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A LINEAR, FUNCTIONAL DIFFERENTIAL EQUATION OF THE FIRST ORDER

BY

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In this paper rational function methods are used to study the analytic nature of a function satisfying a functional equation and related to a linear, functional differential equation of the first order. The solution of this differential equation is of intrinsic interest, since it can be regarded as a generalisation of the exponential function. From the rational functions sequences of approximations to solutions of the differential equation are constructed. Some of these sequences can be used to calculate the solutions for an appreciable range of the independent variable.

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Page

1. The Problem

From the literature it appears that the basic problem has arisen from two distinct sources, namely, from a problem in electrical engineering [Ockendon & Tayler, 1971] and from a problem in the theory of number [Mahler, 1940].

The problem is to solve the functional, differential equation

$$\frac{dy(t)}{dt} = ay(\lambda) + by(t) \tag{1}$$

for t > 0 with y(o) = 1,where we take $\lambda \geq 0$and a,b to be real constants,

For $0 \leq \lambda \leq 1$ the solution is found to be unique, [See Kato & Mcleod, 1971], but for $\lambda > 1$ there exist non-trivial solutions of (1.1) with y(o) = 0, Thus for X >1 eigensolutions exist. For $\lambda > 1$ we shall be interested, particularly, in those solutions which, as functions of λ, are the analytic continuations of solutions found for $0 < \lambda < 1$, but other solutions are also discussed.

Since most of the analysis is carried out in the Laplace transform plane, it is convenient to modify, slightly, the formulation of (1.1) to

$$\frac{dy(t)}{dt} + \alpha\lambda y(\lambda t) = \beta y(t) \tag{1.2}$$

where α,β are real constants, the remaining conditions being unchanged.

A number of particular cases of (1.2) are trivial and soluble immediately.

(i)For $\lambda = 0$, or $\alpha = 0$, the solution is $y(t) = e^{\beta t}$.

(ii) For $\lambda = 1$ the solution is $y(t) = e^{(\beta-\alpha)t}$

(iii) If for integer $n \geq 0$, $\beta = \alpha\lambda^{n+1}$, , the solution is a polynomial of degree n in t, namely

$$y(t) = 1 + \sum_{r=1}^{n} \left\{ \prod_{p=1}^{r} (\beta - \alpha\lambda^p) \right\} \frac{t^r}{r!}.$$

For $0 < \lambda < 1$ this solution is unique, but not for $\lambda > 1$ It is, however, the analytic continuation of the solution valid for $0 < \lambda < 1$.

From now on we assume that, unless stated otherwise, $\alpha \neq 0$, $\lambda \neq 0,1$, and $\beta - \alpha\lambda^{n+1} \neq 0$ for $n = 0,1,2,\ldots$.

The series solution of (1.1)

$$y(t) = 1 + \sum_{r=1}^{\infty} \left\{ \prod_{p=1}^{r} (\beta - \alpha\,\lambda^{p}) \right\} \frac{t^{r}}{r!} \tag{1.3}$$

is absolutely convergent for $0 \leq \lambda \leq 1$ and can be used to calculate y(t) for quite large values of t when α,β are such that, after a few terms, all the coefficients in the series are positive. when the signs of the coefficients in the series alternate it is not suitable. In particular the solution for the case with $\beta\text{-}\alpha^{\lambda 6}$ small mentioned by Fox [1971]

$$\frac{dy}{dt} = -y(0.99t) + 0.95\, y(t) \tag{1.4}$$

can be evaluated from (1.3) for quite large values of t.

2. The Functional Equations

Laplace transforming (1.2) gives the functional equation

$$(s-\beta)Y(s) + \alpha Y\left(\frac{s}{\lambda}\right) = 1 \tag{2.1}$$

where $Y(s) = \int_0^{\infty} e^{-st}\, y(t)dt$, and is a function of four variables s, α, β and λ.

This is a linear equation so we can write

$$Y(s) = Y_o(s) + Y_e(s) \tag{2.2}$$

where $Y_o(s) \rightarrow \frac{1}{s}$ as $|s| \rightarrow \infty$ and satisfies the initial condition $Y_{0=1;}$

$$(s-\beta)Y_0(s) + \alpha\, Y_0\left(\frac{s}{\lambda}\right) = 1. \tag{2.3}$$

and $Y_e(s)$ is such that its original is zero at $t = 0$

$$(s-\beta)Ye(s) + \alpha\, Y_e\left(\frac{s}{\lambda}\right) = 0 \tag{2.4}$$

It is sufficient if these eigerisolutions Y_e (s) are such that, with $\delta>0$,

$$Y_e(s) = 0\left(\frac{1}{s^{1+\delta}}\right) \text{ as } |s| \to \infty \tag{2.5}$$

Such solutions do not arise for $0 < \lambda < 1$, and for $\lambda > 1$ we require the analytic continuations of the $\lambda < 1$ solutions. We will defer further discussion of $Y_e(s)$ until section 7.

A sequence of functions closely related to $Y_o(s)$ can be generated successively by writing for m = 0,1,2, ...

$$sY_m(s) = 1 + (\beta-\alpha\lambda^{m+1})\, Y_{m+1}(s) \tag{2.6}$$

with the condition $Y_m(s) \to \frac{1}{s}$ as $|s| \to .\infty$.

The functional equation satisfied by each Y_m (s) is

$$(s - \beta)\, Y_m(s) + \alpha\lambda^m Y_m\left(\frac{s}{\lambda}\right) = 1 \tag{2.7}$$

and we immediately see that we can write

$$Y_m(s, \alpha) \equiv Y_o(s, \alpha\lambda^m) \ . \tag{2.8}$$

Further from (2.7) we find

$$Y_m(0) = \frac{1}{\alpha\lambda^m - \beta} \tag{2.9}$$

Setting m = 0 $Y_o(O) = \frac{1}{\alpha-\beta}$ which suggests we should be able to

introduce negative values for m by writing

$$sY_{-1}(s) = 1 + (\beta - \alpha)\, Y_o(s) .$$

Substituting in (2.3)

$$(s - \beta)\,[sY_{-1}(s) - 1] + \alpha\,[Y_{-1}(\frac{s}{\lambda}) - 1\ ] = (\beta - \alpha)$$

$$(s - \beta)\, sY_{-1}(s) - s + \frac{\alpha}{\lambda}\, sY_{-1}(\frac{s}{\lambda}) = 0$$

$$(s - \beta)\, Y_{-1}(s) + + \frac{\alpha}{\lambda}\, Y_{-1}(\frac{s}{\lambda}) = 1$$

provided $s \neq 0$

In general (2.6) and (2.7) can be introduced for negative values of m along with the condition $Y_m(s) \to \frac{1}{s}$ as $|s| \to \infty$. (2.8) and (2.9) are also valid for negative m. However we must be cautious in using these relations recursively with m decreasing, in particular at s = 0.

3. The Dual Equations

In (2.8) we observed that $Y_m(s,\alpha) \equiv Y(s, \alpha\lambda^m)$.From now on, for our convenience, we will drop the zero suffix of Y_0. Consequently (2.6) and (2.7) are simply

$$sY(s,\alpha) = 1 + (\beta - \alpha\lambda)\, Y(s,\alpha\lambda) \tag{3.1}$$

$$(s - \beta)\, Y(s,\alpha) + \alpha Y(, \alpha) = 1 \tag{3.2}$$

with a replaced by $\alpha\lambda^m$. Further on rearranging the former in the form

$$(\alpha - \beta)\, Y(s,\alpha) + s\, Y(s, \left(\frac{s}{\lambda}\right)) = 1 \tag{3.3}$$

we see that (3.3) is the dual of the equation (3.2) in that the roles of α and s are interchanged.

It is because of this duality between s and α that we write Y as a function of two variables, $Y(s,\alpha)$, although it is a function of β and λ as well. In fact we will find that it is β and λ that largely determine the analytic behaviour of $Y(s,\alpha)$.

In addition we note that (2.9) becomes

$$Y(0,\alpha) = \frac{1}{\alpha - \beta}. \tag{3.4}$$

The dual result for s is

$$Y(s,0) = \frac{1}{s-\beta}. \tag{3.5}$$

Strictly, however, it is from the equation (3.2) with the condition $Y(s,\alpha) \to \frac{1}{s}$ as $|s| \to \infty$ that we must determine $Y(s,\alpha)$.

To obtain solutions of our original differential equation (1.2) we require the Laplace inverse of $Y(s,\alpha)$. We will therefore concentrate on the s variable and find both series and continued fractions in s for $Y(s,\alpha)$. On inversion these will yield sequences r approximations to the solutions in the original variable.

Solutions of the Functional Equation

We will introduce our solutions in a convenient order for discussing the nature of the singularities of $Y(s,\alpha)$.

Two rational function solutions of the basic equation (3.2),

$$(s-ß)\,Y(s,\alpha) + \alpha Y(-\frac{s}{\lambda},\alpha) = 1$$

with the boundary condition $Y(s,\alpha) = \frac{1}{s}$ as $|s| \to \infty$, suggest themselves.

4.1 The Iterated Series

By iteration, we find

$$Y(s,\alpha) = \frac{1}{(s-\beta)}\left\{1 - \frac{\alpha}{(\frac{s}{\lambda}-\beta)}\left\{1 - \frac{\alpha}{(\frac{s}{\lambda^2}-\beta)}\left\{1 - \dots - \frac{\alpha}{(\frac{s}{\lambda^n}-\beta)}\left\{1 - \alpha Y(\frac{s}{\lambda^{n+1}},\alpha)\right\},\right\}\right\}\right\} \tag{4.1.1}$$

For $0 < X \lambda \ 1$ and $s \neq 0$, $\left|\frac{s}{n+1}\right|$ as $n \to \infty$, so that

$Y(\frac{s}{\lambda^{n+1}},\alpha) \to \frac{\lambda^{n+1}}{s}$, and the last term of (4.1.1)

$$\frac{(-\alpha)^{n+1}\lambda^{n(n+1)/2}\,Y\left(\frac{s}{\lambda^{n+1}},\alpha\right)}{\prod_{p=0}^{n}(s-\beta\lambda^{p})} \to 0 \qquad \text{as } n \to \infty$$

provided $0 < \lambda < 1$ and $s \neq ß^{\lambda P}$ $\qquad p = 0,1,2, \ldots\ldots\ldots\ldots$

Hence for $0 < \lambda < 1$ and for all $\alpha,ß$

$$Y(s,\alpha) = \sum_{r=0}^{\infty} \frac{(-\alpha)^{r}\lambda^{r(r+1)/2}}{\prod_{p=0}^{r}(s-\beta\lambda^{p})} \tag{4.1.2}$$

This result produces the trivial cases (i), (ii) and (iii) of section 1 and further setting $s = 0$ and assuming $|\alpha|<|\beta|$ we have $Y(0,\alpha) = \frac{1}{\alpha-\beta}$. To obtain (iii) of section 1, since (4.1.2) does not terminate if $\beta = \alpha \lambda^{n+1}$ for some n, consider first the case $\beta = \alpha\lambda$, the r^{th} partial sum of $sY(s,\alpha)$ is

$$\frac{s}{s-\alpha\lambda}-\frac{\alpha\lambda s}{(s-\alpha\lambda)(s-\alpha\lambda^2)}+\frac{\alpha^2\lambda^3 s}{(s-\alpha\lambda)(s-\alpha\lambda^2)(s-\alpha\lambda^3)}-..+\frac{(-\alpha)^{r-1}\lambda^{r(r-1)/2}s}{\prod_{p=1}^{r}(s-\alpha\lambda^p)}$$

$$=(1+\frac{\alpha\lambda}{s-\alpha\lambda})-\frac{\alpha\lambda}{s-\alpha\lambda}(1+\frac{\alpha\lambda^2}{s-\alpha\lambda^2})-\ \dots\ +\frac{(-\alpha)^{r-1}\lambda^{r(r-1)/2}}{\prod_{p=1}^{r-1}(s-\alpha\lambda^p)}\left(1+\frac{\alpha\lambda}{s-\alpha\lambda^r}\right)$$

$$=1-\frac{(-\alpha)^{r-1}\lambda^{r(r+1)/2}}{\prod_{p=1}^{r}(s-\alpha\lambda^p)}.$$

Hence for $0<\lambda<1$ $sY(s,\alpha)=1$.

A similar technique when $\beta = \alpha\lambda^{n+1}$ shows that the rth partial sum of the series

$$s^{n+1}Y(s,\alpha)-\sum_{r=0}^{n-1}\left\{\prod_{p=1}^{r}(\beta-\alpha\lambda^p)\right\}s^{n-r} \text{ is } \prod_{p=1}^{n}(\beta-\alpha\lambda^p)\left[1-\frac{(-\alpha)^r\lambda^{r(r+2n+1)/2}}{\prod_{p=n+1}^{n+r}(s-\alpha\lambda^p)}\right]$$

(4,1.3)

This shows not only that (iii) of section 1 is contained in the expansion (4.1.2) but is also gives the error in approximating the L.H.S. of (4.1.3) by a finite number of terms of the iterated series.

The expansion (4.1,2) for $Y(s,\alpha)$ is absolutely convergent under the conditions

i) $0<\lambda<1$; all α,β ; $s\neq\beta\lambda^r$ $r=0,1,2\ \dots$;

ii) $\lambda>1$; $|a|<|\beta|$; $s\neq\beta^{\lambda r}$ $r=0,1,2\ \dots$.

Thus, with $\beta\neq 0$ and $0<\lambda<1$, it appears that in general $Y(s,\alpha)$ is an analytic function of s with simple poles at $s=\beta\lambda^r$, $r = 0,1,2\ \dots$; unless $\beta=\alpha\lambda^{n+1}$ when $Y(s,\alpha)$ has a pole of order $n+1$ at $s=0$. For $0<\lambda<1$ $\beta\lambda^r\to 0$ as $r\to\infty$ so that $Y(s,\alpha)$ has an essential singularity at $s=0$. The function given by (4.1.2) also has simple poles at $s=\beta\lambda^r$, $r = 0,1,2\ ..$ for $\lambda>1$ provided $|\alpha|<|\beta|$, and then has an essential singularity at $s=\infty$.

The absolute convergence of (4.1.2) allows the series to be rearranged and expressed in partial fractions, so that, with $\beta \neq 0$,

$$Y(s, \alpha) = K(\alpha)\left[\frac{1}{s-\beta} + \sum_{r=1}^{\infty} \frac{\left(\frac{\alpha\lambda}{\beta}\right)^r}{\prod_{p=1}^{r}(1-\lambda^p)} \cdot \frac{1}{s-\beta\lambda^r}\right] \tag{4.1.4}$$

where

$$K(\alpha) = 1 + \sum_{r=1}^{\infty} \frac{\lambda^{r(r+1)/2}\left(-\frac{\alpha}{\beta}\right)^r}{\prod_{p=1}^{r}(1-\lambda^p)}.$$

This K(α) may be expressed as an infinite product

$$\left.\begin{aligned} K(\alpha) &= \prod_{p=1}^{\infty}\left(1 - \frac{\alpha\lambda^p}{\beta}\right) && \text{for } 0 < \lambda < 1 \\ &= \prod_{p=0}^{\infty}\left[1 - \frac{\alpha}{\beta\lambda^p}\right]^{-1} && \text{for } \lambda > 1. \end{aligned}\right\} \tag{4.1.5}$$

The product when $0 < \lambda < 1$ is indicating the conflict in the nature of the solution at points where $\beta = \alpha\lambda^P$, $p = 1,2\ldots$. For $\lambda > 1$ the absolute convergence of the expansion (4.1.2) required $|\alpha|<|\beta|$, the product (4.1.5) suggests a singularity when $\alpha = \beta$.

The expansion (4.1.2) is clearly fitting Y(s,α) for |s| large in the sense that

$$Y(s,\alpha) - S_n(s) = 0\left(\frac{1}{s}n+1\right) \quad \text{as} \quad |s| \to \infty$$

where S_n (s) is the n^{th} partial sum of (4.1.2). The original of $S_n(s)$ must, for $0 < \lambda < 1$, match the first n terms in the series for y(t) and hence be useful at least for t small. When $\lambda > 1$, (4.1.2) can be readily expanded for |s| small but not |s| large it does <u>not</u> produce the solution (iii) of section 1 as can be seen from (4.1.3). Consequently it is not the required continuation of the above $0 < \lambda < 1$ solution. However, the fact that a function with the roots $s = \beta\lambda^r$ exists for $\lambda > 1$ and which satisfies (3.2) demands closer investigation; this will be undertaken in section 5.

Successive terms of (4.1.4) fit the poles at $s = \beta\lambda^r \quad r = 0,1,2 \; \ldots$. When $s = \beta$ is the pole with the largest real part which is so when $\beta > 0$ and $0 < \lambda < 1$ the original of the series (4.1.4) will usually provide good approximations to y(t) for t large. When $\lambda > 1$ and $\beta < 0$ the original of the series (4.1.4), although not y(t), will also be found interesting.

By repeated use of the equation (3.1), we find that

$$Y(s,\ \alpha\ ) = \frac{1}{s} + \sum_{r=1}^{m-1}\left\{\prod_{p=1}^{r}(\beta - \alpha\ \lambda^{p})\right\}\frac{1}{s^{r+1}} + \left\{\prod_{p=1}^{m}(\beta - \alpha\lambda^{p})\right\}\frac{Y(s,\ \alpha\lambda^{m})}{s^{m}} \tag{4.1.6}$$

Like $Y(s,\alpha)$, $Y(s,\alpha\lambda^m)$ has simple poles at $s = \beta\lambda^r \quad r = 0,1,2 \; \ldots$

When $\beta = \alpha\lambda^m$ for some m, this series in $\frac{1}{s}$ terminates and corresponds to case (iii) of section 1. Further when $\beta \approx \alpha\lambda^m$ it is advantageous to take out m terms of the series in $\frac{1}{s}$ using (4.1.6) and use (4,1.2) or (4.1.4) for $Y(s,\alpha\lambda^m)$. But changing α to $\alpha\lambda^m$ improves the convergence of both (4.1.2) and (4.1.4) for $0 < \lambda < 1$ and so, even when β is not close to $\alpha\lambda^m$, it will often be useful to take out m terms of the series in $\frac{1}{s}$; m need not be small. When λ is close to 1 there are difficulties in using (4.1,2), nevertheless using (4.1.6) excellent numerical results can be obtained for values of λ less than about 0.95.

4.2 The Continued Fraction Matching Power Series

From (4.1.6) we have

$$Y(s,\ \alpha\ ) = \frac{1}{s} + \sum_{r=1}^{m-1}\left\{\prod_{p=1}^{r}(\beta - \alpha\lambda^{p})\right\}\frac{1}{s^{r+1}} + \left\{\prod_{p=1}^{m}(\beta - \alpha\ \lambda^{p})\right\}\frac{Y(s,\ \alpha\lambda^{m})}{s^{m}} \tag{4.2.1}$$

where $Y(s,\alpha\lambda^m)$ has simple poles at $s = \beta\lambda^r \quad r = 0, 1,2 \; ..$ for $0 < \lambda < 1$.

$Y(s,\alpha)$ satisfies the boundary condition $Y(s,\alpha) \to \frac{1}{s}$ as $|s| \to \infty$ and since all the poles of $Y(s,\alpha)$ lie in $|s| \le |\beta|$ for $0 < \lambda < 1$ the last term $\to 0$ as $m \to \infty$ provided $|s| > |\beta|$. Consequently $Y(s,\alpha)$ has the power series expansion

$$Y(s,\alpha\ ) = \frac{1}{s} + \sum_{r=1}^{\infty}\left\{\prod_{p=1}^{r}(\beta - \alpha\lambda^{p})\right\}\frac{1}{s^{r+1}} \tag{4.2.2}$$

for $0 < \lambda < 1$, $|s| > |\beta|$; the series is absolutely convergent. For

$\lambda > 1$ and $\alpha \neq 0$ the series diverges, unless $\beta = \alpha\lambda^n$ for some integer n when the series terminates. This expansion is in agreement with the particular cases $\lambda = 0$, $\lambda = 1$ and $\beta = \alpha\lambda^n$ quoted in section 1.

This series (4.2.2) is readily converted into the continued fraction

$$\frac{1}{s+}\ \frac{(\alpha\lambda-\beta)}{1\quad+}\ \frac{\lambda(\lambda-1)\alpha}{s\quad+}\ \frac{\lambda(\alpha\lambda^2-\beta)}{1\quad+}\ \cdots\cdots\ \frac{\lambda^{n-1}(\lambda^{n-1}-1)\alpha}{s\quad+}\ \frac{\lambda^{n-1}(\alpha\lambda^n-\beta)}{1\quad+}\ \cdots \qquad (4.2.3)$$

by, for example, applying the Q-D algorithm. Now the elements of this continued fraction occur in pairs and only the even convergents are finite for s small. We consider only the even convergents, or what is equivalent, the convergents of the J fraction that matches the series (4.2.2)

$$\frac{1}{s+\alpha\lambda-\beta+}\ \frac{\lambda(\lambda-1)\alpha(\beta-\alpha\lambda)}{s+\lambda(\lambda-1)\alpha+\lambda(\alpha\lambda^2-\beta)+}\cdots\cdots\ \frac{\lambda^{2n-1}(\lambda^n-1)\alpha(\beta-\alpha\lambda^n)}{s+\lambda^n(\lambda^n-1)\alpha+\lambda^n(\alpha\lambda^{n+1}-\beta)+\ldots} \qquad (4.2.4)$$

It is obtained on contracting (4.2.3)

This continued fraction is fitting the terms in the series in $\frac{1}{s}$ (4.2.2) for $Y(s,\alpha)$ and gives the required solution when $\lambda = 0$ and when $\lambda = 1$. Further when $\beta = \alpha\lambda^n$ it terminates and gives the polynomial solution quoted in section 1 for not only $0<\lambda<1$ but also for $\lambda > 1$.

In section 6 we show that for $0 < \lambda < 1$ the denominator roots of (4.2.4) tend to $\beta\lambda^n$ and the continued fraction converges to $Y(s,\alpha)$ for all and $s \neq 0$. When $\lambda > 1$ the denominator roots do not tend to $\beta\lambda^n$ but instead tend to $-\alpha\lambda^{2n-1}$ The continued fraction still fits the boundary condition as $|s| \longrightarrow \infty$ and it converges; but not to the same function as (4.1.2).

When $\beta \approx \alpha\lambda^n$ for all λ the continued fraction provides an analytic continuation of the polynomial solution in $\frac{1}{s}$. However a more effective analytic continuation is to set $m = n$ in (4.2.1) and use the continued fraction for $Y(s,\alpha\lambda^n)$, or put another way, the series expansion with the terms after the n^{th} replaced by the corresponding continued fraction. In particular the solution of the problem (1.4) is efficiently calculated by this method with $n = 6$.

In general the solution to our problem is to approximate the original of (4.2.1) with the continued fraction (4.2.4), with $\alpha \lambda^m$ in place of a, replacing Y(s, $\alpha \lambda^m$). Denoting the n^{th} convergent of (4.2.4) by Y/n (s,α) the corresponding approximation to y(t) is

$$1 + \sum_{r=1}^{m-1} \left\{ \prod_{p=1}^{r} (\beta - \alpha \lambda^p) \right\} \frac{t^r}{r!} + \prod_{p=1}^{m} (\beta - \alpha \lambda^p) L^{-1} \frac{Y/n(s, \alpha\lambda^m)}{s^m} \tag{4.2.5}$$

The inversion of the continued fraction can be performed in two ways. Indirectly using a method based on the complex inversion formula such as Talbots method [1976]; or directly by evaluating the positions of the poles of (4.2.4) numerically, expanding in partial fractions and then inverting. With this latter method as λ approaches 1 high convergents ($n > 10$) are often required and the zeros of the denominators must be determined very accurately. The use of high convergents can be reduced by taking out more terms of the series when $0 < \lambda < 1$. The advantage, when $0 < \lambda < 1$, in using (4.1.2) or (4.1.4), over the continued fraction, is that the positions of the poles $\beta\lambda^r$ are known and the partial fraction expansion can be performed algebraically.

4.3 Further Solutions

In section 4.1 we generated the solution(4.1.2) by iteration of the equation (3.2). The solution satisfyingthe boundary condition $Y(s,\alpha) \to \frac{1}{s}$ as $|s| \to \infty$ was valid for $0 < \lambda < 1$ and for all α, β.

Now our functional equation (3.2)

$$(s - \beta)\ Y(s,\alpha) + \alpha Y(\frac{s}{\lambda},\alpha) = 1$$

can be iterated in the sense opposite to that which generated (4.1.2). This produces

$$Y(s, \alpha) = \frac{1}{\alpha}\left\{1 + \frac{(\beta - s\lambda\lambda}{\alpha}\left\{1 + \ldots + \frac{(\beta - s\lambda^{n-1})}{\alpha}\left\{1 + (\beta - s\lambda^n)Y(s\lambda^n, \alpha\right\}\right\}\right\}. \tag{4.3.1}$$

The associated series

$$\frac{1}{\alpha} + \sum_{r=1}^{\infty} \left\{ \prod_{p=1}^{r} (\beta - s\lambda^p) \right\} \frac{1}{\alpha^{r+1}} \tag{4.3.2}$$

converges absolutely for all finite s under the conditions $0 < \lambda < 1$ and $|\alpha| > |\beta|$, but clearly does not converge to the same function as (4.1.2) ; for compare (4.3.2) with (4.2.2) in our previous solution s and α are not interchangeable.Similarly when (3.1) is iterated in the sense opposite to that used in generating (4.1.6), we find

$$Y(s, \alpha) = \sum_{r=0}^{n=1} \frac{(-s)^r}{\prod_{p=0}^{r} (\frac{\alpha}{\lambda^p} - \beta)} + \frac{(-s)^n \, Y(s, \frac{\alpha}{\lambda^n})}{\prod_{p=0}^{n=1} (\frac{\alpha}{\lambda^p} - \beta)} \tag{4.3.3}$$

The sereies

$$\sum_{r=0}^{\infty} \frac{(-s)^r}{\prod_{p=0}^{r} (\frac{\alpha}{\lambda^p} - \beta)} \tag{4.3.4}$$

converges when $0 < \lambda < 1$ for all s,β ; $\alpha \neq \lambda^p \beta$ r =0,1,2, ... and is the same function as (4.3.2) when $|\alpha| > |\beta|$. However when $\lambda > 1$, (4.3.4) has an interesting relationship with (4.1.2), they become the same function when both series are absolutely convergent, that is when $|\alpha| < |\beta|$ and $|s| < |\beta|$. It is to this function, with Its interchangeability between the two variables s and α, that we turn our attention in the next section.

5. Symmetric Solution of the Functional Equations for $\lambda > 1$.

If we relax the boundary condition $Y(s,\alpha) \to \frac{1}{s}$ as $|s| \to \infty$, the functional equations (3.2) and (3.3) have an interesting solution symmetrical in the variables s and α. For the variables to be interchangeable, both s and α must have identical arrangements of poles. The iterative solution (4.1.1) displays poles at $s = \beta \lambda^r$ $r = 0,1,2, \ldots$ while the iterative solution (4.3.3) has poles at $\alpha = \beta \lambda^r$ $r = 0,1,2, \ldots$, moreover both are convergent for $\lambda > 1$ for some α, s (as well as for $0 < \lambda < 1$). Let us combine these properties by solving, in a similar iterative manner, the pair of functional equations (3.2) and (3.3) simultaneously.

$$(s - \beta)\, Y(s,\alpha) + \alpha Y(\frac{s}{\lambda}, \alpha) = 1 \tag{3.2}$$

$$(\alpha - \beta)\, Y(s,\alpha) + s\, Y(s, \frac{\alpha}{\lambda}) = 1 \tag{3.3}$$

Multiplying (3.2) by (α -β),

$$(s-\beta)(\alpha-\beta)\ Y(s,\alpha)\ +\alpha[1-\frac{s}{\lambda}Y(\frac{s}{\lambda},\frac{\alpha}{\lambda})]=\ \alpha-ß$$

$$Y(s,\alpha)\ =-\frac{\beta}{(s-\beta)(\alpha-\beta)}[1-\frac{\alpha\, s}{\lambda\beta}Y(\frac{s}{\lambda},\frac{\alpha}{\lambda})] \tag{5.1}$$

Iteration of this equation gives, if $\beta \neq 0$

$$Y(s,\ \alpha\ ) = \frac{-\beta}{(s-\beta)(\alpha\ -\beta\ )}\left[1+\frac{\lambda\alpha\ s}{(s-\beta\,\lambda)(\alpha\ -\beta\,\lambda)}\right[1 + .../$$

$$... + \frac{\lambda\alpha s}{(s-\beta\,\lambda^{n-1})(\alpha-\beta\,\lambda^{n-1})}\left[1-\frac{\lambda\,\alpha\, s}{\lambda^{2n}\,\beta}Y(\frac{s}{\lambda^{n}},\frac{\alpha}{\lambda^{n}})..\right] \tag{5.2}$$

When the boundary conditions are

$$Y(s,0)\ = s-\beta \text{ and } Y(0,\alpha)\ = \frac{1}{\alpha-\beta} \tag{5.3}$$

we see that the solution for A > 1 is

$$Y(s,\ \alpha\ ) = -\beta\sum_{r=0}^{\infty}\frac{(\lambda\,\alpha\, s)^{r}}{\prod_{p=0}^{r}(s-\beta\,\lambda^{p})(\alpha-\beta\lambda^{p})}, \tag{5.4}$$

This series is absolutely convergent, excluding the lines poles at $s = \beta\lambda^{r}$ and $\alpha = \beta\lambda^{r}$ r =0,1,2,..., for all $\lambda>0$. At $\lambda=1$ the series takes the value

$$\frac{1}{s+\alpha-\beta}. \tag{5.5}$$

For the behaviour of this function as $|s| \to \infty$, write

$$1 \equiv \frac{-\beta}{\alpha-\beta}[1-\frac{\alpha}{\beta}] \equiv \frac{-\beta}{\alpha-\beta}[1+\frac{\lambda\alpha}{\alpha-\beta\,\lambda}[1+...+\frac{\lambda\alpha}{\alpha-\beta\,\lambda^{n-1}}[1-\frac{\alpha}{\beta\,\lambda^{n-1}}]..]$$

then the partial sum of n terms of (5.4), for a given λ & β, tends to

$$\frac{1}{s}\left\{1-\frac{\alpha^{n}}{\prod_{p=0}^{n-1}(\alpha-\beta\lambda^{p})}\right\} \tag{5.6}$$

The function (5.4) does not tend to $\frac{1}{s}$ as $|s| \to \infty$ if $\alpha = \beta\lambda^{P}$ for p=0,1,2,..., or if $0 < \lambda < 1$.

Suppose the n iterations of (3.2) and of (3.3) are performed consecutively, rather than simultaneously using (5.1), Then the relation (5.2) is obtained but in a rearranged form connecting both (4.1-1) and (4.3.3).

When the n iterations with (3.3)

$$Y(s,\ \alpha\ ) = \frac{1}{\alpha-\beta}\left[1-\frac{s}{\frac{\alpha}{\lambda}-\beta}\left[1-\ldots\ldots\ldots\ .-\frac{s}{\frac{\alpha}{\lambda^{n-1}}-\beta}\left[1-sY(s,\ \frac{\alpha}{\lambda^{n}})\right]..\right]\right]$$

are followed by the n iterations with (3.2) (5.7)

$$Y(s,\ \frac{\alpha}{\lambda^{n}}) = \frac{1}{s-\beta}\left[1-\frac{\frac{\alpha}{\lambda^{n}}}{\frac{s}{\lambda}-\beta}\left[1-\ldots\ldots\ldots\ -\frac{\frac{\alpha}{\lambda^{n}}}{\frac{s}{\lambda^{n-1}}-\beta}\left[1-\frac{\alpha}{\lambda^{n}}Y(\frac{s}{\lambda^{n}},\frac{\alpha}{\lambda^{n}})\right]..\right]\right]$$

together they form a rearrangement of (5.2). Removing the final term, we readily obtain the partial fraction in s expansion of n terms of (5.4).

The residue at s = ß

$$\underset{S\to\beta}{\mathrm{Lim}}(s-\beta)Y(s,\ \alpha) = \frac{(-\beta)^{n}}{\prod_{p=0}^{n-1}(\frac{\alpha}{\lambda^{p}}-\beta)}\left[1-\frac{\frac{\alpha}{\lambda^{n}\beta}}{(\frac{1}{\lambda}-1)}+\frac{\left[\frac{\alpha}{\lambda^{n}\beta}\right]^{2}}{(\frac{1}{\lambda}-1)(\frac{1}{\lambda^{2}}-1)}\ldots.+\frac{\left[-\frac{\alpha}{\lambda^{n}\beta}\right]^{n-1}}{\prod_{p=1}^{n-1}(\frac{1}{\lambda}-1)}\right]$$

$$= \frac{1}{\prod_{p=0}^{n-1}(1-\frac{\alpha}{\lambda^{p}\beta})}\text{x n terms of the series for } K(\frac{\alpha}{\lambda^{n}}).$$

For $0 < A < 1$ the n terms of the series for $K(\frac{\alpha}{\lambda^{n}})$. is tending to cancel the factors in the denominator, while for $X > 1$ it is tending to introduce extra factors to the denominator. As $n \to \infty$, this residue tends to $K(\alpha)$ as defined in (4.1.5), further the complete partial fraction expansion of (5.4) for $\lambda > 1$ is

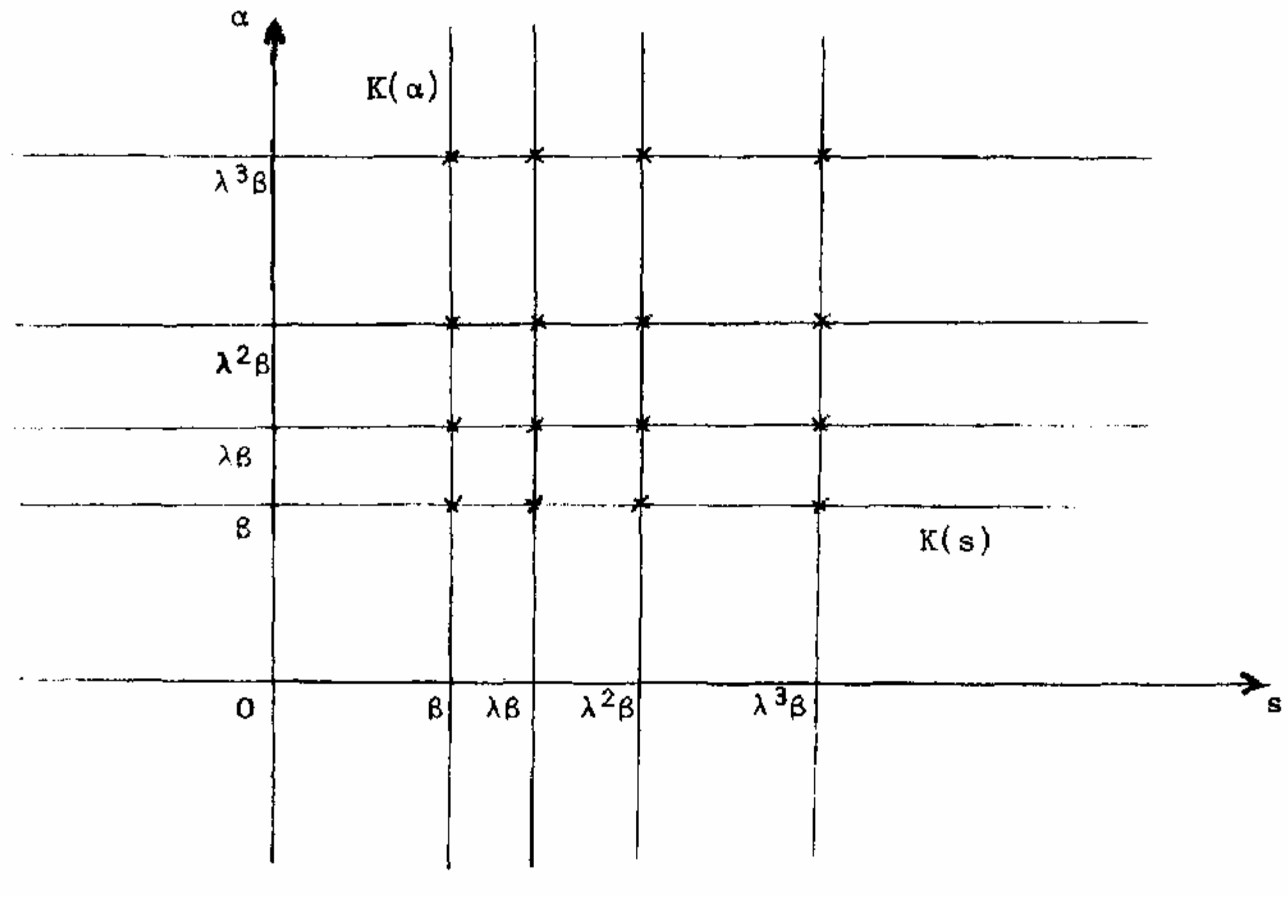
$$Y(s,\alpha) = K(\alpha\ )\left[\frac{1}{s-\beta}+\sum_{r=1}^{\infty}\frac{\left(\frac{\alpha\lambda}{\beta}\right)^{r}}{\prod_{p=1}^{r}(1-\lambda^{p})}.\frac{1}{s-\beta\lambda^{r}}\right] \tag{5.9}$$

and is identical to (4.1.4). For $0 < \lambda < 1$ the precise nature of the partial fraction expansion of (5,4) is not important.

The function (5.4) is identical to the function (4.1.2) for $\lambda>1$ and $|\alpha|<|\beta|$, and is an analytic extension for $\lambda>1$ of this function. (5.4) may also be used to define the function for $0 < \lambda < 1$, but the natural barrier that exists at $\lambda = 1$ makes this choice less obvious. Linking (5.4) for $\lambda > 1$ with our previous solution (4.1.2) for $0 < \lambda < 1$ has much to commend it; in particular (4,1.4) holds for all positive $\lambda \neq 1$, and, if $\lambda > 1, \alpha \neq \beta\lambda^p$ for $p = 0,1,2,\ldots$

Expansion of Symmetric Solution

The function (5.4), with its symmetrical arrangement of the poles in the two variables s and α, can be expanded in terms of the residues at the intersections of these poles.



With $\lambda > 1$, the residue at $s = \alpha = \beta$, from (5.1) is

$$K = \lim_{S\to\beta}\lim_{\alpha\to\beta}(s-\beta)(\alpha-\beta)Y(s,\alpha) = -\beta\left[1-\frac{\beta}{\lambda}Y\left(\frac{\beta}{\lambda},\frac{\beta}{\lambda}\right)\right] = \frac{-\beta}{\prod_{p=1}^{\infty}\left(1-\frac{1}{\lambda^p}\right)}$$

While the residue at the point $s=\beta\lambda^{r}$, $\alpha=\beta\lambda^{m}$ is

$$\lim_{S\to\beta\lambda^r}\ \lim_{\alpha\to\beta\lambda^m}(s-\beta\lambda^{r})(\alpha-\beta\lambda^{m})Y(s,\alpha)=\frac{\lambda^{rm+r+m}}{\prod_{p=1}^{r}(1-\lambda^{p})\prod_{q=1}^{m}(1-\lambda^{q})},\frac{-\beta}{\prod_{p=1}^{\infty}(1-\frac{1}{\lambda^{p}})}$$

Setting $K_{r,m}\equiv\dfrac{\lambda^{rm+r+m}}{\prod_{p=1}^{r}(1-\lambda^{p})\prod_{q=1}^{m}(1-\lambda^{q})}$, (5.4) can be written

$$Y(s,\alpha)=K\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\frac{K_{r,m}}{(s-\beta\lambda^{r})(\alpha-\beta\lambda^{m})} \qquad (5.9)$$

The double summation cannot be split into a product of a sum over r and a sum over m.

6. Continued Fraction Solutions

We write our continued fractions in the form

$$\frac{p_1}{q_1+}\ \frac{p_2}{q_2+}\ \ldots\ldots\ \frac{p_n}{q_n+}\ \ldots$$

and suppose that P_n/Q_n is the nth convergent. Only standard results from the theory of continued fractions are used,
P_n and Q_n both satisfy

$$u_{n+1} = q_{n+1}u_n + p_{n+1}u_{n-1}$$

for n = 1,2,3, . .., the initial values for P being $P_0 = 0$, $P_1 = P_1$ whilst the corresponding values for Q_n are $Q_o = 1$, $Q_1 = q_1$,
From here we deduce the determinant formulae

$$P_{n+1}Q_n - P_nQ_{n+1} = (-1)^n\ p_1p_2\ldots.p_{n+1}\ ,$$
$$P_{n+2}Q_n - p_nQ_{n+2} = (-1)^n\ p_1p_2\ldots..p_{n+1}q_{n+2'}$$

for n=0,1,2,......

If the numerator elements p_n of the continued fraction are non-zero constants and the denominator elements q_n are polynomials in some parameter s, say, then P_n, Q_n are polynomials in s. The first determinant formula shows that any zero of Q_n cannot be a zero of P_n (or Q_{n+1}).

Conventionally a power series is converted into its corresponding continued fraction by some such method as the Q-D algorithm. For the series (4.2.2) we found the corresponding continued fraction

$$\frac{1}{s}+\frac{(\alpha\lambda-\beta)}{1}+\frac{\lambda\,(\lambda-1)\alpha}{s}+\frac{\lambda(\alpha\lambda^2-\beta)}{1}+\ldots+\frac{\lambda^{n-1}(\lambda^{n-1}-1)\alpha}{s}+\frac{\lambda^{n-1}(\alpha\lambda^n-\beta)}{1}+\ldots \tag{6.1}$$

The iterated series (4.1.2) is also in a form suitable for conversion to a continued fraction.

$$\frac{1}{(s-\beta)}+\frac{\alpha\lambda(s-\beta)}{(s-\beta\lambda)}+\frac{\alpha\lambda(\lambda-1)s}{(s-\beta\lambda^2)}+\frac{\alpha\lambda^3(s-\beta\lambda)}{(s-\beta\lambda^3)}+\ldots+\frac{\alpha\lambda^{n-1}(\lambda^{n-1}-1)s}{(s-\beta\lambda^{2n-2})}+\frac{\alpha\lambda^{2n-1}(s-\beta\lambda^{n-1})}{(s-\beta\lambda^{2n-1})+\ldots} \tag{6.2}$$

We start by discussing the location of the roots of the denominator polynomials of the convergents of these two continued fractions. Since the elements of the continued fractions are clearly grouped in pairs, we consider only even order convergents. The denominator polynomial of the 2n convergent of (6.1) is of degree n in s, whilst for the continued fraction (6.2) it is a polynomial of degree 2n in s. Although for these polynomials we cannot, in general, determine the exact position of the roots, we can locate them approximately to within a certain order in A, such as for example, 0(A) as $\lambda \to 0$ or $0(\frac{1}{\lambda})$ as $\lambda \to \infty$. It is necessary to consider six cases in the ranges of the parameters $\alpha, ß, \lambda$. These are

(i) $\beta \neq 0,\ \ \beta \neq \alpha^{\lambda m}$, $0 < \lambda < 1$;

(ii) $\beta \neq 0,\ \ \beta = \alpha\lambda^m$, $0 < \lambda < 1$;

(iii) $\beta = 0,\ \ 0 < \lambda < 1$;

(iv) $\beta \neq 0,\ \ \beta \neq \alpha^{\lambda m}$ or $\beta\,\lambda^m \neq \alpha,\ \ \lambda > 1$;

(v) $\beta \neq 0,\ \ \beta = \alpha\lambda^m$ or $\beta\,\lambda^m = \alpha,\ \ \lambda > 1$;

(vi) $\beta = 0,\ \ \lambda > 1$;

m being a positive integer ≥ 0.

Now the $2n^{th}$ denominator polynomial of (6.1) may be expressed in the form

$$(s-\beta)(s-\beta\lambda)\_.(s-\beta\lambda^{n-1}) + \alpha\lambda^{n}(1+\lambda+\ldots+\lambda^{n-1})(s-\beta)(s-\beta\lambda)\ldots(s-\beta\lambda^{n-2})+\ldots\ldots + \alpha^{n-1}\lambda^{n(n-1)}(1+\lambda+\ldots+\lambda^{n-1})(s-\beta)+\alpha^{n}\lambda^{n2}, \qquad (6.3a)$$

or, alternatively as

$$s^{n} + (1+\lambda+\ldots+\lambda^{n-1})(\alpha\lambda^{n}-ß)s^{n-1}+\ldots+\lambda^{n(n-1)/2}(\alpha\lambda^{n}-\beta)(\alpha\lambda^{n-1}\beta)\ldots(\alpha\lambda-\beta). \qquad (6.3b)$$

If we define $C_r^{(n)}$ by

$$C_r^{(n)} = \frac{(1-\lambda^n)}{(1-\lambda)}.\frac{(1-\lambda^{n-1})}{(1-\lambda^2)}\ldots\frac{(1-\lambda^{n-r+1})}{(1-\lambda^r)}, r = 1,2,3,\ \ldots, c_0^{(n)} \equiv 1$$

the two polynomials (6.3a) and (6.3b) may be expressed as respectively

$$\sum_{r=0}^{n}\alpha^r\lambda^{nr}C_r^{(n)}\left\{\prod_{p=0}^{n-r-1}(s-\beta\lambda^P)\right\} \text{ and } \sum_{r=0}^{n}\lambda^{r(r-1)/2}C_r^{(n)}\left\{\prod_{p=n-r+1}^{n}(\alpha\lambda^p-ß)\right\}s^{n-r},$$

where $\prod_{p=0}^{-1} \equiv \prod_{p=n+1}^{n} \equiv 1$.

(i) $\beta \neq 0$, $\beta \neq \alpha^{\lambda m}$, $0 < \lambda < 1$, $m = 1,2,3, \ldots$.

For these conditions the roots are $s - \beta\lambda^r \{1+0(\lambda)\}$ as $\lambda \rightarrow 0$, $r = 0,1,2,\ldots (n-1)$. The order term is not best possible except for $r = (n-1)$, but it is adequate for our purposes. It appears, in fact,

that the roots are $s = \beta\lambda^r \{1 + 0(^{\lambda(n-r)2}\}$ as $\lambda \rightarrow 0$, so that the denominators are producing the earlier poles of $^{Y}(s,\alpha)$ more closely than the later ones. The order terms do not imply that the roots are necessarily real.

(ii) $\beta \neq 0$, $\beta = \alpha\lambda^m$, $0<\lambda<1$ for some $m = 1,2,3, \ldots$.

Under these conditions it is clear from (6.3b) that the denominator polynomial $Q_{2n}(s)$, say, has m roots at $s = 0$ and the remaining roots at $s = \beta\lambda^r \{1+0(\lambda)\}$ as $\lambda+0$, for $r = 0,1,2,\ldots (n-m-1)$. Of course in this case the continued fraction terminates and it is not necessary to consider $Q_{2n}(s)$ for $n > m$.

On the other hand if $\beta - \alpha\lambda^m \approx 0$ for some $m = 1,2,3,..$ then $Q_{2n}(s)$ has m complex conjugate roots for m even, or (m+1) complex conjugate roots for m odd, all located near s = 0. This result is not inconsistent with (i) but is providing additional information concerning roots near s =0 when $\beta - \alpha\lambda^m \approx 0$,

(iii) $\beta = 0,\ 0 < \lambda < 1$.

In this case (6.3) reduces to

$$s^n + \alpha\lambda^n(1+\lambda+..+\lambda^{n-1})s^{n-1}...+\alpha^{n-1}\lambda^{n(n-1)}(1+\lambda+..+\lambda^{n-1})s+\alpha^n\lambda^{n2}, \qquad (6.4)$$

or

$$\sum_{r=0}^{n} \alpha^r \lambda^{nr} C_r^{(n)} s^{n-r},$$

and all the roots are $0(\lambda^n)$ as $\lambda \to 0$. In fact they all lie on the circle $|s| = |\alpha|\lambda^n$. For n odd, symmetry in the coefficients shows that one root is $s = -\alpha\lambda^n$ exactly,

(vi) $\beta = 0,\ \lambda > 1$.

Having dealt with case (iii) it is convenient to consider next case (vi) For these conditions we find that the roots are located at $s = -\alpha\lambda^{2r-1}\{1 + 0(\frac{1}{\lambda})\}$ as $\lambda \to \infty$ for r - 1,2,..n, and since (6.1) is positive definite [see Wall, 1948] these roots are necessarily real.

(iv) $\beta \neq 0,\ \beta \neq \alpha\lambda^m,\ \lambda > 1$ for $m = 1,2, ... n$.

In this case the roots are at $s = -\alpha\lambda^{2r} -1 \{1+0(\frac{1}{\lambda})\}$ as $\lambda \to \infty$ for r = 1,2, ... n, but need not be real.

(v) $\beta \neq 0\ \ \beta = \alpha\lambda^m,\ \lambda > 1$ for some $m = 1,2 .... n$.

The continued fraction again terminates at the m[th] convergent. However the polynomial (6.3) has m roots at s = 0 and the remaining n - m roots are at $s = -\alpha\lambda^{2r-1}\{1 + 0(\frac{1}{\lambda})\}$ as $\lambda \to \infty$.

We note that when $\lambda = 1$, the denominator polynomials $Q_{2n}(s)$ reduce to

$$\{s + \alpha - \beta\}^n \qquad (6.5)$$

For the continued fraction (6.2) we find that for the 2n denominator polynomial n roots are known exactly, and that this polynomial may be expressed in the alternative forms

$$(s-\beta)(s-\beta\lambda)\ldots(s-\beta\lambda^{n-1})\Big[(s-\beta^{n})\;.\;.\;.\;(s-\beta\lambda^{2n-1}) \;+$$

$$+\,\alpha\lambda^{n}(1+\lambda+\ldots+\lambda^{n-1})(s-\beta\lambda^{n})\;.\;.\;.\;(s-\beta\lambda^{2n-2})+\ldots \qquad (6.6a)$$

$$\ldots+\alpha^{n-1}\lambda^{n(n-1)}(1+\lambda+\ldots+\lambda^{n-1})(s-\beta\lambda^{n})+\alpha^{n}\lambda^{n2}\Big],$$

and

$$(s-\beta)(s-\beta\lambda)\ldots(s-\beta\lambda^{n-1})\Big[s^{n}+(\alpha-\beta)(1+\lambda\ldots+\lambda^{n-1})\lambda^{n}s^{n-1}+\ldots$$

$$\ldots+(\alpha-\beta)(\alpha-\beta\lambda)\ldots(\alpha-\beta\lambda^{n-2})(1+\lambda+\ldots+\lambda^{n-1})\lambda^{n(n-1)}s \;+ \qquad (6.6b)$$

$$+\;(\alpha-\beta)(\alpha-\beta\lambda)\ldots(\alpha-\beta\lambda^{n-1})\lambda^{n2}\Big]$$

The polynomial in the square bracket is the polynomial (6.3) with ß replaced by $\beta\lambda^{h}$.

Using the same notation as for (6.3) the polynomials (6.6a) and (6.6b) may be expressed as

$$\left\{\prod_{p=0}^{n-1}(s-\beta\lambda^{p}\right\}\sum_{r=0}^{n}\alpha^{r}\lambda^{nr}C_{r}^{(n)}\left\{\prod_{p=n}^{2n-r-1}(s-\beta\lambda^{p})\right\}$$

and

$$\left\{\prod_{p=0}^{n-1}(s-\beta\lambda^{p}\right\}\sum_{r=0}^{n}\lambda^{nr}C_{r}^{(n)}\left\{\prod_{p=0}^{r-1}(\alpha-\beta\lambda^{p})\right\}s^{n-r}$$

where

$$\prod_{p=0}^{-1}\equiv\prod_{p=n}^{n-1}\equiv 1$$

(i) $\beta\neq 0$, $\beta\neq\alpha\lambda^{m}$, $0<\lambda<1$, $m=1,2,3,\ldots\ldots\ldots\ldots$
The roots of (6.6) are $s=\beta^{\lambda r}$, $r=0,1,2,..(n-1)$ and $0(\lambda^{n})$ as $\lambda\to 0$ for n remaining roots.

(ii) $\beta\neq 0$, $\beta=\alpha^{m}$, $0<\lambda<1$. for some $m=1,2,3,,..$.
Like the iterated series (4.1.2), the continued fraction (6.2) does not terminate when $ß=\alpha\lambda^{m}$, but tends to the polynomial

solution in the limit. The roots for this case are of the same character as those for (i) above.

(iii) $\beta=0, \quad 0<\lambda<1.$

In this case the polynomial in the square brackets in (6,6) reduces to (6.4),consequently there are n roots at s =0 and n roots that as mentioned before lie on $|s| = |\alpha|\lambda^n$. The roots of (i) and (ii) will tend to these values as $ß \to 0$.

(iv) $\beta\neq 0, \ \alpha\neq\beta\lambda \ , \ \lambda>1$ for some m =0,1,2 ....

The roots of (6.6) are $s = \beta^{\lambda r}$ r =0,1, . .(n-1) ; $s = \beta^{\lambda r}[1+0(\frac{1}{\lambda})]$

as $\lambda\to\infty$ for r =n, (n+1) ,. .,(2n-2) ; $s = (\beta-\alpha)\lambda^{2n-1}\{1+0(\frac{1}{\lambda})\}$

as $\lambda\to\infty$.

(v) $B \neq 0$, $\alpha = ß \ \lambda^m = \lambda>1$ for some m = 0,1 ,2,...

From (6.6b) we find that the denominator polynomial has roots at

$s = \beta^{\lambda r}$, r =0,1,.. (n-1) ; (n-m) roots at s =0; $s = \beta^{\lambda r}\{1+0(\frac{1}{\lambda})\}$

as $\lambda \to \infty$ for r =(2n-m), (2n-m+1) , .. (2n-2) ; $s = (ß-\alpha)\backslash^{2n-1}\{1+0(y)\}$

as $\lambda \to \infty$. In particular when α - ß , the roots are

$s = \beta\lambda^r$, r =0,1,.. .(n-1) and n roots at s = 0.

(vi) $\beta = 0, \ \lambda > 1$.

The roots of (6.6) are n at s =0 and $s = -\alpha\lambda^{2r-1}$ $\{1+0(\frac{1}{\lambda})\}$as $\lambda\to\infty$

for r =1,2,.. n.

For A = 1 we see that (6.6) becomes

$$(s-\beta)^n \ \{s+\alpha-\beta)^n. \qquad (6.7)$$

Summing up we have for the roots of (6.3) and (6.6):

	(6.3)	(6.6)
(i) (ii)	$\beta\lambda^r \ \{1+0(\lambda)\}$, r =0(l)(n-l). $\beta\lambda^r\{1+0(\lambda)\}$, r=0(l)(n-m-l) ; m roots s =0.	$\beta\lambda^r$, r=0(1)(n-1) ; n roots $0(\lambda^n)$ $\beta\lambda^r$, r=0(1)(n-1) ; n roots $0(\lambda^n)$
(v)	n roots $0(\lambda^n)$ on circle $\|\alpha\|\lambda^n$, $-\alpha\lambda^{2r-1}\{1+0(\frac{1}{\lambda})\}$, r =1(1)n.	n roots s =0 ; n roots 0(A) . $\beta\lambda^r$,r=0(1) (n-1) ; $\beta\lambda^r\{1+0)(\frac{1}{\lambda})\}$,r=n(1) (2n-2) ; $(ß=\alpha)\lambda^{2n-1}\{1+0(\frac{1}{\lambda})\}$.
(vi)	m roots at s =0; $-\alpha\lambda^{2r-1}\{1+0(\frac{1}{\lambda})\}$, r=-m+1(1)n	$\beta\lambda^r$, r =0(1) (n-1); (n-m) roots at s =0; $\beta\lambda^r \ \{1+0(\frac{1}{\lambda})\}$, r=(2n-m)(1)(2n-2); $(ß-\alpha)\lambda^{2n-L}\{1+0(\frac{1}{\lambda})\}$.
	$-\alpha\lambda^{2r-1}\{1+0(\frac{1}{\lambda})\}$, r=1(1)n	n roots s =0 ;$-\alpha\lambda^{2r-1}\{1+0(\frac{1}{\lambda})\}$ r=1(1)n.

(6.8)

Observing that when 3 =0, the continued fraction (6.2) reduces to (6.1), we note that the significant difference in the roots tabulated in (6.8) lies in cases (iv) and (v) where $\beta \neq 0$ and $\lambda > 1$. The roots of the denominator polynomials of (6.2) are tending, as expected, to the poles $\beta\lambda^r$ given by (4,1.2), whereas the roots of (6.3) do not. We now investigate the convergence of the continued fraction (6.2). If $P_n(s)/Q_n(s)$ is the n^{th} convergent of (6.2) and R (s) is the 'remainder' function which when subtracted from the nth denominator element makes

$$Y(s) 5 \quad (P_n - R_n P_{n-1}\ ,)/(Q_n - R_n Q_{n-1}\ ,) \tag{6.9}$$

we find that

$$Q_{2n}Y - P_{2n} = (P_{2n}Q_{2n-1} P_{2n-1} Q_{2n}) R_{2n}/(Q_{2n} - R_{2n} Q_{2n-1})$$

$$= \frac{(-1)^n \lambda^{n(3n-1)/2}\left\{\prod_{p=1}^{n-1}(1-\lambda^p)\right\}\left\{\prod_{p=1}^{n-1}(s-\beta\lambda^p)\right\}\alpha^{2n-1}s^{n-1}R_{2n}}{(Q_{2n} - R_{2n}Q_{2n-1})}$$

As noted earlier some of the factors $(s-\beta\lambda^p)$, $p=0,1,2,..$ actually occur in $Q_{2n}.(s)$ and in fact we may writeX

$$\left.\begin{aligned} Q_{2n}(s) &= \left\{\prod_{p=0}^{n-1}(s-\beta\lambda^p)\right\}Q_n^{(e)}(s) \\ Q_{2n-1}(s) &= \left\{\prod_{p=0}^{n-1}(s-\beta\lambda^p)\right\}Q_{n-1}^{(0)}(s) \end{aligned}\right\} \tag{6.10}$$

where $Q_n^{(e)}(s)$, $Q_{n-1}^{(0)}(s)$ are polynomials in s of degrees n,n -1 respectively and associated with the even and odd order polynomials $Q_{2n}(s)$, $Q_{2n-1}(s)$ respectively. The above expression for $(Q_{2n}Y - P_{2n}.)$ thus reduces to

$$Q_{2n}y - P_{2n} = \frac{(-1)^n \lambda^{n(3n-1)/2}\left\{\prod_{p=1}^{n-1}(1-\lambda^p)\right\}\alpha^{2n-1}s^{n-1}R_{2n}}{\left[Q_n^{(e)} - R_{2n}Q_{n-1}^{(o)}\right]} \tag{6.11}$$

The determinant formula for continued fractions shows that the roots of $Q_n^{(e)}$ (s) are distinct from those of $Q_{n-1}^{(0)}$ (s). From (6.2) we see that the function $R_n(s)$ can be expressed as the continued fraction attached to the n^{th} element and so for $0< \lambda<1$ we find that $R_{2n}(s)\to 0$ as $n \to\infty$ Hence for $0 < \lambda <1$ and $s \neq$ root of $Q_n^{(e)}$ (s), the R.H.S.of (6.11) $\to$ 0 as $n \to \infty$. At a root of $Q_n^{(e)}$ (s), the R.H.S. of (6.11) becomes

$$(-1)^{n-1}\lambda^{n(3n-1)/2}\left\{\prod_{p=1}^{n-1}(1-\lambda^p)\right\}\alpha^{2n-1}s^{n-1}/Q_{n-1}^{(o)}$$

and since $Q_{n-1}^{(0)}\neq 0$ at a root of $Q_n^{(e)}$, this expression again tends to zero as $n\to\infty$ Hence for all α,ß,s and $o<\lambda<1$, the continued fraction (6.2) converges.

We now consider the case of $\lambda > 1$.

For $\beta \neq 0$ and $\beta\lambda^m \neq \alpha$ for $m = 0,1,2, \ldots$, $R_{2n}(s) \to +\frac{\alpha s}{\beta}\left\{1+0(\frac{1}{\lambda^n})\right\}$ as $\lambda \to\infty$. To this order, this term is produced by the (2n+l)th partial quotient, but for convenience we consider

$$Y - \frac{P_{2n}}{Q_{2n}} = \left[\frac{P_{2n+n}}{Q_{2n+n}} - \frac{P_{2n}}{Q_{2n}}\right]\left\{1+0(\frac{1}{\lambda^{n+1}})\right\} \text{ as } \lambda \to \infty$$

$$= \frac{\lambda^{n(n=1)/2}\left\{\prod_{p=1}^{n}(\lambda^p-1)\right\}\lambda^{n^2}\alpha^{2n}s^n\left\{\prod_{p=0}^{n-1}(s-\beta\lambda^p)\right\}(s-\beta\lambda^{2n+1})\left\{1+0(\frac{1}{\lambda^{n+1}})\right\}}{Q_{2n}Q_{2n+2}}$$

$$= \frac{o(\lambda^{n(2n+1)})\alpha^{2n}s^n(s-\beta\lambda^{2n+1})}{Q_n^{(e)}Q_{2n+2}} \qquad \textit{as } \lambda \to \infty.$$

The roots of the denominato r range from β to $(\beta-\alpha)\lambda^{2n+1}\{1+0(\frac{1}{\lambda})\}$ as $\lambda\to\infty$.

The residue at $s=\beta$ is $0(\frac{1}{\lambda^{n(3n-1)/2}})$ as $\lambda\to\infty$ and the residue at $s = (\beta-\alpha)\lambda^{2n+1}\left\{1+0(\frac{1}{\lambda})\right\}$ is $0(\frac{1}{\lambda^{n(2n+1)}})$ as $\lambda\to\infty$. The residues at the remaining roots of the denominator lie between these extreme values. Consequently for $\beta\neq0$, $\beta\lambda^m\neq\alpha$ for $m = 0,1,2\ldots$, and $\lambda > 1$ we have $Y - P_{2n}/Q_{2n}\to 0$ as $n\to \infty$.. For $\beta \neq 0$, $\beta\lambda^m= \alpha$ for some m=0,1,2,..., $Q_n^{(e)}$ (s) has (n-m) roots at s=0 and $Q_{n+1}^{(e)}$(s) has (n-m + 1) roots at s=0.

From preceding calculations

$$\frac{P_{2n+2}}{Q_{2n+2}} - \frac{P_{2n}}{Q_{2n}} = \frac{0(\lambda^{n(2n+1)})s^n(s-\beta\lambda^{2n+1})}{(s-\beta)(s-\beta\lambda)\dots(s-\beta\lambda^m)Q_n^{(e)}Q_{n+1}^{(e)}} \quad \text{as } \lambda \to \infty$$

$$= \frac{0(\lambda^{n(2n+1)})(s-\beta\lambda^{2n+1})}{s^{n-2m+1}(s-\beta)(s-\beta\lambda)\dots(s-\beta\lambda^n)\{s^{2m}+\dots\}} \quad \text{as } \lambda \to \infty.$$

if $\beta\lambda^m = \alpha$. Hence, provided $2m < n+1$ there is a pole of order $(n-2m+1)$ at $s = 0$. Using (4.3.7.b) which has a general term $\left\{\prod_{p=0}^{m-1}(\beta\lambda^P - \alpha)\right\}0(\lambda^{m(2n-m)})s^{n-m}$ as $\lambda \to \infty$, we find that in the above experssion the contributi on near $s = 0$ is $0\left\{\lambda^{(n+1)(\frac{3n}{2}+1)-m(4n-m+1)}\right\}\Big/s^{n-2m+1}$ as $\lambda \to \infty$. For $m < 0.4\ n$ this term $\to \infty$ as $n \to \infty$. From this we infer that the continued fraction does not converge if for some integer m=0,1,2,.. $\beta\lambda^m = \alpha$ From (4.1.4) and (4.1.5) this was to be expected,

For ß=0 and $\lambda > 1$, the odd convergents of R_{2n} are $0(\lambda^{2n})$ as $\lambda \to \infty$ and the even convergents are $0(\frac{1}{\lambda})$ as $\lambda \to \infty$. With these values

$$\frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} = \frac{0(\lambda^{n(2n+1)})s^{2n}}{Q_{2n}Q_{2n+1}} \quad \text{as } \lambda \to \infty$$

$$= \frac{0(\lambda^{n(2n+1)})}{sQ_n^{(e)}Q_n^{(o)}} \quad \text{as } \lambda \to \infty.$$

Near the root of smallest modulus,namely s =0 $Q_n^{(e)} = o(\lambda^{n2})$ as $\lambda \to \infty$ and $Q_n^{(o)} = 0(\lambda^{n(n+1)})$ as $\lambda \to \infty$ and so the contribution near s=0 is

0(1)/s as $\lambda \to \infty$

On the other hand

$$\frac{P_{2n+2}}{Q_{2n+2}} - \frac{P_{2n}}{Q_{2n}} = \frac{o(\lambda^{n(2n+1)})}{Q_n^{(e)}Q_{n+1}^{(e)}}$$

and at the root of least modulus of the denominator, namely $s=-\alpha\lambda\{1+0\{(\frac{1}{\lambda})\}$ the residue is $0(\frac{1}{\lambda^{n-1}})$ as $\lambda \to \infty$, whilst at the root of greatest modulus, namely $s=-\alpha\lambda^{2n+1}$ $\{1+0(\frac{1}{\lambda})\}$ the residue is $0(\frac{1}{\lambda^{n(2n+1)}}$.

The residues at the intermediary roots lie between these two residues•

Collecting together the various results we have finally, that (6.2) converges for all α, β, s and $\lambda > 0$ except for $\lambda > 1$ with either $\beta\lambda^m = \alpha$ for some $m = 0, 1, 2, \ldots$ or $\beta = 0$. For $\beta = 0$ and $\lambda > 1$, only the even convergents of (6.2) tend to a limit function. This case will be discussed further when dealing with the convergence of (6.1).

An analysis similar to that just carried out, shows that the continued fraction (6.1) converges for $0 < \lambda < 1$, for all α, β and $s \neq 0$. At $s = 0$ the even convergents of (6.1) converge to the expected value of $\frac{1}{\alpha - \beta}$ provided $|\alpha| < |\beta|$. To prove this we use the result that the $2n^{th}$ convergent of the continued fraction

$$\frac{a_1}{s+}\ \frac{a_2}{1+}\ \frac{a_3}{s+}\ \frac{a_4}{1} + \ldots \ldots$$

has at $s = 0$, the value

$$\frac{a_1}{a_2} + \frac{a_1 a_3}{a_2 a_4} + \ldots + \frac{a_1 a_3 \ldots a_{2n-1}}{a_2 a_4 \ldots a_{2n}} \tag{6.12}$$

Applying this result to (4,3.1), we find at $s=0$, the $2n^{th}$ convergent has the value

$$\left[\frac{1}{(\alpha\lambda - \beta)} + \frac{1}{(\alpha\lambda - \beta)}\cdot\frac{(\lambda - 1)\alpha}{(\alpha\lambda^2 - \beta)} + \ldots + \frac{1}{(\alpha\alpha - \beta)}\cdot\frac{(\lambda - 1)\alpha}{(\alpha\lambda^2 - \beta)}\cdots\frac{(\lambda^{n-1} - 1)\alpha}{(\alpha\lambda^n - \beta)}\right]$$

$$= \frac{1}{(\alpha - \beta)}\left[\left\{1 + \frac{(1-\lambda)\alpha}{(\alpha\lambda - \beta)}\right\} + \frac{(\lambda - 1)\alpha}{(\alpha\lambda - \beta)}\left\{1 + \frac{(1-\lambda^2)\alpha}{(\alpha\lambda^2 - \beta)}\right\} + \ldots \right.$$

$$\left. \ldots + \frac{(\lambda - 1)\alpha}{(\alpha\lambda - \beta)}\cdots\frac{(\lambda^{n-1} - 1)}{(\alpha\alpha^{n-1}\beta)}\left\{1 + \frac{(1-\lambda^n)\alpha}{(\alpha\lambda^n - \beta)}\right\}\right]$$

$$= \frac{1}{(\alpha - \beta)}\left[1 - \frac{\left\{\prod_{p=1}^{n}(1-\lambda^p)\right\}}{\left\{\prod_{p=1}^{n}(1-\alpha\lambda^p/\beta)\right\}}\left[\frac{\alpha}{\beta}\right]^n\right]$$

For $0 < \lambda < 1$ and $|\alpha| < |\beta|$, the products involved converge as $n \to \infty$ and the value of the expression tends to $\frac{1}{\alpha - \beta}$.

For $\lambda > 1$ only the even convergeuts of (6.1) tend to a limit function. The roots of the denominator polynomials tabulated in (6.8) show that for $\beta \neq 0$, this limit function is different to that of (6.2). The reason for this appears to lie in the power series (4.2.2). As a function of λ, this series has a natural boundary at $|\lambda| = 1$ [see Hardy, 1949]. Now the continued fraction (6.1) is not only attempting to analytically continue the power series as a function of s, but also as a function of λ. As a function of s, the continued fraction (6.1) extends the range from $|s| > |\beta|$ for the series for $0 < \lambda < 1$ to the range all $s \neq 0$. While, as a function of λ, (6.1) is giving a continuation for $|\lambda| > 1$. Since $|\lambda| = 1$ is a natural boundary we do not expect this continuation to be unique.

The iterated series converges even for $\lambda > 1$ provided $|\alpha| < |\beta|$ and the corresponding continued fraction (6.2) extends this range to $\lambda > 1$, $\beta \neq 0$. Nevertheless it is the natural boundary at $|\lambda| = 1$ that produces the problems with the iterated series, and in particular the $\beta = 0$ case.

In order that the Laplace transforms should possess original it is necessary for the real parts of the singularities of the transforms to be bounded on the right. For $0 < \lambda < 1$ this is always the case. For $\lambda > 1$ in the continued fraction (6.1), this condition requires $\alpha > 0$. For $\lambda > 1$ and $\beta \neq 0$ in the iterated series (4.1.2) this requires $\beta < 0$ and in the continued fraction (6.2) this condition requires $\beta < 0$ and $\beta - \alpha < 0$.

For $\lambda > 1$ we do not expect the solution of $Y(s,\alpha)$ to be unique, since for this range of λ eigensolutions exist. These are discussed in Section 7.

7. Eigensolutions

We now discuss equation (2.4) which is

$$(s-\beta)Y_e(s) + \alpha Y_e(s/\lambda) = 0 \tag{2.4}$$

and note that for the initial condition $y(0) = 0$ to be satisfied it is sufficient for

$$Y_e(s) = 0(1/s^{1+\delta}) \quad \text{as} \quad |s| \to \infty \quad \delta > 0. \tag{2.5}$$

Setting $s = 0$ in (2.4) we have $(\alpha-\beta)Y_e(0) = 0$. Assuming $Y_e(0) \neq 0$

this gives $(\alpha-\beta) - 0$, and with this condition we have by iteration from (2.4) that

$$Y_e(s) = \frac{Y_e(s/\lambda^{n+1})}{(1 - s/\beta)(1 - s/\beta\lambda)\dots(1 - s/\beta\lambda^n)} \tag{7.1}$$

For $\lambda > 1$, $s/\lambda^{n+1} \to 0$ and the product converges as $n \to \infty$. From (7.1) we thus have the solution

$$Y_e(s) = Y_e(0)/\{\prod_{p=0}^{\infty}(1 - s/\beta\lambda^p)\} \qquad (7.2)$$

for $\lambda > 1$.

Approximating to Y_e (S) by more than a single factor of the product ensures that (2.5) is satisfied. The solution given by (7.2) is an eigensolution to the problem for the case $\alpha = \beta$.

Iterating from (2.4) in the opposite direction produces, for $0 < \lambda < 1$, the solution

$$Y_e(s) = Y_e(0)\{\prod_{p=1}^{\infty}(1 - s\lambda^P/\beta)\} \qquad (7.3)$$

Approximating to Y_e (s) by a finite number of factors of (7.3) the condition $y(0) = 0$ in the original is not satisfied and there is no eigensolution to our problem in this case.

The above analysis may be generalised by assuming that near $s=0$

$$Y_e(s) = K(s/\alpha)^V\{1 + 0(s)\} \quad \text{as} \quad |s| \to 0 \qquad (7.4)$$

where v,K are constants. Substituting in (2.4) shows that this form is possible provided

$$\alpha - \beta\lambda^V \qquad (7.5)$$

For given values of α, β, λ this equation determines v; v may take complex values. Proceeding as before and using condition (7.5) we have

$$Y_e(s) = K(s/\alpha)^V/\{\prod_{p=0}^{\infty}(1 - s/\beta\lambda^p)\} \qquad (7.6)$$

which for $\lambda > 1$ is an eigensolution. With n sufficiently large for the condition (2.5) to hold, approximations to Y(S) may be obtained from (7.6). For prescribed α, β, λ and for values v_i satisfying (7.5), the linear form of (2.4) shows that other eigensolutions are given by

$$Y_e(s) = \{\sum_i K_i(s/\alpha)^{V_i}\}/\{\prod_{p=0}^{\infty}(1 - s/\beta\lambda^p)\} \qquad (7.7)$$

These eigensolutions all possess originals if $\beta < 0$.

The case $\beta = 0$ may be regarded as a limiting case of (7.5) with $\lambda > 1$ and $v \to \infty$.

Writing α/λ^v for β in (7.6) and then setting v+n for v, suggests that we consider the function

$$W_n(s) = K_n (s/\alpha)^{v+n} / \{\prod_{p=-n}^{\infty}(1 - s\lambda^{v-p}/\alpha)\} \tag{7.8}$$

Clearly

$$W_n(s/\lambda)/W_n(s) = (1-s\lambda^{V+n}/\alpha)/\lambda^{V+n}$$

$$\rightarrow -s/\alpha \quad \text{as} \quad n \rightarrow \infty \text{ for } \lambda > 1.$$

Expressing (7.8) in partial fractions, the term in $(1-s\lambda^{v+r}/\alpha)^{-1}$ has coefficient

$$K_n \lambda^{-(v+n)(v+r)} / \{\prod_{p=0}^{\infty}(1 - \lambda^{n-r-p})\}$$

$$(-1)^{n-r} K_n \lambda^{-(v+n)(v+r)-(n-r)(n-r+1)/2} \Big/ \{\prod_{p=1}^{n-r}(1-\lambda^{-p})\}\{\prod_{p=1}^{\infty}(1-\lambda^{-p})\}$$

keeping r fixed and letting $n \rightarrow \infty$, the denominator converges for $\lambda > 1$. Thus with a suitable choice of K_n we are led to consider the series

$$W(s) = K \sum_{r=-\infty}^{\infty} \frac{(-1)^r \lambda^{-(r+v-\frac{1}{2})^2/2}}{(1 - s\lambda^{v+r}/\alpha)} \tag{7.9}$$

v,K being arbitrary constants. This is, essentially, the solution due to Bowen and quoted in Fox et al [1971], This series satisfies

$$\frac{sW(s)}{\alpha} + W(s/\lambda) = K \sum_{r=-\infty}^{\infty} (-1)^r \lambda^{-(r+v-\frac{1}{2})^2/2} \tag{7.10}$$

For suitable choices of v ,K, W(s) satisfies

$$sY(s) + \alpha Y(s/\lambda) = 1 \quad ,$$

but cases occur, where for particular values of v, W(s) satisfies

$$sY(S) + \alpha Y(s/\lambda) = 0$$

So that again eigensolutions exist. An obvious value is v = 0 in which case the series in (7.10) vanishes (v = ±1, ± 2,.. are all equivalent to v = 0) but there are also complex values of v which make the series vanish. We observe that the function W(s) of (7.9) has essential singularities at both s = 0 and s = ∞, and does not possess Taylor series expansions about these two points.

8. Approximations to y(t)

Our principle objective was to solve the functional differential equation

$$\frac{dy(t)}{dt} = ay(\lambda t) + \quad by(t) \tag{1.1}$$

for t > 0 and y(0) = 1; a = -λα and b = 3. We now derive approximations to the function y(t) by finding the originals of the convergents of the continued fraction and of the series of section 4.

Inverting (4.1.6) we find

$$y(t) = -1 + \sum_{r=1}^{m-1} \{\prod_{p=1}^{r} (ß - \alpha\lambda^p)\} \frac{t^r}{r!} + \{\prod_{p=1}^{m} (1 - \frac{\alpha\lambda^p}{ß})\} \mathcal{L}^{-1} \frac{\beta^m Y(s, \alpha\lambda^m)}{s^m} \tag{8.1}$$

The results of section 4 enable us to approximate Y(s,αλ) by sequences of rational functions, each of which can be expressed in partial fractions. To complete the inversion of (8.1), we therefore require the functions

$$e_m(\rho, t) \equiv \mathcal{L}^{-1} \frac{\rho^m}{s^m (s - \rho)} = e^{\rho t} - [1 + (\rho t) + \frac{(\rho t)^2}{2!} + \ldots \frac{(\rho t)^{m-1}}{(m-1)!}] \tag{8.2}$$

When $Y(s, \alpha\lambda^m)$ is approximated by n terms of (4.1.4), the original (8.1) for $0 < \lambda < 1$ and $\beta \neq 0$, becomes

$$y_S(t) = 1 + \sum_{r=1}^{m=1} \{\prod_{p=1}^{r} (\beta - \alpha\lambda^P)\} \frac{t^r}{r!} + K(\alpha)[e_m(\beta, t) + \sum_{r=1}^{n-1} \frac{\left[\frac{\alpha\lambda}{\beta}\right]^r}{\prod_{p=1}^{r} (1 - \lambda^P)} e_m(\beta\lambda^r, r) \tag{8.3}$$

where, as in (4.1.5),

$$K(\alpha) = \prod_{D=1}^{\infty} (1 + \frac{\alpha\lambda^P}{\beta})$$

The weights in (4.1.4) in general increase in magnitude for a certain number of terms and then decrease, some weights can be very large. The weights continual decrease if

$$\left| \frac{\alpha\lambda}{\beta} \right| < 1 - \lambda$$

The change $\alpha \rightarrow \alpha.\lambda^m$ improves the convergence of the series and the condition

$$\left|\frac{\alpha}{\beta}\right|\lambda^{m+1} < 1 - \lambda \tag{8.4}$$

can be usefully used to fix the minimum size of m in (8.3). Notice however that the change $\alpha \rightarrow \alpha\lambda^m$ does not directly improve the coefficients in (8.3). m should not be excessively large otherwise loss of accuracy may be the forfeit.

Now the terms of (4.1.4) fit the poles of $Y(s,\alpha)$ in the order β, $\lambda\beta$, $\lambda^2\beta$... In (8.3) the m series terms are fitting y(t) for small t and for $\beta > 0$ the terms derived from (4.1.4) are fitting y(t) for t large, consequently (8.3) provides excellent approximations to y(t) when m is suitably chosen if $\beta > 0$ and λ is not too close to 1. For some numerical results see table B,

We now use in (8.1) n terms of the iterated series to approximate $Y(s,\alpha\lambda^m)$. For $0 < \lambda < 1$ and $\beta \neq 0$, expanding the first n terms of the iterated series (4.1.2) in partial fractions leads to

$$\frac{K_{n-1}}{(s-\beta)} + \frac{K_{n-2}\left(\frac{\alpha\lambda}{\beta}\right)}{(1-\lambda)(s-\beta\lambda)} + \ldots\ldots + \frac{K_0\left(\frac{\alpha\lambda}{\beta}\right)^{n-1}}{\prod_{p=1}^{n-1}(1-\lambda^p)(s-\beta\lambda^{n-1})}, \tag{8.5}$$

where the coefficients K_n are the partial sums of the series (4.1.5) for $K(\alpha)$

$$K_n(\alpha) = 1 + \sum_{r=1}^{n} \frac{\lambda^{r(r+1)/2}(-\alpha/\beta)^r}{\prod_{p=1}^{r}(1-\lambda^p)} \tag{8.6}$$

The expression (8.5) inverts to

$$\sum_{r=0}^{n-1} \frac{K_{n-1-r}(\alpha)\left(\frac{\alpha\lambda}{\beta}\right)^r e^{\beta\lambda^r t}}{\prod_{p=1}^{r}(1-\lambda^p)} \tag{8.7}$$

The weights in (8.5) also normally increase in magnitude to a maximum, which can be large, before decreasing. For sufficiently large n they tend to decrease if $\left|\frac{\alpha\lambda}{\beta}\right| < 1 - \lambda$.

Thus when (8.5) is inserted into (8.1) for $Y(s,\alpha\lambda^m)$, we obtain the following approximation to y(t)

$$y_1(t)=1+\sum_{r=1}^{m-1}\{\prod_{p=1}^{r}(\beta-\alpha\lambda^p)\}\frac{t^r}{r!}+\{\prod_{p=1}^{m}(1-\frac{\alpha\lambda^p}{\beta})\}\sum_{r=0}^{n-1}\frac{K_{n-t-r}(\alpha\lambda^m)\left(\frac{\alpha\lambda}{\beta}\right)^r e_m(\beta\lambda^r,t)}{\prod_{p=1}^{r}(1-\lambda^p)} \quad (8.8)$$

and the condition (8.4) could again be used to choose the value of m.

Since the iterated series (4.1,2) matches terms in the series of $Y(s,\alpha)$ for |s| large, all the terms in (8.8) are contributing to fitting the function y(t) for small t. With m suitably chosen, for both positive and negative β , (8.8) produces good approximations to y(t) for $\lambda < 0.9$ but the range in t diminishes as λ is increased towards 1. Some numerical results are recorded in table C.(8.8) is a more versatile formula than (8.3), in fact the only virtue of (8.3) is that it can deal with large values of t when $\beta > 0$.

For our third method we approximate $Y(s,\alpha\lambda^m)$ in (8.1) by the nth Convergent $\frac{P_n(s)}{Q_n(s)}$ of the continued fraction (4.2.4), where P_n (s) and Q_n (s) denote the numerator and denominator polynomials respectively.

Assuming $\beta \neq \alpha\lambda^m$ for any positive integer m, the n roots of Q_n (s) are simple and for $0 < \lambda < 1$ tend to the values $\beta\lambda^r$ while for $\lambda > 1$ they tend to values $-\alpha\lambda^{2r+1}$, r = 0,1,2...n-1. The roots are not necessarily real, although they are tending to real values. On determining the roots numerically the nth convergent can be expressed in partial fractions

$$\frac{P_n(s)}{Q_n(s)}=\sum\frac{P_n(\rho_r)}{Q'_n(\rho_r)(s-\rho_r)} \quad (8.9)$$

whereρ_r denotes a root of Q_n (s) . Considerable care must be taken in evaluating $P_n(\rho_r)$.$P_n(\rho_r)$ should not be calculated directly or from the recurrence relations that generate P_n (s). When ρ_r is close to a root of P_n (s) many significant figures may be lost. The determinant formula given in section 6 may often be useful used in this situation.
Since $Q_n(\rho_r) = 0$, we have

$$P_n(\rho_r)Q_{n-1}(\rho_r)=\lambda^{n(n-1)}\prod_{p=1}^{n-1}(1-\lambda^p)\prod_{p=1}^{n-1}(\beta-\alpha\lambda^p)\alpha^{n-1} \quad (8.10)$$

and, provided ρ_r is not close to a root of Q_{n-1} ,(s), the P (ρ_r) can be calculated using only the denominator polynomials. If (8.10) is used to find P (ρ_r) the expansion (8.9) inverts to give

$$\lambda^{n(n-1)} \prod_{p=1}^{n-1} (1-\lambda^p) \prod_{p=1}^{n-1} (\beta - \alpha\lambda^p)\alpha^{n-1} \sum_{r=1}^{n} \frac{e^{\rho_r t}}{Q_{n-1}(\rho_r)Q'_n(\rho_r)} \tag{8.11}$$

But in practice to maintain accuracy it is often necessary to use a mixture of methods in evaluating the P_n (ρ_r) .

When, in (8.1),Y(s,$\alpha\lambda^m$) is replaced by the expansion of its nth convergent

$$\frac{P_n(s, \alpha\lambda^m)}{Q_n(s, \alpha\lambda^m)} = \sum_{r=1}^{n} \frac{P_n(\rho_r, \alpha\lambda^m)}{Q_n(\rho_r, \alpha\lambda^m)(s-\rho_r)} \tag{8.12}$$

we obtain the approximation to y(t)

$$y_c(t) = 1 + \sum_{r=1}^{m-1} \{\prod_{p=1}^{r}(\beta - \alpha\lambda^p)\} \frac{t^r}{r!} + \{\prod_{r=1}^{m}(\beta - \alpha\lambda^p)\} \sum_{r=1}^{n} \frac{P_n(\rho_r, \alpha\lambda^m)}{Q'_n(\rho_r, \alpha\lambda^m)\rho_r{}^m} e_m(\rho_r, t) \tag{8.13}$$

where ρ_r now denotes a root of Q_n (s,$\alpha\lambda^m$) and e_m (ρ,t) is defined in (8,2).

These roots ρ_r need not be real. Nevertheless when α and β are of the same sign complex roots can be avoided by choosing m sufficiently large. Using properties of real J fractions Wall [p.1 1 9 1 , from a consideration of the signs of the partial numerators of (4.2.3), with a replaced by $\alpha\lambda^m$, we conclude that for its even convergents (8.12):

For $0 < \lambda < 1$,

a) α and β both < 0 and $|\beta| > |\alpha|\lambda^{m+1}$, the poles are simple, real and negative, and have positive residues.

b) α and β both > 0 and $\beta > \alpha\lambda^{m+1}$, the poles are simple, real and positive and have negative residues.

c) α and β opposite in sign, the partial numerators of (4.2.3) alternate in sign and the poles of (8.12) may be complex. In practice, as n is increased pairs of complex poles coalesce and shed real poles which then tend to values $\beta\lambda^r$.

For $\lambda > 1$,

d) $\alpha > 0$ and either $\beta > 0$ with $\alpha\lambda^{m+1} > \beta$ or $\beta < 0$, the poles are simple, real and negative and have positive residues.

e) $\alpha < 0$ and either $\beta < 0$ with $\alpha \mid \lambda^{m+1} > [\beta \mid$ or $\beta > 0$, the poles are simple, real and positive and have negative residues.

Thus complex roots need only arise when $0 < \lambda < 1$ and α, β are opposite in sign. When the conditions (e) hold the inversion of (8, 12) is not meaningful, so that when $\lambda > 1$, y_c (t) can only be useful for $\alpha > 0$.

The continued fraction is primarily matching terms in the series in $\frac{1}{s}$ of $Y(s,\alpha\lambda^m)$, consequently the functions y_c (t) approximate y(t) well for small t. However for $0 < \lambda < 1$ the continued fraction progressively fits the poles $\beta, \lambda\beta, \lambda^2\beta$ and can produce good approximations to y(t) for large t, certainly if $\beta > 0$. Good approximations are also obtained when $\lambda > 1$ and $\alpha > 0$. In table D we record some numerical results. The advantages of this method is that it extends the solution to values of λ greater than one when $\alpha > 0$, and further gives solutions when λ is near one.

Solutions of the functional differential equation (1.1).

TABLE A using (1.3)

$\lambda\alpha = -a = + 1.0$ $\beta = 0.95$ $\lambda = 0.99$

t	y(t)	
0	l	
1	0.95099	
5	0.77378	
10	0.59046	
25	0.23707	
50	3.06402	E-02
75	6.86576	E-04
100	7.45578	E-06
120	4.25506	E-05
140	-1.77213	E-03
150	1.39535	E-02
160	1.74588	
180	1.22838	E+05
200	7.18998	E+10
220	1 .71180	E+17
240	1.10332	E+24
250	3.73368	E+27
260	1.47881	E+31
280	3.43702	E+38
300	1.21665	E+46

TABLE B using (4,1.4) in (8.1)

α = 1 3 = 1 A = 0.8
M - 15 N = 5

t	y(t)	
0	1.000000	
1	1.242853	
10	1.500779	E+02
50	1.746529	E+19
100	9.053594	E+40

α = -10 β =1 λ = 0.9
M = 45 N = 10

t	y(t)	
0	1.000000	
1	1.741286	E+03
10	3.038451	E+15
50	1.908673	E+38
100	1 .643197	E+60

TABLE C using (4.1.2) in (8.1)

α = -2 β = -1 λ = 0.8
M = 9 N = 15

t	y(t)	
0	1.000000	
1	1.684644	
10	1.588661	E+01
50	2.755279	E+02
100	1.096282	E+03

α = -1 β = 1 λ = 0.95
M = 50 N = 10

t	y(t)	
0	1.000000	
1	6,728839	
10	1.525186	E+07
50	8.350665	E+27
100	1.545544	E+50

TABLE D using (8.12) in (8.1)

α = -1 6 = 1 λ = 0.9
M = 5 N = 15

t	y(t)	
0	1.000000	
1	6.193152	
10	2.633091	E+06
50	8.521912	E+24
100	4.686615	E+46

λα = -a = +1 β = 0.95 λ = 3.01
M = 0 N = 12

t	y(t)	
0	1.000000	
1	0.951466	
10	0.620896	
50	0.131362	
100	0.309330	E-01

λα = -a - +1, β = 0.95, λ = 0.99

M = 6 N = 20

produces y(t) in table A for t < 150.

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