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An Introduction to Bivariate Uniform Subdivision

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1 Introduction

The generation of curves and surfaces by recursive subdivision is a well known technique in Approximation Theory and CAGD (Computer Aided Geometric Design). Our purpose is not to provide a review of such techniques but rather to provide an introduction to the theory of uniform subdivision which has been developed in recent years. In particular, we will concentrate attention on some of the practical tools which can be used in the study of continuity and differentiability of the limits of bivariate, uniform, binary subdivision schemes.

The work presented here is but a small part of uniform subdivision theory. A much more extensive review of uniform subdivision is given in Dyn[3], where a full bibliography of the subject can be found, and the paper by Dahmen, Cavaretta and Micchelli[l] is a major contribution to the subject. The shorter review article of Caveratta and Micchelli[2] provides another introduction to this area.

Ideally, an introduction to uniform subdivision would begin with a study of the univariate case but, for brevity, we consider only the theory for the bivariate case. The multivariate case is then an immediate obvious generalization and the univariate case is but a simplification. The discussion is also restricted to the case of binary (diadic) subdivision schemes, although the generalization to p-adic schemes is immediately apparent.

Uniform subdivision schemes generate sets of ‘control points’ according to some fixed subdivision rule and, in the bivariate case, we are concerned with whether the points become dense on some continous, and possibly differentiable, limit surface. This concern will be resolved in a very simple way, namely, that for a continuously differentiable limit, divided differences will be converging to a continuous limit, and, for a continuous limit, differences will be converging to zero. After introducing some preliminary notation and theory in Section 2, the theory of differentiability is considered in Section 3 and the analysis for continuous limits is considered in Section 4. The theory is illustrated for the case of box splines in Section 3 and for the case of an interpolatory ‘butterfly’ subdivision scheme in Section 5.

2 Preliminaries

2.1 Binary subdivision scheme

A bivariate, uniform, *'binary subdivision scheme'* generates sets of *'control points'*

$$\mathrm{f}^{k} := \left\{ f_{\alpha}^{k} \in I\!R^{m} : \alpha \in \mathbb{Z}^{2} \right\},\ k = 0,1,2,..., \tag{2.1}$$

according to the rule

$$f_{\alpha}^{k+1} := \sum_{\beta\in\mathbb{Z}^{2}} a_{\alpha-2\beta} f_{\beta}^{k},\ \alpha \in \mathbb{Z}^{2}. \tag{2.2}$$

Here there are really four different rules which can be exhibited as

$$f_{2\alpha+\gamma}^{k+1} := \sum_{\beta\in\mathbb{Z}^{2}} a_{2(\alpha-\beta)+\gamma} f_{\beta}^{k} = \sum_{\beta\in\mathbb{Z}^{2}} a_{\gamma-2\beta} f_{\alpha+\beta,}^{k}\ \alpha \in \mathbb{Z}^{2}, \tag{2.3}$$

where

$$\gamma \in E := \{(0,0),(1,0),(0,1),(1,1)\}. \tag{2.4}$$

The set

$$\mathrm{a} := \left\{ a_{\alpha} \in I\!R : \alpha \in \mathbb{Z}^{2} \right\} \tag{2.5}$$

is called the 'mask' of the scheme, where the 'support'

$$\mathrm{supp(a)} := \left\{ \alpha \in \mathbb{Z}^{2} : \alpha_{\alpha} \neq 0 \right\} \tag{2.6}$$

is assumed finite.

As a simple example, consider the mask with

$$\mathrm{supp(a)} = \{-1,0,1\}^{2}. \tag{2.7}$$

Then the binary subdivision scheme is described, for $(i,j) \in \mathbb{Z}^{2}$, by

$$\left.\begin{aligned}
f_{2i,2j}^{k+1} &:= a_{0,0} f_{i,j}^{k}, \\
f_{2i+1,2j}^{k+1} &:= a_{1,0} f_{i,j}^{k} + a-1{,}0\, f_{i+1,j}^{k}, \\
f_{2i,2j+1}^{k+1} &:= a_{0,1} f_{i,j}^{k} + a_{0,-1} f_{i,j+1}^{k}, \\
f_{2i+1,2j+1}^{k+1} &:= a_{1,1} f_{i,j}^{k} + a_{-1,1} f_{i+1,j}^{k} + a_{1,-1} f_{i,j+1}^{k} + a_{-1,-1} f_{i+1,j+1}^{k}
\end{aligned}\right\} \tag{28}$$

The subdivision scheme (2.2) defines a bounded linear operator S_a on $\ell_{\infty}(\mathbb{Z}^{2})$, namely, for $\mathrm{f} \in \ell_{\infty}(\mathbb{Z}^{2}), S_{\mathrm{a}}\mathrm{f} \in \ell_{\infty}(\mathbb{Z}^{2})$ is defined by

$$(S_{\mathrm{a}}\mathrm{f})_{\alpha} := \sum_{\beta\in\mathbb{Z}^{2}} \alpha_{\alpha-2\beta} f_{\beta}\ . \tag{2.9}$$

The norm of this operator is

$$\|S_{\mathrm{a}}\| := \sup_{\|\mathrm{f}\|_{l_{\infty}}=1} \|S_{\mathrm{a}} f\| \ell_{\infty} = \max_{\alpha\in \mathrm{E}} \sum_{\beta\in\mathbb{Z}^{2}} \left|\alpha_{\alpha-2\beta}\right|. \tag{2.10}$$

The subdivision scheme can now be written as

$$\mathrm{f}^{k+1} := S_{\mathrm{a}}\mathrm{f}^{k} = S_{\mathrm{a}}^{k+1}\mathrm{f}^{0}, \tag{2.11}$$

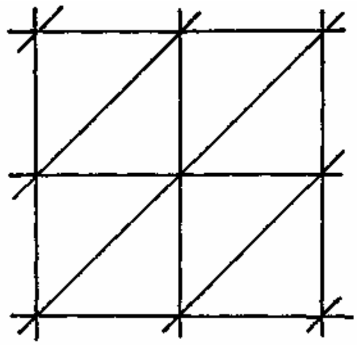

Figure 1: The (1,1) triangulation

where

$$\mathrm{f}^0 := \{ f_\alpha^0 \in \mathbb{R}^m : \alpha \in \mathbb{Z}^2 \ \}. \tag{2.12}$$

denotes the set of given initial control points. Finally, we associate with each subdivision operator S_a the bivariate Laurent polynomial

$$a(z) := \sum_{\alpha \in \mathbb{Z}^2} a_\alpha z^\alpha, \alpha := (i, j), \ z^\alpha := z_1^i z_2^j, \ z_1, z_2 \in C. \tag{2.13}$$

For example, the subdivision scheme (2.8) has the Laurent polynomial

$$a(z) = \sum_{i=-1}^{1} \ \sum_{j=-1}^{1} a_{i,j} z_1^i z_2^j \ . \tag{2.14}$$

This is called the *'generating polynomial'* for the subdivision scheme and provides an extremely useful tool for the analysis.

2.2 The control polygon

The control points $f_\alpha^k \in \mathbb{R}^m$, at level k, are associated with the diadic rectangular grid 'domain points'

$$2^{-k} \alpha = \left(2^{-k} i, 2^{-k} j \right) \in \mathbb{R}^2, \ \alpha = (i, j) \in \mathbb{Z}^2 \tag{2.15}$$

Hence the control points $f_{2\alpha+\gamma}^{k+1}$, $\gamma \in E$, at level k+1 are associated with the domain points

$$2^{-k} \alpha + 2^{-k-1} \gamma, \ \alpha \in \mathbb{Z}^2, \gamma \in \mathrm{E} \tag{2.16}$$

given on the finer diadic grid obtained by binary subdivision. We now consider a particular definition of a 'control polygon' whose vertices are the control points f_α^k, $\alpha \in \mathbb{Z}^2$.

Suppose the grid at level k is triangulated by subdivision along the (1,1) direction, giving triangles T_α^1 and $T_\alpha^2, \alpha \in \mathbb{Z}^2$, with vertices $2^{-k}\{\alpha, \alpha + (1,0), \alpha + (1,1)\}$ and $2^{-k}\{\alpha, \alpha + (1,1), \alpha + (0,1)\}$ respectively, see Figure 1. The piecewise linear interpolant on the (1,1) triangulation is now defined by

$$L_k\left[f^k\right](s,t):=\begin{cases}(1-\theta_i)f_{i,j}^k+(\theta_i-\theta_j)f_{i+1j}^k+\theta_j f_{i+1,j+1}^k, & (s,t)\in T_{I,J,}^1\\(1-\theta_J)f_{i,j}^k+\theta_i f_{i+1,j+1}^k+(\theta_j-\theta_i)f_{i,j+1}^k, & (s,t)\in T_{IJ}^2,\end{cases} \tag{2.17}$$

Where

$$\theta_i := 2^k s - i, \theta_j := 2^k t - j. \tag{2.18}$$

Thus

$$L_k\left[\mathrm{f}^k\right]\left(2^{-k}\alpha\right)=f_\alpha^{\ k}, \alpha\in \mathbb{Z}^2, \tag{2.19}$$

and we define $L_k[\mathrm{f}^k]$ as the control polygon of f^k with respect to the (1,1) triangulation.

The control polygon with respect to the (—1,1) triangulation can be similarly defined. More generally, consider a rectilinear partition of the diadic points $2^{-k}\alpha, \alpha\in \mathbb{Z}^2$, along skew directions, which is then triangulated along either of the diagonals. A control polygon can then be defined with respect to this skew triangulation. Finally, a control polygon can be defined with respect to any rectilinear partition as the piecewise bilinear interpolant on that partition. The choice of the definition of an appropriate control polygon is usually determined a priori in the construction of a particular subdivision scheme.

2.3 The fundamental solution and convergence

Let

$$\varphi^k := S_a^k \varphi^0 \tag{2.20}$$

denote the subdivision scheme applied to the 'cardinal set' of initial scalar data

$$\varphi^0 := \left\{\varphi_\alpha^0 := \delta_{a,(0,0)} : \alpha\in \mathbb{Z}^2\right\}. \tag{2.21}$$

Thus

$$\varphi_\alpha^0 := \begin{cases}\mathrm{I}, \alpha=(0,0),\\ 0, \alpha\in \mathbb{Z}^2\setminus(0,0).\end{cases} \tag{2.22}$$

We then have:

Definition 1 (Uniform convergence.) *The subdivision scheme is said to be uniformly convergent (with respect to the diadic point parameterization (2.15)) if there exists* $\varphi\in C\left(\mathbb{R}^2\right)$ *such that*

$$\lim_{k\to\infty}\sup_{\alpha\in\mathbb{Z}^2}\left|\varphi_\alpha^k-\varphi\left(2^{-k}\alpha\right)\right|=0. \tag{2.23}$$

Equivalently, in terms of the behaviour of the control polygon sequence,

$$\lim_{k\to\infty}\left\|L_k\left[\varphi^k\right]-\varphi\right\|_\infty=0. \tag{2.24}$$

If there exists such a continuous function φ, we call it the *'fundamental solution'* of the subdivision scheme and write

$$\varphi = S_{\mathrm{a}}^\infty \varphi^0. \tag{2.25}$$

This function has the important property of having 'local support', since it can be shown that

$$\mathrm{supp}(\varphi):=\{(s,t)\in \mathbb{R}^2 : \varphi(s,t)\neq 0\}\subset[\mathrm{supp}(\mathrm{a})], \tag{2.26}$$

where [supp(a)] denotes the convex hull in $\mathbb{R}^2$ of supp(a) $\subset$ $\mathbb{Z}^2$. In fact,

$$\operatorname{supp}\left(\mathrm{L}_{\mathrm{k}}\left[\varphi^k\right]\right) \subset \operatorname{supp}\left(\mathrm{L}_{\mathrm{k}+1}\left[\varphi^{k+1}\right]\right) \subset \left[\operatorname{supp}\left(\mathrm{a}\right)\right]. \tag{2.27}$$

The limit for bounded initial data f° can now be defined in terms of translates of the fundamental solution as

$$f(s,t) := \sum_{(i,j)\in\mathbb{Z}^2} f^0_{i,j}\varphi(s-i,t-j). \tag{2.28}$$

The fundamental solution can be characterized in the following way: Observe that

$$\varphi(s,t) = \left(S_{\mathrm{a}}^\infty \varphi^0\right)(s,t) = \left(S_{\mathrm{a}}^\infty \varphi^1\right)(2s,2t) = \left(S_{\mathrm{a}}^\infty \mathrm{a}\right)(2s,2t), \tag{2.29}$$

where it has been observed that, for cardinal initial data,

$$\varphi^1 = S_{\mathrm{a}}\varphi^0 = \mathrm{a}. \tag{2.30}$$

Thus, from (2.28),

$$\varphi(s,t) := \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}\varphi(2s-i,2t-j) \tag{2.31}$$

This is called the 'functional equation' of the subdivision scheme and plays an important role in the study of uniform subdivision, see [1] and [3], although we will not pursue its study here.

A simple consequence of the definition of the binary subdivision scheme as in (2.3) is:

Lemma 2 *A necessary condition for uniform convergence is that*

$$\sum_{\beta\in\mathbb{Z}^2} a_{\gamma\text{-}2\beta} = 1,\ \gamma\in E \tag{2.32}$$

This condition can be characterized in terms of the generating polynomial as

$$a(1,1) = 4,\ a(\text{-}1,\text{-}1) = a(1,\text{-}1) = a(\text{-}1,1) = 0. \tag{2.33}$$

and implies that the subdivision scheme is invariant under affine transformations of the initial data in $\mathbb{R}^m$.

2.4 Examples

We conclude this preliminary section by considering two simple examples of convergent binary subdivision schemes, the first of which will be used as a building block for the theory of Sections 3 and 4.

2..1 Piecewise linear scheme

Consider the scheme (2.8) with

$$a_{0,0} = 1,\ a_{-1,1} = a_{1,-1} = 0, \text{ and } a_{1,0} = a_{-1,0} = a_{0,1} = a_{0,-1} = a_{1,1} = a_{-1,-1} = \tfrac{1}{2}. \quad (2.34)$$

This scheme is symmetric with respect to the (1,1) triangulation and the limit of the scheme is the initial control polygon with respect to the (1,1) triangulation, since

$$L_{k+1}[f^{k+1}] = L_k[f^k] = L_0[f^\circ]. \quad (2.35)$$

The scheme will be required in the later analysis and hence we distinguish its generating polynomial as

$$l(z) := 1 + \tfrac{1}{2}(z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_1 z_2 + z_1^{-1} z_2^{-1}) \quad (2.36)$$

$$= \tfrac{1}{2}(1 + z_1^{-1})(1 + z_2^{-1})(1 + z_1 z_2). \quad (2.37)$$

The fundamental solution φ for this case is the well known Courant hat function, namely the piecewise linear interpolant on the (1,1) triangulation having the value 1 at (0,0) and zero at all the other vertices of $\mathbb{Z}^2$.

One can similarly define a piecewise linear binary subdivision scheme with respect to the (-1,1) triangulation which we leave as an exercise for the reader.

2..2 Piecewise bilinear scheme

Consider the scheme (2.8) with

$$a_{0,0} = 1,\ a_{1,0} = a_{-1,0} = a_{0,1} = a_{0,-1} = \tfrac{1}{2}, \text{ and } a_{1,1} = a_{-1,1} = a_{1,-1} = a_{-1,-1} = \tfrac{1}{4}. \quad (2.38)$$

This scheme is symmetric with respect to the rectangular diadic grid and it is easily seen that the limit of the scheme is the initial bilinear control polygon with respect to the rectilinear partition of $\mathbb{Z}^2$. Thus the scheme has a tensor product structure which is reflected in its generating polynomial factorization

$$a(z) := \left(\tfrac{1}{2} z_1^{-1} + 1 + \tfrac{1}{2} z_1\right)\left(\tfrac{1}{2} z_2^{-1} + 1 + \tfrac{1}{2} z_2\right) \quad (2.39)$$

Here, each factor represents the generating polynomial of a univariate (piecewise linear) binary subdivision scheme and further factorization gives

$$a(z) := \tfrac{1}{4}\left(1 + z_1^{-1}\right)\left(1 + z_1\right)\left(1 + z_2^{-1}\right)\left(1 + z_2\right) \quad (2.40)$$

(A significance of a factorization of the generating polynomial will become apparent in Subsection 3.2.)

3 Differentiable limits and Box splines

3.1 Differentiability

Given the set of control points f^k at level k, l et

$$\Delta_\gamma f^k := \{\Delta_\gamma f_\alpha^k := f_{\alpha+\gamma}^k - f_\alpha^k : \alpha \in \mathbb{Z}^2\} \tag{3.1}$$

define the set of 'differences' and

$$D_\gamma f^k := \{D_\gamma f_\alpha^k := 2^k \Delta_\gamma f_\alpha^k : \alpha \in \mathbb{Z}^2\} \tag{3.2}$$

define the set of 'divided differences' along the direction $\gamma = (m,n) \in \mathbb{Z}^2 \setminus (0,0)$. Also, let

$$\partial_\gamma := m\partial / \partial s + n\partial / \partial t \tag{3.3}$$

define the derivative operator along the direction 7 with respect to different) able functions of (s,t). We now consider the divided difference sequence $\{D_\gamma \varphi^k\}_{k=0}^\infty$ and have the following:

Theorem 3 (Differentiability.) Suppose there exists, $g \in C(\mathbb{R}^2)$, with $\text{supp}(g) \subset [\text{supp}(a)]$, such that

$$\lim_{k\to\infty} \sup_{\alpha \in \mathbb{Z}^2} \left| S^\gamma \varphi_\alpha^k - g(2^{-k}\alpha) \right| = 0. \tag{3.4}$$

Thus the divided differences of the binary subdivision scheme $S_\&$, with cardinal initial data, converge uniformly to a continuous, compactly supported function g (see Definition 1). Then the subdivision scheme S_a is uniformly convergent with fundamental solution

$$\varphi = I_\gamma[g], \tag{3.5}$$

where

$$I_\gamma[g](s,t) := \int_{-\infty}^0 g((s,t) + \theta\gamma) d\theta \tag{3.6}$$

defines the indefinite integral of g along the direction γ . Thus

$$\partial_\gamma \varphi = g \,. \tag{3.7}$$

Proof. Consider, in particular, $\gamma \in G := \{(1,0), (0,1), (1,1)\}$ and let $L_k[\varphi^k]$ denote the piecewise linear interpolant of φ^k with respect to the (1,1) triangulation, see (2.17). We will show that $\{L_k[\varphi^\kappa]$ converges uniformly to $I_\gamma[g]$. Since $\{L_k[\varphi^\kappa]$ is a continuous, piecewise linear function, it can be written as the indefinite integral of its piecewise constant derivative along the 7 direction, that is

$$L_k[\varphi^\kappa] = I_\gamma[\partial_\gamma L_k[\varphi^\kappa]] \,. \tag{3.8}$$

Also, observe that for all bounded, compactly supported functions f (with supp(f) $\subset$ [supp(a)])

$$\left\| I_\gamma[f] \right\|_\infty \leq C \left\| f \right\|_\infty , \tag{3.9}$$

where C is a constant dependent only on the support. We then have

$$
\begin{aligned}
\left\| L_k[\varphi^k] - I_\gamma[g] \right\|_\infty &= \left\| L_k[\partial_\gamma L_k[\varphi^k]] - I_\gamma[g] \right\|_\infty \\
&\le \left\| L_k[\partial_\gamma L_k[\varphi^k]] - I_\gamma[L_k[D_\gamma \varphi^k]] \right\|_\infty + \left\| I_\gamma[L_k[D_\gamma \varphi^k]] - I_\gamma[g] \right\|_\infty \\
&\le C\left\| \partial_\gamma L_k[\varphi^k] - L_k[D_\gamma \varphi^k] \right\|_\infty + C\left\| L_k[D_\gamma \varphi^k] - g \right\|_\infty \\
&\le C \max_{\mu \in G} \left\| \Delta_\mu D_\gamma \varphi^k \right\|_{l_\infty} + C\left\| L_k[D_\gamma \varphi^k] - g \right\|_\infty . \qquad (3.10)
\end{aligned}
$$

Here, the first term on the the right hand side of the last inequality follows from the definition of L_k. For example, with $\gamma = (1,0)$, (2.17) gives

$$
\partial_{1,0} L_k[\varphi^k](s,t) - L_k[D_{1,0}\varphi_k](s,t) = \begin{cases} -\theta_i \Delta_{1,0} D_{1,0} f_{i,j}^k - \theta_j \Delta_{1,0} f_{i,j}^k & (s,t) \in T_{i,j}^1, \\ (1-\theta_j)\Delta_{0,1} D_{1,0} f_{i,j}^k - \theta_j \Delta_{1,0} f_{i,j}^k & (s,t) \in T_{i,j}^2, \end{cases} \qquad (3.11)
$$

and, by symmetry on the (1,1) triangulation, similar relations hold for $\gamma = (0,1)$ and $\gamma = (1,1)$. Both terms on the right hand side of the last inequality of (3.10) tend to zero as $k \to \infty$, by the hypothesis (3.4). This completes the proof for the particular choice of $\gamma \in G$ and for general $\gamma \in \mathbb{Z}^2 \setminus (0,0)$ the above proof can be generalized by defining L_k with respect to a skew triangulation. □

3.2 Divided difference schemes and Box splines

The previous theorem indicates that differentiability of the limits of uniform subdivision schemes is related to the behaviour of their divided differences. We now consider a special case where the divided differences themselves satisfy binary subdivision schemes. An illustration of this case for box spline subdivision schemes is then given.

Proposition 4 (Difference and divided difference schemes.) *Suppose that there exist Laurent polynomials b(z) and a(z) :=2b(z) such that*

$$
a(z) = (1+z^{-\gamma})b(z) = \tfrac{1}{2}(1+z^{-\gamma})\,\hat{a}(z), \qquad (3.12)
$$

where $\gamma \in \mathbb{Z}^2 \setminus (0,0)$. Then

$$
\Delta_\gamma \mathrm{f}^{k+1} = S_b \Delta_\gamma \mathrm{f}^k \text{ and } D_\gamma \mathrm{f}^{k+1} = S_{\hat{a}} D_\gamma \mathrm{f}^k, \qquad (3.13)
$$

that is, the differences and divided differences satisfy binary subdivision schemes with generating polynomials b(z) and $\hat{a}$ (z) respectively.

Proof. From (3.12), $a_\alpha = b_\alpha + b_{\alpha+\gamma}$. Hence, from (2.2),

$$
\begin{aligned}
f_{\alpha+\gamma}^{k+1} - f_\alpha^{k+1} &= \sum_{\beta \in \mathbb{Z}^2} (a_{\alpha+\gamma-2\beta} - a_{\alpha-2\beta}) f_\beta^k, \\
&= \sum_{\beta \in \mathbb{Z}^2} (b_{\alpha+2\gamma-2\beta} - b_{\alpha-2\beta+\gamma}) f_\beta^k, \\
&= \sum_{\beta \in \mathbb{Z}^2} b_{\alpha-2\beta} (f_{\beta+\gamma}^k - f_\beta^k). \qquad (3.14)
\end{aligned}
$$

This is the subdivision scheme for the differences and multiplying both sides by 2^{-k-1} gives the divided difference scheme. □

Remark. In the case of a univariate uniform subdivision scheme, the existence of a difference and a divided difference scheme follows from the univariate form of Lemma 2. In this case a(-1) = 0, which implies that $(1 + z^{-1})$ is a factor of the univariate polynomial a(z). In the bivariate case, however, factorization of the generating polynomial does not necessarily follow from (2.33).

The function g of Theorem 3 can be considered as the limit of the divided difference scheme applied to the initial data $D_\gamma \varphi^0$. Thus

$$g(s,t) = -\hat{\varphi}(s,t) + -\hat{\varphi}((s,t)+\gamma), \tag{3.15}$$

cf. (2.28), where $\hat{\varphi}$ is the fundamental solution of the divided difference scheme. Thus application of (3.6) of Theorem 3 gives:

Corollary 5 *Suppose that there exists a uniformly convergent divided difference scheme, with generating polynomial $a(z)$ satisfying (3.12) and with fundamental solution $\hat{\varphi} \in C(\mathbb{R})$. Then the basic scheme S_a is uniformly convergent with fundamental solution*

$$\varphi(s,t) = \int_0^1 \hat{\varphi}((s,t)+\theta\gamma)d\theta. \tag{3.16}$$

More generally, we have:

Corollary 6 Suppose that

$$a(z) = 2^{-n}\prod_{i=1}^{n}(1+z^{-\gamma i})\hat{a}(z),\ \gamma_i \in \mathbb{Z}^2 \setminus (0,0), \tag{3.17}$$

where $\hat{a}(z)$ is the generating polynomial of a uniformly convergent subdivision scheme with fundamental solution $\hat{\varphi}$. Then the subdivision scheme S_a is uniformly convergent with fundamental solution

$$\varphi(s,t) = \int_0^1 \ldots \int_0^1 \hat{\varphi}((s,t)+\theta_1\gamma_1+\ldots+\theta_n\gamma_n)d\theta_1\ldots d\theta_n. \tag{3.18}$$

Box splines. A simple consequence of Corollary 6 is that it gives a binary subdivision development for the theory of box splines. For example, let $\hat{a}(z) = l(z)$ in (3.17), where $l(z)$ is the generating polynomial of the piecewise linear scheme on the (1,1) triangulation. Then $\hat{\varphi}$ is the piecewise linear Courant hat function on the triangulation with centre the origin. Equation (3.17) then gives the generating polynomial of a bivariate spline subdivision scheme with fundamental solution defined by (3.18). Each integral along a direction γ_i in (3.18) corresponds to an increase by one of the polynomial degree and continuity of the fundamental spline along that direction. The survey paper [2] gives more details of such subdivision schemes. It can also be observed that the factorizations (2.37) and (2.40) reflect the simple fact that piecewise linear and bilinear schemes can be considered as the 'integrals' of piecewise constant schemes, although we have chosen here to define convergence of subdivision schemes with respect to their having continuous limits.

4 A C^0 convergence analysis

We now consider how to determine if a subdivision scheme $S_{\hat{a}}$ is uniformly convergent, in the case where the fundamental solution limit is not known explicitly. Here, $\hat{a}(z)$ may be the generating polynomial of a basic scheme, or may correspond to the special case of schemes having divided difference polynomial factors as in (3.12) or (3.17). In a later subsection, we briefly consider the need to generalize the theory to *'matricial schemes'*, for the case where such special factorizations are not available.

4.1 A preliminary result

Proposition 7 *Let* S_c *be a binary subdivision operator, with finite mask c, such that*

$$\sum_{\beta \in \mathbb{Z}^2} c_{\gamma\text{-}2\beta} = 0 \text{ for } \gamma \in E, \tag{4.1}$$

cf. (2.32). *Then given any two directions* $\lambda, \mu \in \mathbb{Z}^2 \setminus (0,0), \lambda \neq \mu,$ *which generate a rectilinear partition of* $\mathbb{Z}^2$*, there exist (non-unique) finite masks* b^λ *and* b^μ *such that*

$$S_c = S_{\mathrm{b}\lambda}\Delta_\lambda + S_{\mathrm{b}\mu}\Delta_\mu. \tag{4.2}$$

Proof. The subdivision operator S_c is defined for $\mathrm{f} \in l_\infty(\mathbb{Z}^2)$ by

$$(S_c\mathrm{f})_{2\alpha+\gamma} = \sum_{\beta \in \mathbb{Z}^2} c_{\gamma\text{-}2\beta} f_{\alpha+\beta}, \ \alpha \in \mathbb{Z}^2, \ \gamma \in E. \tag{4.3}$$

The proof of the lemma is then based on the observation that, for each $\gamma \in E$, there exists a (non-unique) finite path through the mask c, covering all the non-zero coefficients, each step of which is taken along either the λ or μ direction. The fact that the sum of coefficients is zero then means that the linear combination can be written as a sum of differences along the path, that is

$$(S_c\mathrm{f})_{2\alpha+\gamma} = \sum_{\beta \in \mathbb{Z}^2} b^\lambda_{\gamma\text{-}2\beta}\Delta_\lambda f_{\alpha+\beta} + \sum_{\beta \in \mathbb{Z}^2} b^\mu_{\gamma\text{-}2\beta}\Delta_\mu f_{\alpha+\beta}, \ \alpha \in \mathbb{Z}^2, \ \gamma \in E. \tag{4.4}$$

for some finite masks b^λ and b^μ. □

Remark. The proof of Proposition 7 can be argued in terms of the generating polynomial c(z) as follows: The hypothesis (4.1) is equivalent to the condition

$$c\,(1, 1) = c\,(\text{-}1, \text{-}1) = c\,(1, \text{-}1) = c(\text{-}1, 1) = 0. \tag{4.5}$$

It can then be shown that this condition gives the generating polynomial decomposition

$$c(z) = (-1 + z^{-2\gamma})b^\lambda(z) + (-1 + z^{-2\mu})b^\mu(z), \tag{4.6}$$

for some non-unique Laurent polynomials $b^\lambda(z)$ and $b^\mu(z)$. The result (4.2) now follows by applying the following lemma to each term of (4.6):

Lemma 8 *Suppose that*

$$c(z) = (-1 + z^{-2\gamma})b(z), \ \gamma \in \mathbb{Z}^2 \setminus (0,0), \tag{4.7}$$

for some Laurent polynomials c(z) and b(z). Then

$$S_c = S_{\mathrm{b}}\Delta_\gamma. \tag{4.8}$$

4.2 Uniform convergence

We wish to find conditions for which the scheme S& is uniformly convergent. Consider the control polygon sequence $\{L_k[\hat{\varphi}^k]\}_{k=0}^{\infty}$ where, for example, L_k is the piecewise linear interpolation operator defined by (2.17), and $\hat{\varphi}^k$ denotes the values at level k produced by the subdivision scheme applied to cardinal initial data. Then we seek conditions for which $\{L_k[\hat{\varphi}^k]\}_{k=0}^{\infty}$ is a Cauchy sequence. Proposition 7 leads to:

Lemma 9 *Suppose that* $\hat{a}(z)$ *satisfies the necessary convergence condition*

$$\hat{a}(1,1) = 4, \; \hat{a}(-1,-1) = \hat{a}(1,-1) = \hat{a}(-1,1) = 0, \tag{4.9}$$

see Lemma 2. Then

$$\left\| L_{k+1}[\hat{\varphi}^{k+1}] - L_k[\hat{\varphi}^k] \right\|_\infty \le C \max\{\left\| \Delta_\lambda \hat{\varphi}^k \right\|_{l_\infty}, \left\| \Delta_\mu \hat{\varphi}^k \right\|_{l_\infty}, \tag{4.10}$$

for $\lambda, \mu \in$ *{(1,0), (0,1), (1,1)},* $\lambda \neq \mu$*. More generally, defining* L_k *with respect to a skew triangulation, then (4.10) holds for the* λ, μ *directions defining any rectilinear partition of the points* $\mathbb{Z}^2$*.*

Proof. Observe that, for any $\mathrm{f} \in l_\infty(\mathbb{Z}^2)$,

$$\left\| L_{k+1}[\mathrm{f}] \right\|_\infty = \left\| \mathrm{f} \right\|_{l_\infty}, \tag{4.11}$$

since any piecewise linear interpolant achieves its extreme values at the vertices. We thus have that

$$\begin{aligned} \left\| L_{k+1}[\hat{\varphi}^{k+1}] - L_k[\hat{\varphi}^k] \right\|_\infty &= \left\| L_{k+1}[(S_{\hat{a}} - S_l)\hat{\varphi}^k] \right\|_\infty, \\ &= \left\| (S_{\hat{a}} - S_l)\hat{\varphi}^k \right\|_{l_\infty}, \\ &= \left\| S_{\mathrm{c}} \hat{\varphi}^k \right\|_{l_\infty}, \end{aligned} \tag{4.12}$$

where

$$c(z) := \hat{a}(z) - l(z) \tag{4.13}$$

is a generating polynomial satisfying conditions (4.5). Thus Proposition 7 can be applied and (4.10) follows by expressing S_C in the form (4.2). □

We now make the simplifying assumption that there exist difference schemes for $\Delta_\lambda \hat{\varphi}^k$ and $\Delta_\mu \hat{\varphi}^k$. Thus

$$\hat{a}(z) = (1 + z^{-\lambda})\hat{b}^\lambda(z) \text{ and } \hat{a}(z) = (1 + z^{-\mu})\hat{b}^\mu(z) \tag{4.14}$$

where $\hat{b}^\lambda(z)$ and $\hat{b}^\mu(z)$ are the generating polynomials for the difference schemes (see Proposition 4). Lemma 9 now leads to the following convergence result:

Theorem 10 (Convergence.) *Let* $S_{\hat{a}}$ *define a binary subdivision scheme having difference schemes* $S_{\hat{b}^\lambda}$ *and* $S_{\hat{b}^\mu}$ *where the directions* λ *and* μ *define a rectilinear partition of* $\mathbb{Z}^2$*. Furthermore, suppose that there exists a positive integer L such that the the Lth iterated difference operators have the 'contractive property' that*

$$\left\| S_{\hat{b}^\lambda}^{\mathrm{L}} \right\| < 1 \text{ and } \left\| S_{\hat{b}^\mu}^{\mathrm{L}} \right\| < 1. \tag{4.15}$$

Then $S_{\hat{a}}$ is uniformly convergent.

The proof of Theorem 10 follows from the fact that the differences along the λ and μ directions will be contracting over L steps. This condition, together with Lemma 9, can then be used to show that $\{L_k[\hat{\phi}^k]\}_{k=0}^{\infty}$ Cauchy sequence and hence that the scheme is uniformly convergent.

To apply Theorem 10 we require the Lth iterated operators of the difference schemes, together with their norms. These are given by the following proposition:

Proposition 11 *Let* $\hat{b}(z)$ *be the generating polynomial of a bivariate binary subdivision scheme* $S_{\hat{b}}$ *. Then* $S_{\hat{b}}^{L}$ *is defined by*

$$(S_{\hat{b}}^{L}\,\mathrm{f})_{\alpha} := \sum_{\beta\in\mathbb{Z}^2} \hat{b}_{\alpha-2^L\beta}^{[L]} f_{\beta},\ \ \mathrm{f}\in l_{\infty}(\mathbb{Z}^2), \tag{4.16}$$

with generating polynomial

$$\hat{b}^{[L]}(z) := \hat{b}(z)\hat{b}(z^2)\ldots\hat{b}(z^{2^{L-1}}) \tag{4.17}$$

and norm

$$\left\|S_{\hat{b}}^{L}\right\| := \max_{a\in\{0,\ldots,2^L-1\}^2}\left\{\sum_{\beta\in\mathbb{Z}^2}\left|\hat{b}_{\alpha-2^L\beta}^{[L]}\right|\right\}. \tag{4.18}$$

Proof. (Levin[6]) Define the *z-transform*

$$G_k(z) := \sum_{\alpha\in\mathbb{Z}^2}(S_{\hat{b}}^{k}\,\mathrm{f})_{\alpha} z^{\alpha}. \tag{4.19}$$

Then it is easily shown that

$$G_{k+1}(z) = \hat{b}(z)G_k(z^2) \tag{4.20}$$

and hence that

$$G_L(z) = \hat{b}(z)\ \hat{b}(z^2)\ \ldots\hat{b}(z^{2^{L-1}})\ G_0(z^{2^L}) = \hat{b}^{[L]}(z)G_0(z^{2^L}). \tag{4.21}$$

Equating coefficients then gives the Lth iterated subdivision operator defined by (4.16). The norm of this operator then immediately follows (cf. (2.10)). □

4.3 Matricial schemes

To prove convergence to differentiate limits using the theory of the previous subsection, we must assume that a(z) has divided difference generating polynomial factors $\hat{a}(z)$. Also, in Theorem 10, the simplifying assumption has been made that $\hat{a}(z)$ can be factored appropriately to give difference schemes along directions λ and μ. A generalization of the theory to cover 'matricial schemes' avoids these simplifying assumptions. We thus conclude by briefly showing how matricial schemes arise in the study of differentiable limits by considering a generalization of Proposition 4:

Proposition 12 *Let S_a be a binary subdivision operator with generating polynomial $a(z)$ satisfying the necessary uniform convergence condition (2.33). Also, let λ and μ be two directions defining a rectilinear partition of $\mathbb{Z}^2$. Then given $\gamma \in \mathbb{Z}^2 \setminus (0,0)$, there exist (non-unique) Laurent polynomials $b^{\gamma,\lambda}(z), b^{\gamma,\mu}(z)$ and $\hat{a}^{\gamma,\lambda}(z) := 2b^{\gamma,\lambda}(z), \hat{a}^{\gamma,\mu}(z) := 2b^{\gamma,\mu}(z)$ such that*

$$\Delta_\gamma S_{\mathrm{a}} = S_{\mathrm{b}^{\gamma,\lambda}} \Delta_\lambda + S_{\mathrm{b}^{\gamma,\lambda}} \Delta_\mu, \tag{4.22}$$

$$D_\gamma S_{\mathrm{a}} = S_{\hat{\mathrm{a}}^{\gamma,\lambda}} D_\lambda + S_{\hat{\mathrm{a}}^{\gamma,\lambda}} D_\mu. \tag{4.23}$$

Proof. The operator

$$S_{\mathrm{C}^\gamma} := \Delta_\gamma S_{\mathrm{a}} \tag{4.24}$$

has generating polynomial coefficients $c_a^\gamma := a_{\alpha+\gamma} - a_\alpha$. Thus

$$c^\gamma(z) = (z^{-\gamma} - 1)a(z) \tag{4.25}$$

and hence c(l, 1) = 0. Thus, using (2.33), it follows that c(z) satisfies condition (4.5) and hence

$$c^\gamma(z) = (-1 + z^{-2\lambda})b^{\gamma,\lambda}(z) + (-1 + z^{-2\mu})b^{\gamma,\mu}(z) \tag{4.26}$$

for some non-unique Laurent polynomials $b^{\gamma,\lambda}(z)$ and $b^{\gamma,\mu}(z)$. Proposition 7 then gives (4.22) and multiplying by 2^{-k-1} gives (4.23). □

In the special case where

$$a(z) = (1 + z^{-\gamma})b(z) \tag{4.27}$$

we obtain

$$c^\gamma(z) = (-1 + z^{-2\gamma})b(z) \tag{4.28}$$

in the above proof. Hence Proposition 12 gives

$$\Delta_\gamma S_{\mathrm{a}} = S_{\mathrm{b}} \Delta_\gamma \quad \text{and} \quad D_\gamma S_{\mathrm{a}} = S_{\mathrm{a}} D_\gamma \tag{4.29}$$

where $\hat{a}(z) = 2b(z)$. This is the case of Proposition 4, when there exist difference and hence divided difference schemes. When such divided difference schemes are not available, we can take $\gamma = \lambda$ and $\gamma = \mu$ in Proposition 12 to give the matricial divided difference scheme

$$\begin{bmatrix} D_\lambda \mathrm{f}^{\mathrm{k}+1} \\ D_\mu \mathrm{f}^{\mathrm{k}+1} \end{bmatrix} = \begin{bmatrix} S_{\hat{\mathrm{a}}^{\lambda,\lambda}} & S_{\hat{\mathrm{a}}^{\lambda,\mu}} \\ S_{\hat{\mathrm{a}}^{\mu,\lambda}} & S_{\hat{\mathrm{a}}^{\mu,\mu}} \end{bmatrix} \begin{bmatrix} D_\lambda \mathrm{f}^{\mathrm{k}} \\ D_\mu \mathrm{f}^{\mathrm{k}} \end{bmatrix}. \tag{4.30}$$

This suggests the analysis of matricial schemes per se.

5 Example of the Butterfly subdivision scheme

We conclude this introduction to uniform subdivision by applying the theory to the interpolatory 'butterfly' subdivision scheme described in [4]. This scheme has been analysed by Dyn, Levin and Micchelli [5], who show that there exists an interval for a shape parameter ω for which the scheme converges to a C^1 limit. Here, we give more precise details of the calculation of the norm of the 2nd iterate of the appropriate subdivision operator. This calculation is equivalent to that of Qu [7], who uses a matrix norm approach.

The butterfly scheme is defined with respect to the (1,1) triangulation by

$$\left.\begin{aligned}
f^{k+1}_{2i,2j} &:= f^{k}_{i,j},\\
f^{k+1}_{2i+1,2j} &:= \frac{1}{2}\left(f^k_{i,j}+f^k_{i+1,j}\right)+2\omega\left(f^k_{i,j-1}+f^k_{i+1,j+1}\right)\\
&\qquad -\omega\left(f^k_{i-1,j-1}+f^k_{i+1,j-1}+f^k_{i,j-1}+f^k_{i-2,j-1}\right),\\
f^{k+1}_{2i,2j+1} &:= \frac{1}{2}\left(f^k_{i,j}+f^k_{i,j-1}\right)+2\omega\left(f^k_{i-1,j}+f^k_{i+1,j+1}\right)\\
&\qquad -\omega\left(f^k_{i-1,j-1}+f^k_{i-1,j+1}+f^k_{i+1,j}+f^k_{i+1,j+2}\right),\\
f^{k+1}_{2i+1,2j+1} &:= \frac{1}{2}\left(f^k_{i,j}+f^k_{i+1,j+1}\right)+2\omega\left(f^k_{i+1,j}+f^k_{i,j+1}\right)\\
&\qquad -\omega\left(f^k_{i,j-1}+f^k_{i-1,j}+f^k_{i+2,j+1}+f_{i+1,j+2}\right).
\end{aligned}\right\}\tag{5.1}$$

This scheme is symmetric with respect to the (1,1) triangulation and is interpolatory by definition of the first rule in (5.1). The description of the scheme is derived from the butterfly appearance of the individual masks for the second, third and fourth rules. The parameter ω can be used to control the shape of the limit surface. The case ω = 0 gives the piecewise linear scheme of subsection 2.4.1 and the case $\omega = \frac{1}{16}$ gives a scheme which reproduces cubic polynomials. Here we will indicate that $0 < \omega < \frac{1}{12}$ is a sufficient condition for the scheme to have a C^1 limit.

The generating poynomial a(z), for the subdivision scheme defined by (5.1), has factor $\frac{1}{2}(1+z_1^{-1})$, $\frac{1}{2}(1+z_2^{-1})$, and $\frac{1}{2}(1+z_1^{-1}z_2^{-1})$. Thus there exist uniform subdivision schemes for the divided difference sets $D^{\gamma}\mathrm{f}^{k}$, for 7 = (1,0), (0,1), (1,1). We now seek conditions for which these divided difference schemes have C^0 limits where by symmetry, it is sufficient to consider only γ = (1,0). It will then follow, by Theorem 3, that the butterfly scheme converges to a C^1 limit.

Writing

$$a(z) = \tfrac{1}{2}(1+z_1^{-1})\hat{a}\ (z),\tag{5.2}$$

then, with λ = (-1,0) and μ = (-1,-1),

$$\hat{a}\,(z)\ = (1+z_2^{-1})\,\hat{b}^{\lambda}(z) \text{ and } \hat{a}\,(z)\ = (1+z_1^{-1}z_2^{-1})\,\hat{b}^{\mu}(z),\tag{5.3}$$

where

$$\begin{aligned}\hat{b}^{\lambda}(z) := {}&(1-8\omega)(1+z_1z_2)+4\omega(z_1^{-1}z_2^{-1}+z_1^{2}z_2^{2}+2\omega(z_2^{-1}+z_1^{-1}+z_1^{2}z_2+z_1z_2^{2})\\ &-2\omega(z_1^{-1}z_2^{-2}+z_1^{-2}z_2^{-1}+z_1z_2^{-1}+z_1^{-1}z_2+z_1^{2}+z_2^{2}+z_1^{3}z_2^{2}+z_1^{2}z_2^{3})\end{aligned}\tag{5.4}$$

with a dual expression for $\hat{b}^{\mu}(z)$. The subdivision operator $S_{\hat{b}^{\lambda}}$ has norm $\|S_{\hat{b}^{\lambda}}\| \geq 1$. However, calculation of the generating polynomial $\hat{b}^{\lambda}(z)\,\hat{b}^{\lambda}(z^2)$ for the iterated operator $S^2_{\hat{b}^{\lambda}}$ leads to

$$\|\, S^2_{\hat{b}^{\lambda}}\, \| = \max\{A_1(\omega), A_2(\omega), A_3(\omega), A_4(\omega), A_5(\omega), A_6(\omega)\},\tag{5.5}$$

where

$$\left.\begin{aligned}
A_1(\omega) &:= 16\omega^2+4|4\omega^2+\omega|+8|9\omega^2-\omega|+|72\omega^2-16\omega+1|,\\
A_2(\omega) &:= 44\omega^2+6|6\omega^2-\omega|+2|24\omega^2-\omega|,\\
A_3(\omega) &:= 52\omega^2+4|8\omega^2-\omega|+2|16\omega^2-\omega|+|6\omega^2-\omega|,\\
A_4(\omega) &:= 104\omega^2+4|2\omega^2-\omega|,\\
A_5(\omega) &:= 48\omega^2+8|6\omega^2-\omega|+|12\omega-1|,\\
A_6(\omega) &:= 40\omega^2+4|6\omega^2+\omega|.
\end{aligned}\right\}\tag{5.6}$$

Here, the expected sixteen terms in (5.5), see Proposition 11, reduce to six because of symmetries and repetitions. Also, by symmetry, the expression for $\| S^2_{\hat{b}^\mu} \|$ is identical to (5.5). A careful analysis of the terms (5.6) now leads to

$$\| S^2_{\hat{b}^\lambda} \| = \| S^2_{\hat{b}^\mu} \| < 1 \text{ for } 0 < \omega < \tfrac{1}{12}. \tag{5.7}$$

In particular, $\omega < \frac{1}{12}$ is obtained from the condition $A_4(\omega) < 1$, where $A_4(\omega)$ is the dominant term in (5.5) in the neighbourhood of $\omega = \frac{1}{12}$. It now follows, from Theorem 10, that the subdivision scheme for the divided difference converges to a C° limit, and hence the butterfly scheme converges to a C^1 limit, for $0 < \omega < \frac{1}{12}$.

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