The Mathematical Analysis of Voting and Districting Rules

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Summary of the Ph.D. thesis

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1 Introduction

We analyze two major components of electoral systems, which can be fruitfully analyzed from an axiomatic viewpoint and are also computational problems. We focus on the mathematical analysis of voting rules and the districting problem.

The parts on voting and districting heavily employ the axiomatic method, which allows one to endow the vast space of conceivable rules with useful additional structure: (i) each combination of desirable properties characterizes a specific class of districting rules, and thereby helps one to assess their respective merits; (ii) one may hope that specific combinations of axioms single out a few, perhaps sometimes even a unique districting rule, thus reducing the space of possibilities; and (iii) the axiomatic approach may reveal incompatibility of certain axioms by showing that *no* districting rule can satisfy certain combinations of desirable properties, thereby terminating a futile search.

Since the axiomatic approach does not give us the ultimate answer we also consider the derivation of rules as solutions of optimization problems. In contrast to the known research direction we propose a new way by optimizing the distance to undesirable voting rules, namely, the dictatorial voting rules.

So far the axiomatic method could not be successfully applied to the districting problem. We present a new framework in which we can fruitfully analyze the districting problem. In addition, we determine the computational complexity of certain versions of the districting problem.

2 Voting

2.1 An axiomatic approach

Let $X = \{1, \dots, q\}$ be a universe of social alternatives. By $\mathcal{P}_X$, we denote the set of all linear orderings (strict preference relations) on X, and by $\mathcal{P} \subseteq \mathcal{P}_X$ a generic subdomain of the unrestricted domain $\mathcal{P}_X$. Moreover, denote by $\mathcal{R}$ the set of all weak orderings (preference relations).

Definition 1. A mapping $F : \bigcup_{n=1}^{\infty} \mathcal{P}^n \to \mathcal{R}$ that assigns a social preference ordering $F(\succ_1, ..., \succ_n) \in \mathcal{R}$ to each n-tuple of linear orderings and all n is called a *social choice rule (SCR).*

Definition 2. A SCR F satisfies the *Pareto rule* on $\mathcal{P}$ if, for all $x, y \in X$, all $\succ_i \in \mathcal{P}$ and all n,

$$[x \succ_i y \text{ for all } i = 1, ..., n] \;\Rightarrow\; x \succ y,$$

where $\succ$ is the strict part of the social preference relation $\succeq = F(\succ_1, ..., \succ_n)$.

Definition 3. A SCR F is called *non-dictatorial* on $\mathcal{P}$ if, either $\#\mathcal{P} = 1$ or, for all $n \geq 2$ and all $i = 1, ..., n$, there exist $x, y \in X$ and $\succ_i \in \mathcal{P}$ such that $x \succ_i y$ and $y \succeq x$, where $\succeq = F(\succ_1, ..., \succ_n)$.

Definition 4. A SCR F satisfies *independence of irrelevant alternatives (IIA)* on $\mathcal{P}$ if, for all $x, y \in X$, all n and all $\succ_i, \succ_i' \in \mathcal{P}$,

$$[\succ_i|_{\{x,y\}} = \succ_i'|_{\{x,y\}} \text{ for all } i = 1, ..., n] \;\Rightarrow \succeq|_{\{x,y\}} = \succeq'|_{\{x,y\}},$$

where $\succeq = F(\succ_1, ..., \succ_n)$, $\succeq' = F(\succ_1', ..., \succ_n')$, and $\succ|_{\{x,y\}}$ denotes the restriction of the binary relation $\succ$ to the pair $\{x, y\}$.

Definition 5. We will say that $\mathcal{P}$ is an *Arrovian domain* for the SCR F if F is non-dictatorial and satisfies the Pareto rule as well as IIA on $\mathcal{P}$.

Definition 6. Let $rk[x, \succ]$ denote the *rank* of alternative x in the ordering $\succ$. The SCR denoted by F^B is called the *Borda count* if for all $x, y \in X$, all n and all $\succ_i$, $i = 1, ..., n$,

$$x \succeq y \Leftrightarrow \sum_{i=1}^{n} rk[x, \succ_i] \leq \sum_{i=1}^{n} rk[y, \succ_i],$$

where $\succeq$ is the social preference corresponding to $(\succ_1, ..., \succ_n)$.

Definition 7. A domain $\mathcal{P}$ satisfies the *equal rank difference (ERD)* condition if, for all $x, y \in X$, either all orderings in $\mathcal{P}$ agree on $\{x, y\}$, or if not, then

$$rk[x, \succ] - rk[y, \succ] = rk[x, \succ'] - rk[y, \succ']$$

for all $\succ, \succ' \in \mathcal{P}$ such that $\succ|_{\{x,y\}} = \succ'|_{\{x,y\}}$.

Theorem 1 (Barbie, Puppe and Tasnádi [1]). *A domain is Arrovian for the Borda count if and only if it satisfies the equal rank difference (ERD) condition.*

We are often interested in domains that are "rich" in the sense that any alternative is on top of some preference ordering.

Definition 8. A domain $\mathcal{P}$ is called *rich* if for any $x \in X$ there exists $\succ \in \mathcal{P}$ such that $rk[x, \succ] = 1$.

The set of all cyclic permutations of a fixed ordering $\succ$ is denoted by $\mathcal{Z}(\succ)$.

Theorem 2 (Barbie, Puppe and Tasnádi [1]). *For any linear ordering $\succ$, there is exactly one rich Arrovian domain for the Borda count that contains $\succ$, namely the cyclic permutation domain $\mathcal{Z}(\succ)$.*

Definition 9. A mapping $f : \bigcup_{n=1}^{\infty} \mathcal{P}^n \to X$ that assigns a social alternative to each n-tuple of linear orderings and all n is called a *social choice function (SCF)*.

Definition 10. A SCF f is called *non-manipulable, or strategy-proof* on $\mathcal{P}$ if for all n, all $\succ_i, \succ'_i \in \mathcal{P}$ and all $\succ_{-i} \in \mathcal{P}^{n-1}$,

$$f(\succ_i, \succ_{-i}) \succeq_i f(\succ'_i, \succ_{-i}).$$

Theorem 3 (Barbie, Puppe and Tasnádi [1]). *On the rich domain $\mathcal{P}$ the Borda count is non-manipulable for all tie-breaking rules τ if and only if $\mathcal{P}$ satisfies ERD, i.e. $\mathcal{P}$ is a cyclic permutation domain.*

In a follow-up work (Puppe and Tasnádi [5]) we determined the Maskin-monotonic and Nash-implementable domains for the Borda count. In particular, the monotonic domains emerge in a recursive way from the cyclic permutation domains. The corresponding quite lengthy definitions and results can be found in the thesis.

2.2 An operations research method

Voting rules can be derived as the solution of an optimization problem on the set of social choice functions by minimizing the distance from some plausible criterion, such as unanimity or the Condorcet criterion. In contrast, we propose a new alternative, namely, the optimization of the distance to the undesirable dictatorial voting rules.

Let $\mathcal{F} = X^{\mathcal{P}_X^n}$ be the set of SCFs. A tie-breaking rule $\tau : \mathcal{P}_X^n \to \mathcal{P}_X$ maps preference profiles to linear orderings on X, which will be only employed to resolve ties when a formula does not determine a unique winner. The dictatorial rules will be denoted by $\mathcal{D} = \{d_1, \ldots, d_n\} \subset \mathcal{F}$, where d_i is the dictatorial rule with voter i as the dictator.

We employ in our analysis the distance function

$$\rho(f, g) = \#\{\succ \in \mathcal{P}_X^n \mid f(\succ) \neq g(\succ)\}, \tag{1}$$

where f, g are SCFs and $\rho(f, g)$ stands for the number of profiles on which f and g choose different alternatives.

A possible goal could be to get as close as possible to all dictators at the same time, which could be considered as a kind of neutral or balanced solution with respect to all dictators and, in this sense, as a kind of desirable solution.

Definition 11. We define the set of *balanced rules* for domain restriction $\mathcal{P}_X^n$ by

$$\mathcal{F}_b = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \sum_{i \in N} \rho(f, d_i) \leq \sum_{i \in N} \rho(f', d_i) \right\}.$$

The following SCF will be the balanced one.

Definition 12. The plurality rule $\tilde{f}_\tau$, where τ is an arbitrary tie-breaking rule, is defined in the following way: If there is a unique alternative, ranked first most often, then that alternative is the chosen one. If not, disregard those alternatives that are not ranked first most often, and select the chosen alternative based on the given tie-breaking rule.

Proposition 1 (Bednay, Moskalenko and Tasnádi [2])**.** $\tilde{f}_\tau \in \mathcal{F}_b$. *Furthermore, for any anonymous* $f \in \mathcal{F}_b$ *there exists a tie-breaking rule* τ *such that* $f = \tilde{f}_\tau$.

We specify the set of least dictatorial rules by those ones which are the furthest away from the closest dictatorial rule, which means that we are maximizing the minimum of the distances to the dictators.

Definition 13. We define the set of *least dictatorial rules* by

$$\mathcal{F}_{ld} = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \min_{i \in N} \rho(f, d_i) \geq \min_{i \in N} \rho(f', d_i) \right\}.$$

The least-dictatorial rule will be the following one.

Definition 14. The *reverse-plurality rule* f^*_τ, where τ is an arbitrary tie-breaking rule, is defined in the following way: If there is a single alternative, ranked first least often, then that alternative is the chosen one. If not, disregard those alternatives that are not ranked first least often, and select the chosen alternative based on the given tie-breaking rule.

Proposition 2 (Bednay, Moskalenko and Tasnádi [2])**.** $f^*_\tau \in \mathcal{F}_{ld}$. *Furthermore, for any anonymous* $f \in \mathcal{F}_{ld}$ *there exists a tie-breaking rule* τ *such that* $f = f^*_\tau$.

Though f^*_τ performs well according to our specification of a least dictatorial rule, as it can be easily verified, it can select a Pareto dominated alternative, never selects a unanimous winner, and violates monotonicity among many other desirable properties. Proposition 2 can be interpreted in a way that a reasonable rule must have a 'dictatorial ingredient'.

It is worthwhile mentioning that based on our approach of measuring the distance of a voting rule to the dictatorial rules in a follow-up paper (see Bednay, Moskalenko and Tasnádi [3]) we formulated a non-dictatorship index (NDI). By employing computer simulations, we estimated the NDIs of some well-known social choice functions.

3 Districting

The thesis develops an axiomatic approach to the districting problem and investigates its computational complexity.

3.1 Axiomatic districting

We assume that parties A and B compete in an electoral system consisting only of single member districts, where the representatives of each district are determined by plurality. The parties as well as the independent bodies face the following districting problem.

Definition 15 (Districting problem)**.** A *districting problem* is given by the structure $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$, where (i) the voters are located within a subset X of the plane $\mathbb{R}^2$, (ii) $\mathcal{A}$ is the σ-algebra on X consisting of all districts that can be formed without geographical or any other type of constraints, (iii) the distribution of voters is given by a measure μ on $(X, \mathcal{A})$, (iv) the distributions of party A and party B supporters are given by measures μ_A and μ_B on $(X, \mathcal{A})$ such that $\mu = \mu_A + \mu_B$, (v) t is the given number of seats in parliament, (vi) $G \subseteq \mathcal{A}$, also called *geography*, is a collection of admissible districts satisfying $\mu(g) = \mu(X)/t$ and $\mu_A(g) \neq \mu_B(g)$ for all $g \in G$, and admitting a partitioning of X, i.e there exist mutually disjoint sets $g'_1, \ldots, g'_t \in G$ such that $\cup_{i=1}^t g'_i = X$.

A *districting* for problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is a subset $D \subseteq G$ such that D forms a partition of X and $\#D = t$. We shall denote by $\delta_A(D)$ and $\delta_B(D)$ the number of districts won by party A and party B under D, respectively. We write $\mathcal{D}_\Pi$ for the set of all districtings of problem Π and let $\delta_A(\mathcal{D}) = \{\delta_A(D) : D \in \mathcal{D}\}$ and $\delta_B(\mathcal{D}) = \{\delta_B(D) : D \in \mathcal{D}\}$ for any $\mathcal{D} \subseteq \mathcal{D}_\Pi$. A *solution* F associates to each districting problem Π a non-empty set of chosen districtings $F_\Pi \subseteq \mathcal{D}_\Pi$.

In the summary of the thesis we restrict ourselves to the optimal partisan solution.

Definition 16. The optimal solution O^A for party A determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of those districtings that maximize the number of winning districts for party A, i.e.

$$O^A_\Pi = \arg\max_{D \in \mathcal{D}_\Pi} \delta_A(D).$$

The optimal solution O^B for party B can be defined in an analogous way.

Our first axiom requires that a solution must in fact be "determinate" in the two-district case in the sense that it must not leave open the issue whether there is a draw between the two parties or a victory for one party.

Axiom 1. A solution F satisfies *two-district determinacy* if for any districting problem Π with $t = 2$, the sets $\delta_A(F_\Pi)$ and $\delta_B(F_\Pi)$ are singletons.

Our next axiom requires that a solution behaves "uniformly" on the set of two-district problems in the sense that the solution must treat different two-district problems in the same way, provided they admit the same set of possible distributions of the number of districts won by each party.

Axiom 2. A solution F satisfies *two-district uniformity* if for any districting problems Π and Π' with $t = 2$ such that $\delta_A(\mathcal{D}_\Pi) = \delta_A(\mathcal{D}_{\Pi'})$ (and therefore also $\delta_B(\mathcal{D}_\Pi) = \delta_B(\mathcal{D}_{\Pi'})$) we have $\delta_A(F_\Pi) = \delta_A(F_{\Pi'})$ (and therefore also $\delta_B(F_\Pi) = \delta_B(F_{\Pi'})$).

Our third axiom requires that if a possible districting induces the same distribution of the number of winning districts for each party than some districting chosen by a solution, it must be chosen by this solution as well.

Axiom 3. A solution F satisfies *indifference* if for any districting problem Π we have that $D \in F_\Pi$, $D' \in \mathcal{D}_\Pi$, $\delta_A(D) = \delta_A(D')$ and $\delta_B(D) = \delta_B(D')$ implies $D' \in F_\Pi$.

The following consistency axiom, requiring that a solution to a problem should also deliver appropriate solutions to specific subproblems, plays a central role. Prior to the definition of consistency we have to introduce specific subproblems of a districting problem. For any problem Π, any $D \in F_\Pi$ and any $D' \subseteq D$, let $Y = \cup_{d \in D'} d$ and define the subproblem $\Pi|_Y$ to be $(Y, \mathcal{A}|_Y, \mu|_Y, \mu_A|_Y, \ \mu_B|_Y, \#D', G|_Y)$, where $\mathcal{A}|_Y = \{A \cap Y : A \in \mathcal{A}\}$, $G|_Y = \{g \in G : g \subseteq Y\}$ and $\mu|_Y, \mu_A|_Y, \mu_B|_Y$ stand for the restrictions of measures μ, μ_A, μ_B to $(Y, \mathcal{A}|_Y)$.

Axiom 4. A solution F satisfies *consistency* if for any districting problem Π, any $D \in F_\Pi$ and any $D' \subseteq D$ we have for $Y = \cup_{d \in D'} d$ that

$$D' \in F_{\Pi|_Y}.$$

Our final axiom expresses the symmetric treatment of parties *ex ante*.

Axiom 5. A solution F satisfies *anonymity* if exchanging the distributions of party A and party B voters μ_A and μ_B does not change the set of chosen districtings: for all districting problems $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$,

$$D \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,t,G)} \text{ if and only if } D \in F_{(X,\mathcal{A},\mu,\mu_B,\mu_A,t,G)}.$$

We restrict the family of admissible geographies.

Definition 17. The geography G of a problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is *linked* if for any two possible districtings $D, D' \in \mathcal{D}_\Pi$ there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $\{D_2, \ldots, D_{k-1}\} \subseteq \mathcal{D}_\Pi$, $D' = D_k$, and $\# D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$.

Now we can formulate our main results.

Theorem 4 (Puppe and Tasnádi [8])**.** *The optimal solution O is the only solution that satisfies two-district determinacy, two-district uniformity, indifference and consistency on linked geographies.*

Corollary 1. *There does not exist a two-district determinate, two-district uniform, indifferent, consistent and anonymous solution on linked geographies.*

3.2 The computational complexity of the political districting problem

If the number of districts to be formed is large (e.g. California), finding an optimal partisan or an unbiased districting (i.e. the number of seats won by a party is proportional to its share of votes in the entire population), is still infeasible. We established that two simplified versions of the optimal gerrymandering problem are NP-complete in Puppe and Tasnádi [7] and Fleiner, Nagy and Tasnádi [4]. The NP-completeness of finding an unbiased districting was shown in Puppe and Tasnádi [6].

References (to the candidate's papers)

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