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IMPROVED ORDERS OF APPROXIMATION DERIVED FROM INTERPOLATORY CUBIC SPLINES

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ABSTRACT

Let s be a cubic spline, with equally spaced knots on [a,b], interpolating a given function y at the knots. The parameters which determine s are used to construct a piecewise defined polynomial P of degree four. It is shown that P can be used to give better orders of approximation to y and its derivatives than those obtained from s. It is also shown that the known superconvergence properties of the derivatives of s, at specific points [a,b], are all special cases of the main result contained in the present paper.

1. Introduction

Let s be a cubic spline on [a,b] with equally spaced knots

$$x_i = a + ih ; \quad i=0,1,...., k , \qquad (1.1)$$

where h=(b-a)/k. Then $s \epsilon C^2[a,b]$ and in each of the intervals $[x_{i-1} , x_i]$; i =1,2, ,.,k, s is a cubic polynomial.

Given the set of values y_i ;i=0,1,…,k, where

$$y_i = y(x_i.) ; y \epsilon c^n [a,b], n \geq 4$$

consider the problem of constructing an interpolatory s such that

$$s(x_i) = y_i ; \quad i =0, 1,..., k. \qquad (1.2)$$

If the values $m_i = s^{(1)}(x_i)$; i = 0,1,... , k are known then, by use of Hermite's two point interpolation formula, s is given by

$$s(x) = \frac{(x_i - x)^2\{2(x - x_{i-1}) + h\}}{h^3} y_{i-1} + \frac{(x - x_{i-1})^2\{2(x_i - x) + h\}}{h^3} y_i$$
$$+ \frac{(x_i - x)^2 x - x_{i-1})}{h^2} m_{i-1} - \frac{(x - x_{i-1})^2 (xi - x}{h^2} m_i ,$$
$$x \varepsilon [x_{i-1},x_i] ; i= 1, 2, …., k. \qquad (1.3)$$

Equivalently, if the values $M_i = s^{(2)} (x_i)$; i = 0,1,.. . ,k, are known, s can be obtained in $[x_{i-1} ,x_i]$ by integrating

$$s^{(2)}(x) = \frac{1}{h}\{(x_i - x)M_{i-1} + (x - x_{i-1})M_i\}$$

twice with respect to x and using the interpolation conditions $s(x_{i-1}) = y_{i-1}$, $s(x_i) =y$ for the determination of the two constants of integration. To determine either of the k + 1 parameters m_i or M_i the consistency relations

$$m_{i-1} + 4m_{i+1} = \frac{3}{h}\{y_{i+1} - y_{i-1})\}; \qquad i= 1, 2, ……,k-1, \qquad (1.4)$$

or

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\{y_{i-1} - 2y_i + y_{i+1}\}; \quad i= 1, 2, …….., k-1, \qquad (1.5)$$

are used, these being direct consequences of the continuity constrains on s. Since either of (1.4) or (1.5) provide only k-1 linear equations, it follows that the interpolation conditions (1.2) are not sufficient to determine s uniquely. Two additional linearly independent conditions are always needed for this purpose. These are usually taken to be end conditions, i.e. conditions imposed on s, $s^{(1)}$ or $s^{(2)}$ near the two endpoints a and b.

As might be expected the choice of end conditions plays a critical role on the quality of the spline approximation. It is well-known that if $y\in C^4[a,b]$ and s satisfies appropriate end conditions then

$$\| s^{(r)}-y^{(r)}\|= 0(h^{4-r}) ; \quad r=0,1,2,3, \tag{1.6}$$

where $\|\cdot\|$ denotes the uniform norm on [a,b]. Furthermore, with r=0, (1.4) gives the best order of approximation to y which can be achieved by an interpolatory cubic spline s. It is also known that if $y\epsilon C^5[a,b]$ then, for a variety of end conditions,

$$m_i = y_i^{(1)} + 0(h^4) ; \quad i = 0,l,...,k, \tag{1.7}$$

where $m_i = s^{(1)}(x_i)$ and $y_i^{(1)} = y^{(1)}(x_i)$. More generally, if (1.7) holds then improved orders of approximation to the first three derivatives of y are obtained at certain specific points of [a»b] as follows,

$$s^{(r)}(x_{i-1}+\alpha_r h)=y^{(r)}(x_{i-1}+\alpha_r h) + 0(h^{5-r}): \; r=1,2,3,; \; i=1,2,..,k, \tag{1.8}$$

where the admissible values of a are respectively

$$\alpha_1 =1/2, \; \alpha_2 = (3\pm/3)/6 \text{ and } \alpha_3 = l/2. \tag{1.9}$$

The class of end conditions for which (1.7) holds includes the conditions

$$m_o=y_o^{(1)}, \qquad m_k = y_k^{(1)} \quad , \tag{1.10}$$

$$\Delta m_o= \Delta y_o^{(1)}, \qquad \nabla m_k = \nabla y_k^{(1)} , \tag{1.11}$$

$$\Delta^3 M_o=\nabla^3 M_k= 0, \tag{1.12}$$

$$\Delta^4 M_o= \nabla 4M_k= 0, \tag{1.13}$$

$$\Delta s(a+0.5h) = \Delta y(a +0.5h) , \nabla s(b-0.5h) = \nabla y(b-0.5h) \quad , \tag{1.14}$$

and, for periodic y, the conditions

$$m_o = m_k , M_o = m_k , \tag{1.15}$$

of the periodic spline. (In the above A and V are respectively the usual forward and backward difference operators and, as before $m_i = s^{(1)}(x_i.)$, $M_i = s^{(2)}(x_i)$).

For splines with end conditions (1.10) the result (1.7) has been established by Birkhoff and De Boor C4] , Hall [6] and Kershaw [7]. For periodic splines, (1.7) has been established by Kershaw [7] and, under the assumption that $y \epsilon C^6[a,b]$, by Albasiny and Hoskins [2]. More recently Lucas [8] has established (1.7) and also (1.8) -(1.9) for cubic splines satisfying any of the end conditions (1.10)- (1.14) as well as some other end conditions which are listed in [8]. (We note that if $y \epsilon C^6[a,b]$ then these results of Lucas can also be derived from the results of Daniel and Swartz [5].) Lucas has also shown that when y€C^5Ca,b] and s satisfies end conditions such that (1.7) holds then, 0(h) estimates of the derivatives $y_i^{(r)} = y^{(r)}(x_i)$;r = 2,3,4, at the knots can be obtained in terms of linear functionals of $M_i = s^{(2)}(x_i)$. For i - 1,2,..., k - 1, these estimates are,

$$y_i^{(2)} = \frac{1}{12}\{M_{i-1} + 10M_i + M_{i+1}\} + 0(h^3), \tag{1.16}$$

$$y_i^{(3)} = \frac{1}{2h}\{M_{i+1} - M_{i-1}\} + 0(h^2), \tag{1.17}$$

$$y_i^{(4)} = \frac{1}{h^2}\{M_{i-1} - 2M_i + M_{i+1}\} + 0(h), \tag{1.18}$$

In the present paper we take $y \epsilon C^5[a,b]$ and consider interpolatory cubic splines for which (1.7) holds. We show that, for such splines, formulae of the form (1.16)-(1.18) can be used to produce $0(h^{5-r})$ approximations to $y^{(r)}$; r = 1,2,3,4, not only at the knots but at any point $x \epsilon [a,b]$, For this we construct a piecewise defined polynomial P of degree 4, whose coefficients are given in terms of y_i and $m_i = s^{(1)}(x_i)$, i.e. in terms of the parameters which determine s. We prove that

$$||y^{(r)} - P^{(r)}|| = 0(h^{5-r}) \; ; r=0,1,2,3,4, \tag{1.19}$$

and show that the results (1.8)-(1.9) .and (1.16)-(1.18) are all special cases of (1.19). More generally, our result shows that P can be used with very little additional computational effort to compute, at any point $x \epsilon [a,b]$, more accurate approximations to y and its derivatives than those obtained from s.

2. Improved Orders of Approximation

Given the values $y_i = y(x_i)$; i=0,l,...,k, where x. are the equally spaced points (1.1), let H_i. ; i=l,2,..,,k-l, denote the quartic Hermite polynomials which are such that

$$H_i(x_j) = y_j \qquad ; j = i-1, i, i+1,$$

and

$$H_i^{(1)}(x_j) = y_j(1) \quad ; j = i-1, i,$$

where $y_j^{(1)} = y^{(1)}(x_j)$. Then,

$$H_i(x) = \theta_i(x)y_{i-1} + n_i(x)y_i + v_i(x)y_{i+1}$$
$$+ \varphi_i(x)y^{(1)}_{i-1} + {}^{\psi}{}_i(x)y_i^{(1)} \; ; \qquad i-1,2,\ldots, k-1 , \qquad (2.1)$$

where

$$\left.\begin{aligned}
\theta_i(x) &= -(x-x_i)^2(x-x_{i+1})\{7h+5(x-x_i)\}/4h^4 ,\\
\eta_i(x) &= -(x-x_{i-1})^2(x-x_{i+1})^2/h^4 ,\\
\nu_i(x) &= -(x-x_{i-1})^2(x-x_i)^2/4h^4 ,\\
\phi_i(x) &= -(x-x_{i-1})(x-x_i)^2(x-x_{i+1})/2h^3 ,\\
\psi_i(x) &= -(x-x_{i-1})^2(x-x_i)(x-x_{i+1})/h^3 .
\end{aligned}\right\} \qquad (2.2)$$

Also, if $y \in C^5[a,b]$, then

$$H_i(x) - y(x) = \frac{y^{(5)}(\xi_i)}{5!}(x-x_{i-1})^2(x-x_i)^2(x-x_{i+1}), \quad x\varepsilon[x_{i-1}, x_{i+1}]; \qquad (2.3)$$

where ξ_i. is some point in $[x_{i-1}, x_{i+1}]$; see e.g. Ralston [9 :p65].

It follows that

$$|H_i^{(r)}(x) - y^{(r)}(x)| \le K_r h^{5-r}\left\|y^{(5)}\right\| \; ; r = 0,1,2,3,4,\ x \in [x_{i-1}, x_{i+1}] \; ;$$
$$i = 1, 2, \ldots\ldots, k-1, \qquad (2.4)$$

where, from (2.3),

$$K_0 = 1/240 , \qquad (2.5)$$

and, by Taylor series expansions,

$$K_1 = 99/60,\ K_2 = 484/60,\ K_3 = 1093/60 \text{ and } K_4 = 1128/60. \qquad (2.6).$$

(In (2.4) and throughout this paper || • || denotes the uniform norm on [a,b].)

We point out that the error bounds given by (2.4) and (2.6), for r = 1,2,3,4, are by no means optimal. Sharper values for the K_r

can be found by use of Peano's theorem. However,since our purpose is to establish asymptotic orders of convergence, we prefer the simplicity of Taylor's theorem.

Let s be an interpolatory cubic spline which agrees with a function y at the equally spaced knots (1.1), and denote by p. the quartic polynomials

$$P_i(x) = \theta_i(x)y_{i-1} + n_i(x)y_i + v_i(x)y_{i+1} + \varphi_i(x)m_{i-1} + \psi_i(x)m_i;$$

$$i=1, 2, \ldots\ldots, k-1, \qquad (2.7)$$

where m. = $s^{(1)}(x_i)$ and $\theta_i, n_i, v_i, \varphi_i$ and ψ_i are given by (2.2).

Definition 1. The piecewise defined polynomial

$$P(x) = \begin{cases} p_1(x), & x\varepsilon[x_0, x_1], \\ p_i(x), & x\varepsilon[x_{i-1}, x_i];\ i = 2,3,\ldots\ldots\ldots\ldots, k-1, \\ p_{k-1}(x), & x\varepsilon[x_{k-1}, x_k], \end{cases} \qquad (2.8)$$

where the p. are given by (2.7), will be called the <u>piecewise Hermite quartic induced by s</u>.

It follows at once from the definition that $P\varepsilon C^1[a,b]$.

The main result of this paper is contained in the following theorem.

<u>Theorem 1_0</u> Let s be an interpolatory cubic spline which agrees with the function y ϵ $C^5[a,b]$ at the equally spaced knots (1.1) and satisfies end conditions such that

$$\left|m_i - y_i^{(1)}\right| \le Ah^4;\ i=0,1,.., k, \qquad (2.9)$$

where A is a constant. Let P be the piecewise Hermite quartic induced by s. Then there exist constants C ; r =0,1,2,3,4 such that

$$\left\|P^{(r)} - y^{(r)}\right\| \le C_r h^{5-r};\ r=0,1,2,3,4. \qquad (2.10)$$

<u>Proof</u> From (2.1) and (2.7),

$$\left|P_i^{(r)}(x) - H_i^{(r)}(x)\right| \le \left\{\left|\phi_i^{(r)}(x)\right| + \left|\psi_i^{(r)}(x)\right|\right\} \max_{0\le j\le k}\left|m_j - y_j^{(1)}\right|;\ r\text{-}0,1,2,3,4, \qquad (2.11)$$

$$x\varepsilon[x_{i-1}, x_{i+1}];\ i=1, 2, \ldots\ldots\ldots, k-1,$$

where, by determining the maximum values of $|\varphi^{(r)}(x)$ and $|\psi_i^{(r)}(x)|$,

$$|\varphi^{(r)}(x)| + |\psi_i^{(r)}(x)| \le L_r h^{1-r};\ r=0,1,2,3,4,\ x\varepsilon[x_{i-1}, x_{i+1}], \qquad (2.12)$$

with

$$L_0 = 41/8,\ L_1 = 5,\ L_2 = 21,\ L_3 = 41 \text{ and } L_4 = 36 \qquad (2.13)$$

Hence, from (2.11), (2.12) and (2.9),

$|p^{(r)}(x) - H_i^{(r)}(x)| \leq L_r A h^{5-r}$; r =0,1,2,3,4, x ϵ$[x_{i-1}$] $^{x}{}_{i+1}$'];i = 1,2,...,k-1.

(2.14)

Finally, from (2.4), (2.14) and the definition of P, the result (2.10) follows with

$$C_r = K_r \| y^{(5)} \| + L_r A \; ; r = 0,1,2,3,4. \quad (2.15)$$

The piecewise Hermite quar.tic P, of Definition 1, is defined completely by the parameters which determine the cubic spline s. Thus, under the conditions of Theorem 1, P can be used with very little additional computational effort to produce, at any point Xε[a,b], more accurate approximations to y and its derivatives than those obtained from s.

Theorem 2 below shows that the results (1.8) and (1.16)-(1.18), derived by Lucas [8] are all special cases of the result (2.10) of Theorem 1

Theorem 2. Let s be an interpolatory cubic spline with equally spaced knots (1.1) matching the values y. ;i=0,l,...,k at the knots. Let P be the piecewise Hermite quartic induced by s. Then,

$$P^{(r)}(x_{i-1} + \alpha_r h) = s^{(r)}(x_{i-1}. + a_r\, h) \; ; r=0,1,2,3; \; i-1,2,\ldots\ldots\ldots,k, \quad (2.16)$$

for all choices of the y_i, if and only if

$$\alpha_0 = 0,1,\ \alpha_x = 0,1/2,1,\ \alpha_2 = (3 \pm 3)/6\ ,\ \alpha_3 = 1/2. \quad (2.17)$$

Also

$$P^{(2)}(x_i) = \begin{cases} (13M_0 + 2M_1 + M_2)/12 \; ; \; i = 0 \\ (M_{i-1} + 10M_i + M_{i+1})/12 \; ; \; i = 1,2,\ldots\ldots\ldots,K-1, \\ (M_{k-2} + 2M_{k-1} + 13M_k)/12 \; ; \; i = K, \end{cases} \quad (2.18)$$

$$P^{(3)}(x_i) = \begin{cases} (-3M_0 + 4M_1 - M_2)/2h \; ; \; i = 0 \\ (M_{i+1} - M_{i-1})/2h \; ; \; i = 1,2,\ldots\ldots\ldots,K-1, \\ (M_{k-2} - 4M_{k-1} + 3M_k)/12 \; ; \; i = K, \end{cases} \quad (2.19)$$

$$P^{(4)}(x_i) = \begin{cases} (M_0 - 2M_1 + M_2)/h^2 \; , \; x\varepsilon[x_0, x_1] \; , \\ (M_{i-1} - 2M_i + M_{i+1})/h^2 \; , \; x\varepsilon[x_{i-1}, x_i] \; , i = 1,2,3,\ldots\ldots\ldots,K-1, \\ (M_{k-2} - 2M_{k-1} + M_k)/ \; , \; x\varepsilon[x_{k-1}, x_k] \; , \end{cases} \quad (2.20)$$

where $M_i = s^{(2)}(x_i)$.

Proof At any point $x\epsilon[x_{i-1},x_j];i=1,2,\ldots,k-1$, s is given by (1.3) in terms of y_{i-1} , y_i ' m_{i-1} and m_i , whilst P is given, by (2.7)-(2.8), in terms of these four parameters and also y_{i+1} . The consistency relation (1.4) shows that y_{i+1} cannot, in general, be expressed in terms of y_{i-1} , y_i ,m_{i-1} and m_i only. It follows, from (2.7), that a necessary condition for (2.16) to hold, for all choices of the y., is that the points

$$x_{i-1}+\alpha_r h;\ r^{=0,1,2,3};\ i = 1,2,\ldots,k-1 \tag{2.21}$$

are respectively real zeros in $[x_{i-1}$, x_i] of the polynomials

$v_i^{(r)}$ (x); r =0,1,2,3; i=1,2,...,k - 1. It can be easily shown that such zeros occur only at the points defined by (2.21) and (2.17).

The results

$$\begin{aligned} P(x_i) &= y_i \\ &= s(x_i) \quad ; i=0,1,2,..,k, \end{aligned}$$

and

$$\begin{aligned} P^{(1)}(x_i) &= m_i \\ &= s^{(1)}(x_i) \ ; i=0,1,2,...,k, \end{aligned}$$

follow at once from the definition of P. The other results contained in (2.16) can be established easily from (1.3), (2.7),(2.8) and the consistency relation (1.4). Thus,

$$\begin{aligned} P_i^{(1)}(x_{i-1}+0.5h) &= \frac{3}{2h}(y_i - y_{i-1}) - \frac{1}{4}(m_{i-1}+m_i) \\ &= s^{(1)}(x_{i-1}+0.5h);\ i=1,2,\ldots\ldots,k-1\ , \\ P_{k-i}^{(1)}(x_{k-1}+0.5h) &= \frac{1}{4}(m_{k-2}+\frac{3}{h}y_{k-2}) - \frac{3}{2h}y_{k-1} + \frac{3}{4h}y_k + \frac{3}{4}m_{k-1} \\ &= (-m_{k-1} - \frac{1}{h}m_k + \frac{3}{4h}y_k) - \frac{3}{2h}y_{k-1} + \frac{3}{4h}y_k + \frac{3}{4}m_{k-1} \\ &= \frac{3}{2h}(y_k - y_{k-1}) - \frac{1}{4}(m_{k-1}+m_k) \\ &= s^{(1)}(x_{k-1}+0.5h)\ . \end{aligned}$$

and hence,

$$P^{(1)}(x_{i-1}+0.5h) = s^{(1)}(x_{i-1}+0.5h); \qquad i=1, 2, \ldots,k.$$

Similarly, it can be shown that,

$$P^{(2)}(x_{i-1}+\alpha_2 h) = s^{(2)}(x_{i-1}+\alpha_2 h)\ ; \qquad i=1, 2, \ldots,k,$$

where $\alpha_2 = (3 \pm \sqrt{3})/6$, and

$$P^{(3)}(x_{i-1} + 0.5h) = s^{(3)}(x_{i-1} + 0.5h);\ i=1,2,...,k.$$

This completes the proof of the first part of the theorem.

For the proofs of (2.18) -(2.20) the following cubic spline identity is needed,

$$m_i = \frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{1}{h}(y_{i+1} - y_i)\ ;\ i = 0,1,...,k-1; \tag{2.22}$$

see e.g. Ahlberg, Nilson and Walsh [1]. Using (2.22) and the consistency relation (1.5), the polynomials p. in (2.7) can be written in the form

$$\begin{aligned} P_i(x) = &\{\theta_i(x) - v_i(x) - \frac{1}{h}(\Phi_i(x) + \Psi_i(x))\}y_{i-1} \\ &+ \{\eta_i(x) + 2v_i(x) + \frac{1}{h}(\Phi_i(x) + \Psi_i(x))\}y_i \\ &+ \frac{h}{6}\{hv_i(x) - 2\Phi_i(x) + \Psi_i(x)\}M_{i-1} \\ &+ \frac{h}{6}\{4hv_i(x) - \Phi_i(x) + 2\Psi_i(x)\}M_i \\ &+ \frac{h^2}{6} v_i(x)M_{i+1}\ ;\ i = 1,2,..., k-1. \end{aligned} \tag{2.23}$$

The results (2.18)-(2.20) then follow from the derivatives of (2.23), by direct substitution.

Numerical Results and Discussion

Let s be the cubic spline with knots

$$x_i = 0.05i\ ; \qquad i=0, 1, ...20, \tag{3.1}$$

which interpolates the function

$$y(x) = \exp(x)\ ,$$

at the knots and satisfies the end conditions (1.12), i.e.

$$\Delta^3 M == \nabla 3_M o_0 = 0,\ \ M_i. = s^{(2)}(x_i.).$$

In Table 1 we list values of

$$e^{(r)}(x) = |s^{(r)}(x) - \exp(x)|\ ;\ r-0,1,2,3, \tag{3.2}$$

and

$$E^{(r)}(x) = |P^{(r)}(x) - \exp(x)|\ ;\ r-0,1,2,3,4, \tag{3.3}$$

computed at various points of [0,1] by constructing this s and the corresponding Hermite quartic P induced by s. The results illustrate the improvement in accuracy obtained when P is used instead of s, and confirm some of the theoretical results contained in Theorem 2.

We note that the spline s considered above is an E(3) cubic spline. By the definition of Behforooz and Papamichael [3], an E(α) cubic spline is an interpolatory cubic spline with equally spaced knots (1.1) and end conditions

$$(2-\alpha)^{\Delta 3}M + (9-3\alpha)^{\Delta 2}M_o = 0.,$$
$$(2-\alpha)\ ^{\nabla 3}M_k + (9-3\alpha)^{\nabla 2}M_k=0\ ,$$

For any a in the domain defined by $\alpha < 11/3$ and $\alpha > 19/5$, an E (α) cubic spline exists uniquely and, for $y\varepsilon C^5[a,b]$, it satisfies (1.6). However, the approximation of the first derivative at the knots is, in general, $0(h^3)$ and only the value a -3 yields an E(α) spline for which (1.7) holds; see [3].

We also note that, for $y\in C^5[a,b]$, the interpolatory cubic spline with equally spaced knots (1.1) and end conditions

$$M_0 = y_0^{(2)}\ ,\ M_k = Y_k^{(2)}$$

gives

$$m_i = y_i^{(1)} + 0(h^3)\ ;\quad i=0,l,...,k.$$

However, it has been shown by Kershaw [7] that if h < 1 and k is sufficiently large then there exists an integer p, where

$$-\ \log h/\log(2+\sqrt{3}) \le p < 1 - \log h/\log(2+\sqrt{3}),$$

such that

$$m_i - y_i^{(1)} = 0(h^4)\ ;\quad p \le i \le k-p \tag{3.4}$$

By direct application of the technique of [7], results similar to (3.4) can also be established for other interpolatory cubic splines for which (1.7) does not hold. This occurs when the matrix of the linear system which determines the parameters m_i of the spline can be reduced to the tri-diagonal form

$$\begin{bmatrix} a & 1 & & & \\ 1 & 4 & 1 & & \\ & . & . & . & \\ & & 1 & 4 & 1 \\ & & & 1 & a \end{bmatrix}$$

with a = 2 or 4. In particular results similar to (3.4) hold for the E(α) cubic splines which correspond respectively to the values of α =0,1/4, 1/2, 2,7/2 and 4. It follows, from the proof of Theorem 1, that if P is the piecewise Hermite quantic induced by any of these splines then the result

$$P^{(r)}(x) = y^{(r)}(x) + 0(h^{5-r}) \; ; \; r = 0,l,2,3,4,$$

does not hold in the full range [a,b] but only in intervals bounded away from the two end points. To illustrate this we construct the piecewise Hermite quartic induced by the E(2) cubic spline s interpolating the function y(x) =exp(x) at the equally spaced knots (3.1). In Table 2 we present the values of the errors (3.2) and (3.3) computed, by using these P and s, at various points of [0,1].

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TABLE (1)

	r / x	0	1	2	3	4
e$^{(r)}{}_{(x)}$	0.0375	0.13x10^{-7}	0.14x10^{-5}	0.28x10^{-4}	0.15x10^{-1}	–
E$^{(r)}{}_{(x)}$		0.33x10^{-8}	0.28x10^{-6}	0.24x10^{-6}	0.10x10^{-2}	0.56x10^{-1}
e$^{(r)}{}_{(x)}$	0.2375	0.11x10^{-7}	0.12x10^{-5}	0.30x10^{-4}	0.16x10^{-1}	-
E$^{(r)}$(x)		0.46x10^{-9}	0.24x10^{-7}	0.29x10^{-5}	0.16x10^{-3}	0,16x10^{-1}
e$^{(r)}$(x,	0.3625	0.14x10^{-7}	0.14x10^{-15}	0.40x10^{-4}	0.18x10^{-1}	-
E$^{(r)}$(x)		0.16x10^{-9}	0.44x10	0.14x10	0,72x10	0.55x10
e$^{(r)}{}_{(x)}$	0.4250	0.25x10^{-7}	0.22x10^{-7}	0.16x10-^{3}	0.16x10^{-3}	-
E$^{(r)}$(x),		0.65x10^{-9}	0.22x10^{-7}	0.41x10^{-5}	0.16x10^{-3}	0.39x10^{-1}
e$^{(r)}{}_{(x)}$	0.5875	0.16x10^{-7}	0.17x10^{-5}	0.43x10^{-4}	0.23x10^{-1}	-
E$^{(r)}{}_{(x)}$		0.67x10^{-9}	0.36x10^{-7}	0.41x10^{-5}	0.24x10^{-3}	0.23x10^{-1}
e$^{(r)}$(x)	0.8000	0.12xl0^{-13}	0.88x10^{-7}	0.46X10^{-3}	0.55X10^{-1}	—
E$^{(r)}$(x)		0.12x10^{-13}	0.88x10^{-7}	0.36x10^{-7}	0.49x10^{-3}	0.19x10^{-2}
e$^{(r)}{}_{(x)}$	0.9625	0.15xl0^{-7}	0.18x10^{-6}	0.66x10^{-6}	0.29x10^{-1}	-
E$^{(r)}{}_{(x)}$		0.76x10^{-8}	0.66x10^{-6}	0.94x10^{-6}	0.23x10^{-2}	0.13x10^{0}

TABLE (2)

	x \ r	0	1	2	3	4
$e^{(r)}{}_{(x)}$	0.0375	$0.91xl0^{-7}$	$0.76x10^{-5}$	$0.55x10^{-4}$	$0.28x10^{-1}$	-
$E^{(r)}(x)$		$0.91x10^{-7}$	$0.76x10^{-5}$	$0.55x10^{-4}$	$0.28x10^{-1}$	$0.10x10^{-1}$
$e^{(r)}(x)$	0.2375	$0.11x10^{-7}$	$0.12x10^{-5}$	$0.30x10^{-4}$	$0.16x10^{-1}$	-
$E^{(r,}(x)$		$0.94x10^{-9}$	$0.64xl0^{-7}$	$0.32xl0^{-5}$	$0.31x10^{-3}$	$0.11xl0^{-1}$
$e^{(r)}(x)$	0.3625	$0.14xlo^{-7}$	$0.14x10^{-5}$	$0.40X10^{-4}$	$0.13x10^{-1}$	-
$E^{(r)}(x)$		$0.16xl0^{-9}$	$0.44x10^{-7}$	$0.15x10^{-5}$	$0.73x10^{-3}$	$0.55X10^{-1}$
$e^{(r)}(x)$	0.4250	$0.25x10^{-7}$	$0.22x10^{-7}$	$0.16x10^{-5}$	$0.16x10^{-3}$	-
$E^{(r)}(x)$		$0.65x10^{-9}$	$0.22x10^{-7}$	$0.41x10^{-5}$	$0.16x1O^{-3}$	$0.39x10^{-1}$
$e^{(r)}(x)$	0.5875	$0.16x10^{-7}$	$0.17x10^{-5}$	$0.43x10^{-4}$	$0.23X10^{-1}$	-
$E(^{r)(x)}$		$0.69xl0^{-9}$	$0.36x10^{-7}$	$0.42x10^{-5}$	$0.24x10^{-3}$	$0.23x10^{-1}$
$e^{(r)}{}_{(x)}$	0.8000	$0.12x10^{-13}$	$0.38x10^{-6}$	$0.48x10^{-3}$	$-0.56x10^{-1}$	-
$E^{(r)}(x)$		$0.12x10^{-13}$	$0.38x10^{-6}$	$0.l0x10^{-4}$	$0.12x10^{-2}$	$0.49X10^{-1}$
$e^{(r)}(x)$	0.96 25	$0.22x10^{-6}$	$0.18x10^{-4}$	$0.12x10^{-3}$	$0.69X10^{-1}$	-
$E^{(r)}(x)$		$0.22xl0^{-6}$	$0.18x10\text{-}^{4}$	$0.12x1o^{-3}$	$0.69X10^{-1}$	$0.26x10^{-1}$

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