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NONLINEAR PHYSICS AND MECHANICS

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Nonlinear Gradient Flow of a Vertical Vortex Fluid in a Thin Layer

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A new exact solution to the Navier – Stokes equations is obtained. This solution describes the inhomogeneous isothermal Poiseuille flow of a viscous incompressible fluid in a horizontal infinite layer. In this exact solution of the Navier – Stokes equations, the velocity and pressure fields are the linear forms of two horizontal (longitudinal) coordinates with coefficients depending on the third (transverse) coordinate. The proposed exact solution is two-dimensional in terms of velocity and coordinates. It is shown that, by rotation transformation, it can be reduced to a solution describing a three-dimensional flow in terms of coordinates and a two-dimensional flow in terms of velocities. The general solution for homogeneous velocity components is polynomials of the second and fifth degrees. Spatial acceleration is a linear function. To solve the boundary-value problem, the no-slip condition is specified on the lower solid boundary of the horizontal fluid layer, tangential stresses and constant horizontal (longitudinal) pressure gradients specified on the upper free boundary. It is demonstrated that, for a particular exact solution, up to three points can exist in the fluid layer at which the longitudinal velocity components change direction. It indicates the existence of counterflow zones. The conditions for the existence of the zero points of the velocity components both inside the fluid layer and on its surface under nonzero tangential stresses are written. The results are illustrated by the corresponding figures of the velocity component profiles and streamlines for different numbers of stagnation points.

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The possibility of the existence of zero points of the specific kinetic energy function is shown. The vorticity vector and tangential stresses arising during the flow of a viscous incompressible fluid layer under given boundary conditions are analyzed. It is shown that the horizontal components of the vorticity vector in the fluid layer can change their sign up to three times. Besides, tangential stresses may change from tensile to compressive, and vice versa. Thus, the above exact solution of the Navier – Stokes equations forms a new mechanism of momentum transfer in a fluid and illustrates the occurrence of vorticity in the horizontal and vertical directions in a nonrotating fluid. The three-component twist vector is induced by an inhomogeneous velocity field at the boundaries of the fluid layer.

Keywords: Poiseuille flow, gradient flow, exact solution, counterflow, stagnation point, vorticity

1. Introduction

The Poiseuille flow [33, 34] is a classical exact solution of the Navier – Stokes equation describing the unidirectional motion of a viscous incompressible fluid [3, 17, 40]. The Poiseuille flow is induced by setting a horizontal pressure gradient in the channels of various transverse sections [3, 17, 33, 34, 40]. In hydrodynamics, two classes of Poiseuille flows are considered. The flow of a fluid layer between two planes, nonparallel in the general case, is distinguished [17]. The second type of the Poiseuille flow is the motion of a viscous incompressible fluid in pipes with different transverse sections [21, 27, 39, 43]. In this regard, the range of applicability of the classical exact Poiseuille solution is constantly expanding. The exact Poiseuille solution proposed to describe the movement of blood through vessels [33, 34] has been repeatedly modified and generalized [2, 31, 41, 42, 47, 49]. At present, the classical Poiseuille flow and its variations are widely used in biological hydrodynamics to study pathological processes in the cardiovascular system (for example, stenosis or aneurysm) [28, 29]. The exact Poiseuille solution is used to study the functioning of the mammary gland during lactation [48], as well as to identify models of biological tissues [18, 44]. It is well known that, in solving problems of magnetohydrodynamics, the analog of the Poiseuille solution is the Hartmann flow [20, 23]. The superposition of the Poiseuille and Hartmann flows was studied in [14, 32, 37, 38]. In addition, the exact Poiseuille solution was used in the hydrodynamics of non-Newtonian fluids in [1, 10, 12, 13, 15, 24, 25, 30, 46]. The exact Poiseuille solution is a favorite object for studying the stability of a viscous incompressible fluid under disturbing secondary flows (for various classes of perturbations of the main flow) [11, 16, 37–39, 45, 50]. At present, the classical Poiseuille flow is used as a starting study for describing fluid flows caused by pressure changes along the channel length. As is well known, this flow has a rather limited range of applicability [26]. If the classical Poiseuille flow unsatisfactorily describes new methods for transmitting the angular momentum in a fluid, numerical solutions of the Navier – Stokes equations are used. This is caused by the complexity of integrating the equations of viscous fluid motion and the lack of universal algorithms for finding exact solutions. It is important to remember that exact solutions are an indispensable tool for testing approximate, numerical, and asymptotic methods and allow us to evaluate the applicability of hydrodynamic models most effectively. It was shown in [6, 9, 35] that there is a class of exact solutions describing isothermal and convective inhomogeneous Couette-type flows of a vertical vortex fluid. The constructed exact solution describes a new mechanism of momentum transfer in a fluid. The purpose of this paper is to extend the obtained results to study the solution classes to the isothermal Poiseuille flow of a viscous incompressible fluid between two planes.

2. Problem statement

We write the Navier – Stokes equation describing the isothermal motion of a viscous incompressible fluid, supplemented by the incompressibility equation [27]

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P + \mathbf{F} + \nu \triangle \mathbf{V}; \tag{2.1}$$

$$\nabla \cdot \mathbf{V} = 0. \tag{2.2}$$

The following notation is introduced in Eqs. (2.1) and (2.2): $\mathbf{V}(t, x, y, z) = (V_x, V_y, V_z)$ is the velocity vector; P is the deviation from hydrostatic pressure, taken relative to constant average fluid density ρ; $\mathbf{F} = (0; 0; g)$ is the mass force field density; ν is kinematic viscosity; $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is the three-dimensional Hamilton operator, and $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional Laplace operator [27].

We consider a viscous incompressible fluid flow in a horizontal infinite layer (Fig. 1). The coordinate axis Oz is directed perpendicular to the boundary surfaces of the layer. From below, the fluid layer is bounded by an absolutely solid surface defined by the equation of the plane $z = 0$. The upper boundary of the fluid layer is a nondeformable surface defined by the equation of the plane $z = h$. The boundaries of the fluid layer given by the equations of the planes $z = 0$ and $z = h$ are parallel to the plane Oxy. The variable z varies in the range $[0; h]$, and the value of h is significantly less than the characteristic longitudinal dimensions of the flow, laid off on the axes Ox (abscissa axis) and Oy (ordinate axis). The velocities V_x and V_y are the longitudinal (horizontal) components of the velocity vector, V_z is the transverse (vertical) component. We assume that the flow under study is steady and layered, i.e., all the required functions are independent of time and the vertical component of the velocity vector is $V_z = 0$.

In view of the assumptions made, the system of equations (2.1) projected onto the axes of the Cartesian coordinate system can be rewritten in the following form [4, 9]:

$$\begin{gathered}
V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right); \\
V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \right); \\
\frac{\partial P}{\partial z} = g; \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0.
\end{gathered} \tag{2.3}$$

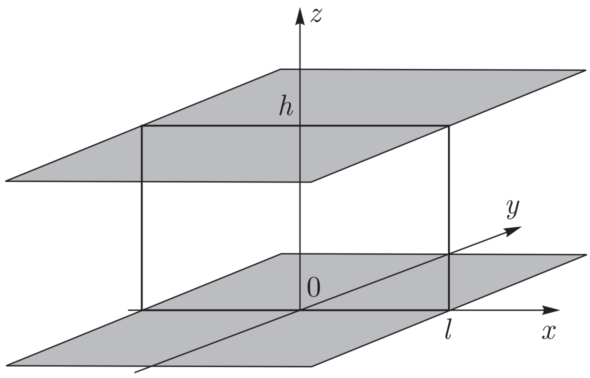

Fig. 1. Problem statement.

The system of equations (2.3) is overdetermined, since there are four equations for determining three functions V_x, V_y and P. In [4, 9], the solvability of this system of equations was investigated and an exact solution different from the trivial one was constructed. We note that the system of Navier – Stokes equations (2.3) describes the equatorial currents in the World Ocean [22] with one Coriolis parameter, with a latitude equal to zero [6].

We find the exact solution to system (2.3) in the following form [6, 7, 9, 35, 36]:

$$\begin{aligned} V_x(y,z) &= U(z) + yu(z), \\ V_y(y,z) &= V(z), \\ P(x,y,z) &= P_0(z) + xP_1(z) + yP_2(z). \end{aligned} \tag{2.4}$$

Generally speaking, the solution (2.4) is two-dimensional in terms of velocity and coordinates; however, the rotation transformation [6, 8, 9]

$$U_1 = u\cos\theta\sin\theta, \quad U_2 = u\cos^2\theta, \quad V_1 = -u\sin^2\theta, \quad V_2 = -u\cos\theta\sin\theta \tag{2.5}$$

can "propagate" it to a solution describing a three-dimensional flow in coordinates and two-dimensional flow in velocities. Here, θ is an arbitrary constant, and the function u satisfies the simplest parabolic equation of the (1+1) dimension

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2}.$$

The solution (2.4) is obtained by substituting the value $\theta = 0$ into the expression (2.5).

It follows from the third equation of (2.3) that the uniform pressure term is determined as

$$P_0(z) = g(z-h) + S, \tag{2.6}$$

where S is the atmospheric pressure specified on the upper free surface of the fluid layer considered. The expression (2.6) defines the pressure field that is used for the hydrostatic approximation of the Navier – Stokes equations. This approximation is most often used in oceanology to describe large-scale solutions.

The components P_1 and P_2 are constant coefficients, pressure gradients along the longitudinal coordinates x and y, respectively. We note that the fluid motion characterized by the pressure function (2.4) is a generalization of the classical Poiseuille flow [3, 17, 33, 34, 40].

We substitute the class of exact solutions (2.4) into the nonlinear system (2.3):

$$uV + P_1 - \nu\left(U'' + yu''\right) = 0, \quad \nu V'' - P_2 = 0. \tag{2.7}$$

Here, the primes denote the derivatives of the functions with respect to the coordinate z. Taking into account the adopted form of solutions (2.4), the last two equations of (2.3) are satisfied identically.

In the first equation of (2.7), we equate to zero the coefficients of an independent variable y and the free terms. We obtain the following system of ordinary differential equations for determining three unknown functions U, u and V, which is written in the order of integration:

$$\begin{aligned} u'' &= 0, \\ \nu V'' &= P_2, \\ \nu U'' &= uV + P_1. \end{aligned} \tag{2.8}$$

The general solution for the system of equations (2.8) is written as

$$u = C_1 z + C_2, \quad V = \frac{z^2}{2\nu} P_2 + C_3 z + C_4,$$
$$U = \frac{z^5}{40\nu^2} P_2 C_1 + \frac{z^4}{4!\nu^2} (P_2 C_2 + 2\nu C_1 C_3) + \frac{z^3}{3!\nu} (C_2 C_3 + C_1 C_4) + \tag{2.9}$$
$$+ \frac{z^2}{2!\nu} (P_1 + C_2 C_4) + z C_5 + C_6.$$

In the general solution thus obtained, spatial acceleration u [19] is a linear function, the homogeneous components of the velocity vector U and V are second- and fifth-degree polynomials, respectively.

3. Boundary-value problem

To determine the integration constants, we write a boundary-value problem. We assume that the horizontal (longitudinal) pressure gradients P_1 and P_2 are given at the upper boundary defined by the equation $z = h$.

Let the no-slip condition be satisfied at the lower boundary of the horizontal layer of a viscous incompressible fluid defined by the plane equation $z = 0$, i.e., the velocity components are defined as follows:

$$V_x(y, 0) = 0, \quad V_y(0) = 0. \tag{3.1}$$

When studying the properties of the flow, we assume the tangential stresses at the upper boundary determined by the equation of the plane $z = h$ to be spatially inhomogeneous rather than constant, by analogy with the boundary conditions discussed in [5, 7, 8, 22]:

$$\nu \frac{\partial V_x}{\partial z}\bigg|_{z=h} = -\tau_1 - \tau_2 y, \quad \nu \frac{\partial V_y}{\partial z}\bigg|_{z=h} = -\tau_3. \tag{3.2}$$

Setting the boundary conditions (3.2) determines the so-called parabolic wind. This approximation of the distribution of the wind stress components is the simplest in oceanology.

Taking into account the structure of the exact solution (2.4), we write the boundary conditions (3.1)–(3.2) in the following form:

$$U(0) = 0, \quad u(0) = 0, \quad V(0) = 0,$$
$$\nu \frac{\partial U}{\partial z}\bigg|_{z=h} = -\tau_1, \quad \nu \frac{\partial u}{\partial z}\bigg|_{z=h} = -\tau_2, \quad \nu \frac{\partial V}{\partial z}\bigg|_{z=h} = -\tau_3. \tag{3.3}$$

The values of the integration constants are determined by substituting the boundary conditions (3.3) into the general solution (2.9) as

$$C_1 = -\frac{\tau_2}{\nu}, \quad C_2 = 0, \quad C_3 = -\frac{P_2 h + \tau_3}{\nu},$$
$$C_4 = 0, \quad C_5 = -\frac{P_1 h + \tau_1}{\nu} - \frac{\tau_2 h^3}{4!\nu^3} (5P_2 h + 8\tau_3), \quad C_6 = 0. \tag{3.4}$$

We substitute the integration constants (3.4) into the general solution (2.9) and obtain the following particular exact solution to the problem (2.8), (3.3):

$$u = -\frac{\tau_2}{\nu} z, \quad V = \frac{z}{2\nu} [P_2 z - 2(P_2 h + \tau_3)],$$
$$U = -\frac{\tau_2 z^5}{40\nu^3} P_2 + \frac{\tau_2 z^4}{12\nu^3} (P_2 h + \tau_3) + \frac{P_1 z^2}{2\nu} - \frac{z}{4!\nu^3} \left[24\nu^2 (P_1 h + \tau_1) + \tau_2 h^3 (5P_2 h + 8\tau_3)\right]. \tag{3.5}$$

4. Exact solution analysis

Let us analyze the solution (3.5). Spatial acceleration u (parallel to the abscissa axis) is a monotonic function, which increases or decreases depending on the sign of the horizontal gradient of the tangential stress τ_2 at the free boundary of the fluid layer specified by the equation $z = h$.

The velocity component V may take zero values for $z = 0$ (due to the no-slip condition), and $z = \dfrac{2(P_2 h + \tau_3)}{P_2}$. Thus, for the function V to vanish on the interval $(0; h)$, the following double inequality must be satisfied:

$$0 < \frac{2(P_2 h + \tau_3)}{P_2 h} < 1. \tag{4.1}$$

Note that the z coordinate value at which the velocity component V vanishes does not depend on the kinematic viscosity of the fluid ν; it depends only on the thickness of the fluid layer thickness h, the longitudinal pressure gradient P_2, and the tangential stress τ_3 specified at the upper nondeformable boundary of the layer. We simplify the double inequality (4.1) and obtain the following condition imposed on the parameters specified by the boundary conditions for the existence of a stagnation point for the velocity component V in the interval $(0; h)$:

$$-1 < \frac{\tau_3}{P_2 h} < -\frac{1}{2}. \tag{4.2}$$

Condition (4.2) makes sense when the tangential stress τ_3 and the longitudinal pressure gradient P_2 have different signs. Taking into account the boundary conditions (3.3), we can conclude that, for the stagnation point of the velocity component V to exist, it is necessary that the pressure gradient along the ordinate axis $\dfrac{\partial P}{\partial y}$ and the tangential stress $\dfrac{\partial V}{\partial z}$ at the upper boundary of the fluid layer be simultaneously either positive or negative, i.e., either compressive or tensile.

Figure 2a shows the profile of the velocity component V with a stagnation point existing inside the thickness of the fluid layer under study. Note that, for the layer thickness

$$h = -\frac{2\tau_3}{P_2},$$

the velocity component V vanishes at the upper boundary of the fluid layer, while the tangential stress τ_3 on it is not equal to zero. The profile of the velocity component V for this case is shown in Fig. 2b.

We reduce the analysis of the velocity component U in the interval $(0; h)$ to the determination of the zero points of the function U, i.e., to finding the roots of the equation $U = 0$ and introduce the dimensionless coordinate $q = z/h$; thus, the area under study is reduced to the interval $q \in (0; 1]$, and the function U can be written as

$$U = q\left(aq^2 + bq^3 - cq - d\right).$$

Here, the constant coefficients are defined as follows:

$$a = -\frac{P_2 \tau_2 h^5}{40\nu^3}, \quad b = \frac{\tau_2 h^4}{12\nu^3}(P_2 h + \tau_3), \quad c = -\frac{P_1 h^2}{2\nu},$$
$$d = \frac{h}{4!\nu^3}\left[24\nu^2 (P_1 h + \tau_1) + \tau_2 h^3 (5P_2 h + 8\tau_3)\right].$$

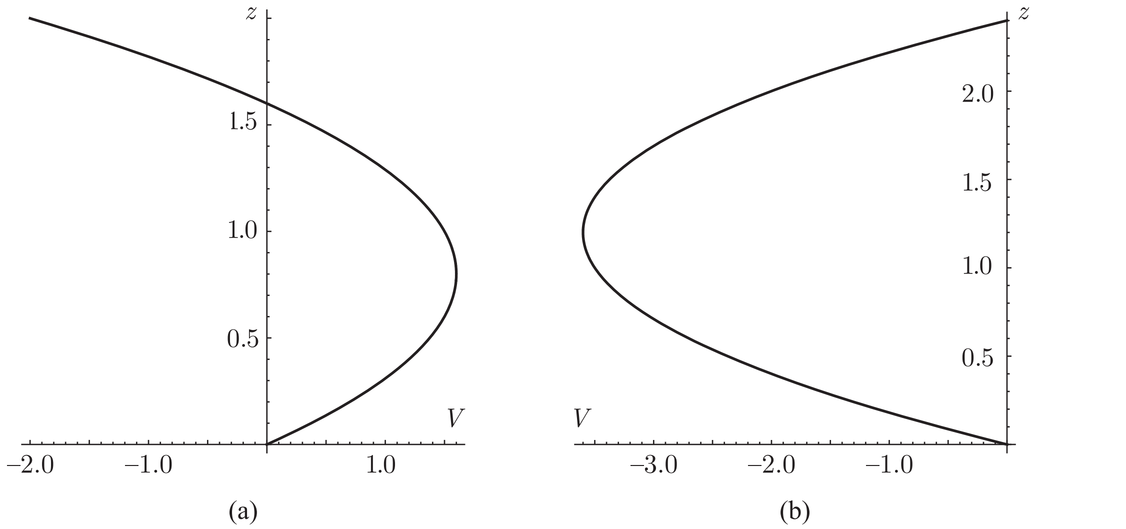

Fig. 2. The profile of the velocity component V for the cases of the stagnation point existing inside the thickness of the fluid layer for $h = 2$ m, $P_2 = -5 \cdot 10^{-6}$ m/s^2, $\tau_3 = 6 \cdot 10^{-6}$ m^2/s^2 (a) and at the upper boundary for $h = 2.4$ m, $P_2 = 5 \cdot 10^{-6}$ m/s^2, $\tau_3 = -6 \cdot 10^{-6}$ m^2/s^2 (b).

Thus, the analysis of the velocity component U reduces to the study of the roots of the equation

$$aq^4 + bq^3 - cq - d = 0. \tag{4.3}$$

We represent Eq. (4.3) in the form of the equality of two functions: $f_2(q) = cq + d$. Thus, the graphs of the functions f_1 and f_2d intersect at points which are zero for the function U. The analysis of the existence of roots for Eq. (4.3) has shown that the functions f_1 and f_2d can intersect in the interval $(0; 1)$ at one, two, three points or even have no intersection points. Figure 3 shows the location of the graphs of the functions f_1 and f_2d in the case of their intersection at three points. Consequently, the velocity component U can have no more than three zero points in the interval $(0; 1)$.

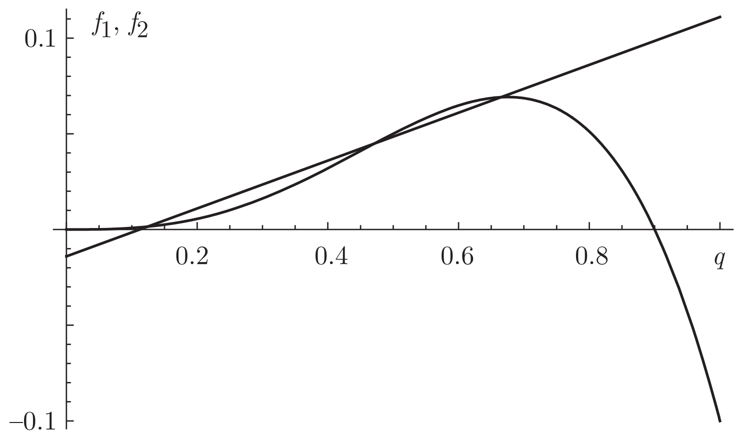

Fig. 3. The graphs of the functions f_1 and f_2 illustrating the existence of three zero points of the velocity component U for $a = -1$, $b = 0.9$, $c = 0.125$, $d = -0.014$.

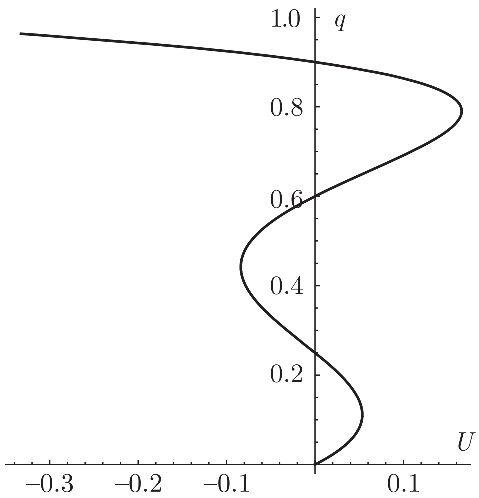

Fig. 4. The profile of the velocity component U in the case of the existence of three zero points in the interval $(0; 1)$.

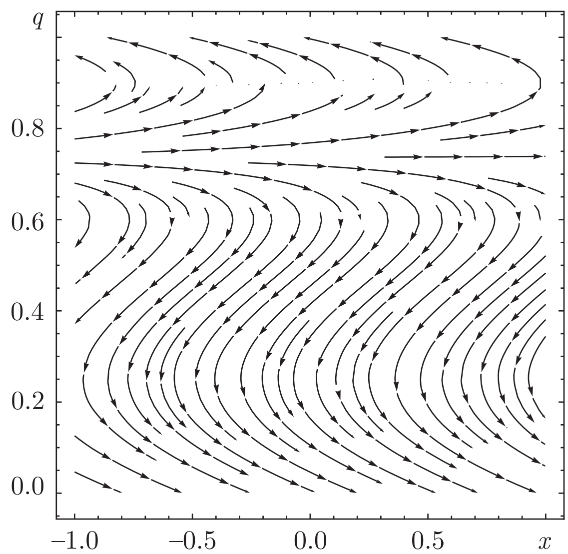

Fig. 5. The streamlines for the following parameter values: $\nu = 10^{-6}$ m^2/s, $h = 100$ m, $\tau_1 = 1.464 \cdot 10^{-6}$ m^2/s^2, $\tau_2 = 10^{-15}$ m/s^2, $\tau_3 = -2.86 \cdot 10^{-8}$ m^2/s^2, $P_1 = -2.82 \cdot 10^{-9}$ m/s^2, $P_2 = 4 \cdot 10^{-10}$ m/s^2.

Figure 4 shows the profile of the velocity component U in the case of the existence of three zero points in the interval $(0; 1)$. Figure 5 shows the corresponding streamlines in the case that the velocity component U has three zero points in the interval $(0; 1)$ and the velocity component V has one.

Figure 6 shows the profile of the velocity component U in the case of the existence of two zero points in the interval $(0; 1)$. Figure 7 shows the corresponding streamlines in the case that the velocity component U has two zero points in the interval $(0; 1)$ and the velocity component V has one.

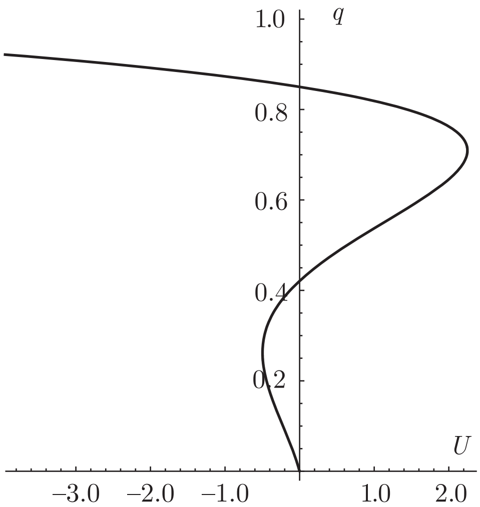

Fig. 6. The profile of the velocity component U in the case of the existence of two zero points in the interval $(0; 1)$.

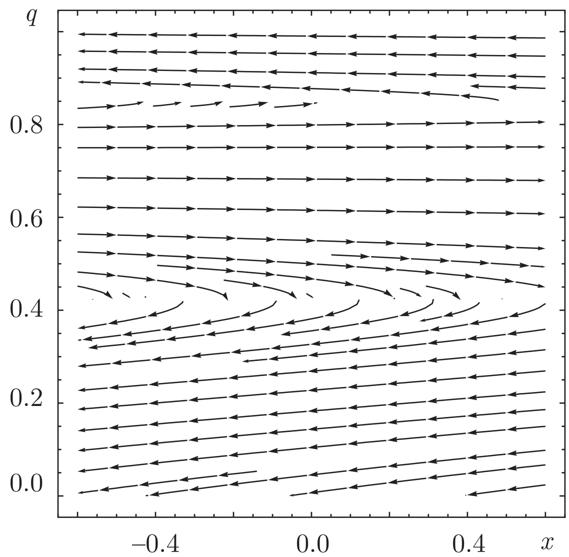

Fig. 7. The streamlines for the following parameter values: $\nu = 10^{-6}$ m^2/s, $h = 100$ m, $\tau_1 = 1.32{\cdot}10^{-6}$ m^2/s^2, $\tau_2 = 10^{-15}$ m/s^2, $\tau_3 = -2.86{\cdot}10^{-8}$ m^2/s^2, $P_1 = -1.02{\cdot}10^{-9}$ m/s^2, $P_2 = 4{\cdot}10^{-10}$ m/s^2.

The specific kinetic energy for the type of velocity obtained, taking into account the dimensionless coordinate q, has the form

$$T = \frac{h^2q^2}{4\nu^2}\left[P_2hq - 2\left(P_2h + \tau_3\right)\right]^2 + \left\{-\frac{hqy\tau_2}{\nu} + \right.$$
$$\left. + q\left[\frac{P_1h^2q}{2\nu} - \frac{h^5\tau_2P_2q^4}{40\nu^3} + \frac{\tau_2h^4q^3}{12\nu^3}\left(P_2h + \tau_3\right) - \frac{h^4\tau_2\left(5P_2h + 8\tau_3\right)}{4!\nu^3} - \frac{h^2P_1 + \tau_1}{\nu}\right]\right\}^2.$$

The total velocity of the fluid flows vanishes at the point where the specific kinetic energy T is zero. Figure 8 shows a graph of the specific kinetic energy function with one zero point in the plane $y = 0$. Figure 9 shows the streamlines for the same parameter values.

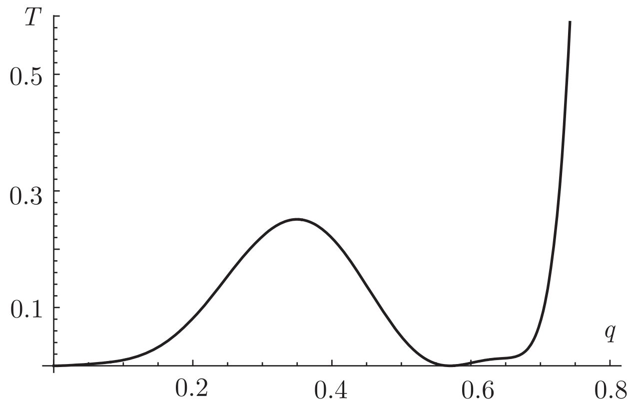

Fig. 8. The graph of the specific kinetic energy function T.

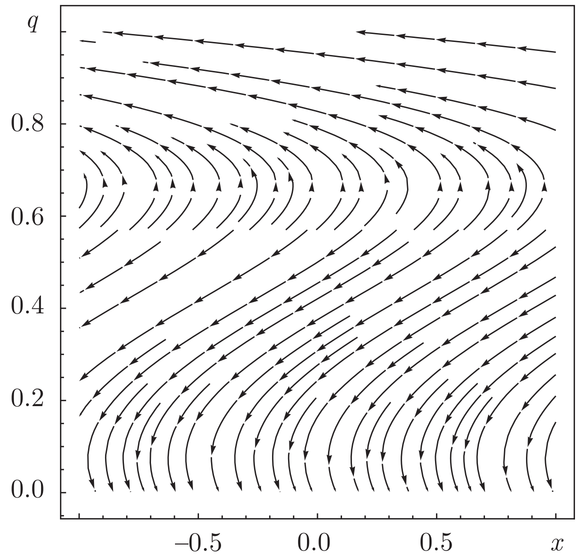

Fig. 9. The streamlines for the following parameter values: $\nu = 10^{-6}$ m^2/s, $h = 100$ m, $\tau_1 = 1.472 \cdot 10^{-6}$ m^2/s^2, $\tau_2 = 10^{-15}$ m/s^2, $\tau_3 = -2.86 \cdot 10^{-8}$ m^2/s^2, $P_1 = -2.82 \cdot 10^{-9}$ m/s^2, $P_2 = 4 \cdot 10^{-10}$ m/s^2.

5. The study of tangential stresses and vorticity

To analyze the solution (3.5), we study the vorticity vector $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ and the tangential stresses arising in the flow of a viscous incompressible fluid under consideration. The components of the vorticity vector and those of the tangential stress tensor are determined as follows:

$$\Omega_x = -\frac{\partial V_y}{\partial z} = -\tau_{yz}, \quad \Omega_y = -\frac{\partial V_x}{\partial z} = -\tau_{xz}, \quad \Omega_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}. \tag{5.1}$$

Substituting the velocity components (3.4) into (5.1), we obtain

$$\Omega_x = \frac{1}{\nu}\left[P_2\left(h - z\right) + \tau_3\right], \quad \Omega_z = \frac{\tau_2 z}{\nu},$$

$$\Omega_y = -\frac{\tau_2 P_2}{8\nu^3} z^4 + \frac{\tau_2}{3\nu^3}\left(P_2 h + \tau_3\right) z^3 + \frac{P_1}{\nu} z - \frac{h^3 \tau_2 \left(5P_2 h + 8\tau_3\right)}{4!\nu^3} - \frac{1}{\nu}\left(hP_1 + \tau_1\right) - \frac{y\tau_2}{\nu}.$$

The component Ω_x monotonically decreases from the solid lower surface to the upper free one and vanishes when $z = h + \dfrac{\tau_3}{P_2}$. Consequently, Ω_x can vanish in the fluid layer with thickness h when the following restriction on the values of the boundary parameters τ_3 and P_2 is fulfilled:

$$0 \leqslant -\frac{\tau_3}{P_2} \leqslant h.$$

The component Ω_z is monotonic, the characteristic of monotonicity being dependent on the sign of the parameter τ_2.

The analysis of the longitudinal component Ω_y of the vorticity vector is completely analogous to the analysis of the functions f_1 and f_2. Therefore, we can conclude that the function Ω_y in the fluid layer $z \in (0; 1)$ can change its sign up to three times. Taking into account the dependences (5.1) relating the vorticity vector components and the tangential stress vector components, we can make the following conclusion: the tangential stress τ_{xz} in the fluid layer can change from tensile to compressive up to three times.

6. Conclusion

An exact solution has been obtained for the isothermal Poiseuille flow of a viscous incompressible fluid in a horizontal infinite layer. It has been demonstrated that, for the boundary conditions considered in this study, up to three counterflow regions can arise in the thickness of the fluid layer, where the velocity reverses direction. The vorticity vector components for the exact solution obtained and the condition of vanishing tangential stresses have been analyzed.

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