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# Nonexistence of some Griesmer codes of dimension 5 *

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## 1 Introduction

A linear code over $\mathbb{F}_q$, the field of $q$ elements, of length $n$, dimension $k$ is a $k$-dimensional subspace $\mathcal{C}$ of the vector space $\mathbb{F}_q^n$ of $n$-tuples over $\mathbb{F}_q$. $\mathcal{C}$ is called an $[n,k,d]_q$ code if it has minimum Hamming weight $d$. A $k \times n$ matrix $G$ whose rows form a basis of $\mathcal{C}$ is a *generator matrix* of $\mathcal{C}$. A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length $n$ for which an $[n,k,d]_q$ code exists for given $q,k,d$ [6, 7]. A natural lower bound on $n_q(k,d)$ is the Griesmer bound:

$$n_q(k,d) \geq g_q(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil ,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$, see [1]. A linear code attaining the Griesmer bound is called a *Griesmer code*. The values of $n_q(k,d)$ are determined for all $d$ only for some small values of $q$ and $k$ [5, 16]. Note that $n_q(k,d) = g_q(k,d)$ for all $d$ when $k = 1$ or 2 [6]. The problem to determine $n_q(k,d)$ for all $d$ has been solved for $k \leq 8$ when $q = 2$, for $k \leq 5$ when $q = 3$, for $k \leq 4$ when $q = 4$ and only for $k = 3$ when $5 \leq q \leq 9$, see [16]. For the case $k = 5$, the following results are known.

**Theorem 1.1** ([2, 9, 10, 15])**.** *For any prime power $q$, $n_q(5,d) = g_q(5,d)$ for*

*(1)* $q^4 - q^3 - q + 1 \leq d \leq q^4 - q^3 + q^2 - q$,

*(2)* $q^4 - 2q^2 + 1 \leq d \leq q^4 + q$,

*(3)* $2q^4 - 3q^3 + 1 \leq d \leq 2q^4 - 3q^3 + q^2$,

*(4)* $2q^4 - 2q^3 - q^2 + 1 \leq d \leq 2q^4 + q^2 - q$,

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*(5)* $3q^4 - 5q^3 + q^2 + 1 \le d \le 3q^4 - 5q^3 + 2q^2$,

*(6)* $d \ge 3q^4 - 4q^3 + 1$.

**Theorem 1.2** ([3, 4, 11, 15, 16])**.** $n_q(5,d) = g_q(5,d) + 1$ *for*

*(1)* $q^4 - q^3 - q^2 + 1 < d \le q^4 - q^3 - q$ *for* $q \ge 3$,

*(2)* $q^4 - 2q^2 - 2q + 1 \le d \le q^4 - 2q^2 - q$ *for* $q \ge 4$,

*(3)* $q^4 - 2q^2 - q + 1 \le d \le q^4 - 2q^2$ *for* $q \ge 3$,

*(4)* $2q^4 - 2q^3 - q^2 - 2q + 1 \le d \le 2q^4 - 2q^3 - q^2$ *for* $q \ge 3$,

*(5)* $3q^4 - 4q^3 - 2q + 1 \le d \le 3q^4 - 4q^3 - q$ *for* $q \ge 11$,

*(6)* $3q^4 - 4q^3 - q + 1 \le d \le 3q^4 - 4q^3$ *for* $q \ge 5$.

Our main result is the following.

**Theorem 1.3.** $n_q(5,d) = g_q(5,d) + 1$ *for* $3q^4 - 4q^3 - 4q + 1 \le d \le 3q^4 - 4q^3 - q$ *for* $q \ge 5$.

## 2 Preliminaries

In this section, we give the geometric method through $\mathrm{PG}(r,q)$, the projective geometry of dimension $r$ over $\mathbb{F}_q$, and preliminary results to prove the main result. The 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats in $\mathrm{PG}(r,q)$ are called *points, lines, planes, solids, secundums* and *hyperplanes*, respectively.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix $G$ of $\mathcal{C}$ can be considered as a multiset of $n$ points in $\Sigma = \mathrm{PG}(k-1,q)$, denoted by $\mathcal{M}_\mathcal{C}$. A point $P$ of $\Sigma$ is an *i-point* if it has multiplicity $m_\mathcal{C}(P) = i$ in $\mathcal{M}_\mathcal{C}$. In other words, $m_\mathcal{C}(P)$ is the number of times which $P$ appears as columns of $G$. Denote by $\gamma_0$ the maximum multiplicity of a point from $\Sigma$ in $\mathcal{M}_\mathcal{C}$. For any subset $S$ of $\Sigma$, *the multiplicity of S with respect to* $\mathcal{M}_\mathcal{C}$, denoted by $m_\mathcal{C}(S)$, is defined as $m_\mathcal{C}(S) = \sum_{P\in S} m_\mathcal{C}(P)$. Then $m_\mathcal{C}$ satisfies $n = m_\mathcal{C}(\Sigma)$ and

$$n - d = \max\{m_\mathcal{C}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}, \tag{2.1}$$

where $\mathcal{F}_j$ denotes the set of $j$-flats of $\Sigma$. Conversely, such a mapping $m_\mathcal{C} : \Sigma \to \mathbb{N}_0 = \{0,1,2,\ldots\}$ as above gives an $[n,k,d]_q$ code in the natural manner, see [1]. For an $m$-flat $\Pi$ in $\Sigma$, we define

$$\gamma_j(\Pi) = \max\{m_\mathcal{C}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\} \ \text{ for } \ 0 \le j \le m.$$

We denote simply by $\gamma_j$ instead of $\gamma_j(\Sigma)$. Then $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$. For a Griesmer $[n,k,d]_q$ code, it is known (see [15]) that

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \quad \text{for} \ \ 0 \le j \le k-1. \tag{2.2}$$

A line $l$ with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane* and so on are defined similarly. Denote by $a_i$ the number of $i$-hyperplanes in $\Sigma$. The list of $a_i$'s is called the *spectrum* of $\mathcal{C}$. We usually use $\tau_j$'s for the spectrum of a hyperplane $\Pi$ of $\Sigma$ to distinguish from the spectrum of $\mathcal{C}$ ($\tau_j$ is the number of $j$-secundums contained in $\Pi$). Let $\theta_j$ be the number of points in a $j$-flat, i.e., $\theta_j = (q^{j+1} - 1)/(q - 1)$. Simple counting arguments yield the following.

**Lemma 2.1** ([17]). *Let $\Pi$ be a $w$-hyperplane through a $t$-secundum $\delta$. Then*

(a) $t \le \gamma_{k-2} - (n - w)/q = (w + q\gamma_{k-2} - n)/q$.

(b) $a_w = 0$ *if an $[w, k-1, d_0]_q$ code with $d_0 \ge w - \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor$ does not exist, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$.*

(c) $\gamma_{k-3}(\Pi) = \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor$ *if an $[w, k-1, d_1]_q$ code with $d_1 \ge w - \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor + 1$ does not exist.*

(d) *Let $c_j$ be the number of $j$-hyperplanes through $\delta$ other than $\Pi$. Then $\sum_j c_j = q$ and*

$$\sum_j (\gamma_{k-2} - j)c_j = w + q\gamma_{k-2} - n - qt. \tag{2.3}$$

(e) *For a $\gamma_{k-2}$-hyperplane $\Pi_0$ with spectrum $(\tau_0, \dots, \tau_{\gamma_{k-3}})$, $\tau_t > 0$ holds if $w + q\gamma_{k-2} - n - qt < q$.*

**Lemma 2.2** ([12]). *Let $\Pi$ be an $i$-hyperplane and let $\mathcal{C}_\Pi$ be an $[i, k-1, d_0]$ code generated by $\mathcal{M}_{\mathcal{C}}(\Pi)$. If any $\gamma_{k-2}$-hyperplane has no $t$-secundum with $t = \left\lfloor \frac{i+q\gamma_{k-2}-n}{q} \right\rfloor$, then $d_0 \ge i - t + 1$.*

**Lemma 2.3.** *The spectrum of an $[n, k, d]_q$ code satisfies $\sum_{i \le u} a_i \le 1$, where*

$$u = \left\lfloor \frac{n - (n-d)(q-1) - 1}{2} \right\rfloor.$$

*Proof.* Assume $a_i > 0$ for an $i \le u$. Then, the right hand side of (2.3) is at most $u + (n-d)q - n$. Since $u < (n - (n-d)(q-1))/2$, we have $n - d - u > u + (n-d)q - n$, which implies that $c_j = 0$ for any $j \le u$. Hence, $a_i = 1$ and $a_j = 0$ for other $j \le u$. □

An $f$-multiset $\mathcal{F}$ on PG$(r, q)$ satisfying

$$m = \min\{m_{\mathcal{F}}(\pi) \mid \pi \in \mathcal{F}_{r-1}\}$$

is called an $(f, m)$-*minihyper*. When an $[n, k, d]_q$ code is projective (i.e. $\gamma_0 = 1$), the set of 0-points forms a $(\theta_{k-1} - n, \theta_{k-2} - (n - d))$-minihyper in PG$(k-1, q)$, and vice versa.

**Lemma 2.4** ([8]). *Every $(x(q+1), x)$-minihyper in PG$(2, q)$ with $q = p^m$, $p$ prime, $m \ge 1$, $1 \le x \le q - q/p$, is a sum of $x$ lines.*

## 3 A sketch of the proof of Theorem 1.3

**Lemma 3.1.** *Let $q \geq 3$ be a prime power.*

(a) *A $[2q^2, 3, 2q^2 - 2q]_q$ code has spectrum $(a_0, a_{2q}) = (1, q^2 + q)$.*

(b) *A $[2q^2 + q + 1, 3, 2q^2 - q]_q$ code has spectrum $(a_{q+1}, a_{2q+1}) = (1, q^2 + q)$.*

(c) *A $[2q^2 + 2q + 1, 3, 2q^2 - 1]_q$ code has spectrum $(a_{2q+1}, a_{2q+2}) = (q + 1, q^2)$.*

(d) *A $[2q^2 + 2q + 2, 3, 2q^2 - 2q]_q$ code has spectrum $a_{2q+2} = q^2 + q + 1$.*

**Lemma 3.2.** *Let $\mathcal{C}_1$ be a Griesmer $[3q^2 - q - 1, 3, 3q^2 - 4q]_q$ code with $q \geq 5$. Then, the spectrum of $\mathcal{C}_1$ is $(a_{2q-1}, a_{3q-1}) = (4, \theta_2 - 4)$ and $\mathcal{M}_{\mathcal{C}_1} = 3\Sigma - (l_1 + l_2 + l_3 + l_4)$, where $\Sigma = \mathrm{PG}(2, q)$ and $l_1, \ldots, l_4$ are four non-concurrent lines.*

*Proof.* Since $\gamma_0 = 3$ from (2.2), the multiset $\mathcal{F} = 3\Sigma - \mathcal{M}_{\mathcal{C}_1}$ forms a $(4\theta_1, 4)$-minihyper. Hence $\mathcal{F}$ is a sum of four lines, say $l_1, \ldots, l_4$, by Lemma 2.4, which are non-concurrent because of $\gamma_0 = 3$. □

Using Lemmas 3.1 and 3.2, one can prove the following.

**Lemma 3.3.** *Let $\mathcal{C}_2$ be a Griesmer $[3q^3 - q^2 - q - a, 4, 3q^3 - 4q^2 - a + 1]_q$ code with $q \geq 5$ and $2 \leq a \leq 4$. Then, the spectrum of $\mathcal{C}_2$ satisfies that $a_i > 0$ implies $2q^2 - q - a \leq i \leq 2q^2 - q - 1$ or $3q^2 - q - a \leq i \leq 3q^2 - q - 1$ and that*

$$\sum_{i \leq 2q^2 - q - 1} a_i = 4. \tag{3.1}$$

**Lemma 3.4** ([14]). *$n_q(4, d) = g_q(4, d) + 1$ for $2q^3 - 3q^2 - q + 1 \leq d \leq 2q^3 - 3q^2$ for $q \geq 4$.*

It is known that $[g_q(5, d) + 1, 5, d]_q$ codes exist for $3q^4 - 4q^3 - 4q + 1 \leq d \leq 3q^4 - 4q^3 - q$ for $q \geq 5$, see [11]. Hence, it suffices to show the following to prove Theorem 1.3.

**Lemma 3.5.** *There exists no $[g_q(5, d), 5, d]_q$ code for $d = 3q^4 - 4q^3 - aq + 1$ with $2 \leq a \leq 4$ for $q \geq 5$.*

*Proof.* We prove the lemma only for $a = 3$. One can prove the lemma similarly for $a = 2, 4$. Let $\mathcal{C}$ be a putative $[g_q(5, d), 5, d = 3q^4 - 4q^3 - 3q + 1]_q$ code with $q \geq 5$. Then, a $\gamma_3$-solid $\Delta_0$ gives a Griesmer $[3q^3 - q^2 - q - 3, 4, 3q^3 - 4q^2 - 2]_q$ code. Since an $i$-solid through a $t$-plane satisfies

$$t \leq \frac{i + q + 2}{q} \tag{3.2}$$

by Lemma 2.1, we have

$$i \geq (2q^2 - q - 3)q - (q + 2) = 2q^3 - q^2 - 4q - 2.$$

Hence, $a_i = 0$ for all $i < 2q^3 - q^2 - 4q - 2$. Applying Lemma 2.1(d), we have $\sum_j c_j = q$ and

$$\sum_j (3q^3 - q^2 - q - 3 - j)c_j = i - qt + q + 2. \tag{3.3}$$

Suppose an $i$-solid $\Delta$ exists for $i = 2q^3 - q^2 - q - 2 + y$ with $0 \le y \le q - 1$. Then, we have $t \le 2q^2 - q - 1$ by (3.2) and Lemma 3.3. Hence, $\Delta$ gives an $[i, 4, 2q^3 - 3q^2 - 1 + y]_q$ code, which does not exist for $y > 1$ by the Griesmer bound. For $y = 0, 1$, $\Delta$ gives a Griesmer code, which does not exist by Lemma 3.4. Hence $a_i = 0$ for $2q^3 - q^2 - q - 2 \le i \le 2q^3 - q^2 - 3$.
Next, suppose an $i$-solid $\Delta$ exists for $i = 2q^3 - q^2 + xq - 2 + y$ with $0 \le x \le q^2 - 5$, $0 \le y \le q - 1$. Then, we have $t \le 2q^2 - q + 1 + x$ by (3.2). Since (3.3) satisfies $c_{n-d} = 0$ for $t = 2q^2 - q + 1 + x$ and $c_{n-d} = c_{n-d-1} = 0$ for $t = 2q^2 - q + x$ by Lemma 3.3, we have $t \le 2q^2 - q - 1 + x$. Hence, $\Delta$ gives an $[i, 4, 2q^3 - 3q^2 + (x+1)q - 1 - x + y]_q$ code, which does not exist by the Griesmer bound. Hence, $a_i = 0$ for $2q^3 - q^2 - 2 \le i \le 3q^3 - q^2 - 4q - 3$. Now, the spectrum of $\mathcal{C}$ satisfies that $a_i > 0$ implies

$$sq^3 - q^2 - 4q - 2 \le i \le sq^3 - q^2 - q - 3 \text{ with } s = 2 \text{ or } 3.$$

Setting $(i, t) = (3q^3 - q^2 - q - 3, 2q^2 - q - 3 + e)$ with $0 \le e \le 2$, the RHS of (3.3) is equal to $q^3 + (3 - e)q - 1$. Hence

$$\sum_{i \le 2q^3 - q^2 - q - 3} a_i = 4 \tag{3.4}$$

by (3.1). Setting $i = 2q^3 - q^2 - q - 3$ in (3.3), (RHS of (3.3))$= 2q^3 - q^2 - 1 - qt$. When $\sum_{j \le 2q^3 - q^2 - q - 3} c_j > 0$, we have $t \le q^2 - q - 1$ from (3.3). It follows from Lemma 2.3 with length $n = i$ and $n - d = 2q^2 - q - 1$ that $u = \lfloor q^2 - \frac{q+5}{2} \rfloor > q^2 - q - 1$. Hence, $\sum_{i \le 2q^3 - q^2 - q - 3} a_i \le 2$, which contradicts (3.4). Similarly, we get $\sum_{i \le 2q^3 - q^2 - q - 3} a_i \le 2$ for $2q^3 - q^2 - 4q - 2 \le i \le 2q^3 - q^2 - q - 4$, which contradicts (3.4) again. Thus, there exists no $[g_q(5, d), 5, d]_q$ code for $d = 3q^4 - 4q^3 - 3q + 1$. □

# References


[1] J. Bierbrauer, Introduction to Coding Theory, Chapman & Hall/CRC, 2005.

[2] E.J. Cheon, Y. Kageyama, S.J. Kim, N. Lee, T. Maruta, Construction of two-weight codes over finite fields and its applications, Bull. Korean Math. Soc. **54** (2017) 731–736.

[3] E.J. Cheon, T. Kato, S.J. Kim, On the minimum length of some linear codes of dimension 5, Design Codes Cryptogr. **37** (2005) 421–434.

[4] E.J. Cheon, T. Kato, S.J. Kim, Nonexistence of a $[g_q(5, d), 5, d]_q$ code for $3q^4 - 4q^3 - 2q + 1 \le d \le 3q^4 - 4q^3 - q$, Discrete Math. **308** (2008) 3082–3089.

[5] M. Grassl, Tables of linear codes and quantum codes (electronic table, online). http://www.codetables.de/.

[6] R. Hill, Optimal linear codes, in: Mitchell C. (ed.) Cryptography and Coding II, pp. 75–104. Oxford Univ. Press, Oxford, 1992.

[7] R. Hill, E. Kolev, A survey of recent results on optimal linear codes, in: Holroyd F.C. et al (ed.) Combinatorial Designs and their Applications, pp.127–152. Chapman and Hall/CRC Press Research Notes in Mathematics CRC Press. Boca Raton, 1999.

[8] R. Hill, H. Ward, A geometric approach to classifying Griesmer codes, Des. Codes Cryptogr. **44** (2007) 169-196.

[9] Y. Inoue, T. Maruta, Construction of new Griesmer codes of dimension 5, Finite Fields Appl. **55** (2019), 231–237.

[10] Y. Kageyama, T. Maruta, On the construction of Griesmer codes of dimension 5, Des. Codes Cryptogr. **75** (2015) 277–280.

[11] Y. Kageyama, T. Maruta, On the geometric constructions of optimal linear codes, Des. Codes Cryptogr. **81** (2016) 469–480.

[12] K. Kumegawa, T. Okazaki, T. Maruta, On the minimum length of linear codes over the field of 9 elements, *Electronic J. Combin.* **24**(1) (2017), #P1.50.

[13] I.N. Landjev, T. Maruta, On the minimum length of quaternary linear codes of dimension five, *Discrete Math.* **202** (1999) 145–161.

[14] T. Maruta, On the minimum length of $q$-ary linear codes of dimension four, Discrete Math. **208/209** (1999) 427–435.

[15] T. Maruta, On the nonexistence of $q$-ary linear codes of dimension five, Des. Codes Cryptogr. **22** (2001) 165–177.

[16] T. Maruta, Griesmer bound for linear codes over finite fields, http://www.mi.s.osakafu-u.ac.jp/~maruta/griesmer/.

[17] M. Takenaka, K. Okamoto, T. Maruta, On optimal non-projective ternary linear codes, Discrete Math. **308** (2008) 842–854.