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End Conditions for Interpolatory Quintic Splines

by

G. H. BEHFOROOZ* & N. PAPAMICHAEL

*Present address: Dept, of Mathematics, University of Tabriz, Tabriz, Iran.

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ABSTRACT

Accurate end conditions are derived for quintic spline interpolation at equally spaced knots. These conditions are in terms of available function values at the knots and lead to $0(h^6)$ covergence uniformly on the interval of interpolation.

1. Introduction

Let Q be a quintic spline on [a,b] with equally spaced knots

$$x_i = a + ih \ ; \qquad i=0,1,\ldots,k, \tag{1.1}$$

where $h = (b-a)/k$. Then $Q \in C^4[a,b]$ and in each of the intervals $[x_{i-1}, x_i]$; $i = 1,2,\ldots,k$, Q is a quintic polynomial. The set of all such quintic splines forms a linear space, of dimension k+5, which we denote by Sp(5,k).

Given the set of values y_i ; $i-0,1,\ldots,k$, where

$$y_i = y(x_i) \ , \ y \in c^n[a,b], \ n \geq 6,$$

we consider the problem of constructing an interpolatory $Q \in Sp(5,k)$ such that

$$Q(x_i) = y_i \ ; \qquad i = 0,1,\ldots,k \ . \tag{1.2}$$

Since dim Sp(5,k) = k+5, the interpolation conditions (1.2) are not sufficient to determine Q uniquely and four additional linearly independent conditions are always needed for this purpose. These are usually taken to be end conditions, i.e. conditions imposed on Q or its derivatives $Q^{(J)}$; $j = 1,2,3,4$, near the two end points a and b.

As might be expected the choice of end conditions plays a critical role on the quality of the spline approximation. It is well known that the best order of approximation which can be achieved by an interpolatory quintic spline Q is

$$\| Q - y \| = 0(h^6) \ ,$$

where $\| . \|$ denotes the uniform norm on [a,b]. Such order of convergence is obtained, if for example, the end conditions

$$Q^{(1)}(x_i) = y_i^{(1)} , Q^{(1)}(x_{k-i}) = y_{k-i}^{(1)} ; i = 0,1,$$

are used. However, these conditions require knowledge of the first derivative of y at the four points x_i, x_{k-i} ; $i + 0,1$, and in an interpolation problem this information is not usually available.
The natural end conditions

$$Q^{(r)}(a) = Q^{(r)}(b) = 0 \; ; \quad r = 3,4,$$

do not require any additional information, but the resulting natural quintic spline does not have $0(h^6)$ convergence uniformly on [a,b].

The purpose of the present paper is to derive end conditions for quintic spline interpolation at equally spaced knots, which depend only on the given function values at the knots and lead to $0(h^6)$ convergence uniformly on [a,b]. We derive a class of such end conditions in Section 3, by generalizing the cubic spline results of Behforooz and Papamichael (1979) to the case of quintic spline interpolation.

2. Preliminary Results

To simplify the presentation we use throughout the abbreviations,

$$m_i = Q^{(1)}(x_i),\ M_i = Q^{(2)}(x_i),\ n_i = Q^{(3)}(x_i) \text{ and } N_i = Q^{(4)}(x_i);$$

$$I = 0,1 \ldots k. \qquad (2.1)$$

The following quintic spline identities are needed for the analysis of Section 3:

$$m_{i-2} + 26m_{i-1} + 66m_i + 26m_{i+1} + m_{i+2}$$

$$= \frac{5}{h}\{-y_{i-2} - 10y_{i-1} + 10y_{i+1} + y_{i+2}\} \; ;$$

$$i = 2, 3, \ldots, k-2, \qquad (2.2)$$

$$M_{i-2} + 26M_{i-1} + 66M_i + 26M_{i+1} + M_{i+2}$$

$$= \frac{20}{h^2}\{ y_{i-2} + 2y_{i-1} - 6y_i + 2y_{i+1} + y_{i+2} \};$$

$$i = 2, 3, \ldots, k-2, \qquad (2.3)$$

$$n_{i-2}+26n_{i-1}+66n_i+26n_{i+1}+n_{i+2} = \frac{60}{h^3}\{-y_{i-2}+2y_{i-1}-2y_{i+1}+y_{i+2}\}; \qquad i=2,3,\dots,k-2, \tag{2.4}$$

$$N_{i-2}+26N_{i-1}+66N_i+26N_{i+1}+N_{i+2} = \frac{120}{h^4}\{y_{i-2}-4y_{i-1}+6y_i-4y_{i+1}+y_{i+2}\}; \qquad i=2,3,\dots,k-2, \tag{2.5}$$

$$N_{i-2}+4N_{i-1}+N_{i+1}-\frac{6}{h^2}\{M_{i-1}-2M_i+M_{i+1}\}=0; \qquad i=1,2,\dots,k-1, \tag{2.6}$$

$$60h\{m_{i-1}+2m_i+m_{i+1}\}-h^3\{3n_{i-1}+14n_{i+1}+3n_{i+1}\} = 120\{y_{i+1}-y_{i-1}\}; \qquad i=1,2,\dots,k-1, \tag{2.7}$$

$$8h\{m_{i+1}-m_{i-1}\}-h^2\{M_{i-1}-6M_i+M_{i+1}\} = 20\{y_{i-1}-2y_i+y_{i+1}\}; \qquad i=1,2,\dots,k,k-1, \tag{2.8}$$

$$m_i=\frac{h}{6}\{2M_i+M_{i-1}\} \quad \frac{h^3}{360}\{8N_i+7N_{i-1}\}+\frac{1}{h}\{y_i-y_{i-1}\}; \qquad i=1,2,\dots,k, \tag{2.9}$$

$$m_i=-\frac{h}{6}\{2M_i+M_{i+1}\}+\frac{h^3}{360}\{8N_i+7N_{i+1}\}+\frac{1}{h}\{y_{i+1}-y_i\}; \qquad i=0,1,\dots,k-1, \tag{2.10}$$

$$m_i=-\frac{h^2}{120}\{n_{i-1}+18n_i+n_{i+1}\}+\frac{1}{2h}\{y_{i+1}-y_{i-1}\}; \qquad i=1,2,\dots,k-1, \tag{2.11}$$

$$M_i=-\frac{h^2}{120}\{N_{i-1}+8N_i+N_{i+1}\}+\frac{1}{h^2}\{y_{i-1}-2y_i+y_{i+1}\}; \qquad i=1,2,\dots,k-1, \tag{2.12}$$

$$M_i = \frac{1}{32h}\{m_{i-2} + 32m_{i-1} - 32m_{i+1} - m_{i+2}\}$$

$$+\frac{5}{32h^2}\{y_{i-2} + 16y_{i-1} - 34y_i + 16y_{i+1} + y_{i+2}\};$$

$$i = 2,3,\ldots,k-2, \tag{2.13}$$

$$N_i = \frac{3}{2h^2}\{M_{i-1} + 18M_i + M_{i+1}\} + \frac{30}{h^4}\{y_{i-1} - 2y_i + y_{i+1}.\};$$

$$i = 1,2,\ldots,K-1. \tag{2.14}$$

The relations (2.2) — (2,14) can be derived from the results of Albasiny and Hoskins (1971), Fyfe (1971), Ahlberg, Nilson and Walsh (1966) and Sakai (1970).

The most convenient way for constructing Q is probably through its B-spline representation

$$Q(x) = \sum_{i=1}^{k+5} c_i \nabla^5 (x_i - x)_+^5$$

where, as usual.

$$(t-x)_+^5 = -\begin{cases} (t-x)^5, & t \geq x, \\ 0, & t < x. \end{cases}$$

However, it is important to observe that the unique existence of Q can be established by showing that any of the four $(k+1) \times (k+1)$ linear systems obtained by using either of the relations (2.2), (2.3) (2.4) or (2.5) together with the four equations derived from the end conditions of $Q_>$is non-singular. For example, if the linear system corresponding to (2.3) has a unique solution M_i ; $i = 0, 1,\ldots,k$, then (2.14) and (2.6) give the parameters N. ; $i = 0,1,\ldots,k$, and Q can be constructed in any interval $[x_{i-1}, x_i]$ by integrating

$$Q^{(4)}(x) = \frac{1}{h}\{N_{i\_1}(x_i - x) + N_i(x - x_{i\_1})\}, \tag{2.15}$$

four times with respect to x and setting

$$Q(x_j) = y_j \quad \text{and} \quad Q^{(2)}(x_j) = M_j; \quad j = i-1, i.$$

for the determination of the four constants of integration. Similarly, if the linear system corresponding to (2.2) has a unique solution m_i ; $i = 0,1,\dots,k$, then (2.13) and (2.8) give the parameters M_i ; $i = 0,1,\dots,k$, and Q can be constructed in any interval $[x_{i-1}, x_i]$ by use of the two point quintic Hermite interpolation formula. Similar arguments establish the unique existence of Q in the two cases where the linear systems corresponding to (2 . 4) and (2.5) are non-singular.

The following two lemmas are also needed for the derivation of the results given in Section 3:

<u>Lemma 2.1.</u> If $y \in C^6[a,b]$ then, for $x \in [x_{i-1}, x_i]$; $i = 1,2,\dots,k$,

$$| Q^{(r)}(x) - y^{(r)}(x) | \le A_r h^{1-r} \max_{0 \le j \le k} | m_j - y_j^{(1)} | + 0(h^{6-r}); \quad r=0,1,\dots,6, \tag{2.16}$$

where the A_r are constants independent of h.

<u>Lemma 2. 2</u>. Let $\lambda_i = m_i - y_i^{(1)}$. If $y \in C^7[a,b]$ then

$$\lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = \beta_i ; \quad i = 2,3,\dots,k-2, \tag{2.17}$$

where

$$| \beta_i | \le \tfrac{11}{21} h^6 \| y^{(7)} \| ; \quad i -2,3,\dots,k-2 . \tag{2.18}$$

Lemma 2.1 can be established, from the representation of Q by means of the two point quintic Hermite interpolation formula, by using a trivial generalization of a result due to Hall (1968: p.214). Lemma 2.2 follows easily from (2.2), by Taylor series expansion about the point x_i.

<u>3, End Conditions</u>

We let Q be a quintic spline interpolating the values $y_i = y(x_i)$; $i = 0, 1,\dots,k$, at the equally spaced knots (1.1), and assume that $k \ge 6$. As before, we use the abbreviations (2.1).

We consider end conditions of the form

$$\left.\begin{aligned} & m_i + \alpha m_{i+1} + \beta m_{i+2} + \gamma m_{i+3} = \frac{1}{60\,h}\sum_{j-i}^{i+5} a_{j-i} y_j, \\ & m_{k-i} + \alpha m_{k-i-1} + \beta m_{k-i-2} + \gamma m_{k-i-3} = -\frac{1}{60\,h}\sum_{j-i}^{i+5} a_{j-i} y_{k-j}; \\ & . \qquad\qquad\qquad\qquad i = 0,1. \end{aligned}\right] - (3,1)$$

and seek to determine the scalars α,β,γ and a_i; $i = 0,1, \ldots, 5$ so that Q exists uniquely and

$$\| Q^{(r)} - y^{(r)}\| = 0\,(h^{6-r}) \; ; \quad r = 0,1,\ldots, 5. \qquad (3.2)$$

For this we let

$$\lambda_i \; -m_i - y_i^{(1)} ; \quad i = 0,1, \ldots, k,$$

and assume that $y \in C^7 [a,b]$. Then, the equations (2.2) and (3.1) give,

$$\left.\begin{aligned} & \lambda_0 + \alpha\lambda_1 + \beta\lambda_2 + \gamma\lambda_3 = \beta_0, \\ & \lambda_1 + \alpha\lambda_2 + \beta\lambda_3 + \gamma\lambda_4 = \beta_1 \\ & \lambda_{i-2} + 26\,\lambda_{i-1} + 66\,\lambda_i + 26\,\lambda_{i+1} + \lambda_{i+2} = \beta_i \quad t = 2,3,\ldots\ldots \; \kappa - 2, \\ & \gamma\lambda_{\kappa-4} + \beta\lambda_{\kappa-3} + \alpha\lambda_{\kappa-2} + \lambda_{\kappa-1} = \beta_{\kappa-1}, \\ & \gamma\lambda_{\kappa-3} + \beta\lambda_{\kappa-2} + \alpha\lambda_{\kappa-1} + \lambda_{\kappa} = \beta_{\kappa}, \end{aligned}\right] - (3.4)$$

where

$$\left.\begin{aligned} & \beta_i = -(\gamma_i^{(1)} + \alpha\gamma_{i+1}^{(1)} + \beta\gamma_{i+2}^{(1)} + \gamma\; y_{i+3}^{(1)}) + \frac{1}{60\,h}\sum_{j=i}^{i+5} a_{j-i} y_j, \\ & \beta_{k-i} = -(\gamma_{k-i}^{(1)} + \alpha\gamma_{k-i+1}^{(1)} + \beta\gamma_{k-i+2}^{(1)} + \gamma\; y_{k-i-3}^{(1)}) + \frac{1}{60\,h}\sum_{j=i}^{i+5} a_{j-i} y_{k-j}; \\ & \qquad\qquad i = 0,1 \end{aligned}\right] - (3.4)$$

$$| ß_i | \le \frac{11}{21}\, h^6 \, \| y\,(7) \| ; \quad i = 2,3, \ldots, k\text{-}2. \qquad (3.5)$$

The matrix of coefficients in (3.3) is the matrix of the $(k + l)\times(k + I)$ linear system which determines the parameters m_i of the quintic spline Q. We denote this matrix by A, let I be the set

$$I = \{(\alpha.\beta.\gamma) \; ; \; \det A \neq 0 \} \qquad (3.6)$$

and for any $(\alpha, \beta, \Upsilon) \in I$ we assume that there exists a number M, independent of h, such that

$$\| A^{-1} \|_{\infty} \leq M.$$

Then, for any $(\alpha, \beta, \Upsilon) \in I$, Q exists uniquely and, since

$$\max_{0 \leq i \leq k} | \lambda_i | \leq \| A^{-1} \|_{\infty} \max_{0 \leq i \leq k} | \beta_i |,$$

Lemma 2.1 shows that

$$\| Q^{(r)} - y^{(r)} \| = 0(h^{6-r}) \; ; \quad r = 0, 1, \ldots, 5,$$

only if $\beta_i = 0(h^5)$; $i = 0,1, k-1, k$.

<u>Theorem 3.1</u>. Let Q be an interpolatory quintic spline which agrees with $y \in C^7[a,b]$ at the equally spaced knots (1.1) and satisfies end conditions of the form (3. 2), where $(\alpha, \beta, \Upsilon) \in I$. Then,

$$\| Q^{(r)} - y^{(r)} \| = 0(h^{6-r}) \; ; \quad i = 0, 1, \ldots, 5,$$

only if, in (3. 1),

$$\left.\begin{aligned} a_0 &= -137 - 12\alpha + 3\beta - 2\gamma \\ a_1 &= 300 - 65\alpha - 30\beta + 15\gamma \\ a_2 &= -300 + 120\alpha - 20\beta - 60\gamma \\ a_3 &= 200 - 60\alpha + 60\beta + 20\gamma \\ a_4 &= -75 + 20\alpha - 15\beta + 30\gamma \\ a_5 &= 12 - 3\alpha + 2\beta - 3\gamma \end{aligned}\right] - \tag{3.7}$$

<u>Proof</u>. By Taylor series expansion we find that

$$\beta_i = 0(h^5); \quad i = 0,1, k-1, k,$$

only if the scalars α, β, γ and a_i; $i = 0, 1, \ldots, 5$ satisfy the relations (3.7). More specifically when the relations (3.7) hold we find that

$$\left.\begin{aligned} \beta_i &= \frac{1}{60}(10 - 2\alpha + \beta - \gamma) h^5 y^{(6)}_{i+2} + F_i h^6, \\ \beta_{k-i} &= \frac{1}{60}(10 - 2\alpha + \beta - \gamma) h^5 y^{(6)}_{k-i-2} + F_{k-i} h^6, \end{aligned}\right] - \tag{3.8}$$

$$i = 0,1,$$

where

$$|F_i| \leq \frac{\|y^{(7)}\|}{302400}\{26880 + 240(|\alpha| + |\gamma|)$$

$$+ |a_1| + |a_3| + 128(|a_o| + |a_4|) + 2187|a_5|\};$$

$$i = 0, 1, k-1, k. \qquad (3.9)$$

Definition 3.1. A quintic spline Q which interpolates the values y_i; $i = 0,1,\ldots, k$ at the equally spaced knots (1.1) and satisfies end conditions of the form (3.1) with the a_i; $i = 0,1,\ldots,5$ given by (3.7), will be called an $E(\alpha, \beta, \Upsilon)$ quintic spline.

For any $(\alpha, \beta, \Upsilon) \in I$ an $E(\alpha, \beta, \Upsilon)$ quintic spline exists uniquely and, by Theorem 3.1, if $y_i = y(x_i)$ where $y \in C^7[a,b]$ then

$$\| Q^{(r)} - y^{(r)} \| = 0(h^{6-r}); \quad r = 0, 01, \ldots, 5.$$

In general for any $(\alpha, \beta, \Upsilon) \in I$,

$$\beta_i = 0(h^5); \quad i - 0,1, k-1, k,$$

and thus

$$| m_i - y_i^{(1)} = 0(h^5); \quad i - 0,1, \ldots, k.$$

However, if the values α, β, Υ are such that

$$10 - 2\alpha + \beta - \Upsilon = 0, \qquad (3.10)$$

then, from (3.8),

$$\beta_i - 0(h^6); \quad i - 0,1, k-1, k.$$

For this reason, the class of $E(\alpha, \beta, \Upsilon)$ quintic splines whose parameters satisfy (3.10) is "best" in the sense that, for any member of this class,

$$| m_i - y_i^{(1)} = 0(h^6); \quad i = 0,1, \ldots, k. \qquad (3.11)$$

Corollary 3.1. The end conditions of an E(α ,ß , Υ) quintic spline can be written as

$$\left.\begin{aligned} m_i + \alpha m_{i+1} + \beta m_{i+2} + \gamma m_{i+3} &= p_i^{(1)}(x_i) + ap_i^{(1)}(x_{i+1}) \\ &\quad + \beta p_i^{(1)}(x_{i+2}) + \gamma p_i^{(1)}(x_{i+3});\, i=0,1 \\ m_i + \alpha m_{i-1} + \beta m_{i-2} + \gamma m_{i-3} &= p_{i-5}^{(1)}(x_i) + ap_{i-5}^{(1)}(x_{i-1}) \\ &\quad + \beta p_{i-5}^{(1)}(x_{i-2}) + \gamma p_{i-5}^{(1)}(x_{i-3});\, i=k-1, k, \end{aligned}\right] - (3.12)$$

where p_i. denotes the quintic polynomial interpolating the values y_i.,y_{i+1}, ... ,y_{i+5} at the points x_i, x_{i+1}, ..., x_{i+5}.

Proof. The proof follows from (3.1),(3,7) and the results,

$$\left.\begin{aligned} p_i^{(1)}(x_i) &= \frac{1}{60h}\{-137y_i + 300y_{i+1} - 300y_{i+2} + 200y_{i+3} - 75y_{i+4} + 12y_{i+5}\}, \\ p_i^{(1)}(x_{i+1}) &= \frac{1}{60h}\{-12y_i - 65y_{i+1} + 120y_{i+2} - 60y_{i+3} + 20y_{i+4} - 3y_{i+5}\}, \\ p_i^{(1)}(x_{i+2}) &= \frac{1}{60h}\{3y_i - 30y_{i+1} - 20y_{i+2} + 60y_{i+3} - 15y_{i+4} + 2y_{i+5}\}, \\ p_i^{(1)}(x_{i+1}) &= \frac{1}{60h}\{-2y_i + 15y_{i+1} - 60y_{i+2} + 20y_{i+3} + 30y_{i+4} - 3y_{i+5}\}, \\ p_i^{(1)}(x_{i+4}) &= \frac{1}{60h}\{3y_i - 20y_{i+1} + 60y_{i+2} - 120y_{i+3} + 65y_{i+4} + 12y_{i+5}\}, \\ p_i^{(1)}(x_{i+5}) &= \frac{1}{60h}\{-12y_i + 75y_{i+1} - 200y_{i+2} + 300y_{i+3} - 300y_{i+4} + 137y_{i+5}\}. \end{aligned}\right] - (3.13)$$

Corollary 3.1 shows that the end conditions of the E(0,0,0) quintic spline are

$$\left.\begin{aligned} m_i &= p_i^{(1)}(x_i);\, i=0,1, \\ m_i &= p_{i-5}^{(1)}(x_i);\, i=k-1, k. \end{aligned}\right] - \qquad (3.14)$$

Clearly $(0,0,0) \in I$ and therefore an E(0,0,0) quintic spline exists uniquely. In fact it is easy to see that, in this case, the (k-4)×(k-4) matrix A of (3.3) is such that

$$\| A^{-1} \|_\infty \le \frac{1}{12}.$$

The corollary also shows than an $E(\infty, \beta, \Upsilon)$ quintic spline can be interpreted as an interpolatory quintic spline with end conditions

$$\left.\begin{aligned} m_i &= p^{(1)}_{i-2}(x_i);\, i=2, \\ m_i &= p^{(1)}_{i-4}(x_i);\, i=k-2, k-1. \end{aligned}\right] \tag{3.15}$$

Similarly the $E(\alpha, \beta, \Upsilon)$ and $E(\alpha, \beta, \infty)$ quintic splines can he interpreted as interpolatory quintic splines with end conditions respectively

$$\left.\begin{aligned} m_i &= p^{(1)}_{i-2}(x_i);\, i=2,3, \\ m_i &= p^{(1)}_{i-3}(x_i);\, i=k-3,\, k-2, \end{aligned}\right] \tag{3.16}$$

and

$$\left.\begin{aligned} m_i &= p^{(1)}_{i-3}(x_i);\, i=3,4, \\ m_i &= p^{(1)}_{i-2}(x_i);\, i=k-4,\, k-3. \end{aligned}\right] \tag{3.17}$$

The unique existence of each of the $E(\infty, \beta, \Upsilon)$, $E(\alpha, \infty, \Upsilon)$ and $E(\alpha, \beta, \infty)$ quintic splines can be established easily by considering the linear systems for the m_i's derived from the consistency relation (2.2) and the end conditions (3.15)- (3.17). However, the existence of the $E(\alpha, \infty, \Upsilon)$) and $E(\alpha, \beta, \infty)$ splines can only be established under the assumptions $k \geq 7$ and $k \geq 9$ respectively.

The corollaries stated below establish various alternative representations for the end conditions of an $E(\alpha, \beta, \Upsilon)$ Quintic spline. They are established by using the quintic spline identities listed in Section 2 and the expressions for the derivatives $p_i^{(k)}(x)$; $k = 2,3,4$ of the interpolating polynomial $p_i(x)$. Although the algebra involved in the derivation of these results is very laborious, the proofs are otherwise elementary and, for this reason, the details are omitted.

<u>Corollary 3.2.</u> The end conditions of an E(α ,ß , ϒ) quintic spline can be written as

$$\left.\begin{aligned} &A_0 \Delta^5 N_i + A_1 \Delta^4 N_i + A_2 \Delta^3 N_i + A_3 \Delta^2 N_i = 0, \\ &A_0 \nabla^5 N_{k-i} - A_1 \nabla^4 N_{k-i} + A_2 \nabla^3 N_{k-i} + A_3 \nabla^2 N_{k-i} = 0; \\ &\qquad i = 0,1, \end{aligned}\right] - \qquad (3.18)$$

where $N_i = Q^{(4)}(x_i)$,

$$\left.\begin{aligned} A_0 &= (12 - 3\alpha + 2\beta - 3\gamma\gamma) \\ A_1 &= 5(69 - 17\alpha + 11\beta - 15\gamma 5\gamma \\ A_2 &= 10(137 - 31\alpha + 17\beta - 19\gamma 9\gamma \\ A_3 &= 120(10 - 2\alpha + \beta - \gamma), \end{aligned}\right] - \qquad (3.19)$$

and Δ, ∇ are respectively the forward and backward difference operators.

Corollary 3.2 shows that the five E(α ,ß , ϒ) quintic splines listed below have particularly simple representations of the form $\Delta^n N_i = \nabla^n N_{k-i} = 0$.

Spline	End conditions	
$E\left(\frac{33}{5}, \frac{21}{5}, \frac{1}{5}\right)$	$\Delta^2 N_i = \nabla^2 N_{k-i} = 0; \quad i = 0, 1$	
E(21,33,5)	$\Delta^2 N_i = \nabla^2 N_{k-i} = 0; \quad i = 1,2$	
E(9, 9, 1)	$\Delta^3 N_i = \nabla^3 N_{k-i} = 0; \quad i = 0,1$	(3.20)
E(17,33, 9)	$\Delta^4 N_i = \nabla^4 N_{k\_i} = 0; \quad i = 0,1$	
E(25,61,21)	$\Delta^5 N_i = \nabla^5 N_{k-i} = 0; \quad i = 0, 1$	

The unique existence of each of these five E(α ,ß , ϒ) quintic splines can be established easily by considering the linear system for the parameters N_i derived by using the end conditions (3.20) and the conaistency relations (2.5). However, the existence of the E(21,33,5), E(17,33,9) and E(25,61,21) splines can be established only under the assumptions $k \geq 7$, $k \geq 7$ and $k \geq 8$ respectively. We note that the parameters of each of the E(9,9, 1), E(17,33,9) and E(25, 61,26) quintic splines satisfy (3.10) and thus, for each of these three splines,

$$\max_{o \leq i \leq k} | m_i - y_i^{(1)} | = 0 (h^n), \tag{3.21}$$

where $n = 6$, rather than $n = 5$.

Corollary 3.3. The end conditions of an E(α ,ß , ϒ) quintic spline can

$$\left.\begin{aligned} &\sum_{j=i}^{i+3} B_{j-i} \{m_j - p_j^{(2)}(x_j)\} = 0, \\ &\sum_{j=i}^{i+3} B_{j-i} \{m_{k-j} - p_{k-5-i}^{(2)}(x_{k-j})\} = 0; i = 0,1, \end{aligned}\right] - \tag{3.22}$$

be written as

where $M_i = Q^{(2)}(x_i)$,

$$\left.\begin{aligned} &B_0 = (-8\alpha + 7\beta - 8\gamma - 23), \\ &B_1 = 3(-77\alpha + 58\beta - 67\gamma + 118), \\ &B_2 = 3(-58\alpha + 77\beta - 118\gamma + 67), \\ &B_3 = (-7\alpha + 8\beta + 23\gamma + 8), \end{aligned}\right] - \tag{3.23}$$

and p_i denotes the quintic polynomial interpolating the values y_i, y_{i+1} . . . . , y_{i+5} at the Points x_i, x_{i+1} ... . x_{i+5}.

When α + 8735/888, ß - 8920/888 and ϒ + 753/888 then, in (3.23), $B_1 = B_2 = B_3 = 0$, It follows, from Corollary 3,3, that the end conditions of the $E\left(\frac{8735}{888}, \frac{8920}{888}, -\frac{753}{888}\right)$ can be written as

$$\left.\begin{aligned} &M_i = p_1^{(2)}(x_i); i = 0, 1, \\ &M_i = p_{i-5}^{(1)}(x_i); i = k-1, k. \end{aligned}\right] - \tag{3.24}$$

The unique existence of this spline follows at once by considering the linear system, for the parameters M. , derived from the consistency relations (2.3) and the end conditions (3.24).

<u>Corollary 3.4</u>. The end conditions of an E(α ,ß , ϒ) quintic spline can be written as

$$\left.\begin{aligned} &\sum_{j=i}^{i+3} C_{j-i}\{n_j - p_j^{(3)}(x_j)\} = 0, \\ &\sum_{j=i}^{i+3} C_{j-i}\{n_{k-j} - p_{k-5-i}^{(3)}(x_{k-j})\} = 0;\ i = 0,1, \end{aligned}\right] \quad (3.25)$$

where $n_i = Q^{(3)}(x_i)$

$$\left.\begin{aligned} C_0 &= \alpha - \gamma - 8, \\ C_1 &= 18\alpha + \beta - 26\gamma - 65, \\ C_2 &= \alpha + 18\beta - 65\gamma - 26, \\ C_1 &= \beta - 8\gamma - 1, \end{aligned}\right] \quad (3.26)$$

and p_i denotes the quintic polynomial interpolating the values $y_i, y_{i+1}, \ldots, y_{i+5}$ at the points, $x_i, x_{i+1}, \ldots, x_{i+5}$.

When α = 2743/728, ß - 1040/728 and ϒ = 39/728 then, in (3.26), $C_1 = C_2 = C_3 = 0$. It follows, from Corollary 3.4, that the end conditions of the $E\left(\frac{2743}{728}, \frac{1040}{728}, \frac{39}{728}\right)$ spline can be written as

$$\left.\begin{aligned} n_i &= p_i^{(3)}(x_i); i=0,1, \\ n_i &= p_{i-5}^{(3)}(x_i); i=k-1,k. \end{aligned}\right] \quad (3.27)$$

The unique existence of this spline follows at once by considering the linear system, for the parameters n_i, derived from the consistency relations (2.4) and the end conditions (3.27).

Corollary 3.5. The end conditions of an E(α ,ß , ϒ) quintic spline can be written as

$$\left.\begin{array}{l} \sum_{j=i}^{i+3} D_{j-i}\{N_j - p_j^{(4)}(x_j)\} = 0, \\ \sum_{j=i}^{i+3} D_{j-i}\{N_{k-j} - p_{k-5-i}^{(4)}(x_{k-j})\} = 0;\ i = 0,1, \end{array}\right] - \tag{3.28}$$

where $N_i = Q^{(4)}(x_i)$,

$$\left.\begin{array}{l} D_o = (8\alpha + \beta + 2\gamma - 19), \\ D_1 = 3(11\alpha - 8\beta + 17\gamma - 36), \\ D_2 = 3(8\alpha + 11\beta + 36\gamma - 17), \\ D_1 = (\alpha - 2\beta + 19\gamma - 2), \end{array}\right] - \tag{3.29}$$

and p_i denotes the quintic polynomial interpolating the values y_i, $y_{i+1}, \ldots$, y_{i+5} at the points x_i $x_{i+1}, \ldots$, x_{i+5}.

When α = 391/83, ß - 179/83 and ϒ = 7/83 then in (3.29), $D_1 = D_2 = D_3 = 0$. It follows, from Corollary 9,5, that the end conditions of the $E\left(\frac{391}{83},\frac{179}{83},\frac{7}{83}\right)$ can be written as

$$\left.\begin{array}{l} N_i = p_i^{(4)}(x_i);\ i = 0,1, \\ N_i = p_{i-5}^{(4)}(x_i);\ i = k-1,\ k. \end{array}\right] - \tag{3.30}$$

The unique existence of this spline follows at once by considering the linear system for the N_i derived from the consistency relations (2.5) and the end conditions (3.30).

Let d_i denote the jump discontinuity of $Q^{(5)}$at the knot x_i. Then, from (2.15),

$$\begin{aligned} d_i &= Q^{(5)}(x_i+) - Q^{(5)}(x_i-) \\ &= \frac{1}{h}\Delta^2 N_{i-1} \\ &= \frac{1}{h}\nabla^2 N_{i+1};\quad i = 1,2,\ \ldots,\ k-1. \end{aligned} \tag{3.31}$$

It follows at once, from (3.2 0), that the fifth, derivatives of the $E\left(\frac{33}{5},\frac{21}{5},\frac{1}{5}\right)$ and E(21, 33, 5) quintic splines are continuous respectively at the knots x_j ; $j = 1, 2, k-2, k-1$ and x_j ; $J = 2, 3, k-3, k-2$. These properties are the special cases $G_2 = G_3 = G_4 = 0$ and $G_1 = G_3 = G_4 = 0$ of the general result contained in the following corollary. This result is established easily by using (3.31) and the result of Corollary 3.2.

Corollary 3.6. Let Q be an $E(\alpha, \beta, \Upsilon)$ quintic spline and let d_i denote the jump discontinuity of $Q^{(5)}$ at $x = x_i$ Then,

$$\left.\begin{aligned} G_1 d_j + G_2 d_{j+1} + G_3 d_{j+2} + G_4 d_{j+3} &= 0;\ i = 1,2, \\ G_1 d_j + G_2 d_{j-1} + G_3 d_{j-2} + G_4 d_{j-3} &= 0;\ j = k-2,\ k-1, \end{aligned}\right] - \qquad (3.32)$$

where,

$$\left.\begin{aligned} G_1 &= 163 - 12\alpha + 3\beta - 22\gamma 2 \\ G_2 &= 716 - 149\alpha + 66\beta - 49\gamma 9 \\ G_3 &= 309 - 76\alpha + 49\beta - 66\gamma 6 \\ G_4 &= 36 - 9\alpha + 6\beta - 9\gamma\gamma \end{aligned}\right] - \qquad (3.33)$$

A number of properties of special interest which emerge from the result of Corollary 3.6 are listed below.

Spline	Property satisfied by the d_i's
$E\left(\frac{33}{5},\frac{21}{5},\frac{1}{5}\right)$	$d_1 = d_2 = d_{k-2} = d_{k-1} = 0$
E(21, 33, 5)	$d_2 = d_3 = d_{k-3} = d_{k-2} = 0$
E(9, 9, 1)	$d_1 = d_2 = d_3$ and $d_{k-1} = d_{k-2} = d_{k-3}$
E(17, 33, 9)	$\Delta^2 d_i = \nabla^2 d_{k-1}. = 0;\ i = 1,2$
E(25, 61, 21)	$\Delta^3 d_i = \nabla^3 d_{k-i} = 0;\ i = 1,2$

(3.34)

4. Numerical results

In Table .1. we present numerical results obtained by taking

$y(x) = \exp(x)$,

$$x_i = 0.05i; \quad i = 0, 1, \ldots, 20, \tag{4.1}$$

and constructing various $E(\alpha, \beta, \gamma)$ quintic splines. The splines considered are the E(0,0,0) quintic spline and the five splines of (3.2.0). The results listed are values of the absolute error

$$|\exp(x) - Q(x)|, \tag{4.2}$$

computed at various points between the knots. For comparison purposes we also list results computed by constructing the natural quintic spline (N.Q.S.) with knots (4.1) interpolating the function $y(x) = \exp(x)$ at the knots.

The numerical results indicate the serious damaging effect that the natural end conditions

$$Q^{(r)}(x_0) - Q^{(r)}(x_{20}) = 0; \quad r = 3,4, \tag{4.3}$$

have upon the quality of the approximation, and demonstrate the considerable improvement in accuracy obtained by using end conditions of the type considered in the present paper, instead of (4.3). The results also show that, as predicted by the theory, the 'best' $E(\alpha, \beta, \gamma)$ quintic splines correspond to values α, β, γ that satisfy (3.10).

TABLE 1

Values of $|\exp(x) - Q(x)|$

X	N.Q.S.	E(0,0,0)	$E\left(\frac{33}{5},\frac{21}{5},\frac{1}{5}\right)$	E(21, 33,5)	E(9,9,1)	E(17,33,9)	E(25,61, 21)
0.01	0.29×10^{-5}	0.17×10^{-9}	$0.21X10^{-9}$	$0.31x10^{-8}$	0.84×10^{-11}	0.70×10^{-11}	$0.17x10^{-11}$
0.02	0.33×10^{-5}	0.78×10^{-9}	0.23×10^{-3}	$0.28x10^{-8}$	0.84×10^{-11}	$0.13x10^{-11}$	$0.25X10^{-11}$
0.07	$0.12^{x}10^{-5}$	$0.72X10^{-9}$	$0.56x10^{-10}$	$0.24x10^{-9}$	$0.31x10^{-11}$	$0.94x10^{-12}$	$0.69x10^{-12}$
0.09	$0.52X10^{-6}$	$0.33X10^{-9}$	$0.24xl0^{10}$	$0.76\times l0^{-10}$	$0,13\times l6^{-11}$	$0.35xl0^{-12}$	$0.26xl0^{-12}$
0.22	0.92×10^{-7}	$0.59X10^{-10}$	$0.54x10^{-11}$	$0.92X10^{-11}$	$0.l0\times l0^{-11}$	$0;11\times10^{-11}$	$0.12X10^{-11}$
0.36	$0.55X10^{-8}$	$0.40X10^{-11}$	$0.32xl0^{-12}$	$0.12\times l0^{-11}$	$0.56xl0^{12}$	$0.56xl0^{-12}$	$0.55xl0^{-12}$
0.62	$0.16X10^{-8}$	$0.98X10^{-11}$	$0.ll\times l0^{-11}$	$0.31X10^{-11}$	0.17×10^{-11}	$0.17X10^{-11}$	$0.17X10^{-11}$
0.93	$0.31X10^{-5}$	0.14×10^{-8}	$0.13xl0^{-9}$	$0.49\times l0^{-9}$	$0.22\times l0^{-11}$	$0.20\times l0^{-11}$	$0.24X10^{-11}$
0.96	$0.35X10^{-5}$	$0.12^{x}10^{-8}$	$0.16x10^{-9}$	$0.12xl0^{8}$	0.71×10^{-11}	$O.14\times l0^{-11}$	$0.94xl0^{-12}$
0.98	0.92×10^{-5}	0.15×10^{-8}	$0.51xl0^{-9}$	$0.58x10^{-8}$	$0.23x10^{-10}$	$0.38x10^{-11}$	$0.24\times l0^{-11}$
0.99	$0.77X10^{-5}$	0.29×10^{-9}	$0.45\times l0^{-9}$	$0.63xl0^{-8}$	$0.20xl0^{-10}$	$0.23X10^{-11}$	$0.l0\times l0^{-11}$

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