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INTERPOLATION TO BOUNDARY ON SIMPLICES

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1 . Introduction.

The finite dimensional problem of constructing Lagrange and Hermite interpolants, which natch function and derivative values at a finite number of points on a simplex in $\underset{\sim}{R}^n$, has been considered by a number of authors [4,9,10,11,12]. More recently, Mansfield [8] has considered the infinite dimensional problem of constructing blending function interpolants which match function and derivative values on the entire boundary of a tetrahedron and, more generally, an n-simplex.

Mansfield's work generalizes a scheme for interpolating on triangles first described in Barnhill, Birkhoff and Gordon [1]. In the present paper we develop a new scheme for blending function interpolation on n—simplices which is a generalization of an interpolant for triangles described in [7]. The essential feature of the scheme is that it is a moving average or 'blend' of interpolants, each of which matches function and derivative values on all but one of the faces of the simplex.

The paper begins, in Section 3, with the development of an explicit representation of a finite dimensional Hermite interpolation polynomial for the simplex. This interpolant is a natural generalization of Hermite two point Taylor interpolation in one variable and includes the bivariate tricubic polynomial interpolant of Birkhoff [3] as a special case. The existence of the interpolant in the general case is suggested by Mansfield [8]. It should be stressed that the piecewise application of the interpolant over a union of non-overlapping simplices in $\underset{\sim}{R}^n, n > 1$, gives a C^0 global function, even though derivatives across the vertices of the simplicial complex are continuous.

The importance of the Hermite interpolant to this paper is that its basis functions are used to construct the weight functions of the blending function scheme. This scheme is described in general terms in Section 4 and polynomial and rational examples of the scheme are developed in Sections 5 and 6. These blending function schemes, unlike the Hermite interpolant, give C^N global functions when they are piecewise applied over a simplicial complex.

All the interpolation schemes described in this paper define bounded idempotent linear operators, i.e. projectors , on some appropriate function space. Thus the schemes are able to reproduce all functions in the range of the interpolation projector. The range is thus called the <u>precision set</u> of the interpolant. The subset of all polynomials of a certain degree which can be contained in the precision set is important in determining the accuracy of each interpolant. Such polynomial sets are considered in detail for each interpolation scheme of the paper.

The paper begins with a summary of notation in Section 2. In particular, the barycentric coordinate system for an n—simplex is introduced since each interpolant will be described in terms of this invariant system.

2, Preliminary Notation.

Let

$$s_n = \{x = \sum_{j=1}^{n+1} \lambda_j v_j \,/\, 0 \le \lambda_j \le 1 \,,\ \sum_{j=1}^{n+1} \lambda_j = 1\} \,, \tag{2.1}$$

where

$$x = (x_1 ,\ldots,x_n) \in \underset{\sim}{R}^n, \tag{2.2}$$

define a simplex in $\underset{\sim}{R}^n$ with vertices

$$v_j = (v_{1j}, \ldots, v_{nj}) \in \underset{\sim}{R}^n, \quad j=1,2,\ldots, n+1 . \tag{2.3}$$

We assume that the simplex is non-degenerate so that the vertices do not lie in an m-dimensional hyperplane of $\underset{\sim}{R}^n$, $m<n$ .In this case, the representation

$$x = \sum_{j=1}^{n+1} \lambda_j v_j \text{ , where } \sum_{j=1}^{n+1} \lambda_j = 1 , \tag{2.4}$$

uniquely defines the barycentric coordinate system $\lambda_j = \lambda_j (x)$, $j=1 ,2,\ldots,n+1$. Let E. denote a point on the face $\lambda_i = 0$, i.e. the face opposite the vertex V_i . Then E. can be represented as

$$E_i = \sum_{j=1}^{n+1} \lambda_j v_j \text{ , where } \lambda_j = 0 , \sum_{j=1}^{n+1} \lambda_j = 1 . \tag{2.5}$$

The barycentric coordinate system can be interpreted as defining an affine transformation between $(x_1, \ldots, x_n) \in \underset{\sim}{R}^n$,and $(\lambda_1, \ldots, \lambda_n) \in \underset{\sim}{R}^n$ This transformation takes S_n , with vertices V_j , $j = 1,2,\ldots,n+1$, onto a standard simplex $\tilde{S}_n$, ith vertices $\tilde{V}_j = e_j$, $j = 1,2,\ldots,n$, and $\tilde{v}_{n+1} = \underset{\sim}{0}$ respectively, where e_j ,$j = 1,\ldots,n$, denotes the canonical basis of $\underset{\sim}{R}^n$

We define a derivative operator along the edge joining the vertices V_i and V_j by

$$D_{ij} = \sum_{k=1}^{n} (v_{kj} - v_{ki}) \partial / \partial x_k , \quad i \neq j, \quad i,j=1,2,\ldots,n+1 , \tag{2.6}$$

and a product of such operators along all edges which meet at V_i by

$$D_i = \prod_{\substack{j=1 \\ j\neq 1}}^{n+1} D_{ij} , i=1,2,\ldots,n+1 . \tag{2.7}$$

Furthermore, if $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ denote n+1 non-negative integers and

$$\underset{\sim}{\alpha}_i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n+1}) , \quad i = 1, 2, \ldots, n+1 \quad , \tag{2.8}$$

Denotes a multi-index of N of these integers, then we define

$$D_i^{\underset{\sim}{\alpha}_i} = \prod_{\substack{j=1 \\ j \neq i}}^{n+1} D_{ij} \; , i = 1, 2, \ldots, n+1 \quad . \tag{2.9}$$

Finally, with

$$\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_n) \quad , \tag{2.10}$$

we define

$$D^{\underset{\sim}{\alpha}} = \prod_{j=1}^{n} (\partial / \partial x_j)^{\alpha_j} \quad . \tag{2.11}$$

(2.11)

We conclude this section with a lemma which will be useful in subsequent work:

<u>Lemma 2.1</u>. Let $f : \underset{\sim}{R}^n \rightarrow \underset{\sim}{R}$ be defined by

$$f(x) = g(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \quad , \tag{2.12}$$

Where g is a real dif ferentiable function of n+1 variables and λ_j , j = 1,2,...,n +1 , are defined by (2.4). Then

$$D_{ij} f = (\partial/\partial\lambda_j - \partial/\partial\lambda_i) \quad i \neq j \; , \; i, j = 1,2,\ldots,n+1 \; . \tag{2.13}$$

<u>Proof</u>. Let

$h(x_1, x_2, \ldots, x_{n+1}) = g(\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$,

where $x_1, x_2, \ldots, x_{n+1}$ are defined by

$$\sum_{j=1}^{n+1} \lambda_j v_{ij} = x_i \; , i = 1, 2, \ldots, n+1 \text{ and } v_{n+1} = 1 \; , j = 1, 2, \ldots, n+1 \; .$$

This transformation is non-singular since the simplex is non-degenerate and (2.4) is the particular case $x_{n+1} \equiv 1$. Now

$$\partial/\partial\lambda_j - \partial/\partial\lambda_i)g = \sum_{k=1}^{n+1} (v_{kj} - v_{ki})\partial h/\partial x_k$$

$$= \sum_{k=1}^{n} ((v_{kj} - v_{ki})\partial h/\partial x_k \ .$$

Thus, from (2.6), the lemma is established, where

$$f(x) = h(x_j ,\dots,x_n,1) \ .$$

3. A Hermite Interpolant for the Simplex .

Theorem 3.1 . Given the non-negative integer N , let f be a real valued function defined on S_n which is such that

$$\left(D_i^{\underset{\sim}{\alpha}_i} f\right)(V_i) \ , \quad i = 1,2,\dots,n+1 \ , \quad \underset{\sim}{\alpha}_i \in \underset{\sim}{N}^n \ , \tag{3.1}$$

are well defined values, where

$$\underset{\sim}{N} = \{0 \ , 1,\dots,N\} \ . \tag{3.2}$$

Then there exists an interpolation polynomial p, explicitly defined by

$$p(x) = \sum_{i=1}^{n+1} \lambda_i^{N+1} \sum_{\underset{\sim}{\alpha}\in\underset{\sim}{N}^n} P_{\underset{\sim}{\alpha}_i}(\lambda)\left(D_i^{\underset{\sim}{\alpha}i}(f/\lambda_i^{\ N+1})\right)(V_i) \ , \tag{3.3}$$

where

$$p_{\underset{\sim}{\alpha}_i}(\lambda) = \prod_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j^{\ \alpha_j}/\alpha_j! \ , \quad \lambda = (\lambda_1,\lambda_2,\dots,\lambda_{n+1}) \ , \tag{3.4}$$

which is such that

$$\left(D_i^{\underset{\sim}{\alpha}_i} p\right)(V_j) = \left(D_i^{\underset{\sim}{\alpha}j} f\right) \quad j = 1,2,\dots,n+1 \ , \quad \underset{\sim}{\alpha}_j \in \underset{\sim}{N}^n \ . \tag{3.5}$$

Proof. Since

$$\left(D_i^{\underset{\sim}{\alpha}_i} \lambda_i^{N+1}\right)(V_j) = 0 \ , \ i \neq j \ , \ i = 1,2,\dots,n+1 \ , \ \underset{\sim}{\alpha}_j \in \underset{\sim}{N}^n \ ,$$

it follows from (3.3) that

$$\left(D_j^{\alpha_j} p\right)(V_j) = \left(D_j^{\alpha_j} \{ \lambda_j^{N+1} \sum_{\alpha'_j \in N^n} p_{\alpha'_j} \left(D_j^{\alpha_j}(f/\lambda_j^{N+1})\right)(V_j) \}\right)(V_j)$$

for all $\alpha_j \in N^n$. Application of Leibniz's rule then gives that

$$\left(D_j^{\alpha_j} p\right)(V_j) = \sum_{\beta_j \le \alpha_j} \left(\binom{\alpha_j}{\beta_j} \{D_j^{\alpha_j - \beta_j} \lambda_j^{N+1}\} \sum_{\alpha'_j \in N^n} (D_j^{\beta_j} p_{\alpha'_j}) \{ D^{\alpha'_j} (f/\lambda_j^{N+1}) \} \right)(V_j)$$

Now, using Lemma 2.1,

$$\left(D_j^{\beta_j} p_{\alpha'_j}\right)(V_j) = \left(\{ \prod_{\substack{k=1 \\ k \ne j}}^{n+1} (\partial/\partial\lambda_k - \partial/\partial\lambda_j)^{\beta_k} \} \{ \prod_{\substack{k=1 \\ k \ne j}}^{n+1} \lambda_k{}^{\alpha'_k}/\alpha'_k! \} \right)(V_j)$$

$$= \begin{cases} 1 & \text{if } \alpha'_j = \beta_j , \\ 0 & \text{otherwise} . \end{cases}$$

Hence, substituting and reconstituting the summation using Leibnitz's rule gives

$$\left(D_j^{\alpha_j} p\right)(V_j) = \sum_{\beta_j \le \alpha_j} \left(\binom{\alpha_j}{\beta_j} \{D_j^{\alpha_j - \beta_j} \lambda_j^{N+1}\} \{D_j^{\beta_j} (f/\lambda_j^{N+1})\} \right)(V_j)$$

$$= \left(D_j^{\alpha_j} (\lambda_j^{N+1} f/\lambda_j^{N+1}) \right)(V_j)$$

$$= \left(D_j^{\alpha_j} f \right)(V_j)$$

which cpmpletes the proof of the theorem.

<u>Example</u> the interpolant defined by (3.3) is a natural generalization of Hermite two point Taylor interpolation in one variable, where, with $n = 1$ and $x = \lambda_1 V_1 + \lambda_2 \lambda_2 \in R$ $\lambda_2 + \lambda_2 = 1$, we have

$$p(x) = \lambda_1^{N+1} \sum_{i=0}^{N} \{ \lambda_2{}^i / i! \} \left(D_{12}{}^i (f/\lambda_1{}^{N+1})\right)(V_1)$$

$$+ \lambda_2^{N+1} \sum_{i=0}^{N} \{ \lambda_1{}^i / i! \} \left(D_{12}{}^i (f/\lambda_1{}^{N+1})\right)(V_2), \qquad (3.6)$$

see Davis [5, p.37]. Equation (3.6) can be expressed in the cardinal basis form

$$p(x) = \sum_{i=0}^{N} h_i(\lambda_2)\left(D_{12}^{\,i} f\right)(V_1) + \sum_{i=0}^{N} h_i(\lambda_2)\left(D_{12}^{\,i} f\right)(V_2) , \qquad (3.7)$$

where

$$h_i(\lambda) = (1-\lambda)^{N+1} \sum_{k=i}^{N} \lambda^k (N+k-i)!/N!i!(k-i)! . \qquad (3.8)$$

When n = 2 and N=1 in (3.3) the tricubic interpolant on a triangle of G. Birkhoff [3] is obtained.

The piecewise application of the interpolant (3.3) over a bounded domain , which consists of the union of non-overlapping simplices in $\underset{\sim}{R}^n$ gives a C global function, except in the case n=1 when it is C^N. The blending function interpolants of the following sections will, however, give C^N global functions for all n .

It follows from the theory of finite dimensional interpolation, and the explicit representation given by (3.3) and (3.4), that the linear functionals defined by (3.1) are linearly independent over the (n+l)(N+l) dimensional polynomial space defined by

$$T = \{ \lambda_i^{N+1} \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \lambda_j^{\alpha_j} / 0 \le \alpha_j \le N , i = 1,2,\dots,n+1 \} . \qquad (3.9)$$

Also, the linear operator P defined by

$$P[f](x) = p(x) , f \in C^{nN}(S_n) , \qquad (3.10)$$

where p is given by (3.3), is a projector on $C^{nN}(S_n)$ with range T . Thus

$$P^2[f] - P[f] \qquad (3.11)$$

and

$$P[f] = f \text{ for all } f \in T \, . \tag{3.12}$$

The following theorem gives more insight into the nature of T .

Theorem 3.2. T is the space of polynomials whose restriction along an edge joining any two vertices V_i and V_j , $i \neq j$, is a polynomial of degree 2N+1 , i.e.

$$T = \{ \prod_{k=1}^{n+1} \lambda_k^{\alpha_k} / 0 \leq \alpha_i + \alpha_j \leq 2N+1 \, , i \neq j \} \, . \tag{3.13}$$

Proof. Clearly, from (3.9),

$$T \subset \{ \prod_{k=1}^{n+1} \lambda_k^{\alpha_k} / 0 \leq \alpha_i + \alpha_j \leq 2N+1 \, , i \neq j \} \, .$$

Thus we require to prove that if

$$f = \{ \prod_{k=1}^{n+1} \lambda_k^{\alpha_k} \text{, where } 0 \leq \alpha_i + \alpha_j \leq 2N+1 \, , i \neq j \} \, , \tag{3.14}$$

then $f \in T$, as defined by (3.9). The proof of this comprises two parts:

(i) Suppose $\alpha_k \geq N+1$ for $k=i$. Then, since $\alpha_i + \alpha_k \leq 2N+1$, it follows that $\alpha_k \leq N$ for , $k \neq i$ $k=1,2,...,n+1$. Now (3.14) can be written as

$$f = \lambda_i^{N+1} \lambda_i^{\alpha i - N-1} \prod_{\substack{k=1 \\ k \neq i}}^{n+1} \lambda_k^{\alpha k}$$

$$= \lambda_i^{N+1} \Big(1 - \prod_{\substack{k=1 \\ k \neq i}}^{n+1} \lambda_k \Big)^{\alpha i - N-1} \prod_{\substack{k=1 \\ k \neq i}}^{n+1} \lambda_k^{\alpha k} ,$$

and since $\alpha_i - N - 1 + \alpha_k \leq N$ it follows that $f \in T$.

(ii) Suppose $\alpha_k \leq N$ for all $k=1 ,2,...,n+1$. Then $\sum_{i=1}^{n+1} \alpha_k \leq (n+1)N$.

Assume further the inductive hypothesis that f ∊ γ for all $M \le \sum_{k=1}^{n+1} \alpha_K \le (N+1)N$, where $1 \ge M \le (N+1)N$. Now if $\sum_{k=1}^{n+1} \alpha_K = M-1$ then (3.1 4) can be written as

$$f = \left(\sum_{i=1}^{n+1} \lambda_i\right)\left(\prod_{k=1}^{n+1} \lambda_k^{\alpha_k}\right)$$

$$= \sum_{i=1}^{n+1} \prod_{k=1}^{n+1} \lambda_k^{\alpha_{ki}} ,$$

where $\sum_{k=1}^{n+1} \alpha_{ki} = M$ and $\alpha_{ki} \le N+1$. Thus f ∊ γ using either the inductive hypothesis if $\alpha_{ki} \le N$ or part (i) if $\alpha_{ki} - N+1$. The inductive hypothesis is true for M =(n+1)N since in this case $\alpha_k=N$ for all k=l , 2,...,n+1 and then (3.14) can be written as

$$f = \left(\sum_{i=1}^{n+1} \lambda_i\right)\left(\prod_{k=1}^{n+1} \lambda_k^{N}\right)$$

$$= \sum_{i=1}^{n+1} \lambda_i^{N+1} \prod_{k=1}^{n+1} \lambda_k^{N} ,$$

so that f ∊ T . Hence, by induction, the hypothesis is true for all $0 \le M \le (n+l)N$.

This completes the proof of the theorem.

The following corollary follows immediately from (3.13)

<u>Corollary 3.1</u>. The following inclusions hold:

$$P_{2N+1} \subset T \subset P_{(n+1)N+1} , P2N+1\ (n+l)N+l \quad (3.15)$$

where

$$P_k = \{ \prod_{i=1}^{n} x_i^{\alpha i} \ / \ x \in \underset{\sim}{R}^n , 0 \le \sum_{i=1}^{n} \alpha_i \le K \} , \quad (3.16)$$

is the set of polynomials of degree ≤ K .

It should be noted that

$$P_{2N+2} \not\subset T \text{ and } T \not\subset P_{(n+1)N} ,$$

An Algebraic Identity. We are now in a position to derive an identity which will be essential for the blending function schemes which follow. Let $f(x) \equiv 1$. Then $f \in T$ and, using (3.12), the following identity can be derived:

$$\sum_{k=1}^{n+1} a_i\ (x) = 1 \text{ for all } x \in \underset{\sim}{R}^n \qquad (3.17)$$

where $a_i \in T$ are polynomial functions defined by

$$a_i(x) = \lambda_i^{N+1} \sum_{\underset{\sim}{\alpha}_i \in \underset{\sim}{N}^n} P_{\underset{\sim}{\alpha}_i} (\lambda\lambda(N+|\underset{\sim}{\alpha}_i|)!/N!\ ,\ |\underset{\sim}{\alpha}_i| = \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \alpha_j\ , \qquad (3.18)$$

and the $P_{\underset{\sim}{\alpha}_i}$ are given by (3.4) .

4. A General Scheme for Blending Function Interpolation.

In this section we define a general scheme for interpolating function and derivative values given all faces of the simplex S_n . Two particular implementations of the scheme will be given in subsequent sections. The interpolation scheme is defined in the following theorem.

Theorem 4. 1. Let f and P_i [f], i=l ,2,...,n+l , where the P_i are linear operators, be real valued functions defined on S_n which are such that

$$\left(D^{\underset{\sim}{\alpha}} P_i[f]\right)(E_j) = \left(D^{\underset{\sim}{\alpha}} f\right)(E_j), j \neq i\ , j = 1,2,...,n+1\ ,$$

$$\text{for all } |\underset{\sim}{\alpha}| = \sum_{k=1}^{n} \alpha_k \leq N. \qquad (4.1)$$

Thus P_i [f] interpolates f and its derivatives of order N and less on all faces of the simplex excluding the face $\lambda_i = 0$'. Then

$$P[f](x) - \sum_{i=1}^{n+1} a_i(x)\ P_i[f](x),\ x \in S_n\ , \qquad (4.2)$$

where the a_i are given by (3.18) , defines a linear operator P which is such that

$$\left(D^{\alpha} P[f]\right)(E_j) = \left(D^{\alpha} f\right)(E_j) \text{ for all } \quad j = 1,2,\ldots,n+1, |\alpha| \le N\ . \qquad (4.3)$$

Proof. The proof is almost self evident, relying on (3.17) and (3.18). More formally, applying Leibnitz's rule gives

$$\left(D^{\alpha} P[f]\right)(E_j) = \left(D^{\alpha} \sum_{i=1}^{n+1} a_i P_i[f]\right)(E_j)$$

$$= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \sum_{i=1}^{n+1} \left(D^{\beta} a_i\right)(E_j) \left(D^{\alpha-\beta} P[f]\right)(E_j)$$

$$= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \sum_{\substack{i=1 \\ i \ne j}}^{n+1} \left(D^{\beta} a_i\right)(E_j) \left(D^{\alpha-\beta} f\right)(E_j)\ ,$$

where, since (3.18) contains the factor λ_i^{N+1}, we have used the fact

That

$$\left(D^{\beta} a_j\right)(E_j) = 0 \text{ for all } |\beta| \le N\ .$$

Furthermore, from (3.17),

$$\sum_{\substack{i=1 \\ i \ne j}}^{n+1} \left(D^{\beta} a_i\right)(E_j) = \begin{cases} 1 & \text{if } |\beta| = 0\ , \\ 0 & \text{if } 0 < |\beta| \le N\ , \end{cases}$$

and hence (4.3) follows.

Remarks, Suppose H is a subspace of bounded real valued functions defined on S_n which is such that $P_i : H \to H$, i=l ,2,...,n+l , and which is such that the derivatives defined by (4.1) exist and are bounded on H . Suppose further that

$$P_i\,[g\,](x) \equiv 0\ ,\ i=1,2,\ldots,\ n+1, \tag{4.4}$$

for all $g \in H$ such that

$$\left(D^{\underset{\sim}{\alpha}} g\right)(E_j) = 0\ ,\ |\underset{\sim}{\alpha}| \le N\ ,\ \ j=1,2,\ldots,n+1\ . \tag{4.5}$$

Then it follows that

$$P_i\,(I-P_i)[f](x) \equiv 0 \text{ for all } f \in H\ , \tag{4.6}$$

where I is the identity operator, and, moreover, that

$$P(I-P)\,[f]\,(x) \equiv 0 \text{ for all } f \in H\ . \tag{4.7}$$

Thus P_i , $i = 1,2,\ldots,n+1$, and P define bounded idempotent linear operators, i.e. <u>projectors</u>, on H . Also if

$$P_i\,[f] = f \text{ for all } f \in \mathcal{H}_i\ , \tag{4.8}$$

i.e. if H_i is the precision set of the operator P_i , then, using (3.17),it follows that

$$P[f] = f \text{ for all } f \in \bigcap_{i=1}^{n+1} \mathcal{H}_i\ , \tag{4.9}$$

i.e. the precision set of P contains the intersection of the $\mathcal{H}_i$.

5 . <u>Polynomial Blending Function Interpolation Scheme</u>.

We now consider an example of the general blending function scheme defined in Theorem 4.1, where projectors P_i are defined by Boolean sums of polynomial Taylor interpolation projectors. Let

$$E_j^j = \sum_{\substack{k=1\\k\neq i,j}}^{n+1} \lambda_k V_k + (\lambda_i + \lambda_j)\,V_i\ ,\ \ i \neq j\ ,\ \sum_{k=1}^{n+1} \lambda_k = 1\ , \tag{5.1}$$

be the point of intersection of the face $\lambda_j = 0$ with the line through x which is parallel to the edge joining V_i and V_j .Also let $C_{\underset{\sim}{i}}^{N^n}(S_n)$, i=1,2,...,n+1 , be the function spaces

$$C_{\underset{\sim}{i}}^{N^n}(S_n) = \{ f \,/\, D_i^{\underset{\sim}{\alpha} i} f \in C^0(S_n) \text{ for all } \underset{\sim}{\alpha}_i \in \underset{\sim}{N}^n \} . \tag{5.2}$$

Then Taylor interpolation projectors T_j^i can be defined on $C_{\underset{\sim}{i}}^{N^n}(S_n)$ by

$$T_j^i[f](x) = \sum_{k=0}^{N} (\lambda_j{}^k/k!) \left(D_{ij}{}^k f\right)(E_j^i) , j \neq i , j = 1,2,..., n+1 . \tag{5.3}$$

Some properties of these projectors are given in the following lemma:

<u>Lemma 5.1</u>. The Taylor projectors defined by (5.3) have the interpolation properties that

$$\left(D_{ij}{}^k T_j^i[f]\right)(E_j) = \left(D_{ij}{}^k f\right)(E_j) , i \neq j, k = 0,1,..., N, f \in C_{\underset{\sim}{i}}^{N^n}(S_n) \tag{5.4}$$

and the precision set property that

$$T_j^i[f] = f \text{ for all } f \in H_j^i , \tag{5.5}$$

where

$$H_j^i = \{ \lambda_j{}^k \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{n+1} g_\ell(\lambda_\ell) \,/\, g_\ell \in C_{\underset{\sim}{i}}^{N^n}(S_n) , 0 \leq k \leq N \} . \tag{5.6}$$

<u>Proof.</u> Since $\lambda j = 0$ at Ej and since $(\partial/\partial\lambda_j - \partial/\partial\lambda_i)\, g(E_j^i) = 0$ for any differentiable function g,it follows by use of Lemma 2.1 that

$$\left(D_{ij}{}^k T_j^i[f]\right)(E_j) = \left((\partial/\partial\lambda_j = \partial/\partial\lambda_i)^k \sum_{k'=0}^{N} (\lambda_j{}^{k'}/k'!) \left(D_{ij}{}^{k'} f\right)(E_j^i)\right)(E_j)$$
$$= \left(\left(D_{ij}{}^k f\right)(E_j^i) \right)(E_j) , k = 0,1,...,N .$$

Now when $x = E_j$ we have $E_j^i = E_j$ and hence the interpolation properties (5.4) follow. Also, if

$$f(x) = \lambda_j^{k'} \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{n+1} g_\ell(\lambda_\ell) , \quad 0 \le k' \le N ,$$

then it follows from (5.1) and Lemma 2.1 that

$$\left(D_{ij}^{k} f\right)(E_j^i) = \left((\partial/\partial\lambda_j - \partial/\partial\lambda_i)^k f\right)(E_j^i)$$

$$= \begin{cases} k'! \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{n+1} g_\ell(\lambda_\ell) \text{ if } k = k', \\ 0 \text{ if } k \text{ is otherwise} \end{cases} .$$

Thus substituting into (5.3) gives the desired precision set result (5.5), which completes the proof of the lemma.

The proj e etors T_j^i ,$j \neq i$, $j=1 ,2 ,...,n+1$, are commutative over $C_{\tilde{i}}^{N^n}(S_n)$. The proof of the following theorem is then easily supplied by induction using the Boolean sum theory of Gordon [6].

<u>Theorem 5.1</u>. The n fold Boolean sum

$$P_i = \bigoplus_{\substack{j=1 \\ j \neq i}}^{n+1} T_j^i , \tag{5.7}$$

where

$$T_j^i \oplus T_{j_2}^i \equiv T_{j_1}^i + T_{j_2}^i - T_{j_1}^i T_{j_2}^i , \tag{5.8}$$

defines a projector on $C_{\tilde{i}}^{N^n}(S_n)$ which is such that

$$\left(D_{ij}^{k} p_i[f]\right)(E_j) = \left(D_{ij}^{k} f\right)(E_j) \text{ for all } k = 0,1,..., N$$
$$\text{and } j \neq i , j = 1,2,..., n+1 . \tag{5.9}$$

Furthermore

$$p_i[f] = \text{ for all } f \in H_i = \bigcup_{\substack{j=1 \\ j \neq i}}^{n+1} H_j^i . \tag{5.10}$$

Equations (5-9) imply that

$$\left(D^{\underset{\sim}{\alpha}} P_i[f]\right)(E_j) = \left(D^{\underset{\sim}{\alpha}} f\right)(E_j), j \neq i, j = 1, 2, \ldots, n+1 ,$$

$$\text{for all } \left|\underset{\sim}{\alpha}\right| \leq N .$$

Thus the following corollary follows immediately from Theorems 5.1 and 4.1 :

Corollary 5.1

$$P[F](X) = \sum_{i=1}^{n+1} a_i(x) P_i[f](x) , x \in S_n , f \in \sum_{i=1}^{n+1} C^{\underset{\sim}{N}_i^n}(S_n), \tag{5.11}$$

defines a blending function interpolant on S_n . Moreover, since $P_{n(N+1)-1} \subset H_i$, $i = 1, 2, \ldots, n+1$, then (4.9) implies that

$$P[f] = f \text{ for all } f \in P_{n(N+1)-1} . \tag{5.12}$$

We refer to (5.11) as the polynomial blending function interpolant since the P. and P involve polynomial weights.

Remark. The derivatives $D_i^{\underset{\sim}{\alpha}i} f$, $\left|\underset{\sim}{\alpha}_i\right| \in \underset{\sim}{N}^n$, are 'compatible' on $C^{\underset{\sim}{N}_i^n}(S_n)$. By this we mean that the derivatives do not depend on the order in which the differentiation is performed. This condition allows the commutativity of the projectors T_j^i , $j \neq i$, j=1,2,...,n+1. The projectors defined in the following section do not require such stringent compatibility conditions. However, this involves the introduction of rational terms and the blending function interpolant which results has, in general, less polynomial precision.

6. Rational Blending Function Interpolation Scheme.

In this section we consider a rational definition of the projectors P_i for the general blending function scheme of Theorem 4.1. The definition is inductive, where Theorem 4.1 is used to define a projector on a k-simplex as a weighted average of projectors on (k-1) dimensional simplices. In order to define the scheme the following notation is introduced:

Let

$$I_k = \{\nu = \{\nu_1, \nu_2, \ldots, \nu_{k+1}\} \;/\; \nu_i \in \{1,2,\ldots,n+1\},\ \nu_i \neq \nu_j \text{ for all } i \neq j\}$$

$$k=1,2,\ldots,n, \qquad (6.1)$$

be the $\binom{n+1}{k+1}$ dimensional set of combinations of $\{1,2,\ldots,n+1\}$ taken k+1 at a time without repetitions. Also let S^ν, $\nu \in I_k$, be the k dimensional simplex given by the intersection of S_n with the n−k hyperplanes through x parallel to the faces $\lambda_i = 0$, $i \in \nu'$, where

$$\nu' = \{1,2,\ldots,n+1\} - \nu \qquad (6.2)$$

The simplex S^ν has vertices

$$V^\nu_{\nu j} = \sum_{i \in \nu}{}' \lambda_i \upsilon_i + \{\sum_{i \in \nu} \lambda_i\} V_{\upsilon j}, \quad V_j \in \nu, \qquad (6.3)$$

and it is easily shown that

$$x = \sum_{j=1}^{k+1} \lambda^\nu_{\nu j} V^\nu_{\nu j}, \qquad (6.4)$$

where

$$\lambda^\nu_{\nu j} = \lambda_{\nu j} / \{\sum_{i \in \nu} \lambda_i\} = \lambda_{\nu j} / \{\sum_{i \in \nu}{}' \lambda_i\}, \quad \nu_j \in \nu, \qquad (6.5)$$

define the barycentric coordinates of S^ν with respect to the vertices $V^\nu_{\nu j}$. Consider now the particular case $\nu = \{\nu_1, \nu_2\} \in I_1$, where I_1 is an (n+1)n/2 dimensional set. Then the one dimensional simplex S^ν is the line segment joining the two vertices

$$\left.\begin{aligned} V^\nu_{\nu 1} &= \sum_{j \in \nu}{}' \lambda_j V_j + \{\lambda_{\nu_1} + \lambda_{\nu_2}\} V_{\nu_1}, \\ V^\nu_{\nu 2} &= \sum_{j \in \nu}{}' \lambda_j V_j + \{\lambda_{\nu_1} + \lambda_{\nu_2}\} V_{\nu_2}, \end{aligned}\right\} \quad , \qquad \nu = \{V_1, V_2\}. \qquad (6.6)$$

In this case

$$x = \lambda^{v}_{v_1} V^{v}_{v_1} + \lambda^{v}_{v_2} V^{v}_{v_2} , v \{v_1, v_2\} , \tag{6.7}$$

where

$$\lambda^{v}_{v_1} = \lambda_{v_1} / \{\lambda_{v_1} + \lambda_{v_2}\} \text{ and } \lambda^{v}_{v_2} = \lambda_{v_2} / \{\lambda_{v_1} + \lambda_{v_2}\} . \tag{6.8}$$

The line segment S^{v} , $v=\{v_1, v_2\}$, can be interpreted as the intersection of the simplex S_n with the line through x which is parallel to the edge joining V_{v1} and V_{v2} . Thus, with the notation of Section 5,

we have $V^{v}_{v_1} = E^{v_1}_{v_2}$ and $V^{v}_{v_2}$ $E^{v_2}_{v_1}$.

Let $C^{N}_{v_1,v_2}(S_n)$ be the function space defined by

$$C^{N}_{v_1,v_2}(S_n) = \{ f / f \in C^{N}(S_n) \text{ and } D_{v_1 v_2}{}^{i} f \in C^{N}(E_{v_1}) \cap C^{N}(E_{v_2}) , 0 < i < N \}. \tag{6.9}$$

Then a Hermite two point Taylor operator P^{v} , $v=\{v_1, v_2\}$,can be defined along the line segment S^{v} by

$$P^{v}[f](x) = \sum_{i=0}^{N} h_i (\lambda_{v_2} / \{\lambda_{v_1} + \lambda_{v_2}\})(\lambda_{v_1} + \lambda_{v_2})^{i} \left(D_{v_1 v_2}{}^{i} f \right)(V^{v}_{v_1})$$

$$+ \sum_{i=0}^{N} h_i (\lambda_{v_1} / \{\lambda_{v_1} + \lambda_{v_2}\})(\lambda_{v_1} + \lambda_{v_2})^{i} \left(D_{v_2 v_1}{}^{i} f \right)(V^{v}_{v_2})$$

$$v = (v_1, v_2) , \quad f \in C^{N}_{v_1,v_2}(S_n) , \tag{6.10}$$

whe re the h_i are defined by (3.8).

If we exclude, for the moment, the singularity $\lambda_{v_1} = \lambda_{v_2} = 0$, then P^{v} , $v =\{v_1, v_2\}$ defines an interpolation projector $C^{N}_{v_1,v_2}(S_n)$ on and we have the following lemma:

Lemma 6.1. The Hermite projectors defined by (6.10) have the interpolation properties that

$$\left.\begin{aligned} \left(D_{v_1 v_2}{}^k P^v[f]\right)(E_{v_2}) &= \left(D_{v_1 v_2}{}^k f\right)(E_{v_2}) \\ \left(D_{v_2 v_1}{}^k P^v[f]\right)(E_{v_1}) &= \left(D_{v_2 v_1}{}^k f\right)(E_{v_1}) \end{aligned}\right\} \quad k=0,1,\ldots,N,\ f\in C^N_{v_1,v_2}(S_n), \tag{6.11}$$

and the precision set property that

$$P^v[f] = f \text{ for all } f \in H^v \,' \tag{6.12}$$

where

$$H^v = \{ \lambda_{vi}{}^k \prod_{\ell\in v'} g_\ell(\lambda_\ell) \,/\, i = 1 \text{ or } 2,\ 0 \le k \le 2N+1,\ g_\ell \in C^N_{v_1,v_2}(S_n) \} \tag{6.13}$$

Proof. The interpolation properties follow directly from the theory of Section 3 in the special case n=l . We also know from Section 3 that

$$P^v[(\lambda_{v_i}/\{\lambda_{v_1}+\lambda_{v_2}\})^k] = (\lambda_{v_i}/\{\lambda_{v_1}+\lambda_{v_2}\})^k,\ i=1,2,0\le k\le 2N+1.$$

The precision set result is then proved by writing

$$\begin{aligned} f &= \lambda_{vi}{}^k \prod_{\ell\in v'} g_\ell(\lambda_\ell) \\ &= (\lambda_{v_i}/\{\lambda_{v_1}+\lambda_{v_2}\})^k \{I - \sum_{\ell\in v'} \lambda_\ell\}^k \prod_{\ell\in v'} g_\ell(\lambda_\ell),\ i=1,2, \end{aligned}$$

and noting that λ_ℓ , $\ell\in v'$, is a scalar with respect to the linear operator P^v .

In Lemma 6.1 we have ignored the problem of the singularity $\lambda_{v1} = \lambda_{v2} = 0$. We now show, in Lemma 6.3, that this singularity is removable by considering the behaviour of the Hermite projector P^v , $v=\{v_1, v_2\}$, in a neighbourhood of the point $\tilde{V}\in S_n$, where

$$\tilde{V} = \sum_{\substack{i=1 \\ i \neq v_1, v_2}}^{n+1} \tilde{\lambda}_i V_i \;, 0 \leq \tilde{\lambda}_i \leq 1 \;, \sum_{\substack{i=1 \\ i \neq v1, v2}}^{n+1} \tilde{\lambda}_i = 1 \;. \tag{6.14}$$

In order to prove Lemma 6.3 we consider first a preliminary lemma which essentially relies on the operator defined by the left hand side of (6.16) annihilating polynomials of a certain degree. Standard results for such operators prove difficult to apply because of the nature of particular operator and function space involved. We thus give a direct proof of this lemma:

<u>Lemma 6.2</u>. Let $f \in C^N_{\nu_1, \nu_2}(S_n)$ and let

$$p(x) = \sum_{j=0}^{N} (\lambda_{\nu_2}{}^j / j!) \left(D_{\nu_1 \nu_2}{}^j f \right) (V^\nu_{\nu_1}) \tag{6.15}$$

be the Taylor interpolant to f about $V^\nu_{\nu_1}$. then

$$D^{\underset{\sim}{\gamma}} \{ \left(D_{\nu_1 \nu_2}{}^i (P-f) \right) (V^\nu_{\nu_2}) \} = (\lambda_{\nu_1} + \lambda_{\nu_2})^{\underset{\sim}{N-|\gamma|\, i}} \; n_{\chi, i}(\underset{\sim}{\gamma}) \tag{6.16}$$

for all $|\underset{\sim}{\gamma}| \leq N-i$, where $n_{\underset{\sim}{\gamma}, i} \in C(S_n$

$$\lim_{x \to v} n_{\chi,i}(x) = 0, \; x \in S_n \;. \tag{6.17}$$

Proof. For simplicity and without loss of generality we consider the case v—{1,2} with the derivative operator

$$D_k^{\underset{\sim}{\gamma} k} = \prod_{\substack{j=1 \\ j \neq k,1,2}}^{n+1} D_{kj}{}^{\gamma_j} \;, \underset{\sim k}{\gamma} = (\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_{n+1}) \;, \; k \neq 1,2, \tag{6.18}$$

(see (2.9)) , where $|\underset{\sim}{\gamma}| = |\underset{\sim}{\gamma}| \leq N-i$. ($D^{\tilde{\gamma}}$ can be expressed as a linear combination of such derivatives.) Now if $h \in C^1(S_n)$

$$D_{kj}\{h(V^\nu_2)\} = \begin{cases} \left(D_{kj} \, h \right) (V^\nu_2) \text{ if } j \neq 1,2 \,, \\ \left(D_{kj} \, h \right) (V^\nu_2) \text{ if } j = 1,2 \,. \end{cases} \tag{6.19}$$

Thus, since $p, f \in C^N(S_n)$, $|\gamma| + i \le N$, and substituting $D_{k2} = D_{k1} + D_{12}$ it follows that

$$D_k^{\gamma_k}\{\left(D_{12}^{\,i}(p-f)\right)(V_2)\}$$

$$= \left(D_{12}^{\,i}\, D_{k2}^{\,\gamma_1+\gamma_2} \prod_{\substack{j=1\\ j\neq k,1,2}}^{n+1} D_{kj}^{\,\gamma_j}\,(p-f)\right)(V_2^v)$$

$$= \sum_{\ell=0}^{\gamma_1+\gamma_2} \binom{\gamma_1+\gamma_2}{\ell} \left(D_{12}^{\,i}\, D_{k1}^{\,\gamma_1+\gamma_2-\ell}\, D_{12}^{\,\ell} \prod_{\substack{j=1\\ j\neq k,1,2}}^{n+1} D_{kj}^{\,\gamma_j}\,(p-f)\right)(V_2^v)$$

$$= \sum_{\ell=0}^{\gamma_1+\gamma_2} \binom{\gamma_1+\gamma_2}{\ell} \left(D_{12}^{\,i+\ell}\,(q-g)\right)(V_2^v) \tag{6.20}$$

where

$$g = D_{k1}^{\,\gamma_1+\gamma_2-\ell} \prod_{\substack{j=1\\ j\neq k,1,2}}^{n+1} D_{kj}^{\,\gamma_j} f\,, \quad g \in C^{N-|\gamma_k|+\ell}(S_n)$$

and, from (6.15) and the dual of (6.19),

$$q = D_{k1}^{\,\gamma_1+\gamma_2-\ell} \prod_{\substack{j=1\\ j\neq k,1,2}}^{n+1} D_{kj}^{\,\gamma_j}\, p$$

$$= \sum_{j=0}^{N-|\gamma_k|+\ell} (\lambda_2^{\,j}/j!)\left(D_{12}^{\,j}\, g\right)(V_1^v) + \lambda_2^{\,N-|\gamma_k|+\ell+1} R\,,$$

$$R \in C^{N-|\gamma_k|+\ell}(S_n)\,.$$

Finally, since $\ell+i \le N-|\gamma_k|+\ell$, a Taylor expansion $D_{12}^{\,i+\ell} g$ about V_1^v gives

$$D_{12}^{\,i+\ell}(q-g) = \lambda_2^{\,N-|\gamma_k|-i}\,\mu_\ell\,,$$

whe re $m_\ell \in C(S)$ and $\lim_{x\to V_1^v} \mu_\ell(x) = 0$. Substitution of this result in (6.20) completes the proof of the lemma.

<u>Lemma 6. 3</u>. Let $f \in C^N_{V_1,V_2}(S_n)$ and let P^V be defined by the Hermite projector (6.10). Then

$$\lim_{x\to V} \left(D^{\alpha} P^{V}[f]\right)(X) = \left(D^{\alpha} f\right)(\tilde{V}) \text{ for all } |\alpha|. \le N , x \in S_n . \qquad (6.21)$$

Proof, Let p be defined by the Taylor interpolant (6.15).Then $p \in H^{v}$ and thus

$$p(x) - P^{V}[f](x) = P^{v}[p - f](x)$$

$$= \sum_{i=0}^{N} h_i(\lambda_{v_1}/\{\lambda_{v_1} + \lambda_{v2}\})(\lambda_{v_1} + \lambda_{v_2})^i \left(D_{v_1 v_2}{}^i (p-f)\right)(V_{v_2}^{v})$$

Thus, by application of Leibnitz's rule.

$$\left(D^{\alpha} (P^{V}[f])\right)(x)$$

$$= \sum_{i=0}^{N} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^{\beta} [h_i(\lambda_{v_1}/\{\lambda_{v_1} + \lambda_{v2}\})(\lambda_{v_1} + \lambda_{v_2})^i]D^{\alpha-\beta}[\left(D_{v_1 v_2}{}^i (p-f)\right)(V_{v_2}^{v})]$$

$$= \sum_{i=0}^{N} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\lambda_{v_1} + \lambda_{v_2})^{i-|\beta|} K_{\beta}(x) D^{\alpha-\beta}[\left(D_{v_1 v_2}{}^i (p-f)\right)(V_{v_2}^{v})], (6.22)$$

where it can be shown that K_{β} is bounded on S_n by use of the fact

that

$$0 \le \lim_{x\to V} \lambda_{v_1}/\{\lambda_{v_1} + \lambda_{v_2}\} \le 1 \text{ for all } x \in S_n \quad .$$

It thus follows that

$$\lim_{x\to V} \left(D^{\alpha} (P - P^{V}[f]\right)(X) = 0 \text{ for } x \in S_n ,$$

where if $|\beta| \ge i$ in (6.22) then $|\alpha - \beta| = |\alpha| - |\beta| \le N-i$, so that

by Lemma 6.2 we may substitute

$$D^{\alpha-\beta} \left(D_{v1v2}{}^i(p-f)\right)(V_{v2}^{v}) = (\lambda_{v1} + \lambda_{v2})^{N-|\alpha|+|\beta|-i} n_{\alpha-\beta,i}(x) .$$

The proof of the lemma is then completed by noting that

$$\lim_{x\to V}\left(D^{\underset{\sim}{\alpha}} P\right)(X) = \left(D^{\underset{\sim}{\alpha}} f\right)(V).$$

Remark. Lemma 6.3 has been proved under slightly weaker conditions than those used in Theorem 2.2 of [2] for the special case n=2 of Hermite projectors on the triangle.

Having defined an interpolation projector P^{v}, $v=\{v_1, v_2\} \in I_1$, with interpolation properties on E_{v1} and E_{v2}, we now consider the the general case $v=\{v_1, v_2, \ldots, v_{k+1}\} \in I_k$, $k=2, \ldots, n$. In this case a projector P^{v} is inductively defined by the following theorem:

Theorem 6.1. Let

$$P^{V}[f](x)=\sum_{i=1}^{k+1} a_i^{V}(x)P^{v-\{v_i\}}[f](x),\ f\in \bigcap_{\substack{i,j\in v\\ i\neq j}} C^{N}_{i,j}(S_n), \tag{6.23}$$

define a projector P^{v}, $v \in I_k$, $k=2, \ldots, n$. where

$$a_i^{V}(x)=(\lambda_{vi}^{V})^{N+1} \sum_{\underset{\sim}{\alpha} i\in N^k} \{\prod_{\substack{j=1\\ j\neq i}}^{k+1} (\lambda_{vj}^{V})\alpha_j/\alpha_j!\}(N+\left|\underset{\sim i}{\alpha}\right|)!/N!, \tag{6.24}$$

$$\underset{\sim i}{\alpha} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{k+1}),$$

(cf. (3.18)), and P^{V}, $v \in I_1$, is defined by (6.10). Then

$$\left(D^{\underset{\sim}{\alpha}} p^{V}[f]\right)(E_j) = \left(D^{\underset{\sim}{\alpha}} f\right)(E_j),\quad j\in v,\ \text{for all } \left|\underset{\sim}{\alpha}\right| \leq N, \tag{6.25}$$

and

$$P^{V}[f] = f \text{ for all } f \in f \in \bigcap_{\substack{i,j\in \nu\\ i\neq j}} H^{\{i,j\}}, \tag{6.26}$$

where $H^{\{i,j\}}$ is defined by (6.13)

Proof. Assume the inductive hypothesis that (6.25) is true for all $v \in I_{k-1}$. Then (6.23) is simply an application of Theorem 4.1

on the k dimensional simplex S^{ν} . Thus (6.25) holds for all $\nu \in I_k$, where the singularities of the a_i^{ν} on $\bigcap_{j\in\nu} E_j$ are easily shown to be removable, since $P^{\nu-\{\nu i\}}$, i=1, 2,...,k+1 , have common interpolation properties on this set and $\sum a_i^{\nu} \equiv 1$. Now, by Lemma 6,1, the inductive hypothesis is true for all $\nu \in I_1$ and hence the first part of the theorem is proved. The precision set property follows directly from (6.12) and (4.9). This completes the proof.

A blending function scheme for the simplex S_n is defined in the following corollary as a special case of Theorem 6.1. We refer to this scheme as the rational blending function interpolant.

Corollary 6.1

$$P[f](x) = P^{\nu}[f](x),\ \nu=\{1,2,...,n+1\},\ x\in s_n,\ f\in \bigcap_{\substack{i,j\in\nu \\ i\neq j}} C^N_{i,j}(S_n), \quad (6.27)$$

defines a blending function interpolant on $S_n = S^{\nu}$. Moreover, since $P_{2N+1} \subset H^{\{i,j\}}$ for all $i,j \in \nu$, $i \neq j$, then

$$P[f] = f \text{ for all } f\ \nu\ P_{2N+1} \quad (6.28)$$

Remark. The polynomial precision set (6.28) is contained in the range of each Hermite two point Taylor projector because of the simple additive form of the inductive scheme. Mansfield [8] considers an additive and product composition of the Hermite projectors which gives a higher precision rational scheme on the tetrahedron. This scheme, although of a more complex form, can be generalized to the n-simplex. Finally, Mansfield has proved that the product composition of all the Hermite projectors (6.10) maps a function f onto the finite dimensional interpolant defined by Theorem 3.1 of this paper.

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