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Author(s)	Sato, Takamichi
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Isomorphism between the stabilizers of finite sets of numbers in the R. Thompson group F

Takamichi Sato

Graduate School of Fundamental Science and Engineering, Waseda University

1. Introduction

This article is an extended version of the talk given at the RIMS Meeting on Set Theoretic and Geometric Topology held in Kyoto University from June 5 to June 7, 2019.

Thompson's group F was discovered by Richard Thompson in 1965. It is usually defined as a group of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable everywhere except at finitely many dyadic rational numbers (i.e., numbers from $\mathbb{Z}[1/2]$) and such that on the intervals of differentiability the derivatives are integer powers of 2. The group F has many fascinating properties which are surveyed in [1]. It is a finitely presented torsion free group and does not contain a nonabelian free subgroup. One of the most interesting open problems about this group is whether F is amenable. From a motivation to study the amenability of F, in [3, 4], D. Savchuk constructed Schreier graphs of subgroups H_U of F, which are the stabilizers of finite sets of real numbers $U \subset (0,1)$. He proved that the Schreier graphs of H_U are amenable and also showed that if U consists of one number, then H_U is an infinite index maximal subgroup of F. In [2], G. Golan and M. Sapir studied the subgroups H_U for arbitrary finite U. Let $U = \{\alpha_1, \ldots, \alpha_n\} \subset [0,1]$, where $\alpha_j < \alpha_{j+1}$, and define a type $\tau(U)$ as the word of length n in the alphabet $\{1,2,3\}$ as follows: for every i, the ith letter in $\tau(U)$ is 1 if α_i is a dyadic rational, 2 if α_i is rational but not a dyadic rational, and 3 if α_i is irrational. They described the algebraic structure of H_U for finite $U \subset (0,1)$ and also proved that H_U is finitely generated if and only if U does not contain irrational numbers. Moreover, it was proved that if $\tau(U) \equiv \tau(V)$ for finite sets U, $V \subset (0,1)$, then H_U and H_V are isomorphic ($p \equiv q$ denotes letter-by-letter equality of words p, q) and that the converse of the statement does not hold. They stated [2, Subsection 4.1] that finding a necessary and sufficient condition for H_U and H_V to be isomorphic is still an open problem.

In this article we are going to summarize the problem and give a necessary and sufficient condition for H_U and H_V to be isomorphic.

2. Notation, Terminology, and Previous Results

Recall that the R. Thompson group F is the group of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable everywhere except at finitely many dyadic rational numbers (i.e., numbers from $\mathbb{Z}[1/2]$) and such that on the intervals of differentiability the derivatives are integer powers of 2. The group operation is composition of homeomorphisms. Composition and evaluation of functions in F will be in word order. That is, for any two elements f, g in F and any $t \in [0,1]$, $tf = f(t)$ and $fg = g \circ f$. Basic facts about this group can be found in [1]. In particular, it is known that the commutator subgroup $[F,F]$ is simple, and also proved that F is generated by two homeomorphisms x_0 and x_1 given by

$$ax_0 = \begin{cases} 2a & \text{if} \quad 0 \le a \le \frac{1}{4}, \\ a + \frac{1}{4} & \text{if} \quad \frac{1}{4} \le a \le \frac{1}{2}, \\ \frac{a}{2} + \frac{1}{2} & \text{if} \quad \frac{1}{2} \le a \le 1, \end{cases} \qquad ax_1 = \begin{cases} a & \text{if} \quad 0 \le a \le \frac{1}{2}, \\ 2a - \frac{1}{2} & \text{if} \quad \frac{1}{2} \le a \le \frac{5}{8}, \\ a + \frac{1}{8} & \text{if} \quad \frac{5}{8} \le a \le \frac{3}{4}, \\ \frac{a}{2} + \frac{1}{2} & \text{if} \quad \frac{3}{4} \le a \le 1. \end{cases}$$

Note that our generators x_0 and x_1 are the inverses of the generators in [1].

The *support* of an element f in F is the subset $\mathrm{Supp}(f) = \{x \in [0,1] \mid xf \neq x\}$. We can easily see that $\mathrm{Supp}(f)$ is a finite union of disjoint open intervals. Each of these open intervals will be called an *orbital* of f. The complement set of the support of f in F is denoted by $\mathrm{Fix}(f)$.

We say that $f \in F$ *has closure of support in an interval* J if the closure in $[0,1]$ of $\mathrm{Supp}(f)$ is contained in J. Let F_J be the set of all functions from F with closure of support in J. Then F_J is a subgroup of F. We note that $F_{(0,1)}$ is the set of all functions from F with slope 1 both at 0^+ and at 1^-. It is proved [1] that the commutator subgroup $[F,F]$ of F is exactly the subgroup $F_{(0,1)}$. It is also known [2] that for any $a, b \in [0,1]$ with $a < b$, $F_{(a,b)}$ is isomorphic to $F_{(0,1)}$.

For any finite subset X of $(0,1)$, let H_X be the stabilizer of X in F. That is, $H_X = \{f \in F \mid xf = x \text{ for each } x \in X\}$. Any finite subset Y of $[0,1]$ is subdivided into three subsets: $Y_1 = Y \cap \mathbb{Z}[1/2]$, $Y_2 = Y \cap (\mathbb{Q} \setminus \mathbb{Z}[1/2])$, and $Y_3 = Y \cap (\mathbb{R} \setminus \mathbb{Q})$.

Let $Y = \{r_1, \ldots, r_n\} \subset [0,1]$ with $r_j < r_{j+1}$, $r_1, r_n \notin Y_2$ and

$$B_Y = F_{[r_1,r_n]} \cap H_{Y \setminus \{r_1,r_n\}}.$$

The following proposition is a slight generalization of [2, Theorem 3.2] and can be proved in much the same way as it.

Proposition 1. *The group B_Y is isomorphic to a semidirect product*

$$B_Y \cong [F,F]^{n-1} \rtimes \mathbb{Z}^{2|(Y \setminus \{r_1,r_n\})_1| + |Y_2| + |\{r_1,r_n\}_1|}.$$

Corollary 1. *Let $U = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_j < \alpha_{j+1}$ and $\alpha_1, \alpha_n \notin U_2$, and $V = \{\beta_1, \ldots, \beta_m\}$, where $\beta_j < \beta_{j+1}$ and $\beta_1, \beta_m \notin V_2$. If the subgroups B_U*

and B_V are isomorphic, then $n = m$ and $2|(U \setminus \{\alpha_1, \alpha_n\})_1| + |U_2| + |\{\alpha_1, \alpha_n\}_1| = 2|(V \setminus \{\beta_1, \beta_m\})_1| + |V_2| + |\{\beta_1, \beta_m\}_1|$.

Let $U = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_j < \alpha_{j+1}$. We define a type $\tau(U)$ as the word of length n in the alphabet $\{1, 2, 3\}$ as follows: for every i, the ith letter in $\tau(U)$ is 1 if α_i is a dyadic rational, 2 if α_i is rational but not a dyadic rational, and 3 if α_i is irrational.

3. Isomorphism between stabilizers of finite sets

Recall that every finite subset U of $(0,1)$ is subdivided into three subsets $U = U_1 \sqcup U_2 \sqcup U_3$, where $U_1 = U \cap \mathbb{Z}[1/2]$, $U_2 = U \cap (\mathbb{Q} \setminus \mathbb{Z}[1/2])$, and $U_3 = U \cap (\mathbb{R} \setminus \mathbb{Q})$. Write $U_1 \cup U_3 = \{r_1, \ldots, r_n\}$, where $r_j < r_{j+1}$ and $|U_1 \cup U_3| = n$. Let $r_0 = 0$, $r_{n+1} = 1$, and $U_{2,k} = \{q \in U_2 \mid r_k < q < r_{k+1}\}$. Then $U_2 = \bigsqcup_{k=0}^{n} U_{2,k}$. Recall (see subsection 4.2 in [2] for details), that

$$H_U = B_{\{r_0, r_1\} \cup U_{2,0}} \times \cdots \times B_{\{r_n, r_{n+1}\} \cup U_{2,n}}.$$

For any word $w_1 w_2 \in \{11, 13, 33\}$ and $j \in \{0, \ldots, |U_2|\}$, let $\Lambda_{U, w_1 w_2, j} = \{i \in \{0, \ldots, n\} \mid \tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_1 2^j w_2$ or $\tau(\{r_i, r_{i+1}\} \cup U_{2,i}) \equiv w_2 2^j w_1\}$.

THEOREM 1. *Let U and V be finite sets of numbers in $(0,1)$. Then the following statements are equivalent.*

1. *H_U and H_V are isomorphic.*
2. *$|U_2| = |V_2|$, and $|\Lambda_{U, w_1 w_2, j}| = |\Lambda_{V, w_1 w_2, j}|$ for each $w_1 w_2 \in \{11, 13, 33\}$ and each $j \in \{0, \ldots, |U_2|\}$.*

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Present Address:
TAKAMICHI SATO
GRADUATE SCHOOL OF FUNDAMENTAL SCIENCE AND ENGINEERING,
WASEDA UNIVERSITY,
3-4-1 OKUBO, SHINJUKU-KU, TOKYO, 169-8555, JAPAN.
e-mail: t.sato@aoni.waseda.jp