TR/12/94   July 1994

APPROXIMATE SOLUTION OF SECOND KIND INTEGRAL EQUATIONS ON INFINITE CYLINDRICAL SURFACES

A.T. Peplow and S.N. Chandler-Wilde

APPROXIMATE SOLUTION OF SECOND KIND INTEGRAL EQUATIONS ON INFINITE CYLINDRICAL SURFACES

ANDREW T. PEPLOW* AND SIMON N. CHANDLER-WILDE[†]

Abstract. The paper considers second kind integral equations of the form $\phi(x) = g(x) + \int_S k(x, y)\phi(y)ds(y)$ (abbreviated $\phi = g + K\phi$), in which S is an infinite cylindrical surface of arbitrary smooth cross-section. The "truncated equation" (abbreviated $\phi_a = E_ag + K_a\phi_a$), obtained by replacing S by S_a, a closed bounded surface of class C^2, the boundary of a section of the interior of S of length $2a$, is also discussed. Conditions on k are obtained (in particular, implying that K commutes with the operation of translation in the direction of the cylinder axis) which ensure that $I-K$ is invertible, that $I - K_a$ is invertible and $(I — K_a)^{-1}$ uniformly bounded for all sufficiently large a, and that ϕ_a converges to ϕ in an appropriate sense as $a \to \infty$. Uniform stability and convergence results for a piecewise constant boundary element collocation method for the truncated equations are also obtained.

A boundary integral equation, which models three-dimensional acoustic scattering from an infinite rigid cylinder, illustrates the application of the above results to prove existence of solution (of the integral equation and the corresponding boundary value problem) and convergence of a particular collocation method.

Key words. second kind integral equations, Wiener-Hopf equations, boundary element method, Helmholtz equation, collocation method

AMS subject classifications. 65R20, 45E10, 65N38, 35J05

1. Introduction. We are concerned in this paper with second kind integral equations of the form

(1) $$\phi(x) = g(x) + \int_S k(x, y)\phi(y)\, ds(y), \qquad\qquad x \in S,$$

and their numerical solution, in the case when S is an infinite cylindrical surface with arbitrary cross-section in three-dimensional space. In equation (1) $g \in BC(S)$ (the space of bounded continuous functions on S) is assumed known and $\phi \in BC(S)$ is to be determined. We abbreviate (1) in operator form as

(2) $$\phi = g + K\phi$$

and make sufficient assumptions on the smoothness of the surface S and on the behaviour of the kernel k (k is continuous or weakly singular) so that $K : BC(S) \to BC(S)$ is bounded but not compact. In particular, we suppose that

$$S = \{(x_1, x_2, x_3) : (x_1, x_2) \in \Gamma, x_3 \in \mathbf{R}\}$$

where $\Gamma \subset \mathbf{R}^2$ is a Jordan curve of class C^2.

Integral equations of the form (1) frequently arise when reformulating linear elliptic boundary value problems in the interior or exterior of S as boundary integral equations. We consider an example of this type in Section 4 of the paper, in which acoustic scattering by an infinite rigid cylinder is investigated, with

(3) $$k(x, y) = 2\frac{\partial}{\partial n(y)} G(x, y),$$

_____

 * Structural Dynamics Group, Institute of Sound and Vibration Research, University of Southampton, Southampton S09 5NH, UK (A.T.Peplow@isvr.soton.ac.uk).
  †Department of Mathematics and Statistics, Brunel University, Uxbridge UB8 3PH, UK (Simon.Chandler-Wilde@brunel.ac.uk)

where $n(y)$ is the normal to S at y, directed into the exterior of S, and

$$(4) \qquad G(x,y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$$

is a fundamental solution of the Helmholtz equation, $\Delta u + \kappa^2 u = 0$.

When solving (1) numerically it is convenient, as a preliminary stage, to truncate the infinite surface S. Let $\widetilde{S}_+$ and $\widetilde{S}_-$ denote the half cylinders denned by

$$\widetilde{S}_\pm := \{(x_1, x_2, x_3) \in S : x_3 \gtrless 0\}$$

and, for $a \geq 2$, let

$$\widetilde{S}_a := \{(x_1, x_2, x_3) \in S : |x_3| \leq a\}.$$

Let E_- be the surface

$$(5) \qquad E_- := \{(x_1, x_2, x_3) : (x_1, x_2) \in \overline{\Omega}, x_3 = f(x_1, x_2)\}$$

where Ω is the interior of Γ, and f is any given continuous function on $\overline{\Omega}$ satisfying

$$f(x_1, x_2) \quad \begin{array}{ll} > 0, & (x_1, x_2) \in \Omega, \\ = 0, & (x_1, x_2) \in \Gamma, \end{array}$$

and such that $S_- := \widetilde{S}_- \cup E_-$ is a smooth surface of class C^2 (see Figure 1). Let $E_+ := \{(x_1, x_2, x_3) : (x_1, x_2) \in \overline{\Omega}, x_3 = f(x_1, x_2)\}$ and $S_+ := \widetilde{S}_+ \cup E_+$.
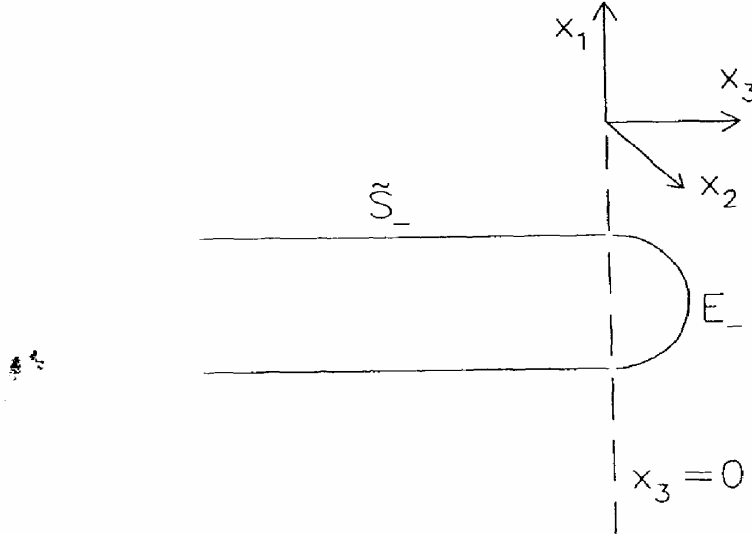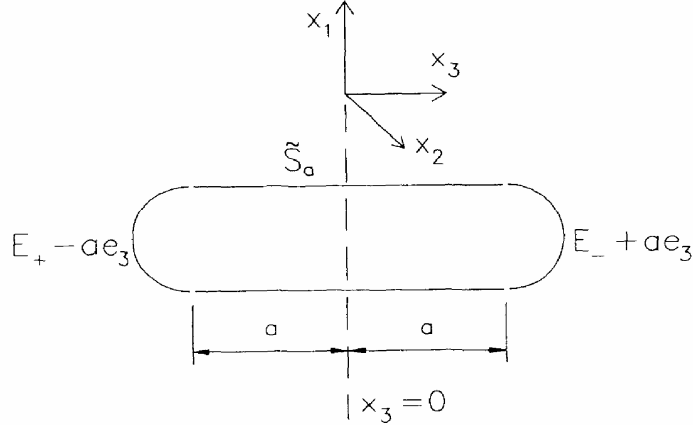

FIG. 1. *Cross-section through the surface S_-.*

For $V \subset \mathbb{R}^3$ and $x \in \mathbb{R}^3$, let $V + x$ denote the translation of the set V by the vector x, and let $e_3 \in \mathbb{R}^3$ be the unit vector in the x_3 direction. Then, for $a \geq 2$, define

$$S_a := \widetilde{S}_a \cup (E_- + ae_3) \cup (E_+ - ae_3) = \widetilde{S}_a \cup \{(x_1, x_2 x_3) : (x_1, x_2) \in \overline{\Omega}, x_3 = \pm(a + f(x_1, x_2))\}$$ (see Figure 2). Note that S_a is a smooth closed surface of class C^2.

Let $\Sigma := \{S, S_+, S_-\} \cup \{S_a + be_3 : a \geq 2, b \in \mathbb{R}\}$ and, for $T \subset S^* \in \Sigma$, define the integral operator K_T on $BC(T)$ by

$$(6) \qquad K_T \psi(x) = \int_T k(x,y)\psi(y)\,ds(y) \quad x \in T.$$

FIG. 2. *Cross-section through the surface S_a.*

Let $K_\pm : K_{S_\pm}$ and $K_a := K_{S_a}$, for $a \geq 2$.

A crucial requirement for the theory developed is a translation invariant assumption on the kernel function k, that, for all $(x, y) \in I := \{(x, y) \in T : T \in \Sigma\}$,

$$k(x + t e_3, y + t e_3) = k(x, y), \quad t \in \mathbf{R}.$$

We analyse, in Section 2 of this paper, the approximation to (1) obtained by replacing the infinite surface S by the finite closed surface S_a and the convergence to ϕ, of the solution $\phi_a \in BC(S_a)$ of the approximate equation, as $a \to \infty$. To make precise the definition of ϕ_a and the sense in which $\phi_a \in BC(S_a)$ approximates $\phi \in BC(S)$, introduce mappings between these two spaces. For $a \geq 1$, let $F_a \in C^\infty(\mathbf{R})$ be an even function satisfying $0 \leq F_a(t) \leq 1$, $t \in \mathbf{R}$, and

$$F_a(t) = \begin{cases} 1, & 0 \leq t \leq a, \\ 0, & t \geq a + 1. \end{cases}$$

For $a \geq 2$ define $E_a : BC(S) \to BC(S_a)$ by

$$(7) \qquad E_a \psi(x) := \begin{cases} F_{a-1}(x_3) \psi(x), & x = (x_1, x_2, x_3) \in \tilde{S}_a, \\ 0, & x \in S_a \setminus \tilde{S}_a, \end{cases}$$

and $R_a : BC(S_a) \to BC(S)$ by

$$(8) \quad R_a \psi(x) := \begin{cases} \psi(x), & x \in \tilde{S}_a, \\ \psi((x_1, x_2, a)) F_a(x_3), & x = (x_1, x_2, x_3) \in S \setminus \tilde{S}_a, \ x_3 \geq a, \\ \psi((x_1, x_2, -a)) F_a(x_3), & x = (x_1, x_2, x_3) \in S \setminus \tilde{S}_a, \ x_3 \leq -a. \end{cases}$$

Then $\phi_a \in BC(S_a)$ is defined by

$$(9) \qquad \phi_a = E_a g + K_a \phi_a,$$

which we will refer to as the "truncated" version of (1).

In Section 2 we construct a partial theory of the solvability of equation (1) and of the truncated equation (9). Theorem 2.8 suggests that the existence of a solution

to the truncated equation for all sufficiently large a depends not just on the unique solvability of the original integral equation (1) but also on that of the "half cylinder" equations obtained by replacing S by $S\pm$. Specifically, it shows that $I - K_a$ is invertible and uniformly bounded for all sufficiently large a provided $I - K$, $I - K_+$ and $I - K_-$ are injective. These conditions also ensure the invertibility of the original operator $I - K$ (Corollary 2.9), so that the spectrum of K is contained in the union of $\{0\}$ and the sets of eigenvalues of K, K_+, and K_-. This result may be powerful for establishing existence of a solution to equation (1) and, in the case when (1) is a boundary integral equation, for establishing existence of solution for the corresponding boundary value problem formulation. These points are illustrated by the example in Section 4.

In Section 3 we consider the numerical solution of the integral equation on S_a, defining $\phi^{(n)} \in BC(S_{a_n})$ by

$$(10) \qquad\qquad \phi^{(n)} = E_{a_n} g + K^{(n)} \phi^{(n)},$$

where $K^{(n)} = K_{a_n} P_n$ and P_n is an interpolatory projection operator onto a space of piecewise constant functions on a finite element mesh on S_{a_n} Note that $P_n \phi^{(n)}$ is a piecewise constant collocation method approximation to ϕ_{a_n}. and $\phi^{(n)}$ is the iterated collocation method approximation of Sloan [12]. The results of Section 2 are extended to show that the operators $(I - K^{(n)})^{-1}$ are uniformly bounded for all sufficiently large n and that $R_{a_n} \phi^{(n)}$ converges to ϕ uniformly on compact subsets of S provided that $a_n \to +\infty$ and $h_n \to 0$ as $n \to \infty$, where h_n is the diameter of the largest element of the mesh on S_{a_n}. Further, if $\phi(x) \to 0$ as $x \to \infty$, $R_{a_n} \phi^{(n)}$ converges to ϕ uniformly on S,

The integral equation (9) may seem a perverse choice as approximation for (1): the approximation obtained by replacing S by $\tilde{S}_a$ in (1) may seem more obvious: indeed, this alternative approximation can be analysed in a similar (in fact simpler) manner. However, the resulting theory appears to be inapplicable in practical situations in which (1) is a boundary integral equation. In such applications it is generally the case that (1) with S replaced by $S_\pm$ is still a boundary integral equation so that the injectivity of $I - k_\pm$ can be established in a similar manner to that of $I - K$ (cf. Section 4). Equation (1) with S replaced by $\tilde{S}_\pm$ is generally not a boundary integral equation, and it is not clear, in practical cases, how the injectivity of $I \quad k_{\tilde{S}_\pm}$ (which the alternative theory requires) might be established.

Integral equations on smooth closed bounded surfaces in $\mathbb{R}^3$ and their numerical treatment have a wide literature: see Colton and Kress [9,10] for that part relevant to the acoustic scattering example of Section 4. The piecewise constant collocation method discussed in Section 3 is the most commonly used boundary element method (Brebbia *et al.* [6]). For integral equations of the class discussed in this paper on smooth bounded surfaces, the stability and convergence of this boundary element method can be analysed using standard results from the collectively compact operator theory of Anselone [1],

To the best of our knowledge, this paper is the first attempt to develop a theory for integral equations of the form (1) and their numerical solution. Our arguments generalise ones in collectively compact operator theory [1]. Our assumptions, results, and methods of proof are closest to those of Atkinson [5], Anselone and Sloan [2-4], and Chandler-Wilde [8], who consider the approximate solution of integral equations on the real line.

2. The Original and Truncated Equations.
Let $k_x(y) = k(x, y), (x, y) \in I$. We suppose throughout that, for all $a \geq 2$ and $x \in S_a, k_x \in L_1(S_a)$, and that k

satisfies the following assumptions, A1-A4:

A1. For all $(x, y) \in I = \{(x, y) \in T : T \in \Sigma\}$, $\quad k(x + te_3, y + te_3) = k(x, y)$, $\quad t \in \mathbb{R}$.

A2. $c := \sup\limits_{a \geq 2,\ x \in S_a} \int_{S_a} |k(x, y)| ds(y) < \infty$.

A3. $\Delta(h) := \sup\limits_{a \geq 2,\ x, x' \in S_a,\ |x - x'| \leq h} \int_{S_a} |k(x, y) - k(x', y)| ds(y) \to 0$ as $h \to 0$.

A4. For $x = (x_1, x_2, x_3) \in S_\pm$, $\int_{E_\pm} |k(x, y)| ds(y) \to 0$ as $x_3 \to \pm\infty$.

Let $B_T := \{\psi \in L_\infty(T) : \|\psi\|_\infty \leq 1\}$, for all $T \in \Sigma$. Note that Assumptions A1-A3 imply that

(11) $$\bigcup_{T \in \Sigma} K_T B_T \text{ bounded and equicontinuous,}$$

in that

$$\sup_{T \in \Sigma,\ \psi \in B_T} \|K_T \psi\|_\infty \leq c$$

and

$$\sup_{T \in \Sigma,\ \psi \in B_T,\ x, x' \in T,\ |x - x'| \leq h} |K_T \psi(x')| \geq \Delta(h).$$

Thus, if Assumptions A1-A3 are satisfied, then K_T is a bounded operator from $L\infty(T)$ to $BC(T)$ for each $T \in \Sigma$ (indeed, K_T is compact if T is bounded), and

(12) $$\sup_{T \in \Sigma} \|K_T\| \leq c.$$

From A1-A3 we have also the following technical lemma:

LEMMA 2.1. *Define* $\Phi_+(t)$, $\Phi_-(t)$, *for* $t \geq 0$, *by*

$$\Phi_\pm(t) := \sup_{x \in \widetilde{\Gamma} \mp te_3} \int_{\widetilde{S}_\pm} |k(x, y)| ds(y),$$

Where $\widetilde{\Gamma} := \{(x_1, x_2, 0) : (x_1, x_2) \in \Gamma\}$, *and* $\Phi(t)$ *by* $\Phi(t) := \max(\Phi_+(t), \Phi_-(t))$. *Then* $\Phi(t) \to 0$ *as* $t \to +\infty$.

Proof. Consider $\Phi_+(t)$. From Assumption A2, $\sup_{x \in S} \int_S |k(x, y)| ds(y) < \infty$, so that Φ_+ is well denned and

$$\int_{\widetilde{S}_+ + te_3} |k(x, y)| ds(y) \to 0$$

as $t \to +\infty$ with x fixed. Applying A1, it follows that

(13) $$\int_{\widetilde{S}_+} |k(x, y)| ds(y) \to 0$$

as $t \to +\infty$ with $x = (x_1, x_2, -t)$ and $(x_1, x_2) \in \Gamma$ fixed. But, by Assumption A3 $\int_{\widetilde{S}_+} |k(x, y)| ds(y)$ uniformly continuous on S. It follows that the convergence (13)

is uniform in $(x_1, x_2) \in \Gamma$, so that $\Phi_+(t) \to 0$ as $t \to +\infty$. Similarly, $\Phi_-(t) \to 0$ as $t \to +\infty$. □

Assumption A4 gives rise to a similar result concerning the integral over $E_\pm$.

LEMMA 2.2. *Define* $\Psi_+(t)$, $\Psi(t)$, *for* $t \geq 0$, *by*
$$\Psi_\pm(t) := \sup_{x \in \widetilde{\Gamma} \pm te_3} \int_{E_\pm} |k(x,y)| ds(y),$$
and $\Psi(t)$ *by* $\Psi(t) := \max (\Psi_+(t), \Psi_-(t))$. *Then* $\Psi(t) \to 0$ *as* $t \to +\infty$.

Proof. From Assumption A2, Ψ is well defined and, from Assumptions A2 and A3, $\int_{E_\pm} |k(x,y)| ds(y)$ is uniformly continuous on $S_\pm$. It follows that $\Psi(t) \to 0$ from Assumption A4.

□

We shall employ principally the following notions of convergence. For a sequence $\{T_n\} \subset \Sigma$ and $T \in \Sigma$ we shall write $T_n \to T$ if, for all $A > 0$, $\{x \in T_n : |x| \leq A\} = \{x \in T : |x| \leq A\}$ for all sufficiently large n. For a sequence $\{\psi_n\}$, with $\psi_n \in L_\infty(T_n)$, we shall write $\psi_n \to \psi \in L_\infty(T)$ if $T_n \to T$, $\sup_n \|\psi_n\|_\infty < \infty$, and, for all $A > 0$,
ess $\sup_{x \in T, |x| \leq A} |\psi_n(x) - \psi(x)| \to 0$ as $n \to \infty$.

REMARK 2.1. *If* $T_n \to T$ *then, for all sufficiently large n,* $T_n \setminus T \subset (E_+ - b_n) \cup (E_- + c_n)$, *where* $b_n, c_n \geq 0$ *and* $b_n, c_n \to +\infty$ *as* $n \to \infty$. *Thus, if T is bounded,* $T_n = T$ *for all sufficiently large n, so that, if* $\psi \in L_\infty(T)$, *then* $\psi_n \to \psi \Leftrightarrow \|\psi_n - \psi\|_\infty \to 0$.

REMARK 2.2. *If* $\psi_n \in BC(T_n)$ *for each n then* $\psi \in BC(T)$.

REMARK 2.3. *If* $\psi_n \to \psi$ *then* $\|\psi\|_\infty \leq \sup_n \|\psi_n\|_\infty$.

REMARK 2.4. *If* $\psi_n \in BC(T)$ *and* $\psi_n \in BC(T)$. *for each n, then the convergence* $\psi_n \to \psi$ *is strict convergence in the sense of Buck [7].*

REMARK 2.5. *A useful test of convergence is:* $\psi_n \to \psi \Leftrightarrow$ *every subsequence of* $\{\psi_n\}$ *has a sub*sequence that converges to ψ.

Our next two results are, respectively, a collective compactness and a convergence property of the operator sequence $\{K_{T_n}\}$, in the case $T_n \to T$.

LEMMA 2.3. *Suppose that* $\{T_n\} \subset \Sigma, T_n \to T \in \Sigma, \psi_n \in L_\infty(T_n)$ *for each n, and* $\sup_n \|\psi_n\|_\infty < \infty$. *Then, for some subsequence* $\{\psi_{n_m}\}, K_{T_{n_m}} \psi_{n_m} \to \psi \in BC(T)$.

Proof. Define $T_A^* := \{(y_1, y_2, y_3) \in T; |y_3| \leq A\}$, for $A > 0$, and let $x_n = K_{T_n} \psi_n$. Note that, for all $A > 0$, $T_A^* \subset T_n$ for all sufficiently large n.

By (11), $\{x_n\}$ is bounded and equicontinuous. By the Arzela-Ascoli theorem and the above remarks, it follows that $\{x_n\}$ has a subsequence, $\{x_n^{(1)}\}$ which is a Cauchy sequence on T_1^*. Similarly, $\{x_n^{(m-1)}\}$ has a subsequence, $\{x_n^{(2)}\}$, which is a Cauchy sequence on T_2^*. Continuing the argument, we may construct, for each $m \in$ N, a subsequence $\{x_n^{(m)}\}$ of $\{x_n^{(m-1)}\}$ which is a Cauchy sequence on T_m^*. Then $\{x_n^{(n)}\}$ is a Cauchy sequence on T_m^* for each $m \in$ N and thus converges to an element of $BC(T)$. □

LEMMA 2.4. *Suppose that* $\{T_n\} \subset \Sigma$, $T_n \to T \in \Sigma$, $\psi_n \in L_\infty(T_n)$ *for each n,* $\psi \in L_\infty(T)$, *and* $\psi_n \to \psi$. *Then* $K_{T_n} \psi_n \to K_T \psi$.

Proof. If T is bounded then, by Remark 2.1, the above is no more than a statement that K_T is continuous. Suppose that T is unbounded. Since, by (11), $\{K_{T_n} \psi_n\}$ is bounded and equicontinuous, to show $K_{T_n} \psi_n \to K_T \psi$ we need only show pointwise convergence of $K_{T_n} \psi_n$ to $K_T \psi$

Let C := $\sup_n \|\psi_n\|_\infty$, choose $x = (x_1, x_2, x_3,) \in T$, and define T_A^* as in the previous proof. For all $A > |x_3|$ and provided $T_A^* \subset T_n$ (true for all sufficiently large

n),

$$\left|K_{T_n}\psi_n(x) \quad K_T\psi_n(x)\right| \le \left|\int_{T_A^*} k(x,y)(\psi_n(y)) \quad \psi_n(y)ds(y)\right|$$

$$+ C\int_{T_n\setminus T_A^*} |k(x,y)|\,ds(y) + C\int_{T\setminus T_A^*} |k(x,y)|\,ds(y)$$

$$\le c\sup_{|y_3|\le A} \left|\psi_n(y) \quad \psi(y)\right| + C\int_{(E_+ \ b_n)\cup(E_-+c_n)} |k(x,y)|\,ds(y)$$

$$+ 2C\int_{T\setminus T_A^*} |(x,y)|\,ds(y)$$

for all sufficiently large n, where $b_n, c_n \ge 0$ and $b_n, c_n \to +\infty$ (To obtain this last inequality we have used Assumption A2, that, $T_n\setminus T_A^* \subset (T_n\setminus T)\cup(T\setminus T_A^*)$, and that, by Remark 2,1, $T_n\setminus T$ C $(E_+ - b_n) \cup (E_- + c_n)$ for all sufficiently large n.) Applying Assumption Al and recalling the definitions of Ψ and Φ we obtain

$$\left|K_{T_n}\psi_n(x) \quad K_T\psi(x)\right| \le c\sup_{|y_3|\le A}\left|\psi_n(y) \quad \psi(y)\right| + C(\Psi(b_n+x_3) + \Psi(c_n \quad x_3))$$

$$+2C(\Phi(A+x_3) + \Phi(A \quad x_3)).$$

Given $\epsilon > 0$, by Lemma 2.1, the final term on the right hand side is $\le \epsilon/2$ if A is chosen sufficiently large enough. Then, since $\psi_n \to \psi$ and by Lemma 2.2, the remaining terms are $\le \epsilon/2$ for all sufficiently large n. Thus $\left|K_{T_n}\psi_n(x) \quad K_T\psi_n(x)\right| \to 0$ as $n\to\infty$ for every fixed x. $\quad\square$

We apply the above compactness and continuity properties first of all to give a condition for the continuous dependence of ϕ on g in equation (1), in the case $g\in (I \quad K)BC(S)$.

THEOREM 2.5. *If I - K is injective then $(I — K)^{-1}$ exists and is a bounded operator on $(I - K)BC(S)$.*

Proof. Suppose that the theorem is false. Then there exists a sequence $\{\psi_n\}\subset BC(S)$ with $\|\psi_n\|_\infty = 1$ for each n such that $\|\psi_n \quad K\psi_n\|_\infty \to 0$.. Since $\|\psi_n\|_\infty = 1$, we can find a sequence $\{a_n\} \subset R$ such that $\sup_{x\in\widetilde{\Gamma}}\|\chi_n(x)\| \ge \frac{1}{2}$, where $\{\chi_n\}\subset BC(S)$ is defined by $\chi_n(x) = \psi_n(x \quad a_n e_3), x\in S$. From Assumption Al it follows that $K\psi_n(x \quad a_n e_3) = K\chi_n(x)$. Thus

(14) $$\|\chi_n \quad K\chi_n\|_\infty = \|\psi_n \quad K\psi_n\|_\infty \to 0$$

as $n\to\infty$.

Since $\{\chi_n\}$ is bounded in $BC(S)$, by Lemma 2.3 we can choose a subsequence $\{\chi_{n_m}\}$ and $\chi \in BC(S)$ such that $K\chi_{n_m} \to \chi$. From (14) it follows that $K\chi_{n_m} \to \chi$ and then, from Lemma 2.4, that $K\chi_{n_m} \to K\chi$. Thus $\chi = K\chi$. Since $\|\chi\|_\infty \ge \inf_{x\in\widetilde{\Gamma}}|\chi_n(x)| \ge \frac{1}{2}$ and $I - K$ is injective we have a contradiction. $\quad\square$

In the next two theorems we commence an examination of conditions for $(I - K)BC(S) = BC(S)$ and of the convergence of ϕ_a, the solution of the integral equation (9) on S_a, to ϕ as $a\to\infty$

THEOREM 2.6. *Suppose that I - K is injective and that, for some $A > 0$ and all $a \ge A, (I \quad K_a)^{-1}$ exists and is a bounded operator on $BC(S_a)$, with $C :=$ $\sup a \ge A\|(I \quad K_a)^{-1}\| < \infty$. Then $(I — K)^{-1}$ exists as an operator on $BC(S)$ with $\|(I \quad K)^{-1}\| \le C$ Moreover, if $g \in BC(S)$, $\{a_n\} \subset R^+$ and $a_n \to \infty$, then $(I - K_{a_n})^{-1}E_{an}g \to (I \quad K)^{-1}g$.*

Proof. Let $g \in BC(S)$ and define the sequence $\{\phi_n\}$, where $\phi_n \in BC(S_{a_n})$ for each n, by $\phi_n = (I - K_{a_n})^{-1} E_{a_n} g$. Then

$$(15) \qquad \phi_n = K_{a_n} \phi_n + E_{a_n} g.$$

Since $\{\phi_n\}$ is a bounded sequence, by Lemma 2.3 there exists a subsequence $\{\phi_{n_m}\}$ and $\phi \in BC(S)$ such that

$$K_{a_{n_m}} \phi_{n_m} + E_{a_{n_m}} g \to \phi.$$

Thus, from (15), $\phi_{n_m} \to \phi$ Hence, from Lemma 2.4, $K_{a_{n_m}} \phi_{n_m} \to K\phi$, and thus $K_{a_{n_m}} \phi_{n_m} + E_{a_{n_m}} g \to K\phi + g$. Thus $\phi = K\phi + g$ and so $g \in (I - K)BC(S)$ and we have shown that $I - K$ is surjective, so that $(I - K)^{-1}$ exists as an operator on $BC(S)$.

By the same argument, given $g \in BC(S)$, every subsequence of $(I-K)^{-1} E_{a_n} g$ has a subsequence converging to $(I - K_{a_n})^{-1} g$ Thus, by Remark 2.5,

$$(I - K_{a_n})^{-1} E_{a_n} g \to (I - K)^{-1} g$$

for every $g \in BC(S)$. From this and Remark 2.3 it follows that $\|(I - K)^{-1}\| \leq C$.
□

THEOREM 2.7. *Suppose that the conditions of the previous theorem are satisfied. Define* $\phi_n := (I - K_{a_n})^{-1} E_{a_n} g$ *and* $\phi := (I - K)^{-1} g$. *If* $\phi(x) \to 0$ *as* $x \to \infty$ *then* $\left\| \phi \quad R_{a_n} \phi_n \right\|_{\infty} \to 0$ *as* $n \to \infty$

Proof. By definition,

$$(16) \qquad \phi_n \quad K_{a_n} \phi_n = E_{a_n} g$$

and $\phi \quad K\phi = g$, so that

$$(17) \qquad E_{a_n} \phi \quad K_{a_n} \phi_n = E_{a_n} g.$$

From (16) and (17) it follows that

$$\phi_n - E_{a_n} \phi = (I - K_{a_n})^{-1} x_n,$$

Where $x_n = K_{a_n} E_{a_n} \phi - E_{a_n} K\phi$, so that

$$(18) \qquad \|\phi_n - E_{a_n} \phi\|_{\infty} \leq C \|X_n\|_{\infty}.$$

For each n, let $A_n := (a_n - 1)/2$. By Assumptions Al and A2, and from the definition of Φ,

$$\sup_{x \in S_{a_n} \backslash \bar{S}_{a_{n-1}}} \left| E_{a_n} K(x) \right| \leq \sup_{x \in \bar{S}_{a_n} \backslash \bar{S}_{a_{n-1}}} \left| K \phi(x) \right|$$

$$\leq \sup_{x \in \bar{S}_{a_n} \backslash \bar{S}_{a_{n-1}}} \int_{S_{A_n}} |k(x,y)| ds(y) \|\phi\|_{\infty} + c \sup_{y \in S \backslash \bar{S}_{A_n}} |\phi(y)|$$

$$\leq \sup_{t \geq A_n} \Phi(t) \|\phi\|_{\infty} + c \sup_{y \in S \backslash \tilde{S}_{An}} \|\phi(y)\|.$$

The same upper bound applies to $\sup_{x \in S_{a_n} / \tilde{S}_{a_{n-1}}} \left| K_{a_n} E_{a_n} \phi(x) \right|$ so that, for $x \in S_{a_n} \backslash \bar{S}_{a_{n-1}}$

$$\left| X_n(x) \right| \leq 2 \sup_{t \geq A_n} \Phi(t) \|\phi\|_{\infty} + 2c \sup_{y \in S \backslash \tilde{S}_{An}} |\phi(y)|.$$

For, $x \in \widetilde{S}_{a_{n-1}}$,

$$X_n(x) = \int_{S_{a_n} \setminus \overline{S}_{a_{n-1}}} k(x,y)(E_{a_n}\phi)(y)ds(y) \quad \int_{S \setminus \overline{S}_{a_{n-1}}} K(x,y)\phi(y)ds(y),$$

So that

$$\left| X_n(x) \right| \le 2c \sup_{y \in S \setminus \overline{S}_{a_{n-1}}} \left| \phi(y) \right| \le 2c \sup_{y \in S \setminus \widetilde{S}_{A_n}} \left| \phi(y) \right|$$

Thus

$$\left\| X_n \right\|_\infty \le 2 \sup_{t \ge A_n} \Phi(t) \|\phi\|_\infty + 2c \sup_{y \in S/S_{A_n}} \left| \phi(y) \right|,$$

so that, by Lemma 2.1, $\left\| X_n \right\|_\infty \to 0$ and, by (18), $\left\| \phi_n - E_{a_n}\phi \right\|_\infty \to 0$ as $n \to \infty$

Now,

$$\left\| \phi - R_{a_n}\phi_n \right\|_\infty \le \sup_{x \in S \setminus \overline{S}_{a_n}} \left| \phi(x) \right| + \sup_{x \in S \setminus \overline{S}_{a_n}} \left| R_{a_n}\phi_n(x) \right| + \sup_{x \in S \setminus \overline{S}_{a_n}} \left| \phi(x) - \phi_n(x) \right|$$

$$\le 2 \sup_{x \in S \setminus \overline{S}_{a_n}} \left| \phi(x) \right| + 2 \sup_{x \in \overline{S}_{a_n}} \left| \phi(x) - \phi_n(x) \right|$$

$$\le 3 \sup_{x \in S \setminus \widetilde{S}_{a_{n-1}}} \left| \phi(x) \right| + 2 \left\| + \phi_n - E_{a_n}\phi \right\|_\infty$$

Since $\left| \phi(x) - E_{a_n}\phi(x) \right| \le \left| \phi(x) \right|, x \in \overline{S} \setminus \overline{S}_{a_{n-1}}$ Thus $\left\| \phi - R_{a_n}\phi_n \right\|_\infty \to 0$ as $n \to \infty$

□

The next theorem is a criterion for the existence of a solution to equation (9) for all sufficiently large $a \ge A$ and the uniform stability of the approximate inverse operators.

THEOREM 2.8. *If* $I-K, I-K_+,$ *and* $I-K$ *are injective, then* $(I-K_a)^{-1}$ *exists and is a bounded operator on BC(S$_a$) for all sufficiently large* $a \ge A$ *and*

$$C := \sup_{a \ge A} \left\| (I - K_a)^{-1} \right\| < \infty.$$

Proof. Note that if $(I-K_a)^{-1}$ exists (i.e. $I - K_a$ is injective) then, since K_a is compact, $(I - K_a)^{-1}$ is a bounded operator on BC(S$_a$) by the Fredholm Alternative.

Suppose now that the theorem is false. Then there exist sequences $\{a_n\} \subset R^+$ and $\{\psi_n\}$ such that $a_n \to \infty, \psi_n BC(S_{a_n})$ and $\left\| \psi_n \right\|_\infty = 1$ for each n, and

(19) $$\left\| \psi_n - K_{a_n}\psi_n \right\|_\infty \to 0.$$

Since $\left\| \psi_n \right\|_\infty = 1$ we can find a sequence $\{x^{(n)}\}$ such that $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}) \in S_{a_n}$ for each n and $\left| \psi_n(x^{(n)}) \right| \ge \frac{1}{2}$.

There are two cases to consider: $(a) a_n \left| x_3^{(n)} \right| \to \infty$ as $n \to \infty$; $(b) \{a_n \left| x_3^{(n)} \right|\}$ is bounded.

CASE (a). For each n, define $T_n := S_{a_n} - x_3^{(n)}$ and $X_n \in BC(T_n)$ by

(20) $$x_n(x) = \psi_n(x + x_3^{(n)}e_3), \quad x \in T_n.$$

Then, by Assumption Al, it is easy to see that

(21) $$K_{a_n}\psi_n(x + x_3^{(n)}e_3) = K_{T_n}X_n(x), \quad x \in T_n.$$

CASE (b). We choose subsequences $\{a_{n_m}\}$ and $\{x_3^{(nm)}\}$ such that either $\{a_{n_m}-x_3^{(nm)}\}$ is bounded or $\{a_{n_m}+x_3^{(nm)}\}$ is bounded. Define $T_m := S_{a_{n_m}}Ta_{n_m}$ in the case $\{a_{n_m}\mp x_3^{(nm)}\}$ bounded, and x_m, for each m, by

(22) $x_m(x) = \psi_{n_m}(x \pm a_{n_m m}e_3,)\ x \in T_m.$

Then

(23) $K_{a_{n_m}}\psi_{n_m}(x \pm a_{n_m}) = K_{T_m}X_m(x), x \in T_m.$

Clearly, in both cases (a) and (b), $\|X_m\|_\infty = 1$ for each m and, from equations (20) and (21) in Case (a), (22) and (23) in Case (b),

(24) $\|X_m - K_{T_m}X_m\|_\infty = \|\psi_{n_m} - K_{a_{n_m}}\psi_{n_m}\|_\infty \to 0$

as $m \to \infty$. Note also that $T_m \to T$, where $T = S$ (Case (a)) or $T = S\pm$ in the case. $\{a_{n_m} \mp x_3^{(n_m)}\}$ bounded (Case (b)), and that, for some $A > 0$,

(25) $\sup_{x \in T_m,|x| \le A} |X_m(x)| \ge \tfrac{1}{2}$

for all m.

Applying Lemma 2.3, $\{K_{T_m}X_m\}$ has a subsequence $\{K_{T_m}X_{mj}\}$ such that $K_{T_{mj}}X_{mj} \to X \in BC(T)$. From (24) it follows that $X_{mj} \to X$ and then, from Lemma 2.4, that $K_{T_{mj}}X_{mj} \to K_TX$. Thus $X = K_TX$ but, by (25), $\|X\|_\infty \ge \tfrac{1}{2}$. This is a contradiction since $T = S, S_+,$ or $S-,$ so that $I - K_T$ is injective.   □

Combining these results, Theorems 2.6-2.8, we have immediately the following corollaries, conditions for the solvability of equation (1) and of (9) and for the convergence of ϕ_a to ϕ.

COROLLARY 2.9. *If* $I-K, I-K_+,$ *and* $I-K$ *are injective, then* $(I-K)^{-1}$ *exists and is a bounded operator on BC(S), so that equation (1) has a unique solution ϕ for all g ∈ BC(S).*

COROLLARY 2.10. *If* $I - K, I-K_+,$ *and* $I-K_-$ *are injective, then equation (9) has a solution, ϕ_a for all sufficiently large $a \ge A$. Moreover, $\phi_a \to \phi$ the solution of equation (1), as a $\to \infty$. Further, if $\phi(x) \to 0$ as $x \to \infty$, then $\|R_a\phi_a - \phi\|_\infty \to 0$.*

3. Numerical Solution of the Truncated Equations. In this section we extend the arguments of Section 2 to investigate the convergence of a simple piecewise constant finite element collocation method applied to the truncated equation (9) in the limit $a \to \infty$ and $h \to 0$, where h is the diameter of the largest element of the mesh.

For $T \in \sum$ bounded we call $\Pi = \{\gamma^{(1)},..... \gamma^{(N)}\}$ a mesh on T if, for each i, $\gamma^{(i)} \subset T$ is open in $T, \gamma^{(i)} \cap \gamma^{(j)} = \theta, i \ne j$, and $\bigcup_{i=1}^{N}\overline{\gamma^{(i)}} = T$ For a mesh II let $h(\Pi)$ denote the diameter of the largest element of Π.

Given a sequence $\{T_n : n \in N\} \subset \Sigma$ (each T_n bounded) and a mesh $\Pi_n = \{\gamma_n^{(1)},...., \gamma_n^{(N_n)}\}$ on T_n, for each n, one can construct a sequence of piecewise constant interpolation operators $\{P_n : n \in N\}$, where $P_n : BC(T_n)\ P_n : BC(T_n) \to L_\infty(T_n)$ defined by

(26) $P_n\psi(x) = \psi(x_n^{(i)}), x \in \gamma_n^{(i)}, i = 1,.... N_n,$

where $x_n^{(i)}, i = 1,...N_n, n = 1,2,....,$ is a given interpolation point in $\gamma_n^{(i)}$.

REMARK 3.1. *If $T_n \to T \in \Sigma$, $\psi_n \in BC(T_n)$ for each n, $\psi_n \to \psi \in BC(T)$, and $h_n := h(\Pi_n) \to 0$, then, for each $A > 0$ and all sufficiently large n,*

$$\sup_{x \in T, |x| \le A} \left| P_n \psi_n(x) \quad \psi(x) \right| \le \sup_{x \in T, |x| \le A+h_n} \left| \psi_n(x) \quad \psi(x) \right|$$

$$+ \sup_{x,x' \in T, |x|,|x'| \le A+h_n, |x-x'| \le h_n} \left| \psi(x) - \psi(x') \right| \quad \to \quad 0$$

as $n \to \infty$, so that $P_n \psi_n \to \psi$.

We have the following lemma, analogous to Lemma 2.4, a continuity property of the operator sequence $\{K^{(n)}\}$, where $K^{(n)} : BC(T_n) \to BC(T_n)$ is denned by

(27) $$K^{(n)} = K_{T_n} P_n.$$

LEMMA 3.1. *Suppose that $\{T_n\} \subset \Sigma$, $T \in \Sigma$, $\psi_n \in BC(T_n)$ for each n, $\psi \in BC(T)$, $\{P_n\}$ is the sequence of piecewise interpolation operators given by (26), $\psi_n \to \psi$ and $h_n \to 0$. Then $K^{(n)} \psi_n \to K_T \psi$.*

Proof. Apply Lemma 2.4 and Remark 3.1 to the sequence $\{P_n \psi_n\}$. □

Throughout the remainder of the section we suppose that the following additional assumption is satisfied:

A5. $\{P_n\}$ and $\{K^{(n)}\}$ are denned by (26) and (27), with $T_n = S_{a_n}$ for $n \in N$, and $\{a_n\} \subset [2,\infty)$. Also, $a_n \to \infty$, $h_n := h(\Pi_n) \to 0$, as $n \to \infty$.

Suppose that Assumption A5 is satisfied and, for $n \in N$, define $\tilde{\phi}^{(n)}$, a piecewise constant finite element collocation method approximation to ϕ_{a_n}, by

$$P_n \tilde{\phi}^{(n)} = P_n E_{a_n} g + P_n K_{a_n} \tilde{\phi}^{(n)}, \qquad \tilde{\phi}^{(n)} \in P_n BC(S_{a_n})$$

Sloan [12] proposed using the iterated collocation solution rather than the collocation method approximation. The iterated collocation solution, $\phi^{(n)} \in BC(S_{a_n})$, is obtained by applying one Neumann iteration to $\tilde{\phi}^{(n)}$: thus

$$\phi^{(n)} : E_{a_n} g + K_{a_n} \tilde{\phi}^{(n)}.$$

Note that $\tilde{\phi}^{(n)} = P_n \phi^{(n)}$, so that $\phi^{(n)}$ and $\tilde{\phi}^{(n)}$ agree at the collocation points, and $\phi^{(n)}$ satisfies equation (10). From the results of Sloan [12] it can be expected that $\phi^{(n)}$ will approximate ϕ_{a_n} more accurately than $\tilde{\phi}^{(n)}$ and we concentrate on proving the convergence of ϕ_{a_n} in the remainder of this section. Note that, since $\tilde{\phi}^{(n)} = P_n \phi^{(n)}$,

$$\tilde{\phi}^{(n)} \to \phi \quad if \quad \phi^{(n)} \to \phi,$$

by Remark 3.1.

We now present results on the stability and convergence to ϕ of the iterated collocation solution, $\phi^{(n)}$. Our results are discrete versions of Theorems 2.6-2.10 of the last section.

THEOREM 3.2. *Suppose that $I - K$ is injective and that, for all sufficiently large $n \ge N$, $(I - K^{(n)})^{-1}$ exists and is a bounded operator on $BC(S_{a_n})$ and $\sup_{n \ge N} \|(I - K^{(n)})^{-1}\| < \infty$. The for all $g \in BC(S)$, $\phi^{(n)} := (I^{(n)})^{-1} E_{a_n} g \to \phi := (I-K)^{-1} g$ as $n \to \infty$. If also $\phi(x) \to \infty$, then $\|\phi - R_{a_n} \phi^{(n)}\|_\infty \to 0$.*

Proof. Let $g \in BC(S)$ and define the sequence $\{\phi^{(n)}\}$ as above. Then

(28) $$\phi^{(n)} = K^{(n)} \phi^{(n)} + E_{a_n} P_n \phi^{(n)} + E_{a_n} g.$$

Since $\sup_{n \geq N} \| P_n \phi^{(n)} \| = \sup_{n \geq N} \| \phi^{(n)} \| < \infty$, by Lemma 2.3 there exists a subsequence $\{\phi^{(n_m)}\}$ and $\phi \in BC(S)$ such that

$$K^{(n_m)} \phi^{(n_m)} + E_{a_{n_m}} g \; \to \; \phi$$

Thus, from (28), $\phi^{(n_m)} \to \phi$. The proof that $\phi^{(n)} \to \phi$ now follows that of Theorem 2.6, utilising Lemma 3.1 in place of Lemma 2.4. Similarly, the proof that $\| \phi - R_{a_n} \phi^{(n)} \|_\infty \to 0$ as $n \to \infty$ follows that of Theorem 2.7. $\quad\square$

THEOREM 3.3. *If $I - K$, $I - K_+$, and $I - K_-$ are infective, then $(I - K^{(n)})^{-1}$ exists and is a bounded operator on $BC(T_n)$ for all sufficiently large $n \geq N$ and $\sup_{n \geq N} \|(I - K^{(n)})^{-1}\| < \infty$.*

Proof. Suppose that the theorem is false. Then, arguing as in the proof of Theorem 2.8, it follows that there exists a subsequence $\{K^{(n_m)}\}$ and a sequence $\{\psi_m\}$ with $\psi_m \in BC(S_{a_{n_m}})$ and $\| \psi_m \|_\infty = 1$, $m \in \mathbb{N}$, such that

$$\| \psi_m \quad K^{(n_m)} \psi_m \|_\infty \; = \; \| \psi_m \quad K_{a_{n_m}} P_{n_m} \psi_m \|_\infty \; \to \; 0.$$

Since $\| \psi_m \|_\infty = 1$, , we can find a sequence $\{x^{(m)}\}$ such that $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, x_3^{(m)})$ $\in S_{a_{n_m}}, m \in \mathbb{N}$, and $\left| \psi^{(m)}(x^{(m)}) \right| \geq \frac{1}{2}$.

Now, arguing as in proof of Theorem 2.8, one can construct a sequence $\{T_m\} \subset \Sigma$ such that $T_m \to T \in \{S, S_+, S_-\}$, a sequence $\{\chi_m\}$, with $\chi_m \in BC(T_m)$ and $\sup_{x \in T_m, |x| \leq A} |\chi_m(x)| \geq \frac{1}{2}$ for some $A > 0$ and each m, and a sequence of piecewise constant interpolation operators, $\{P_m^*\}$, where $P_m^* : BC(T_m) \to L_\infty(T_m)$ for each m, such that

$$\| \chi_m \quad K_{T_m} P_m^* \chi_m \|_\infty \; \to \; 0, \quad h(\Pi_m^*) \to 0,$$

as $m \to \infty$, where Π_m^* is the mesh on T_m corresponding to the operator P_m^*. For example, in Case (a) of the proof of Theorem 2.8 ($a_{n_m} \quad |x_3^{(m)}| \to \infty \; as \; m \to \infty$),

$$T_m := S_{a_{n_m}} \quad x_3^{(m)},$$

$$\chi_m(x) := \psi_m(x + x_3^{(m)} e_3), \; x \in T_m,$$

$$\Pi_m^* := \Pi_m \quad x_3^{(m)},$$

and, for $\psi^* \in BC(T_m)$,

$$P_m^* \psi^*(x) := P_n \psi(x + x_3^{(m)} e_3), \quad x \in T_m,$$

where $\psi \in BC(S_{a_{n_m}})$ is defined by $\psi(x) := \psi^*(x \quad x_3^{(m)} e_3)$. A contradiction is now obtained as in the last paragraph of the proof of Theorem 2.8, utilising Lemma 3.1 in place of Lemma 2.4. $\quad\square$

Combining the two theorems, we obtain the following, our main convergence result for the iterated collocation solution.

COROLLARY 3.4. *If $I - K$, $I - K_+$, and $I - K_-$ are injective, then, for all sufficiently large $n \geq N$, equation (10) has a solution, and $\phi^{(n)} \to \phi$, the solution of (1), as $n \to \infty$. Further, if $\phi(x) \to 0$ as $x \to \infty$, then $\| R_{a_n} \phi^{(n)} \quad \phi \|_\infty \to 0$.*

4. An Application to Acoustic Scattering. We apply the results of the previous sections to the case in which S is the surface of an acoustically rigid cylinder embedded in a homogeneous fluid occupying D, the exterior of S.

For $T \in \Sigma$ let $n(x)$ denote the normal to T at x, directed into the exterior of T, and let D_T denote either the interior or the exterior of T. Following Colton and Kress [9], let $\Re(D_T)$ denote the linear space of all complex functions $\Psi \in C^2(D_T) \cap BC(\overline{D_T})$ such that:

(i) the normal derivative on the boundary exists in the sense that the limit

$$(29) \qquad \frac{\partial u(x)}{\partial v} = \lim_{h \to 0,\, h > 0} \nabla u(x + hv(x)).v(x), \quad x \in T,$$

exists uniformly on compact subsets of T, where $v(x) = n(x)$ if D_T is the exterior domain, $v(x) = -n(x)$ if D_T is the interior domain;

(ii)

$$(30) \qquad \sup_{x \in T,\, 0 < h < \in} |\nabla u(x + hv(x)).v(x)| < \infty$$

for some $\in > 0$;

(iii)

$$(31) \qquad \sup_{x \in D_T,\, dist(x,T) \geq \in} |\nabla u(x)| < \infty$$

for all $\in > 0$,

Note that if $u \in \Re(D_T)$ then, by *(i)* and *(ii)*, $\dfrac{\partial u}{\partial n} \in BC(T)$.

Consider the scattering of an incident acoustic wave, bounded and continuous on S, and of angular frequency ω ($e^{i\omega t}$ time dependence). This gives rise to the following boundary value problem for the space dependent part of the scattered field.

BVP1 *Find $u \in \Re(D)$ such that*

$$(32) \qquad \Delta u + \kappa^2 u = 0 \quad in \quad D,$$

$$(33) \qquad \frac{\partial u}{\partial n} = F \quad on \quad S.$$

In equation (33), $F \in BC(S)$ is given ($F = -\dfrac{\partial u^i}{\partial n}$, where u^i is the incident field) as is $\kappa \in C$. We will assume throughout that $Im\ \kappa > 0$, so that the medium of propagation is lossy. The results we obtain apply only partially to the case $Im\ \kappa > 0$.

This boundary value problem can be reformulated as a boundary integral equation. Define $G(x,y)$ by equation (4). For $R > 0$ let $B_R := \{x \in R^3 : |x| < R\}$. For $h > 0$ sufficiently small, let $S^h := \{x + hn(x) : x \in S\}$, so that S^h is parallel to and distance h from S and of class C^1 [9, p.37], and let D^h be the region exterior to S^h. Applying Green's representation theorem [9, p. 68] in $D^h \cap B_R$, letting $R \to \infty$, noting that $G(x, y)$ decays exponentially as $|x - y| \to \infty$, and that u, are bounded in $\overline{D^h}$ by (31), we obtain

$$u(\mathrm{x}) = \int_{S^h} u(y) \frac{\partial G(x, y)}{\partial n(y)} - \frac{\partial u(y)}{\partial n} G(x, y) \ ds(y), \quad x \in D^h.$$

Letting $h \to 0$, and utilising equations (29), (30), and (33), we see that

$$(34) \qquad u(x) = U(x) + \int_S \frac{\partial G(x, y)}{\partial n(y)} u(y) ds(y), \quad x \in D,$$

Where

$$(35) \qquad U(x) := - \int_S F(y)G(x,y)ds(y), \qquad x \in \overline{D}.$$

Equations (34) and (35) represent u in D as the combination of a single- and a double-layer acoustic surface potential. Prom standard properties of these potentials [9] it follows that $U \in BC(\overline{D})$ (in fact $U \in \Re(D)$) and that the double-layer in (34) is continuous up to the boundary S, its limiting value given by [9, Thm 2.13]. Thus, letting x in (34) approach S, we find that $\phi := u \,|\, s \in BC(S)$ satisfies the following boundary integral equation:

$$(36) \qquad \phi(x) = g(x) + 2 \int_S \frac{\partial(x,y)}{\partial n(y)} \phi(y) ds(y), \qquad x \in S,$$

where $g := 2\, U|s \in BC(S)$. This equation is of the form (1) with $k(x,y)$ given by (3). Using a mixture of elementary and standard arguments (e.g. [9, p. 50]) it can be seen that k, given by (3), satisfies Assumptions A1-A4 (for details see Peplow [11]).

In order to apply the results of the previous sections we need also to consider the same integral equation but with S replaced by S_+, S_- or S_a, for some $a > 0$. We will show uniqueness of solution of equation (36), and of the same equation with S replaced by S_+ or S_-, by using the following uniqueness result for corresponding boundary value problems. Note that this result, in particular, shows that BVP1 has at most one solution.

THEOREM 4.1. *Suppose that $T \in \Sigma$, D_T is either the interior or exterior of T, and $v \in \Re(D_T)$ satisfies*

$$(37) \qquad \Delta v + \kappa^2 v = 0 \quad in \quad D_T$$

and either

$$(38) \qquad v = 0$$

or

$$(39) \qquad \frac{\partial v}{\partial n} = 0$$

on T. Then $v = 0$.

Proof. For $h > 0$, define $T^h = \{x + v(x)h : x \in T\}$. Then T^h is the parallel surface distance h from T lying in D_T Let D_T^h be the region exterior (interior) to T^h if D_T is exterior (interior) to T. For $\in > 0$, define $F_\in \in C^\infty$ $(\mathR^3)$ by $F_\in(x) = \exp(-\in \sqrt{1 + |x|^2})$.

Applying Green's first theorem to $\overline{v}$ and $v F_\in$ in the region $B_R \cap D_T^h$, utilising equation (37), then letting $R \to \infty$, we obtain

$$(40) \qquad \int_{D_T^h} F_\in \left\{ \overline{\kappa}^2 |v|^2 + |\nabla v|^2 \right\} dx + \int_{\partial D_T^h} F_\in v \frac{\partial \overline{v}}{\partial v} ds = - \int_{D_T^h} v \nabla F_\in . \nabla \overline{v} dx$$

Letting $h \to 0$ and utilising (38) or (39) we find that the integral over ∂D_T^h vanishes and equation (40) holds with D_T^h replaced by D_T .Multiplying this equation by κ,

and taking the imaginary part, we obtain

$$(41) \qquad Im \ \kappa(P_\in^2 + q_\in^2) = Im \ \{\kappa \int_{D_T} v \nabla F_\in . \nabla \bar{v} \ dx\}$$

where $p_\in := \{\int_{D_T} F_\in |\kappa|^2 |v|^2 \ dx\}^{\frac{1}{2}}, q_\in := \{\int_{D_T} F_\in |\nabla v|^2 \ dx\}^{\frac{1}{2}}$. Noting that $|\nabla F_\in| \le \in F_\in$ and applying the Cauchy-Schwarz inequality, we obtain

$$(42) \qquad Im \ \kappa(P_\in^2 + q_\in^2) \le \ \in p_\in \ q_\in.$$

It follows that $Im \ \kappa \ p_\in \le \in q_\in$ and $Im \ \kappa \ q_\in \le \in p_\in$ so that $(Im \ \kappa)^2 \ p_\in \ q_\in$. Thus, for $\in < Im \ \kappa$, $p_\in = 0$. Thus $v = 0$ in D_T. $\square$

Using the above result, the following theorem shows that both equation (1) and equation (9), with k defined by (3), have at most one solution. Note that in this next theorem and through to the end of this section K_T is defined by equation (6) with k defined by (3). We give a fairly brief version of the proof of Theorem 4.2 — the result is in any case standard in the case T bounded [9] — for further detail see Peplow [11].

THEOREM 4.2. *For all $T \in \Sigma$, $I - K_T$ is injective on $BC(T)$.*

Proof, Suppose that $T \in \Sigma$, $\psi \in BC(T)$ and $\psi = K_{T\psi}$. From [9, Theorem 2.30] (which applies immediately if T is bounded, with some additional argument, if $T = S, S_+$ or S_-), K_T maps $BC(T)$ onto the Hölder space $C^{0,\alpha}(T)$, and $C^{0,\alpha}(T)$ onto $C^{1,\alpha}(T)$. It follows that $\psi \in C^{1,\alpha}(T)$.

Let D_T denote the exterior of T and define

$$v(x) := \int_T \frac{\partial G(x,y)}{\partial n(y)} \psi(y)ds(y), \qquad x \in D_T,$$

$$w(x) := \int_T \frac{\partial G(x,y)}{\partial n(y)} \psi(y)ds(y), \qquad x \in \mathsf{R}^3 \setminus \overline{D_T}.$$

Clearly

$$(43) \qquad \Delta v + \kappa^2 v = 0 \quad in \quad D_T,$$

$$(44) \qquad \Delta w + \kappa^2 w = 0 \quad in \quad R^3 \setminus \overline{D_T}.$$

From [9, Theorem 2.13 and 2.23], and noting that $\int_T \frac{\partial G(x,y)}{\partial n(y)} w(y)ds(y) = \frac{1}{2}\psi(x)$, $x \in T$ (since $\psi = K_T\psi$), it follows that $v \in \quad (D_T)$, $w \in \quad (R^3 \setminus \overline{D_T})$ and

$$(45) \qquad v = \psi,$$
$$(46) \qquad w = 0,$$

and

$$(47) \qquad \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n}$$

on T. Noting (44) and (46) and applying Theorem 4.1, it follows that $\omega = 0$ in $\mathsf{R}^3 \setminus \overline{D_T}$, so that $\frac{\partial w}{\partial n} = 0$ on T. Then, noting (43) and (47) and applying Theorem 4.1, it follows that $v = 0$ in $\overline{D_T}$ and, from (45), that $\psi = 0$. We have shown that $I - K_T$ is injective. $\square$

Having shown that the homogeneous version of (1), with S replaced by T, has no non-trivial solution, for all $T \in \Sigma$, we can now obtain existence of solution to

equation (1) and equation (9), convergence of $\phi_a \to \phi$ as $a \to \infty$, and convergence of the numerical approximation $\phi^{(n)}$, denned by equation (10). Applying Corollary 2.9 and Theorem 4.2 we have immediately

COROLLARY 4.3. *The integral equation (36) has precisely one solution.*

From Theorem 2.8, Corollary 2.10 and Theorem 4.2 we have

COROLLARY 4.4. *Equation (9), with k defined by (3), has a solution, ϕ_a, for every $a \geq 2$. For some $A > 0$, $\sup_{a \geq A} \|(I - K_a)^{-1}\| < \infty$. Further, $\phi_a \to \phi$ as $a \to \infty$, and, if $\phi(x) \to 0$ as $x \to \infty$, then $\|R_a\phi_a - \phi\|_\infty \to 0$ as $a \to \infty$.*

Finally, from Theorem 3.3, Corollary 3.4 and Theorem 4.2 we have

COROLLARY 4.5. *Suppose that Assumption A5 is satisfied. Then equation (10), with k defined by (3), has a solution, $\phi^{(n)}$ for all sufficiently large $n \geq N$, and $\sup_{n \geq N} \|(I - K^{(n)})^{-1}\| < \infty$. Further, $\phi^{(n)} \to \phi$ as $n \to \infty$, and, if $\phi(x) \to 0$ as $x \to \infty$, then $\|R_{a_n}\phi^{(n)} - \phi\|_\infty \to 0$ as $n \to \infty$.*

We remark that the proof of Theorem 4.2, which shows, *inter alia,* that, if ϕ satisfies equation (36) with $g = 0$, then v satisfies BVP1 with $F = 0$, can be extended (see Peplow [11]) to show that BVP1 and the integral equation (36) (with $g := 2 U|s$) are equivalent, in that, if ϕ satisfies (36), then u, defined on S by $u|s = \phi$, and in D by (34), satisfies BVP1. (We have already shown conversely that if u satisfies BVP1 then $u|s$ satisfies (36).) Thus Corollary 4.3 also establishes, by integral equation methods, unique existence of solution of BVP1.

REFERENCES

[1] P. M. ANSELONE, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations,* Prentice Hall, Englewood Cliffs, NJ 1971.

[2] P. M. ANSELONE AND I. H. SLOAN, *Integral equations on the half-line*, J. Integral Eq., 9 (1985), pp. 3-23.

[3] ———, *Numerical solutions of integral equations on the half-line. II. The Wiener-Hopf case,* J. Integral Eq. Appl., 1 (1988), pp. 203-225.

[4] ———, *Spectral approximations for Wiener-Hopf operators,* J. Integral Eq. Appl., 2(1990),PP. 237-261.

[5] K. E. ATKINSON, *The numerical solution of integral equations on the half-line,* SIAM J. Numer. Anal., 6 (1969), pp. 375-397.

[6] C. A. BREBBIA, J. C. F. TELLES, AND L. C. WROBEL, Boundary Element Techniques, Springer-Verlag, Berlin, 1984.

[7] R. C. BUCK, *Bounded continuous functions on a, locally compact space,* Michigan Math. J., 5 (1958), pp. 95-104.

[8] S. N. CHANDLER-WILDE, *Some uniform stability and convergence results for integral equations on the real line and projection methods for their solution,* IMA J. Numer. Anal., 13 (1993), pp. 509-535.

[9] D. COLTON AND R. KRESS, *Integral Equation Methods in Scattering Theory*, Wiley-Interscience Publication, New York, 1983.

[10] ———, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, 1992.

[11] A. T. PEPLOW, *Boundary Integral Equation Methods in Outdoor Sound Propagation,* PhD thesis (in preparation), University of Bradford, UK.

[12] I. H. SLOAN, *Superconvergence and the Galerkin Method for Integral Equations of the Second Kind,* in Numerical Treatment of Integral Equations, C.T.H. Baker and G.F. Miller, eds., Academic Press, London, 1982.