The Penrose type twistor correspondence for the exceptional simple Lie group G_2

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1 Introduction

The following diagram is known:

$$
\begin{array}{ccccccc}
 & & G_2/U(2)_- & & & & G_2/U(2)_+ \\
 & \overset{\varpi}{\underset{\mathbb{C}P^2}{\swarrow}} & & \overset{\pi_-}{\underset{S^2}{\searrow}} & & \overset{\pi_+}{\underset{S^2}{\swarrow}} & \\
G_2/SU(3) & & & & G_2/SO(4) & &
\end{array}
\qquad (1.1)
$$

Here, $U(2)_\pm$ are two types of $U(2)$ embedded in G_2. As well known, $G_2/SU(3)$ is isomorphic to S^6, and S^6 is equipped with a natural non-integrable almost complex structure. It is also well known that $G_2/SO(4)$ is a 8-dimensional Riemannian symmetric space equipped with a quaternion Kähler structure. The fibration $\pi_+ : G_2/U(2)_+ \to G_2/SO(4)$ is the *twistor fibration* of the quaternion Kähler structure. The map $\varpi : G_2/U(2)_- \to G_2/SU(3)$ is also known as a *twistor fibration* with respect to the almost complex structure on S^6.

On the other hand, on the diagram (1.1), the double fibration given by ϖ and π_- is considered as the "Penrose type" twistor correspondence which is summarized as follows. Let Z be a complex 3-fold. This Z is called the *twistor space*. If Z contains a rational curve Y with normal bundle holomorphically isomorphich to $\mathcal{O}(1) \oplus \mathcal{O}(1)$, such rational curve is called *twistor line*. In general, the set of twistor lines consists a complex 4-fold M with naturally defined self-dual complex conformal structure. This M is called the *space-*

time. Then we obtain the following double fibration:

$$\begin{array}{ccc} & F & \\ {\scriptstyle\varpi}\swarrow & & \searrow{\scriptstyle\pi} \\ Z^3 & & M^4 \end{array} \tag{1.2}$$

For each $p \in Z$, the set $\pi(\varpi^{-1}(p))$ is 2-dimensional complex submanifold on M in general. Such complex surfaces are called β-surfaces, and the family of β-surfaces characterizes the self-dual structure of M.

In this article, we show that the double fibration by ϖ and π_- on the diagram (1.1) actually have an analogous structure with the Penrose's double fibraion. We show that for each $p \in S^6 \simeq G_2/SU(3)$, the subset $\mathfrak{S}_p = \pi_-(\varpi(p))$ is a totally geodesic, totally quaternionic 4-dimensional submanifold on $G_2/SO(4)$ (Theorem 6.3). Further, we show that there exists a symmetric 3-form γ, which satisfies certain integrable condition (Theorem 6.4). In the way to prove these theorem, we study the detail structure of the symmetric space $G_2/SO(4)$, for example, we describe explicitly the tangent space.

Here we remark about the recent work given by Enoyoshi-Tsukada [4]. They notice to the following another double fibration

$$\begin{array}{ccc} & G_2/SO(3) & \\ \swarrow & & \searrow \\ G_2/SU(3) & & G_2/SO(4) \end{array} \tag{1.3}$$

This double fibration is related to the *special Lagrangian submanifold* (or *totally real submanifold*) of S^6. The idea of Penrose type twistor correspondence also takes an important role of this theory. We, however, do not investigate in this theory in this article.

2 Construction of the fibration

2.1 quaternion and G_2

Let $\mathbb{H}$ be the quaterenions generated by $\{1, i, j, k\}$ where $i^2 = j^2 = k^2 = -1$ and $k = ij = -ji$. We write $Sp(1) = \{q \in \mathbb{H} \mid |q| = 1\}$. Let

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon = \mathrm{Span}_{\mathbb{R}}\langle 1, i, j, k, i\varepsilon, j\varepsilon, k\varepsilon\rangle = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{O} \tag{2.1}$$

be the Cayley numbers. The multiplication on $\mathbb{O}$ is defined by $(a + b\varepsilon)(c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon$. The inner product on $\mathbb{O}$ is $\langle x, y\rangle = \mathrm{Re}(x\bar{y})$. The 14-dimensional compact Lie group G_2 is defined as the aoutomorphism group of $\mathbb{O}$, that is

$$G_2 = \{g \in GL(\mathbb{O}) \mid g(xy) = g(x)g(y) \text{ for any } x, y \in \mathbb{O}\}. \tag{2.2}$$

Its Lie algebra $\mathfrak{g}_2$ is given by

$$\mathfrak{g}_2 = \{X \in \mathrm{End}(\mathbb{O}) \mid X(xy) = X(x)y + xX(y) \text{ for any } x, y \in \mathbb{O}\}. \tag{2.3}$$

As well known, $G_2 \subset SO(\mathrm{Im}\,\mathbb{O}) \simeq SO(7)$ and consequently $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. We define an inner product on $\mathfrak{g}_2$ by

$$\langle X, Y\rangle = -\mathrm{Tr}\, XY \qquad (X, Y \in \mathfrak{g}_2). \tag{2.4}$$

2.2 almost complex structure on S^6

Let $S^6 = \{p \in \mathrm{Im}\,\mathbb{O} \mid |p| = 1\}$ be the set of *imaginary units*. The tangent space at $p \in S^6$ is $T_pS^6 = \{u \in \mathrm{Im}\,\mathbb{O} \mid \langle u, p\rangle = 0\}$. A natural almost complex structure J on S^6 is defined by

$$J_p : T_pS^6 \to T_pS^6, \qquad J_p(u) = pu. \tag{2.5}$$

It is well-known that the almost complex structure J is not integrable.

The group G_2 acts transitively on S^6 and the isotropy subgroup at $i \in S^6$ is $SU(3)$ (see [5]). Hence $S^6 \simeq G_2/SU(3)$.

2.3 associative Grassmannian

A 3-dimensional subspace $V \subset \mathrm{Im}\,\mathbb{O}$ is called an *associative 3-plane* if and only if $(xy)z = x(yz)$ holds for any $x, y, z \in V$. We put

$$\mathbb{H}_V = \mathbb{R} \oplus V. \tag{2.6}$$

Then the 3-plane V is associative if and only if $\mathbb{H}_V \subset \mathbb{O}$ is a quaternion subspace, i.e. $\mathbb{H}_V$ is a subalgebra of $\mathbb{O}$ and is isomorphic to $\mathbb{H}$.

Let $Gr_3^+(\mathrm{Im}\,\mathbb{O})$ be the Grassmann manifold of oriented 3-planes on $\mathrm{Im}\,\mathbb{O}$. We write

$$Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O}) = \{V \in Gr_3^+(\mathrm{Im}\,\mathbb{O}) \mid V \text{ is associative}\}, \tag{2.7}$$

and we call $Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$ as *associative Grassmannian*. The following properties hold (see [5]).

Proposition 2.1. (i) *If $x, y \in \mathrm{Im}\,\mathbb{O}$ and $x \perp y$, then $\{x, y, xy\}$ spans an associative 3-plane. Any associative 3-plane is written in this way. Consequently, any associative 3-plane has a natural orientation.*
(ii) *G_2 acts transitively on $Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$. The isotropy subgroup at* $\mathrm{Im}\,\mathbb{H}$ *is $SO(4)$. Hence $Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O}) \simeq G_2/SO(4)$ and $Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$ is an 8-dimensional Riemannian symmetric space.*

Further, $Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O}) \simeq G_2/SO(4)$ has a *quaternion Kähler structure* which we will explain in Section 5 (see also [2]). We also describe the isotropy subgroup $SO(4) \subset G_2$ explicitly in section 3.

2.4 associative calibration

The *associative calibration* φ is the 3-linear form on $\mathrm{Im}\,\mathbb{O}$ defined by

$$\varphi(x, y, z) = \langle x, yz \rangle. \tag{2.8}$$

The following is known.

Proposition 2.2 ([5])**.** (i) *Let $V \in Gr_3^+(\mathrm{Im}\,\mathbb{O})$ and $\{v_1, v_2, v_3\}$ is an oriented orthonormal basis on V. Then*

$$\varphi(V) = \varphi(v_1, v_2, v_3) \tag{2.9}$$

is independent of the choice of the basis.
(ii) $\varphi(\overline{V}) = -\varphi(V)$, *where* $\overline{V}$ *is the orientation reversing of* V.
(iii) $|\varphi(V)| \leq 1$. *In particular* $\varphi(V) = 1$ *if and only if* V *is associative.*

Consequently, we can write

$$Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}) = \{V \in Gr^+_3(\operatorname{Im}\mathbb{O}) \mid \varphi(V) = 1\}. \tag{2.10}$$

2.5 flag manifold $F^+_{1,\text{ass}}(\operatorname{Im}\mathbb{O})$

We have the following double fibration

$$\begin{array}{ccc} & Gr^+_2(\operatorname{Im}\mathbb{O}) & \\ \swarrow^{\varpi} & & \searrow^{\pi_-} \\ S^6 & & Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}), \end{array} \tag{2.11}$$

where ϖ and π_- is defined as follows: let $\xi \in Gr^+_2(\operatorname{Im}\mathbb{O})$ and $\{v_1, v_2\}$ be an oriented orthonormal basis of ξ, then

$$\varpi(\xi) = v_1 v_2 \in S^6, \qquad \pi_-(\xi) = \operatorname{Span}_{\mathbb{R}}\langle v_1, v_2, v_1 v_2\rangle \in Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}). \tag{2.12}$$

The oriented 2-plane $V = \{v_1, v_2\}$ is one-to-one corresponds with the pair $(p, V) \in S^6 \times Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O})$ satisfying $p \in V$ so that $p = v_1 v_2$ and $V = \operatorname{Span}_{\mathbb{R}}\langle v_1, v_2, v_1 v_2\rangle$. Hence the Grassmann manifold $Gr^+_2(\operatorname{Im}\mathbb{O})$ is naturally identified with the flag manifold

$$Fl^+_{1,\text{ass}}(\operatorname{Im}\mathbb{O}) = \{(p, V) \in S^6 \times Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}) \mid p \in V\}. \tag{2.13}$$

Hence we can replace (2.11) by

$$\begin{array}{ccc} & Fl^+_{1,\text{ass}}(\operatorname{Im}\mathbb{O}) & \\ \swarrow^{\varpi} & & \searrow^{\pi_-} \\ S^6 & & Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}), \end{array} \tag{2.14}$$

In this notation, $\varpi(p, V) = p, \pi_-(p, V) = V$ are the natural projections.

The group G_2 acts $Fl^+_{1,\text{ass}}(\operatorname{Im}\mathbb{O})$ transitively, and the isotorpy subgroup at $(i, \operatorname{Im}\mathbb{H})$ is

$$U(2)_- = SU(3) \cap SO(4) = \{g \in G_2 \mid g(i) = i, g(\operatorname{Im}\mathbb{H}) = \operatorname{Im}\mathbb{H}\}. \tag{2.15}$$

This group is isomorphic to $U(2)$, which we see in the next section. In this way we obtain

$$Gr_2^+(\mathrm{Im}\,\mathbb{O}) \simeq Fl_{1,\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O}) \simeq G_2/U(2)_-. \tag{2.16}$$

2.6 submanifolds in S^6 and $Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$

The following proposition means π_- is a $\mathbb{CP}^1$-bundle, while ϖ is a $\mathbb{CP}^2$-bundle.

Proposition 2.3. (i) *For each* $V \in Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$, $Y_V = \varpi(\pi^{-1}(V))$ *is a puedo-holomorphic* $\mathbb{CP}^1$ *in* S^6.
(ii) *For each* $p \in S^6$, $\mathfrak{S}_p = \pi(\varpi^{-1}(p))$ *has a natural complex structure and is biholomorphic to* $\mathbb{CP}^2$.

Proof. We have $Y_V = \{p \in V \mid |p| = 1\} = S^6 \cap V \simeq S^2$. For each $p \in Y_V$, we can write $V = \mathrm{Span}_{\mathbb{R}}\langle p, x, J_p x\rangle$ for some $x \in T_pS^6$. Then $T_pY_V = \mathrm{Span}_{\mathbb{R}}\langle x, J_p x\rangle$ is a complex line in $T_pS^6 \simeq \mathbb{C}^3$. Thus Y_V is a psuedo-complex $\mathbb{CP}^1$ in S^6. So (i) is proved.

Next, for $p \in S^6$, we have

$$\mathfrak{S}_p = \{V \in Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O}) \mid p \in V\}.$$

When $p \in V \in Gr_{\mathrm{ass}}^+(\mathrm{Im}\,\mathbb{O})$, we can write $V = \mathrm{Span}_{\mathbb{R}}\langle p, x, J_p x\rangle$ for some $x \in T_pS^6$. Such V one-to-one corresponds with the complex line $\mathrm{Span}_{\mathbb{R}}\langle x, J_p x\rangle \subset T_pS^6 \simeq \mathbb{C}^3$. Hence $\varpi^{-1}(p)$ is naturally identified with the complex projectivization of $T_pS^6 \simeq \mathbb{C}^3$. □

3 Explicit description of the subgroups

3.1 $SO(4) \subset G_2$

For $(q_1, q_2) \in Sp(1) \times Sp(1)$, we define

$$\rho(q_1, q_2)(a + b\varepsilon) = q_1 a \bar{q}_1 + (q_2 b \bar{q}_1)\varepsilon \qquad (a \in \mathrm{Im}\,\mathbb{H}, b \in \mathbb{H}).$$

It is known that ρ defines an homomorphism $Sp(1) \times Sp(1) \to G_2$. In a matrix style, we can write

$$\rho(q_1, q_2) = \begin{pmatrix} \mathrm{Ad}_{q_1} & O \\ O & L_{q_2} R_{\bar{q}_1} \end{pmatrix} \tag{3.1}$$

with respect to the decomposition $\mathrm{Im}\,\mathbb{O} \simeq \mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$. Since the kernel of ρ is $\mathbb{Z}_2 \simeq \{\pm(1,1)\}$, ρ defines an embedding $SO(4) \simeq (Sp(1) \times Sp(1))/\mathbb{Z}_2 \to G_2$. Further, we have the following (see [5])

$$SO(4) = \left\{ \begin{pmatrix} * & O \\ O & * \end{pmatrix} \in G_2 \right\} = \{ g \in G_2 \mid g(\mathrm{Im}\,\mathbb{H}) = \mathrm{Im}\,\mathbb{H} \} \tag{3.2}$$

3.2 $U(2)_\pm$ and $SU(3)$

Two subgroups of G_2 are defined by

$$U(2)_+ = \rho(Sp(1) \times U(1)), \qquad U(2)_- = \rho(U(1) \times Sp(1)), \tag{3.3}$$

where $U(1) = \{ q \in \mathbb{C} \subset \mathbb{H} \mid |q| = 1 \} \subset Sp(1)$. Though both subgroups are abstractly ismorphic to $U(2)$, the embeddings are not equivalent to each other. Actually, for example, the homotopy types of $G_2/U(2)_\pm$ are different (see [7]).

Another subgroup is defined by

$$SU(3) = \{ g \in G_2 \mid g(i) = i \}. \tag{3.4}$$

The subgroups $SO(4)$, $U(2)_-$, $SU(3)$ are simply characterized by the block decomposition of 7×7 matrices, and we easily see $U(2)_- = SU(3) \cap SO(4)$.

4 Twistor correspondence

We compare our double fibration (2.14) with the Penrose's twistor correspondence.

4.1 The idea of Penrose's twistor correspondence

Penrose's theory ([8]) concerns with the correspondence between a complex 3-fold Z (called the *twistor space*) and a self-dual complex 4-fold M (called the *space-time*). The correspondence is constructed in the following way.

Let Z be a complex 3-fold. We notice to the family *twistor lines* $\{Y_t\}_{t\in M}$, that is, the family of rational curves (i.e. $Y_t \simeq \mathbb{CP}^1$) in Z such that the normal bundle N is biholomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. By the deformation theory, such family is parametrized by a complex 4-fold M. If we put $F = \{(z,t) \in Z \times M \mid z \in Y_t\}$, we obtain the double fibration

$$\begin{array}{ccccc} & & F & & \\ & \stackrel{\varpi}{\swarrow} & \mathbb{CP}^1 & \stackrel{\pi}{\searrow} & \\ Z & & & & M \end{array} \tag{4.1}$$

where ϖ and π are natural projection.

For each $t \in M$, the corresponding object in Z is by definition $\varpi(\pi^{-1}(t)) = Y_t$, which is a holomorphic $\mathbb{CP}^1$ in Z.

On the other hand, for each $z \in Z$, the corresponding object in M is $\mathfrak{S}_z = \pi(\varpi^{-1}(z))$. Each $\mathfrak{S}_z$ is, if not empty, a 2-dimensional complex submanifold in M and is called *β-surface*. There is a unique complex conformal structure $[g]$ on M satisfying $g|_{\mathfrak{S}_z} = 0$ for any $z \in Z$. We can prove that this conformal structure $[g]$ is *self-dual* (i.e. half conformally flat).

4.2 Twistor correspondence for $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$

Our double fibration (2.14) is quite similar to the Penrose's double fibration (4.1) in the following sense.

The correspondence spaces F and $Fl^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ are both the total space of $\mathbb{CP}^1$-bundle over the "space-time" M and $Gr^+_{\mathrm{ass}}(\mathrm{Im}(\mathbb{O})$.

The twistor space Z is a complex 3-fold while S^6 is a real 6-dimensional manifold with an almost complex structure. Z has a family of twistor lines $\{Y_t\}$ ($Y_t \simeq \mathbb{CP}^1$) while S^6 has a family of psuedo holomorphic curves $\{Y_V\}$ ($Y_V \simeq \mathbb{CP}^1$).

The space-time M is a complex 4-fold while $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ is a real 8-dimensional quaternion Kähler manifold. M has a family of β-surfaces $\{\mathfrak{S}_z\}$ while $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ has a family of submanifolds $\{\mathfrak{S}_p\}$ ($\mathfrak{S}_p \simeq \mathbb{CP}^2$).

	Penrose's case	Our case
corresp. sp.	F $\mathbb{CP}^1$-bundle over M	$Fl^+_{1,\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ $\mathbb{CP}^1$-bundle over $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$
twistor space	Z (complex 3-fold) twistor lines $\{Y_t\}$	S^6 (almost complex 6-fold) psued-holo. curves $\{Y_V\}$
space-time	M (complex 4-fold) self-dual β-sufaces $\{\mathfrak{S}_z\}$	$Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ (q. Kähler 8-fold) ?? submanifolds $\{\mathfrak{S}_p\}$

In this comparison, it seems natural to expect that $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ has some extra geometric structure corresponding with the self-dual structure on M. We investigate this geometric structure in Section 5 and 6.

5 Explicit description of the tangent space

5.1 Tangent space of $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$

Proposition 5.1. *There is a natural identification*

$$T_oGr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \simeq \left\{ f \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{H}, \mathbb{H}) \;\middle|\; f(i)i + f(j)j + f(k)k = 0 \right\}. \tag{5.1}$$

where $o = \mathrm{Im}\,\mathbb{H}$ *is the base point on* $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$.

Proof. We have $T_oGr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \simeq T_oG_2/SO(4) \simeq \mathfrak{g}_2/\mathfrak{so}(4) \simeq \mathfrak{p}$, where $\mathfrak{g}_2 = \mathfrak{so}(4) \oplus \mathfrak{p}$ is the Cartan decomposition for $G_2/SO(4)$. In the matrix style,

$$\mathfrak{so}(4) = \left\{ \begin{pmatrix} * & O \\ O & * \end{pmatrix} \in \mathfrak{g}_2 \right\}, \qquad \mathfrak{p} = \left\{ \begin{pmatrix} O & -f^* \\ f & O \end{pmatrix} \in \mathfrak{g}_2 \right\}.$$

So we check that $X = \begin{pmatrix} O & -f^* \\ f & O \end{pmatrix}$ ($f \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{H}, \mathbb{H})$) is contained in $\mathfrak{p}$ if and only if f satisfies the condition $f(i)i + f(j)j + f(k)k = 0$.

For each $x \in \mathrm{Im}\,\mathbb{H}$ we have $X(x) = f(x)\varepsilon$. On the other hand, for $x, y \in \mathrm{Im}\,\mathbb{H}$, we obtain

$$X(xy) = X(x)y + xX(y)$$

by the definition of $\mathfrak{g}_2$. Hence

$$f(xy)\varepsilon = (f(x)\varepsilon)y + x(f(y)\varepsilon) = (f(x)\bar{y})\varepsilon + (f(y)x)\varepsilon,$$

that is,

$$f(xy) = f(x)\bar{y} + f(y)x.$$

Putting $x = j, y = k$, we obtain $f(i)i + f(j)j + f(k)k = 0$. Thus

$$T_o Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \subset \left\{ f \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\,\mathbb{H}, \mathbb{H}) \;\middle|\; f(i)i + f(j)j + f(k)k = 0 \right\}.$$

Both vector spaces have real dimension 8, so these are equal. □

5.2 The quaternion Kähler structure on $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$

Let $V \in Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ and we define

$$\mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) = \left\{ f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{H}_V) \;\middle|\; f(e_1)e_1 + f(e_2)e_2 + f(e_3)e_3 = 0 \right\}, \tag{5.2}$$

where $\mathbb{H}_V = \mathbb{R} \oplus V$ is the quaternion subalgebra of $\mathbb{O}$ and $\{e_1, e_2, e_3\}$ is an oriented orthonormal basis of V. Then, as a consequence of (5.1), we obtain the identification

$$T_V Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O}) \simeq \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V). \tag{5.3}$$

The vector space $\mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V)$ has a natural $\mathbb{H}_V$-module structrue defined by the left multiplication. This is the quaternion Kähler structure on $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$.

5.3 Infinitesimal deformation

A tangent vector $X \in T_V Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ is considered as an infinitesimal deformation of associative 3-plane in the following way.

For the simplicity, we assume $V = o = \mathrm{Im}\,\mathbb{H}$. Let $c(t)$ be a smooth curve on $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$ satisfying $c(0) = o$. We can take a curve $g(t)$ on G_2 so that $c(t) = g(t) \cdot o$ and $g(0) = I$. Then the differential $g'(0)$ is determined uniquely up to $\mathfrak{so}(4)$. This means that the infinitesimal deformation $c'(0)$ can be written as

$$c'(0) = g'(0) + \mathfrak{so}(4) \quad \in \quad \mathfrak{g}_2/\mathfrak{so}(4). \tag{5.4}$$

5.4 The submanifold $\mathfrak{S}_p$

Lemma 5.2. *Let $p \in S^6$ and $V \in \mathfrak{S}_p$ (i.e. $p \in V \in Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$). Then*

$$T_V\mathfrak{S}_p = \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(p) = 0\}\,. \tag{5.5}$$

Proof. We assume $V = o = \mathrm{Im}\,\mathbb{H}$ for the simplicity. For a tangent vector $X \in T_o\mathfrak{S}_p$, let us take a smooth curve $c(t) = g(t) \cdot o$ on $\mathfrak{S}_p$ so that $g(t) \in G_2$, $g(0) = I$ and $c'(0) = X$.

By definition, $p \in g(t) \cdot o$ for any t. Changing the choice of $g(t)$ if needed, we can assume $g(t) \cdot p = p$. Then $g'(0) \cdot p = 0$. If $f \in \mathrm{Hom}_{\mathrm{ass}}(o, \mathbb{H})$ be the corresponding linear map with $X = c'(0) = g'(0) + \mathfrak{so}(4)$, we obtain $f(p) = 0$. □

Corollary 5.3. *Let $p \in S^6$. Then $\mathfrak{S}_p$ is a real 4-dimensional totally quaternionic submanifold of $Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})$.*

Proof. Direct calculation. □

6 The cone field and the symmetric 3-form

6.1 The cone field

In the Penrose's twistor theory, the self-dual structure (more precisely, the self-dual complex conformal structure) $[g]$ is defined so that its *null cone* is tangent to β-surfaces everywhere.

Similarly in our case, we notice to the *cone field* $\mathcal{C}$ defined by

$$\mathcal{C}_V := \bigcup_{V \in \mathfrak{S}_p} T_V\mathfrak{S}_p \qquad (V \in Gr^+_{\mathrm{ass}}(\mathrm{Im}\,\mathbb{O})). \tag{6.1}$$

Then

$$\begin{aligned}
\mathcal{C}_V &= \bigcup_{p \in S(V)} \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(p) = 0\} \\
&= \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(p) = 0 \text{ for some } p \in S(V)\} \\
&= \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid \mathrm{rank}_{\mathbb{R}}\, f < 2\} \\
&= \{f \in \mathrm{Hom}_{\mathrm{ass}}(V, \mathbb{H}_V) \mid f(e_1) \times f(e_2) \times f(e_3) = 0\}
\end{aligned}$$

where $\{e_1, e_2, e_3\}$ is the oriented orthonormal basis of V and

$$x \times y \times z = \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x)) \tag{6.2}$$

is the *triple cross product.*

6.2 The symmetric 3-form

Let us define a *cubic form* $P : T_V Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}) \to \mathbb{H}_V$ by

$$P(f) = f(e_1) \times f(e_2) \times f(e_3) \tag{6.3}$$

which is independent of the choice of the oriented orthonormal basis $\{e_1, e_2, e_3\}$ on V. Since any polynomial one-to-one corresponds with a symmetric tensor, we can define $\mathbb{H}_V$-valued symmetric 3-form γ such that

$$P(f) = \gamma(f, f, f) \tag{6.4}$$

for any $f \in T_V Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O})$. By definition, we obtain

$$\mathcal{C}_V = \left\{ f \in T_V Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}) \mid \gamma(f, f, f) = 0 \right\}. \tag{6.5}$$

6.3 Main results

The associative Grassmannian $Gr^+_{\text{ass}}(\operatorname{Im}\mathbb{O}) \simeq G_2/SO(4)$ is equipped with the natural Riemannian metric h. Let ∇, R be the Riemannian connection and the Riemannian curvature tensor of h.

Theorem 6.1. *The symmetric 3-form γ is parallel, i.e. $\nabla\gamma = 0$.*

Proof. Let $\varrho : SO(4) \to SO(\mathfrak{p})$ be the isotropy representation of $G_2/SO(4)$ at the base point. Then by the property of the triple cross product, we obtain

$$P(\varrho(g)f) = g \cdot P(f). \tag{6.6}$$

Thus we obtain

$$\gamma(\varrho(g)\varphi, \varrho(g)\psi, \varrho(g)\chi) = g \cdot \gamma(\varphi, \psi, \chi). \tag{6.7}$$

Taking the differential, we obtain

$$\gamma(\varrho_*(A)\varphi, \psi, \chi) + \gamma(\varphi, \varrho_*(A)\psi, \chi) + \gamma(\varphi, \psi, \varrho_*(A)\chi) = A \cdot \gamma(\varphi, \psi, \chi). \quad (6.8)$$

for $A \in \mathfrak{so}(4)$. This means

$$\gamma(\nabla\varphi, \psi, \chi) + \gamma(\varphi, \nabla\psi, \chi) + \gamma(\varphi, \psi, \nabla\chi) = \nabla\gamma(\varphi, \psi, \chi) \quad (6.9)$$

i.e. ∇ is parallel. □

Lemma 6.2. *Let $p \in S^6$ and $V \in \mathfrak{S}_p$.*
(i) $\gamma(\varphi, \psi, \chi) = 0$ *for any $\varphi, \psi, \chi \in T_V\mathfrak{S}_p$.*
(ii) *Let φ, ψ be the complex basis of $\mathfrak{S}_p \simeq \mathbb{CP}^2$. Then $\chi \in T_V\mathfrak{S}_p$ if and only if $\gamma(\chi, \varphi, \psi) = 0$.*

Proof. This is directly checked when $V = \mathrm{Im}\mathbb{H}$ and $p = i$. Then the statement follows by the G_2-symmetricty. □

Theorem 6.3. *For any $p \in S^6$, the submanifold $\mathfrak{S}_p$ is real 4-dimensional, totally quaternionic and totally geodesic.*

Proof. By Corollary 5.3, we only need to show $\mathfrak{S}_p$ is totally geodesic.

For vector fields $v, w \in \mathfrak{X}(\mathfrak{S}_p)$, we have $[v, w] \in \mathfrak{X}(\mathfrak{S}_p)$. By $\gamma(v, v, v) = 0$, we obtain $0 = \nabla_w\gamma(v, v, v) = 3\gamma(\nabla_w v, v, v)$. Hence by $\gamma(v, v, w) = 0$,

$$2\gamma(\nabla_v v, v, w) = -\gamma(v, v, \nabla_v w) = -\gamma(v, v, \nabla_w v + [v, w]) = 0.$$

By Lemma 6.2, if we take v, w to be the complex basis, $\nabla_v v \in \mathfrak{X}(\mathfrak{S}_p)$.

On the other hand, by $\gamma(v, w, w) = 0$,

$$2\gamma(v, \nabla_v w, w) = -\gamma(\nabla_v v, w, w) = 0.$$

Hence $\nabla_v w \in \mathfrak{X}(\mathfrak{S}_p)$. Thus $\mathfrak{S}_p$ is totally geodesic. □

Theorem 6.4. *Let $p \in S^6$ and $V \in \mathfrak{S}_p$. Then, for any tangent vectors $\varphi, \psi \in T_V\mathfrak{S}_p$,*

$$\gamma(R(\varphi, \psi)\varphi, \varphi, \psi) = 0. \quad (6.10)$$

Proof. We can assume $\{\varphi, \psi\}$ is the complex basis. Extending φ, ψ to a vector field, we obtain

$$R(\varphi, \psi)\varphi = \nabla_\varphi \nabla_\psi \varphi - \nabla_\psi \nabla_\varphi \varphi - \nabla_{[\varphi,\psi]}\varphi \quad \in \quad \mathfrak{X}(\mathfrak{S}_p). \tag{6.11}$$

Hence we obtain (6.10). □

Remark 6.5. Theorem 6.4 is an analogy of the self-duality. Actually, a Riemannian manifold (M, g) is self-dual if and only if

$$g(R(X, Y)X, Y) = 0$$

for any tangent vector X, Y (see [6]).

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