}$. We remark that for the union $\mathcal{B}(\widetilde{\mathcal{C}})$ of color classes of size at least two of $\widetilde{\mathcal{C}}$ in Π, $\mathcal{B}(\widetilde{\mathcal{C}}) \subseteq \widetilde{\mathcal{B}}$, but equality does not follow immediately from the definitions. In the sequel, we will work with $\widetilde{\mathcal{B}}$ using the property that every line of Π intersects it in at least two equicolored points, yet we make the following, slightly stronger observation.

Proposition 6.3.16. *The coloring $\widetilde{\mathcal{C}}$ with the weight function $\widetilde{w}_{\mathcal{B}}$ is a proper weighted coloring of Π; that is, every line of Π contains some points of the same color whose weights add up to at least two.*

Proof. Let U be the k-space of $\mathrm{PG}(2k, q)$ corresponding to a line ℓ of Π, and let $S = U \cap H$. If $\widetilde{w}_{\mathcal{B}}(S) \geq 2$, we are done. If $\widetilde{w}_{\mathcal{B}}(S) \leq 1$ (whence $\{P, Q\} \not\subset S$ follows), then, as $\mathcal{C}$ is proper, U contains two distinct points of the same color with respect to $\mathcal{C}$, say, P and Q; note that $\{P, Q\} \subset \mathcal{B}$. If both P and Q are affine points (which is the case if $\widetilde{w}_{\mathcal{B}}(S) = 0$), then we are also done. Suppose now $P \in S$, $Q \notin S$ and $\widetilde{w}(S) = 1$. Then, as P is the unique point of $S \cap \mathcal{B}$, the color of S at $\widetilde{\mathcal{C}}$ is the same as the color of P at $\mathcal{C}$, and so S and Q are two points of ℓ having the same color at $\widetilde{\mathcal{C}}$. □

It is clear from Definition 6.3.15 that $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}'}$ (that is, the weighted point set in Π obtained from $\mathcal{B}'$) are weighted double blocking sets in Π of size (total weight) $|\widetilde{\mathcal{B}}| = |\mathcal{B}|$ and $|\widetilde{\mathcal{B}'}| = |\mathcal{B}'|$; however, $\widetilde{\mathcal{B}'}$ may not be minimal. Let $\widehat{\mathcal{B}}$ be the unique minimal weighted double blocking set contained in $\widetilde{\mathcal{B}}$ (cf. Result 2.4.3); then $\widehat{\mathcal{B}} \subset \widetilde{\mathcal{B}'}$ follows.

Proposition 6.3.17. *If $\widehat{\mathcal{B}}$ is monochromatic at $\widetilde{\mathcal{C}}$, then $\mathcal{C}$ is trivial.*

Proof. Clearly, $|\widehat{\mathcal{B}}| \geq 2(\widetilde{q}+1) = 2q^k + 2$. Suppose that each point of $\widehat{\mathcal{B}}$ is, say, green at $\widetilde{\mathcal{C}}$. As $\widehat{\mathcal{B}}$ is minimal, each ideal point $S \in \widehat{\mathcal{B}}$ has weight at most two (the affine points of $\widehat{\mathcal{B}}$ have weight exactly one). An ideal point $S \in \widehat{\mathcal{B}}$ as a $(k-1)$-dimensional subspace in $\mathrm{PG}(2k, q)$ must contain at least one green point (with respect to $\mathcal{C}$). By the choice of H we know that $|\mathcal{B}' \cap H| \leq 2.2\theta_{k-1}$. Therefore $\mathcal{B}'$ contains at least $|\widehat{\mathcal{B}}| - \frac{2.2\theta_{k-1}}{2} \geq 2q^k + 2 - 1.1\theta_{k-1}$ green points. We recall our general assumptions $|\mathcal{B}'| \leq |\mathcal{B}| \leq 2.2\theta_k - 4q^{k-1} + 2$ and suppose to the contrary that $\mathcal{C}$ is not trivial. Then $\mathcal{B}'$ contains a point that is not green but, say, red. As $\mathcal{B}'$ is minimal, this red point is essential and simple thus Lemma 6.3.7 claims that the number of red simple points is more than $3q^k - |\mathcal{B}'|$, whenceforth $|\mathcal{B}'| > 2q^k + 2 - 1.1\theta_{k-1} + 3q^k - |\mathcal{B}'|$, that is, $|\mathcal{B}'| \geq 2.5q^k - 0.55\theta_{k-1}$ follows, a contradiction. Hence $\mathcal{B}'$ is all green, thus $\mathcal{C}$ is trivial. □

By Proposition 6.3.17, it is enough to show that $\widehat{\mathcal{B}}$ is monochromatic. We will do this along the same main ideas as in [14, Proposition 3.14]; however, the ideas must have been adapted to the presence of weights. We need the following lemma.

Lemma 6.3.18. *Let $P \in \mathcal{B}' \setminus H$. Then P is essential for $\widetilde{\mathcal{B}'}$ in Π; consequently, $P \in \widehat{\mathcal{B}}$.*

Proof. Suppose to the contrary. Then every line of Π through P intersects $\widetilde{\mathcal{B}'}$ in at least three points (with respect to $\widetilde{w}_{\mathcal{B}'}$). This yields that for every $S \in \mathcal{S}$, the

k-space $\langle P, S\rangle$ of PG$(2k, q)$ intersects $\mathcal{B}'$ in at least three and thus, by Result 2.4.4, in at least $p+2$ points. As the q^k+1 distinct k-spaces of form $\langle P, S\rangle$, $S \in \mathcal{S}$, pairwise intersect in P only, we get $2.5q^k \geq |\mathcal{B}| \geq |\mathcal{B}'| \geq (p+1)(q^k+1)+1$, a contradiction. □

Suppose that $2\theta_k+2d+2 \leq 2(\tilde{q}+1)+X$ for some $X \in \mathbb{R}$. As $|\mathcal{B}| \leq 2\theta_k+2d+2 \leq 2\theta_k+0.2q^k-4q^{k-1}+2 \leq 2.2q^k$, we may assume $X \leq 0.2\tilde{q}$. By our indirect assumption we know that $\mathcal{B}'$ contains both red and green simple affine points. By Lemma 6.3.18 and the definition of $\widetilde{\mathcal{C}}$ we see that $\widehat{\mathcal{B}}$ also contains both red and green affine (and hence single) points.

If there are other color classes in $\mathcal{C}$ containing more than two points, replace their color by red. In this way we obtain a nontrivial proper coloring $\mathcal{C}'$ such that $\widetilde{\mathcal{B}}(\mathcal{C}) = \widetilde{\mathcal{B}}(\mathcal{C}')$; thus it is enough to restrict our attention for colorings admitting only two color classes of size more than one. Then the points of $\widehat{\mathcal{B}}$ are also either red or green and both colors actually occur in the affine part. Denote the set of red points of $\widehat{\mathcal{B}}$ by $\widehat{\mathcal{B}}_r$ and the set of green ones by $\widehat{\mathcal{B}}_g$.

By Result 2.4.4, every line meets $\widehat{\mathcal{B}}$ in 2 (mod p^e) points, where $e \geq 1$ is the largest integer for which this property holds. Write $|\widehat{\mathcal{B}}| = 2(\tilde{q}+1)+c$. Note that since $\widehat{\mathcal{B}}$ is a minimal double blocking set in Π, every point of it has weight at most two; moreover, by its definition, all double points of $\widehat{\mathcal{B}}$ are on the ideal line. It is easy to see that if $P \in \widehat{\mathcal{B}}$ is a single point, then there are at least $\tilde{q}+1-\frac{\tilde{q}+c}{p^e}$ bisecants through it and if P is a double point, then there are at least $\tilde{q}+1-\frac{2\tilde{q}+c}{p^e}$ bisecants through it. If $P \in \widehat{\mathcal{B}}$ is an affine single point, then at least $\tilde{q}+1-\frac{\tilde{q}+c}{p^e}-(X-c)$ of the bisecants through P to $\widehat{\mathcal{B}}$ are bisecants to $\widetilde{\mathcal{B}}$ as well. Since $\widetilde{\mathcal{C}}$ is proper, the points on these bisecants must have the same color as P. As there are both red and green affine (and hence single) points of $\widehat{\mathcal{B}}$, we find

$$|\widehat{\mathcal{B}}_r| \geq (\tilde{q}+2)-\frac{\tilde{q}+c}{p^e}-(X-c), \tag{6.3.4}$$

$$|\widehat{\mathcal{B}}_g| \geq (\tilde{q}+2)-\frac{\tilde{q}+c}{p^e}-(X-c), \tag{6.3.5}$$

which also immediately gives

$$|\widehat{\mathcal{B}}_r| = |\widehat{\mathcal{B}}|-|\widehat{\mathcal{B}}_g| \leq 2(\tilde{q}+1)+c-\left(\tilde{q}+2-\frac{\tilde{q}+c}{p^e}-(X-c)\right) = \tilde{q}+\frac{\tilde{q}+c}{p^e}+X. \tag{6.3.6}$$

Our aim now is to show that one of the color classes, say, the red class, contains even more points than what was shown above, leading to a lower bound on $|\widehat{\mathcal{B}}|$ large enough to get a contradiction. To this end we want to find an affine single red point in $\widehat{\mathcal{B}}$ that has many non-bisecant lines through it on which there are more red points than green.

For a line ℓ of Π, let $n_\ell = |\ell \cap \widehat{\mathcal{B}}|$, $n_\ell^r = |\ell \cap \widehat{\mathcal{B}}_r|$, $n_\ell^g = |\ell \cap \widehat{\mathcal{B}}_g|$. Clearly, $n_\ell^r + n_\ell^g = n_\ell$ holds for all line ℓ. We denote the affine part of $\widehat{\mathcal{B}}$ by $\widehat{\mathcal{B}}^a$ and for a line ℓ different from $\mathcal{S}$, define $\bar{n}_\ell$, $\bar{n}_\ell^r$, $\bar{n}_\ell^g$ similarly as above but with respect to $\widehat{\mathcal{B}}^a$. Again, $\bar{n}_\ell^r + \bar{n}_\ell^g = \bar{n}_\ell$ holds for every affine line ℓ. Clearly, $n_\ell - 2 \le \bar{n}_\ell \le n_\ell$ also holds. We recall Result 2.4.4 and $p^e \ge 3$. Observe that if $\bar{n}_\ell = 0$, then ℓ must meet the ideal line in a double point of $\widehat{\mathcal{B}}$; if $\bar{n}_\ell = 1$, then ℓ must meet the ideal line in a single point of $\widehat{\mathcal{B}}$; and if $\bar{n}_\ell = 2$, then ℓ must meet the ideal line outside of $\widehat{\mathcal{B}}$. Also, $\bar{n}_\ell > 2 \Leftrightarrow n_\ell > 2$. Let us denote the set of single and double points of $\mathcal{S}$ by $\mathcal{S}^1$ and $\mathcal{S}^2$, respectively. With these notations one can find the inequalities

$$\sum_{\ell \in \mathcal{L} \setminus \ell_\infty,\ \bar{n}_\ell = 1} \bar{n}_\ell \le |\mathcal{S}^1| \tilde{q} \qquad \text{and} \qquad \sum_{\ell \in \mathcal{L} \setminus \ell_\infty,\ \bar{n}_\ell = 2} \bar{n}_\ell \le 2 \cdot (\tilde{q} + 1 - |\mathcal{S}^1| - |\mathcal{S}^2|) \tilde{q}.$$

Clearly, we have $\sum_{\ell \in \mathcal{L} \setminus \ell_\infty} \bar{n}_\ell = |\widehat{\mathcal{B}}^a| \cdot (\tilde{q} + 1)$. Let $\Delta = |\mathcal{S}^1| + 2|\mathcal{S}^2| = |\widehat{\mathcal{B}}| - |\widehat{\mathcal{B}}^a|$. Then

$$\sum_{\ell \in \mathcal{L} \setminus \ell_\infty,\ n_\ell > 2} \bar{n}_\ell = \sum_{\ell \in \mathcal{L} \setminus \ell_\infty,\ \bar{n}_\ell > 2} \bar{n}_\ell = \sum_{\ell \in \mathcal{L} \setminus \ell_\infty} \bar{n}_\ell - \left(\sum_{\ell \in \mathcal{L} \setminus \ell_\infty,\ \bar{n}_\ell = 1} \bar{n}_\ell + \sum_{\ell \in \mathcal{L} \setminus \ell_\infty,\ \bar{n}_\ell = 2} \bar{n}_\ell \right) \ge$$

$$\ge \sum_{\ell \in \mathcal{L} \setminus \ell_\infty} \bar{n}_\ell - \tilde{q}(2\tilde{q} + 2 - \Delta) = (2(\tilde{q}+1) + c - \Delta)(\tilde{q}+1) - \tilde{q}(2\tilde{q} + 2 - \Delta) =$$

$$= (c+2)\tilde{q} + (c + 2 - \Delta).$$

We will refer to a line ℓ as a long secant if $n_\ell > 2$ holds. Let $\mathcal{L}^r$ be the set of affine long secants with $\bar{n}_\ell^r > \bar{n}_\ell^g$; define $\mathcal{L}^g$ and $\mathcal{L}^=$ analogously. Without loss of generality we may assume that $\sum_{\ell \in \mathcal{L}^r} \bar{n}_\ell^r \ge \sum_{\ell \in \mathcal{L}^g} \bar{n}_\ell^g$, therefore

$$(c+2)\tilde{q} + (c+2-\Delta) \le \sum_{\ell \in \mathcal{L} \setminus \ell_\infty,\ n_\ell > 2} \bar{n}_\ell = \sum_{\ell \in \mathcal{L}^r} (\bar{n}_\ell^r + \bar{n}_\ell^g) + \sum_{\ell \in \mathcal{L}^g} (\bar{n}_\ell^r + \bar{n}_\ell^g) + \sum_{\ell \in \mathcal{L}^=} (\bar{n}_\ell^r + \bar{n}_\ell^g) \le$$

$$\le \sum_{\ell \in \mathcal{L}^r} 2\bar{n}_\ell^r + \sum_{\ell \in \mathcal{L}^g} 2\bar{n}_\ell^g + \sum_{\ell \in \mathcal{L}^=} 2\bar{n}_\ell^r \le \sum_{\ell \in \mathcal{L}^r \cup \mathcal{L}^=} 4\bar{n}_\ell^r.$$

We call an affine long secant ℓ with $\bar{n}_\ell^r \ge \bar{n}_\ell^g$ an almost red line. From the last inequality we get that there is a red affine point $P \in \widehat{\mathcal{B}}_r$ such that the number of almost red lines through P is at least

$$\frac{(c+2)\tilde{q} + (c+2-\Delta)}{4|\widehat{\mathcal{B}}_r^a|} \ge \frac{(c+2)\tilde{q} + (c+2-\Delta)}{4|\widehat{\mathcal{B}}_r|},$$

where $\widehat{\mathcal{B}}_r^a$ is the affine part of $\widehat{\mathcal{B}}_r$.

By the $t \pmod p$ result (Result 2.4.4) we know that for a long secant ℓ, $n_\ell \ge p^e + 2$ and since $n_\ell - 2 \le \bar{n}_\ell$, we can deduce that $\bar{n}_\ell^r + \bar{n}_\ell^g \ge p^e$ for all long secants. Hence, since P is an affine single red point, any almost red long secant through P contains at least $\frac{p^e}{2} - 1$ red points of $\widehat{\mathcal{B}}$ different from P on it (counted with weights). Taking into account the number of red points located on the bisecants through P (6.3.4) and also the upper bound (6.3.6) on $\widehat{\mathcal{B}}_r$, this yields

$$(\widetilde{q}+2) - \frac{\widetilde{q}+c}{p^e} - (X-c) + \frac{(c+2)\widetilde{q} + (c+2-\Delta)}{4|\widehat{\mathcal{B}}_r|} \cdot \left(\frac{p^e}{2} - 1\right) \leq |\widehat{\mathcal{B}}_r| \leq \widetilde{q} + \frac{\widetilde{q}+c}{p^e} + X.$$

Rearranging the above inequality gives

$$2X + \frac{2(\widetilde{q}+c)}{p^e} - c - 2 \geq \frac{(c+2)\widetilde{q} + (c+2-\Delta)}{4|\widehat{\mathcal{B}}_r|} \cdot \left(\frac{p^e}{2} - 1\right).$$

From Result 2.4.4 we know that $c + 2 \geq \frac{\widetilde{q}}{p^e+1} - 1$. Applying this and also $|\widehat{\mathcal{B}}_r| \leq \widetilde{q} + \frac{\widetilde{q}+c}{p^e} + X$ we get

$$2X + \frac{2(\widetilde{q}+c)}{p^e} - \frac{\widetilde{q}}{p^e+1} + 1 \geq \frac{\left(\frac{\widetilde{q}}{p^e+1} - 1\right)\widetilde{q} + \left(\frac{\widetilde{q}}{p^e+1} - 1 - \Delta\right)}{4 \cdot \left(\widetilde{q} + \frac{\widetilde{q}+c}{p^e} + X\right)} \cdot \left(\frac{p^e}{2} - 1\right)$$

Let us write $X = \gamma\widetilde{q}$. After that multiply both sides with the whole denominator of the right (note that this is surely a positive number) and arrange everything to the left side to get the following due to a lengthy computation:

$$\begin{aligned} &\widetilde{q}^2 p^e \left(-1 + 16\gamma + 16\gamma^2\right) + 2\widetilde{q}p^{2e} + \widetilde{q}^2\left(10 + 40\gamma + 16\gamma^2\right) + \\ &+ \frac{\widetilde{q}^2}{p^e}(24 + 32\gamma) + \widetilde{q}c(16 + 32\gamma) + 16\frac{\widetilde{q}^2}{p^e} + \widetilde{q}p^e(6 + 8\gamma) + \\ &+ \frac{\widetilde{q}c}{p^e}(40 + 32\gamma) + \widetilde{q}(16 + 8\gamma) + 32\frac{\widetilde{q}c}{p^{2e}} + 16\frac{c^2}{p^{2e}} + 16\frac{c^2}{p^e} + \\ &\quad + 8c + 8\frac{\widetilde{q}}{p^e} + 8\frac{c}{p^e} - 2 - 2\Delta + 2p^{2e} + 2\Delta p^{2e} \geq 0 \end{aligned} \tag{6.3.7}$$

If $p^e = \widetilde{q}$, then every line which is not a 2-secant to $\widehat{\mathcal{B}}$ is contained completely in $\widehat{\mathcal{B}}$ (and the ideal point of it has weight two) since the affine points are single ones and an ideal point has weight at most two. Hence if there exists a double point in the ideal line then $\widehat{\mathcal{B}}$ has to be the union of two complete lines and otherwise every line is a 2-secant to $\widehat{\mathcal{B}}$. In the first case we get a contradiction with Proposition 6.3.9 and in the latter case we get that the number of lines has to be equal to $\binom{|\widehat{\mathcal{B}}|}{2}$, but now $|\widehat{\mathcal{B}}| = 2(\widetilde{q}+1)$. Hence $p^e = \widetilde{q}$ is not possible.

If $p^e < \widetilde{q}$, then the leading term in expression (6.3.7) is $\widetilde{q}^2 p^e(-1 + 16\gamma + 16\gamma^2)$. If γ is chosen so that $-1 + 16\gamma + 16\gamma^2 < 0$ and q and p^e are large enough, then the leading term overflow the remaining ones, hence we will get a contradiction and conclude that our coloring $\mathcal{C}$ must be trivial. The coefficient is negative if $\frac{1}{\gamma} > 8 + 4\sqrt{5} \approx 16,944$ but then the remaining terms can be quite large. Thus at this point we make a rather arbitrary choice of the parameters in our likes and, in case someone would need a differently set result, we will make a remark on the other possible choices.

Let us consider the non-negative expression on the left side of (6.3.7) as a function $f = f(\widetilde{q}, p^e, \gamma, c, \Delta)$. Clearly, f is increasing in c, Δ and γ. By the definitions of Δ and c we immediately see that $\Delta \leq 2(\widetilde{q}+1)$ and $c \leq X = \gamma\widetilde{q}$, thus $g(\widetilde{q}, p^e, \gamma) := f(\widetilde{q}, p^e, \gamma, \gamma\widetilde{q}, 2(\widetilde{q}+1)) \geq 0$ follows. Let us fix the value of $\gamma = \frac{1}{100}$. It means that $d \leq \delta$, because $|\widetilde{B}| \leq 2\theta_k + 2d + 2 \leq 2(\widetilde{q}+1) + X = 2(\widetilde{q}+1) + \gamma\widetilde{q}$. Now $p^{2e} \cdot g\left(\widetilde{q}, p^e, \frac{1}{100}\right) =$

$$(6\widetilde{q}+6)p^{4e} + \left(-\frac{524}{625}\widetilde{q}^2 + \frac{152}{25}\widetilde{q}\right)p^{3e} + \left(\frac{6603}{625}\widetilde{q}^2 + \frac{304}{25}\widetilde{q} - 6\right)p^{2e} +$$

$$+\left(\frac{25453}{625}\widetilde{q}^2 + \frac{202}{25}\widetilde{q}\right)p^e + \frac{201}{625}\widetilde{q}^2 \geq 0$$

Since $p^e \neq \widetilde{q}$, we know that $p^e \leq \frac{\widetilde{q}}{p}$ holds, and on the other hand, from Result 2.4.4 one can deduce that $p^e \geq \frac{\widetilde{q}}{\gamma q+3} - 1$, which is equivalent to $p^e \geq 99 - \frac{30000}{\widetilde{q}+300}$. Since $\widetilde{q} \geq 239$ and the characteristic $p \geq 11$, we can increase the terms with positive coefficients by changing p^e to $\frac{\widetilde{q}}{11}$ or by multiplying with $\frac{\widetilde{q}}{239}$. Moreover, we can decrease the terms with negative coefficients by changing p^e to 47, since the lower bound on p^e also increases as $\widetilde{q}$ increases, therefore $p^e \geq 99 - \frac{30000}{239+300} \approx 43,341$. With these three elementary observations one can give an upper bound $p^{2e} \cdot g(\widetilde{q}, p^e, \frac{1}{100}) \leq$

$$(6\widetilde{q}+6)p^{3e} \cdot \frac{\widetilde{q}}{11} + \left(-\frac{524}{625}\widetilde{q}^2 + \frac{152}{25}\widetilde{q}\right)p^{3e} + \left(\frac{6603}{625}\widetilde{q}^2 + \frac{304}{25}\widetilde{q} - 6\right)p^{2e} +$$

$$+\left(\frac{25453}{625}\widetilde{q}^2 + +\frac{202}{25}\widetilde{q}\right)p^e + \frac{201}{625}\widetilde{q}^2 = \left(-\frac{2014}{6875}\right)p^{3e}\widetilde{q}^2 + \frac{1822}{275}p^{3e}\widetilde{q} + \frac{6603}{625}p^{2e}\widetilde{q}^2 +$$

$$+\frac{304}{25}p^{2e}\widetilde{q} - 6p^{2e} + \frac{25453}{625}p^e\widetilde{q}^2 + \frac{202}{25}p^e\widetilde{q} + \frac{201}{625}\widetilde{q}^2 \leq \left(-\frac{2014}{6875}\right)p^{3e}\widetilde{q}^2 + \frac{1822}{275}p^{3e}\frac{\widetilde{q}^2}{239} +$$

$$+\frac{6603}{625}p^{2e}\widetilde{q}^2 + \frac{304}{25}p^{2e}\widetilde{q} - 6p^{2e} + \frac{25453}{625}p^e\widetilde{q}^2 + \frac{202}{25}p^e\widetilde{q} + \frac{201}{625}\widetilde{q}^2 = \left(-\frac{435796}{1643125}\right)p^{3e}\widetilde{q}^2 +$$

$$+\frac{6603}{625}p^{2e}\widetilde{q}^2 + \frac{304}{25}p^{2e}\widetilde{q} - 6p^{2e} + \frac{25453}{625}p^e\widetilde{q}^2 + \frac{202}{25}p^e\widetilde{q} + \frac{201}{625}\widetilde{q}^2 \leq \left(-\frac{435796}{1643125}\right)p^{2e}\widetilde{q}^2 \cdot 47 +$$

$$+\frac{6603}{625}p^{2e}\widetilde{q}^2 + \frac{304}{25}p^{2e}\widetilde{q} - 6p^{2e} + \frac{25453}{625}p^e\widetilde{q}^2 + \frac{202}{25}p^e\widetilde{q} + \frac{201}{625}\widetilde{q}^2 = \left(-\frac{4997}{2629}\right)p^{2e}\widetilde{q}^2 +$$

$$+\frac{304}{25}p^{2e}\widetilde{q} - 6p^{2e} + \frac{25453}{625}p^e\widetilde{q}^2 + \frac{202}{25}p^e\widetilde{q} + \frac{201}{625}\widetilde{q}^2 \leq \left(-\frac{4997}{2629}\right)p^e\widetilde{q}^2 \cdot 47 + \frac{304}{25}p^e\frac{\widetilde{q}^2}{11} - 6p^{2e} +$$

$$+\frac{25453}{625}p^e\tilde{q}^2+\frac{202}{25}\frac{\tilde{q}^2}{11}+\frac{201}{625}\tilde{q}^2 \;=\; \left(-\frac{78054538}{1643125}\right)p^e\tilde{q}^2-6p^{2e}+\frac{7261}{6875}\tilde{q}^2 \;\leq$$
$$\leq\; \left(-\frac{78054538}{1643125}\right)\cdot 47\tilde{q}^2-6p^{2e}+\frac{7261}{6875}\tilde{q}^2 \;=\; -\frac{3666827907}{1643125}\tilde{q}^2-6p^{2e}<0,$$

which is a contradiction, hence the coloring must be trivial. Thus we finished the proof of Theorem 6.1.8. □

Remark 6.3.19. *If one would like to choose a suitable γ, so that $\frac{1}{\gamma} > 8+4\sqrt{5} \approx 16,944$ still holds, then the conditions on the lower bound on q and on the characteristic may change, which would lead to the corresponding lower bound on p^e. After properly adjusting the assumption on the characteristic of the field, then one can get a contradiction by an analogous argument. For example we computed the conditions on the variables in order to get a contradiction for $\gamma = \frac{1}{20}$ and for $\gamma = \frac{1}{50}$.*

$$\gamma=\frac{1}{20}: \quad \textit{if}\ p\geq 151, q\geq p^2 \qquad\qquad \gamma=\frac{1}{50}: \quad \textit{if}\ p\geq 17, q\geq 479.$$

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Summary

The thesis treats problems from graph theory and most of them have some connections with finite geometry. Throughout the thesis we use some well-known results such as the Godsil-McKay switching, the Cauchy-Davenport theorem, the Bose-Skolem construction of Steiner triple systems, and the so-called $t \pmod{p}$ result of Ferret, Storme, Sziklai and Weiner.

In Chapter 3, we construct a family of infinitely many examples for two b-regular, cospectral graphs so that exactly one of them has a perfect matching. It proves a conjecture of Willem H. Haemers via the use of the Godsil-McKay switching. The results of this chapter are joint with Jay Cummings and Willem H. Haemers.

In the last decade, the investigation of (semi-)resolving sets significantly accelerated. However, a similarly defined notion, the partition dimension, attracted less interest. Together with Zoltán Lóránt Nagy, we determined the order of magnitude of the partition dimension of the incidence graph of finite projective planes in Chapter 4.

The motivation for the results in Chapter 5 is the following embeddability question. If some collinearity constraints are given on the triples of a point set, can then we embed this prescribed structure into a projective plane of order q? It turns out that the answer for this question depends on some kind of connectivity properties of hypergraphs. These observations lead to the study of Steiner triple systems containing no nontrivial subsystems, and to the existence of such Steiner triple systems which are almost 1-expanders. The results of this chapter are joint with Zoltán Lóránt Nagy.

Together with Tamás Héger and Tamás Szőnyi, we generalised their result (together with Gábor Bacsó) concerning the upper chromatic number of $\mathrm{PG}(2,q)$ in Chapter 6. We investigate the upper chromatic number of the hypergraph formed by the points and the k-dimensional subspaces of $\mathrm{PG}(n,q)$; that is, the most number of colors that can be used to color the points so that every k-subspace contains at least two points of the same color. Clearly, if one colors the points of a double

$(n-k)$-blocking set with the same color, the rest of the points may get mutually distinct colors. This gives a trivial lower bound, and we prove that it is sharp in many cases. Furthermore, we prove that a stability phenomenon occurs here, because no matter how we color the points of $\mathrm{PG}(n,q)$ with slightly less number of colors it must contain a monochromatic double $(n-k)$-blocking set, too. Due to this relation with double blocking sets, we also prove that for $t \leq \frac{3}{8}p+1$, a small t-fold (weighted) $(n-k)$-blocking set of $\mathrm{PG}(n,p)$, p prime, must contain the weighted sum of t not necessarily distinct $(n-k)$-spaces.

Összefoglalás

A disszertációban többségében olyan gráfelméleti problémákkal foglalkozunk, amik kapcsolatban állnak a véges geometriával. A tézis során többek között olyan ismert eredményeket fogunk felhasználni, mint a Godsil-McKay switching, a Cauchy-Davenport tétel, Bose és Skolem konstrukciója Steiner hármasrendszerekre, és az úgynevezett $t \pmod p$ eredménye a Ferret, Storme, Sziklai, Weiner szerzőnégyesnek.

A harmadik fejezetben végtelen sok példát adunk olyan gráfpárokra, amelyek b-regulárisak, az adjacencia mátrixra vonatkozó spektrumuk megegyezik, mégis csak pontosan az egyikük tartalmaz teljes párosítást. A bizonyítás a Godsil-McKay switching alkalmazásán múlik, és így igazolja Willem H. Haemers korábbi sejtését. A fejezetben szereplő eredmények Jay Cummings-szal és Willem H. Haemers-szel közösek.

Az elmúlt évtizedben a (félig-)megoldóhalmazok vizsgálata jelentősen felgyorsult, miközben a hasonlóan definiált partíció dimenzió fogalmával kevesebbben foglalkoztak. A negyedik fejezetben a Nagy Zoltán Lóránttal közös eredményünket tárgyaljuk. Sikerült meghatároznunk a véges projektív síkok illeszkedési gráfjának partíció dimenziójának nagyságrendjét.

Az ötödik fejezetbeli eredményekhez a következő beágyazhatósági kérdésen keresztül vezetett az út. Amennyiben adott néhány kollinearitási feltétel ponthármasokon, akkor ezt az előírt struktúrát be tudjuk-e ágyazni egy q-adrendű véges projektív síkba? Az derült ki, hogy ezen kérdés megválaszolásához jobban meg kellene értenünk hipergráfok bizonyos összefüggőségi tulajdonságait. Ezek a megfigyelések vezettek az olyan Steiner hármasrendszerek vizsgálatához, amik nem tartalmaznak nemtriviális részrendszert, továbbá olyan Steiner hármasrendszer létezéséhez, amelyek majdnem 1-expanderek. A fejezetbeli eredmények Nagy Zoltán Lóránttal közösek.

Héger Tamással és Szőnyi Tamással közösen általánosítottuk a korábbi Bacsó Gáborral közös eredményüket véges testre épített projektív síkok felső kromatikus számáról. Mi annak a hipergráfnak a felső kromatikus számát vizsgáljuk, aminek csúcsai megfelelnek $\mathrm{PG}(n,q)$ pontjainak és a hiperélek pedig a k-dimenziós altereknek; vagyis szeretnénk meghatározni azt a legnagyobb színszámot, amivel megszínezhetők $\mathrm{PG}(n,q)$ pontjai úgy, hogy minden k-dimenziós altérben legyen legalább két darab azonos színű pont. Nyilvánvaló, hogy ha egy kétszeres $(n-k)$-lefogó ponthalmazt egyszínűre színezünk, akkor a többi pont lehet páronként különböző színű. Ebből kaphatunk egy triviális alsó korlátot, és bizonyítjuk, hogy számos esetben ez éles is. Sőt igazoljuk, hogy itt egy stabilitási jelenség is megjelenik, ugyanis ha csak egy kicsivel kevesebb színt használhatunk, akkor is biztosan lesz a színezésünkben egy egyszínű kétszeres $(n-k)$-lefogó ponthalmaz. A kétszeres lefogó ponthalmazokkal kapcsolatban sikerült belátnunk, hogy ha $t \leq \frac{3}{8}p+1$ és p prím, akkor egy kicsi t-szeres (súlyozott) $(n-k)$-lefogó ponthalmaz $\mathrm{PG}(n,p)$-ben mindenképp tartalmazza t darab nem feltétlenül diszjunkt $(n-k)$-dimenziós altér súlyozott összegét.

ADATLAP

a doktori értekezés nyilvánosságra hozatalához*

I. A doktori értekezés adatai

A szerző neve: Blázsik Zoltán..

MTMT-azonosító: 10055050..

A doktori értekezés címe és alcíme: Graphs and finite geometries....................................

DOI-azonosító46: 10.15476/ELTE.2019.110 ....

A doktori iskola neve: Matematika Doktori Iskola...

A doktori iskolán belüli doktori program neve: Elméleti matematika ....

A témavezető neve és tudományos fokozata: Szőnyi Tamás, D.Sc.......................................

A témavezető munkahelye: ELTE TTK Számítógéptudományi Tanszék ....

II. Nyilatkozatok

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b) kérem, hogy a mellékelt kérelemben részletezett szabadalmi, illetőleg oltalmi bejelentés közzétételéig a doktori értekezést ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;
c) kérem, hogy a nemzetbiztonsági okból minősített adatot tartalmazó doktori értekezést a minősítés (*dátum*)-ig tartó időtartama alatt ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;
d) kérem, hogy a mű kiadására vonatkozó mellékelt kiadó szerződésre tekintettel a doktori értekezést a könyv megjelenéséig ne bocsássák nyilvánosságra az Egyetemi Könyvtárban, és az ELTE Digitális Intézményi Tudástárban csak a könyv bibliográfiai adatait tegyék közzé. Ha a könyv a fokozatszerzést követőn egy évig nem jelenik meg, hozzájárulok, hogy a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban.
2. A doktori értekezés szerzőjeként kijelentem, hogy
a) az ELTE Digitális Intézményi Tudástárba feltöltendő doktori értekezés és a tézisek saját eredeti, önálló szellemi munkám és legjobb tudomásom szerint nem sértem vele senki szerzői jogait;
b) a doktori értekezés és a tézisek nyomtatott változatai és az elektronikus adathordozón benyújtott tartalmak (szöveg és ábrák) mindenben megegyeznek.
3. A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének plágiumkereső adatbázisba helyezéséhez és plágiumellenőrző vizsgálatok lefuttatásához.

Kelt: Budapest, 2019.05.20.

Blázsik Zoltán
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a doktori értekezés szerzőjének aláírása

**ELTE SZMSZ SZMR 12. sz. melléklet*