Bender-Knuth transformation from a perspective of hives

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Abstract

Kostka numbers $K_{\lambda\mu}$ are non-negative integers indexed by two partitions λ and μ. They equal to the number of semistandard tableaux of shape λ and weight μ. Also they equal to the number of K-hives with boundary edge label determined by λ and μ. Their basic property is $K_{\lambda\mu} = K_{\lambda\sigma(\mu)}$ for any element σ of the symmetric group. This is proved by constructing a bijection called the Bender-Knuth transformation between semistandard tableaux. In this paper, we give a perspective of the Bender-Knuth transformation through the hive model.

1 Introduction

Kostka numbers are important and classical numbers in combinatorics and representation theory. Let λ be a partition of $n \in \mathbb{N}$ and μ be a composition of n which is a partition not assuming the decreasing order. In combinatorics, Kostka number $K_{\lambda\mu}$ equals to the number of semistandard tableaux of shape λ and weight μ. Also, when one expresses the Schur function s_λ as a linear combination of monomial symmetric function m_μ, Kostka numbers appear as the coefficients so that $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$, where λ is a partition of n and the sum is over all partitions μ. In representation theory, Kostka numbers appear as a multiplicity of irreducible representations. More precisely, let S^λ be Specht modules which are irreducible modules of the symmetric group and M^λ be the permutation module. Then we have $M^\mu = \bigoplus_\lambda K_{\lambda\mu} S^\lambda$.

A fundamental property of Kostka number is that $K_{\lambda\mu} = K_{\lambda\sigma(\mu)}$ holds for any element σ of the symmetric group. In fact, this is shown by constructing a bijection between semistandard tableaux. The bijection is called the Bender-Knuth transformation [2] which was also used to prove the Littlewood-Richardson rule [12].

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On the other hand, the hive model was introduced by A.Knutoson and T.Tao [9]. It is a triangular graph like (2). If a hive's boundary edge label is determined and the hive satisfies some conditions, then it is called K-hive. And we see that a K-hive 1-1 corresponds to a semistandard tableau. Hence, Kostka number equals to the number of some K-hives.

In this paper, we give a perspective of the Bender-Knuth transformation from the hive model, namely, we construct a bijection between K-hives corresponding to the Bender-Knuth transformation.

2 Definitions

2.1 Kostka number and related notation

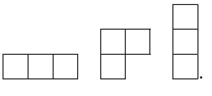
In this section, we will define Kostka numbers and explain the Bender-Knuth transformation. Throughout this paper, $\mathfrak{S}_n$ is the symmetric group of degree n .

Definition 1
$\lambda = (\lambda_1, \dots, \lambda_k)$ is called a partition of $n \in \mathbb{N}$ if it satisfies $\lambda_i \in \mathbb{Z}_{\geq 0}\,(i = 1, \dots, k)$, $\lambda_1 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. Also, each λ_i is called a part. We denote the sum of all parts by $|\lambda|$. If λ satisfies conditions $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 + \cdots + \lambda_k = n$ (but not necessarily $\lambda_1 \geq \cdots \geq \lambda_k$) then it is called a composition.

We define an action of $\mathfrak{S}_n$ on compositions as follows. Let $\mu = (\mu_1, \dots, \mu_k)$ be a composition of n. Then, for $\sigma \in \mathfrak{S}_n$, $\sigma \cdot \lambda := (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k)})$.

Definition 2
Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n. The Young diagram of shape λ is an array of n boxes having k left-justified rows with row i containing λ_i boxes for $1 \leq i \leq k$.

Example 1
If $n = 3$, partitions of 3 are (3), $(2, 1)$, $(1, 1, 1)$. Then the corresponding Young diagram are respectively



A Young diagram can be filled with a number per box.

Definition 3
Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n and $\mu = (\mu_1, \dots, \mu_s)$ be a composition of n. T is a semistandard tableau of shape λ and weight μ if it is an array obtained by filling in the boxes with positive integers such that the number of i's' in T is μ_i, its rows weakly increase and its columns strictly increase.

Definition 4
Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n and $\mu = (\mu_1, \dots, \mu_s)$ be a composition of n. Kostka number $K_{\lambda\mu}$ is defined as the number of semistandard tableaux of shape λ and weight μ, namely

$$K_{\lambda\mu} = \{\textit{Semistandard tablueaux of shape } \lambda \textit{ and weight } \mu\}$$

Example 2

Let $\lambda = (3,2)$ *and* $\mu = (2,1,2)$*. Then Semistandard tableaux of shape* λ *and weight* μ *are*

$$\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 3&3\\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3\\ \cline{1-2}\end{array}$$

Then Kostka numbers $K_{\lambda\mu} = K_{(3,2),(2,1,2)} = 2$.

Also, we take another weight $\mu' = (2,2,1)\,(= (2\,3)\cdot\mu)$*, that is we consider the samistandard tableaux of shape* $\lambda = (3,2)$ *and weight* $\mu' = (2,2,1)$*. Then they are*

$$\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&3\\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2\\ \cline{1-2}\end{array}$$

Thus $K_{\lambda\mu'} = K_{(3,2),(2,2,1)} = 2$.

This example is not a special case. The following proposition shows this property of Kostka numbers.

Proposition 5 ([2], [11])

Let λ *be a partition of* n *and* μ *be a composition of* n*. Then for all* $\sigma \in \mathfrak{S}_n$ *we have* $K_{\lambda\mu} = K_{\lambda\sigma(\mu)}$.

Proof We will construct a bijection between semistandard tableaux of shape λ and weight μ and semistandard tableaux of shape λ and weight $\sigma(\mu)$. It suffices to show the proposition in the case $\sigma = (i\,i+1)$.

Take a semistandard tableau T of shape λ and weight μ. First, we fix blocks which are pairs i and $i+1$ in each column. We call the pairs *fixed* and all other occurrences of i or $i+1$ *free*. In each row, switch the number of free i's and $i+1$'s, more precisely if the row consists of k free i's followed by l free $i+1$'s then replace them by l free i's followed by k free $i+1$'s. Clearly, this is a bijection and the map yields a semistandard tableau. The map is called the *the Bender-Knuth transformation.* ∎

Example 3

We give an example of the Bender-Knuth transformation. Suppose $\lambda = (4,3)$, $\mu = (1,3,2,1)$ *and* $\sigma = (1,2)$*. Then* $\sigma(\mu) = (3,1,2,1)$*. The left of (1) is a semistandard tableau of shape* λ *and weight* μ*, and the right of (1) is a semistandard tableau corresponding to the left of (1) by the the Bender-Knuth transformation. We can see that its shape is* λ *and its weight is* $\sigma(\mu)$.

$$\begin{array}{|c|c|c|c|}\hline 1&2&2&3\\ \hline 2&3&4\\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\ \hline 2&3&4\\ \cline{1-3}\end{array} \tag{1}$$

Recall weight μ *controls the multiplicity of entries. We can check switching* 1 *and* 2.

2.2 Hive

Now we introduce the hive model, see [1] and [7].

Definition 6

In the vertex representation, An n*-hive graph is a labelling of an vertices of equilateral triangular graph.*

The following is the example of 4-hive graph in the vertex representation.

$$\begin{array}{ccccccc} & & & a_{03} & & & \\ & & \diagup & & \diagdown & & \\ & & a_{02} & — & a_{13} & & \\ & \diagup & \diagdown & \diagup & \diagdown & & \\ & a_{01} & — & a_{12} & — & a_{23} & \\ \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \\ a_{00} & — & a_{11} & — & a_{22} & — & a_{33} \end{array} \tag{2}$$

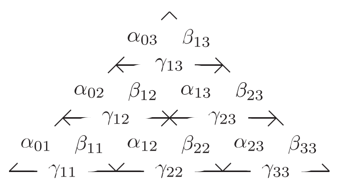
A hive graph has three different type rhombus (3) which are called the *elementary rhombus*.

$$\begin{array}{ccc} \begin{array}{ccc} a & — & b \\ \diagdown & \diagup & \diagdown \\ & c & — \; d \end{array} & \begin{array}{ccc} & a & \\ \diagup & & \diagdown \\ b & — & c \\ \diagdown & & \diagup \\ & d & \end{array} & \begin{array}{ccc} & b & — \; a \\ \diagup & \diagdown & \diagup \\ d & — & c \end{array} \\ \text{left-learning} & \text{upright} & \text{right-learning} \end{array} \tag{3}$$

Also, for the elementary rhombus (3) the *rhombus inequality* takes the following form (4).

$$b + c \geq a + d. \tag{4}$$

A hive graph has other expressions which are very useful. The *edge representation* is a labelling of all edges of a hive graph satisfying triangular conditions $\gamma = \alpha + \beta$ for elementary triangles (6) and betweenness condition (7). If $n = 4$, hive graph in the edge representation is below



(5)

Elementary triangles is below.

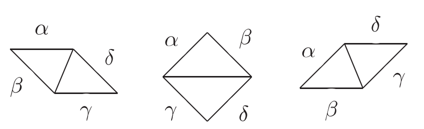
(6)

Also, betweenness conditions are:

$$\alpha_{i-1,j-1} \geq \alpha_{ij} \geq \alpha_{i-1,j}, \quad \beta_{ij} \geq \beta_{i,j-1} \geq \beta_{i+1,j}, \quad \gamma_{i,j-1} \geq \gamma_{ij} \geq \gamma_{i+1,j} \tag{7}$$

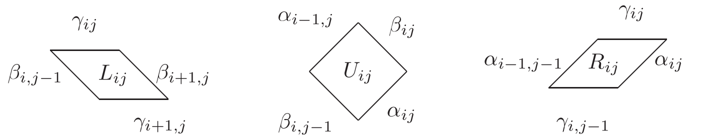
We can change the vertex representation to the edge representation in the following manner. In each edge between neighbouring vertices labelled a and b, edge label determined by means of the difference $b - a$ if b is on the right of a. In the illustration of the edge representation (5), parameters α_{ij}, β_{ij}, γ_{ij} is defined by $\alpha_{ij} = a_{ij} - a_{i\,j-1}$, $\beta_{ij} = a_{ij} - a_{i-1\,j}$, $\gamma_{ij} = a_{ij} - a_{i-1\,j-1}$.

In the edge representation, the rhombus inequality takes the form of $\alpha \geq \gamma$, $\beta \geq \delta$ in the labelling of below (8).



(8)

To introduce another expression of the hive graph, we introduce some notations. For three different elementary rhombi (9)



(9)

we define parameters L_{ij}, U_{ij} and R_{ij} by

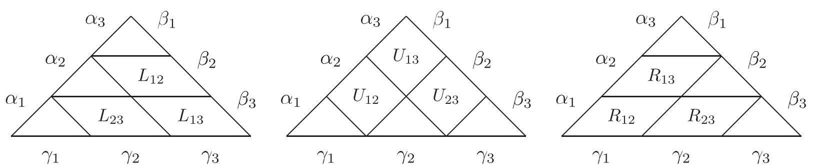
$$L_{ij} = \beta_{i,j-1} - \beta_{i+1,j} = \gamma_{ij} - \gamma_{i+1,j}, \tag{10}$$

$$U_{ij} = \alpha_{ij} - \alpha_{i-1,j} = \beta_{ij} - \beta_{i,j-1}, \tag{11}$$

$$R_{ij} = \alpha_{i-1,j-1} - \alpha_{ij} = \gamma_{i,j-1} - \gamma_{ij}. \tag{12}$$

They are called the *gradient* of the corresponding *left-leaning*, *upright* and *right-leaning* rhombi, respectively. Note that we can see $L_{ij} = \sum_{k=1}^{j-1} U_{i,k} - \sum_{k=1}^{j} U_{i+1,k}$.

A *gradient representation* is a labelling of boundary edges and gradients of giving the gradients of one or other of its three sets of right-leaning, upright or left-leaning elementary rhombi.



(13)

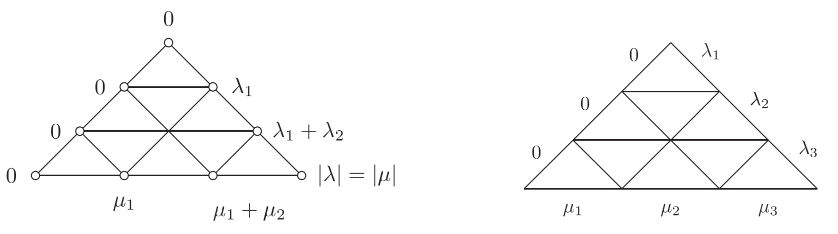
In a gradient representation, the rhombus inequalities change to $L_{ij} \geq 0, U_{ij} \geq 0, R_{ij} \geq 0$, respectively.

Now we define K-hive which is a hive graph satisfying some condition. As will be seen, it 1-1 corresponds to a semistandard tableau. If vertexes of a hive graph are integers we call it *integer hive graph*.

Definition 7

Let λ be a partition with $l(\lambda) \leq n$ and μ be a composition with $l(\mu) \leq n$, also $|\lambda| = |\mu|$. In the vertex representation, a K-hive is an integer hive graph satisfying the rhombus inequality (4) for left-leaning and upright (but not right-learning) with boundary edge labels determined by $a_{0i} = 0 \quad (i = 0, \ldots, n)$, $a_{ii} = \mu_1 + \cdots + \mu_i \quad (i = 1, \ldots, n)$, $a_{in} = \lambda_1 + \cdots + \lambda_i \quad (i = 1, \ldots, n)$. We denote by $\mathcal{H}(\lambda, \mu)$ the set of K-hives with boundary edge labels determined by λ and μ.

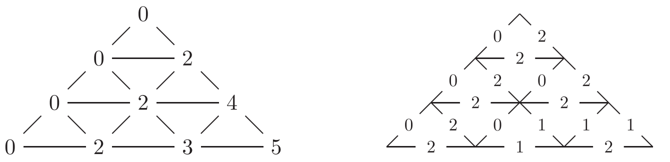
If $n = 4$, K-hives take the right form of (14) also in the edge representation it takes the left form of (14).



(14)

Example 4

If $\lambda = (2,2,1)$ and $\mu = (2,1,2)$, the left of (15) is a example of K-hive with boundary edge label determined by λ and μ. Also, in the edge representation, we have the right of (15).



(15)

3 Relationship between Kostka numbers and hives

In this section, we show that the number of semistandard tableaux of shape λ and weight μ equal to the number of K-hive with boundary edge label determined by λ and μ. Also, we explain the gradient representations in detail. We will start with the following Lemma.

Lemma 8

$$U_{ii} = \lambda_i - \sum_{i<k} U_{ik} = \mu_i - \sum_{i>k} U_{ki}$$

Proof We take a K-hive H and consider the semistandard tableau T corresponding to H. Since T is semistandard, entries of ith row of T are grater than i. Then $\lambda_i = \sum_{i\leq k} U_{ik}$. Thus we get $U_{ii} = \lambda_i - \sum_{i<k} U_{ik}$. Also Since T is semistandard again, the entries i is in from the top to the ith row, namely $\mu_i = \sum_{k\geq i} U_{ki}$. Then we get $U_{ii} = \mu_i - \sum_{k>i} U_{ki}$. ∎

Proposition 9

Let λ be a partition and μ be a composition of n with $l(\lambda), l(\mu) \leq n$. Then we have $K_{\lambda\mu} = \#\mathcal{H}(\lambda,\mu)$.

Proof To prove it, we construct a bijection between semistandard tableaux of shape λ and weight μ and K-hives with boundary edge labels determined by λ and μ. We define a map from tableau T to a hive graph H by

$$U_{ij} = \text{the number of } j \text{ in } i\text{th row} \tag{16}$$

and adding boundary edge label λ and μ. Also, the inverse map is constructed in the following manner. λ and μ determine shape and weight respectively. We can find entries of ith row as follows. From $i+1$ to n can be seen directly from U_{ij} $(i < j \leq n)$. Since T is semistandard, the entries of ith row is greater than i. Thus, it suffices to find the number of i, this is Lemma 8. Then, we arrange the entries in the weakly increasing order from left to right.

Then, we have only to show that the map from T to H is a map from semistandard tableaux to K-hives and its inverse is a map from K-hives to semistandard tableaux. Recall that U_{ij} implies the number of j in ith row and

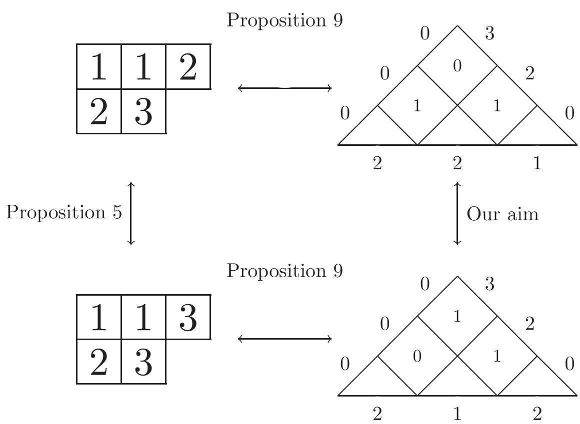
$$\begin{aligned} L_{ij} &= \sum_{k=1}^{j-1} U_{i,k} - \sum_{k=1}^{j} U_{i+1,k} \\ &= (\#\text{ of entries } \leq j-1 \text{ in row } i) - (\#\text{ of entries } \leq j \text{ in row } i+1). \end{aligned}$$

Take T and the image H of the map, we show H is a K-hive, namely H satisfies rhombus inequality for left-learning and upright. By definition, $U_{ij} \geq 0$. Since T is semistandard, $L_{ij} \geq 0$.

Conversely, take K-hive H and the image T of the inverse map. Since $U_{ij} \geq 0$, each row of T contains non-negative numbers of each distinct entry. Also since $L_{ij} \geq 0$, the number of entries in ith row above j's in $i+1$th row is less than j. Then T is semistandard. ∎

4 Bender-Knuth transformation from a perspective of hives

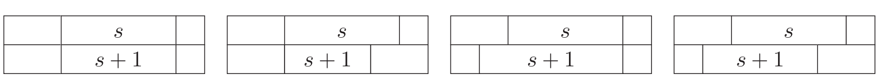
Now we give a perspective of the Bender-Knuth transformation from the hive model, namely, we give another proof of Proposition 5. Concretely, our aim is to construct a bijection between hives. Consider the situation such as:



(17)

In the Bender-Knuth transformation, it is constructed by swapping the number of entries in each row after fixing some blocks. To reproduce this bijection, we will start by identifying the number of fixed blocks from a K-hive.

Take $\sigma = (s\, s+1) \in \mathfrak{S}_n$ and we consider the ith and $i+1$th rows which have fixed blocks. Now possible situations for positions of s and $s+1$ are:



We call them type 1, type 2, type 3 and type 4 from left to right. The number of boxes up to a first fixed box is $\sum_{k=1}^{s-1} U_{ik} = \sum_{k=1}^{s} U_{i+1,k}$ in type 1 and type 2 and $\sum_{k=1}^{s-1} U_{ik}$ in type 3 and type 4, respectively. On the other hand, the number of boxes up to a last fixed box is $\sum_{k=1}^{s} U_{ik} = \sum_{k=1}^{s+1} U_{i+1,k}$ in type 1 and type 3 and $\sum_{k=1}^{s+1} U_{i+1,k}$ in type 2 and type 4, respectively. Now we denote by $F_{\sigma}^{i\,i+1}$ the number of fixed boxes between the ith row and $i+1$th row. If there is a fixed boxes between the ith row and $i+1$th row, then $F_{\sigma}^{i\,i+1}$ can be expressed by $\sum_{k=1}^{s+1} U_{i+1,k} - \sum_{k=1}^{s-1} U_{ik}$. Otherwise, $F_{\sigma}^{i\,i+1}$ is defined as 0.

Definition 10

Set $\sigma = (s, s+1) \in \mathfrak{S}_n$. Let L_{ij}, U_{ij} and R_{ij} be gradients of a $H(\lambda,\mu) \in \mathcal{H}^{(n)}(\lambda,\mu)$ and L'_{ij}, U'_{ij} and R'_{ij} be gradients of a $H'(\lambda,\mu') \in \mathcal{H}^{(n)}(\lambda,\mu)$. Then the map $\phi\colon \mathcal{H}^{(n)}(\lambda,\mu) \to \mathcal{H}^{(n)}(\lambda,\sigma(\mu))$ is defined as follows: $\mu' := \sigma \cdot \mu$ and

$$U'_{ij} = \begin{cases} U_{i,j+1} + F_\sigma^{i,i+1} - F_\sigma^{i-1,i} & (j = s) \\ U_{i,j-1} - F_\sigma^{i,i+1} + F_\sigma^{i-1,i} & (j = s+1) \\ U_{ij} \quad (j \neq s, s+1) \end{cases} \tag{18}$$

Our main result is as follows.

Theorem 11

$\phi\colon \mathcal{H}^{(n)}(\lambda,\mu) \to \mathcal{H}^{(n)}(\lambda,\sigma(\mu))$ is an involution.

Proof Clearly, ϕ is an involution, so we will check only the well-definedness, namely $H(\lambda,\mu)$ is just a K-hive. By definition, $\mu' = \sigma(\mu)$. Then it suffices to show that $H'(\lambda,\mu)$ satisfies the rhombus inequality for upright and left-leaning. We have

$$U'_{is} = U_{i,s+1} + F_\sigma^{i,i+1} - F_\sigma^{i-1,i} = U_{i,s+1} + F_\sigma^{i,i+1} - (\sum_{k=1}^{s+1} U_{i,k} - \sum_{k=1}^{s-1} U_{i-1,k}) \tag{19}$$

$$= F_\sigma^{i,i+1} + \sum_{k=1}^{s-1} U_{i-1,k} - \sum_{k=1}^{s} U_{i,k} = F_\sigma^{i,i+1} + L_{i-1,s} \geq 0 \tag{20}$$

Similarly, we can get $U_{i,s+1} \geq 0$. Also

$$L_{ij} = \sum_{k=1}^{j-1} U'_{i,k} - \sum_{k=1}^{j} U'_{i+1,k} \tag{21}$$

$$= \sum_{k\neq s,s+1}^{j-1} U_{i,k} + U'_{is} + U'_{i,s+1} - (\sum_{k\neq s,s+1}^{j} U_{i+1,k} + U'_{i+1,s} + U'_{i+1,s+1}) \tag{22}$$

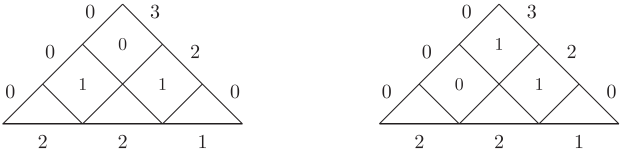
$$= \sum_{k=1}^{j-1} U'_{i,k} - \sum_{k=1}^{j} U'_{i+1,k} \geq 0 \tag{23}$$

Thus $H'(\lambda,\mu')$ is a K-hive ∎

Here let us calculate an involution ϕ in detail.

Example 5

Suppose that $\lambda = (3,2,0)$, $\mu = (2,2,1)$ and $\sigma = (2\,3) \in \mathfrak{S}_3$. Take the left of (24), then the corresponding K-hive is the right of (24).



(24)

where $F_\sigma^{12} = 0$, $F_\sigma^{23} = 0$, $U'_{12} = U_{13} = 0$, $U'_{13} = U_{12} = 1$, $U'_{23} = U_{22} = \lambda_2 - U_{23} = 1$. We can see that it agrees with (17).

5 Conclusion

We gave a perspective of the Bender-Knuth transformation through the hive model (11). More precisely, we constructed a bijection between K-hives. We have dealt with K-hives which are the hive models corresponding to semistandard tableaux. On the other hand, the hive is related not only Kostka numbers but also Littlewood-Richardson coefficients. It is called *LR-hive*. There is a relationship between Kostka numbers and Littlewood-Richardson coefficients. If the relationship is described using the hive model, we may get a good understanding of the relationship and the hive model.

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