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ON THE WEAK SOLUTION OF MOVING BOUNDARY PROBLEMS

by

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ABSTRACT

The weak formulation of moving boundary problems with possibly vanishing specific heat, that is governed by parabolic and/or elliptic differential equations, is developed. The uniqueness of the resulting weak solution is then proved. This approach is used to obtain numerical solutions to some physical examples, which arise in electrochemical machining processes, and in saturated / unsaturated flow in porous media.

ON THE WEAK SOLUTION OF MOVING BOUNDARY PROBLEMS

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§ 1. Introduction

In this paper the extension of the weak formulation of the classical Stefan problem to more general moving boundary problems with parabolic or elliptic differential equations is discussed. This formulation of the classical Stefan problem with a parabolic differential equation is described by Oleinik (1960). The theory of weak solutions for the classical Stefan problem has proved most helpful, leading as it does to a fixed domain formulation of this moving boundary problem, which is particularly advantageous in the search for numerical solutions in multi-dimensional situations. This approach to the computation of solutions to the Stefan problem was pioneered by Rose (1960). Here we prove the uniqueness of the weak solution for the case in which the governing differential equation may be elliptic in one or more regions, but in which the problem remains time-dependent. In the paper by Oleinik (1960), the proof of the uniqueness of the weak solution is restricted to the case where the differential equation is parabolic everywhere in the domain. In the last section some simple examples of considerable physical interest, which lie outside the scope of the earlier results, are solved numerically.

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Before proceeding further, we outline the weak formulation of the classical Stefan problem of melting ice in order to establish some terminology. In each phase $i = 1, 2$ (where 1 denotes ice) the governing differential equation is

$$c_i \frac{\partial u_i}{\partial t} = \nabla (k_i \nabla u_i)$$

where $c > 0$ is the specific heat, u the temperature, $k > 0$ the thermal conductivity, and the density, assumed constant, is taken as unity. At the moving boundary, $s(\underline{x},t) = 0$, between the ice and water the boundary conditions are that the temperature is the melting temperature, u_M,

$$u = u_M \quad \text{on} \quad s(\underline{x}, t) = 0$$

and that heat is conserved,

$$L \frac{\partial s}{\partial t} = (k \nabla u . \nabla s)_2 - (k \nabla u . \nabla s)_1 \quad \text{on} \quad s(\underline{x}, t) = 0,$$

where $L > 0$ denotes the latent heat, the density being again taken as unity, and $(k\nabla u . \nabla s)_i$. denotes the limit of $k\nabla u . \nabla s$ as the boundary is approached from phase i. Appropriate initial and boundary conditions are also given.

By introducing the variables

$$H = \int_{u_M}^{u} c(v)dv \quad (+ L \text{ if } u > u_M)$$

usually called the enthalpy, and

$$\phi = \int_{u_M}^{u} k(v)dv$$

these equations may be reformulated to give

$$\frac{\partial H}{\partial t} = \nabla^2 \phi$$

where this equation holds in a weak sense. In the classical Stefan problem where $c > 0$, $k > 0$, the enthalpy H is a strictly monotone function of ϕ, while ϕ is a single-valued function of H. However in a more general moving boundary problem any of c_1, c, L may be zero, in which case the temperature function ϕ is no longer a single-valued function of the enthalpy H.

As mentioned above, in the classical Stefan problem the specific heat c is required to be strictly positive. Two physical problems in which this is not so are described and solved in section 5. The first of these is a model of the electro-chemical machining process, in which the metal to be shaped is used as the anode in an electrolytic cell (see for example McGeough & Rasmussen (1974), Fitz-Gerald and McGeough (1968)). Here the 'specific heat' is zero in both phases resulting in elliptic differential equations, but with a Stefan type boundary condition ($L > 0$). This model also describes the flow of an incompressible fluid in a Hele-Shaw cell (Richardson (1972)). The second problem described is one where the specific heat vanishes in one phase only, yielding an elliptic equation in this phase, with a parabolic equation in the other. This situation, considered by Hornung (1978), arises in the study of saturated/unsaturated flow in porous media, and has aroused interest because of its relevance to irrigation problems.

Moving boundary problems in general, and the classical Stefan problem in particular have attracted much attention in recent years. Kamenomostskaja (1961) showed that a numerical scheme based on the weak formulation converged to the unique weak solution. Another approach which has proved successful is the reformulation of Stefan problems as variational inequalities (Duvaut (1975), Elliott (1977)), and monotonicity methods (Brezis (1971)) have also proved powerful in showing the existence and uniqueness of solutions to such problems. However, most effort has been devoted to cases where the so-called temperature function ϕ is a single valued function of the enthalpy. For the reasons explained above, we are, in this paper, particularly interested in situations where, because the specific heat vanishes over some temperature range, this is not true.

§2. The weak formulation

We consider a parabolic/elliptic differential equation, which is to be satisfied in some region Ω x[0,T], of the form

$$c_i(u_i)\frac{\partial u_i}{\partial t} = \nabla(k_i(u_i)\nabla u_i) + q_i \qquad (2.1)$$

where $k_j(u_j) > 0 \ \forall \ u_i$, $c_i \ (u_i) \geq 0 \ \forall \ u_i$, q_i is a bounded source or sink term, and the subscript i denotes the i th phase. We shall consider a two-phase problem, with the first phase being defined, after suitable choice of origin, by $u < 0$, and the second by $u > 0$. Across the moving boundary $u = 0$, or $s(\underline{x},t) = 0$, the boundary condition

$$L\frac{\partial s}{\partial t} = (k_2\nabla u_2 - k_1\nabla u_1).\nabla s \qquad (2.2)$$

is to be satisfied, where $L \geq 0$ is called the latent heat by analogy with the classical Stefan problem outlined inn the previous section.

Proceeding as in the case of the classical Stefan problem, and following the techniques of Oleinik we introduce the functions

$$\varphi = \int_0^u k(v)\, dv, \tag{2.3a}$$

$$H = \begin{cases} \int_0^u c(v)\, dv \le 0, & u < 0 \\ \in [0, L], & u = 0 \\ \int_0^u c(v)dv + L \ge L\ ; & u > 0 \end{cases} \tag{2.3b}$$

and

$$Q = Q(H) \begin{cases} q(u) & u < 0 \\ \in [q_1(0),\ q_2(0)] & u = 0 \\ q(u) & u > 0 \end{cases} \tag{2.3c}$$

where $k(v) = k_j(v)$, if v is in phase i, and c(v), q(v) are defined similarly. For the uniqueness proof which follows we require that Q be a smooth function of the enthalpy H.

The differential equation (2.1) then reduces to

$$\frac{\partial H}{\partial t} = \nabla^2 \phi + Q \tag{2.4}$$

which holds in a weak sense throughout the fixed domain Ω x [0,T]. We shall refer to H, as generalised enthalpy and temperature functions respectively. ϕ is a strictly monotone function of u, and thus may be inverted to yield u. H is a monotone function of u and hence of ϕ, but since we permit one or other of c_1, c_2 to vanish for a range of values of u neither H nor ϕ is necessarily sufficient to determine the other. This contrasts with the classical Stefan problem where the temperature is a well-defined function of the enthalpy .

Let the initial condition be given as

$$\phi\ (\underline{x},0) = \phi_0\ (\underline{x}), \tag{2.5a}$$

and consider the case of Dirichlet boundary conditions

$$\phi\ |_{\partial\Omega} = f\ (\underline{x}, t) \tag{2.6a}$$

Since $H(\phi)$ is not single-valued at $\phi = 0$, we consider that

$$H\ (\underline{x},0) = \phi_1\ (\underline{x}) \tag{2.5b}$$

$$H\,|_{\partial\Omega} = f_1(\underline{x}, t) \tag{2.6b}$$

are also given, compatible with (2.5a), (2.6a).

A weak or generalized solution of (2.3) - (2.6) is defined as a pair of bounded measurable functions $\{H,\Phi\}$, such that (2.3) holds for all $\underline{x}$, t ,

$$\int_0^T \int_\Omega H\frac{\partial\Psi}{\partial t} + \varphi\nabla^2\Psi + \varphi Q\ d\Omega dt$$

$$= -\int_{\Omega(0)} \varphi_1(\underline{x})\Psi\Psi(x,0d\Omega + \int_0^T \int_{\partial\Omega} f\frac{\partial\Psi}{\partial n}dSdt \tag{2.7}$$

for all suitable test functions $\{\psi\}$. The set $\{\psi\}$ consists of bounded continuous functions for which $\frac{\partial\Psi}{\partial t}, \nabla\Psi, \nabla^2\Psi$ exist and are continuous, and which satisfy the boundary conditions

$$\Psi(\underline{x}, T) = 0$$

$$\Psi\ |\ \partial\Omega = 0\ \forall\ \ t \in [0,\ T]. \tag{2.8}$$

In the following sections it is shown that the weak solution here defined is unique, and extensions of the proof for other boundary conditions are indicated. Then in the final section two physical examples are described, recast into the form (2.4), and hence solved numerically.

§3. The uniqueness of the weak solution

In this section we demonstrate that the weak solution $\{H, \phi,\}$ defined in section 2 is unique. Suppose that there exist two weak solutions $\{H_1, \phi_1\}$, $\{H_2, \phi_2\}$. Then since both solutions satisfy (2.7) by definition, we have by subtraction

$$\int_0^T \int_\Omega (H_1 - H_2)\frac{\partial \Psi}{\partial t} + (\phi_1 - \phi_2)\nabla^2\Psi + (Q_1 - Q_2)\psi \, d\Omega dt = 0 \qquad (3.1)$$

The structure of the uniqueness proof to be described here is similar in style to that given by Oleinik. However the analysis is complicated by the fact that neither H nor ϕ is a strictly monotone function of the other. Thus the ratio $(\phi_1 - \phi_2)/(H_1 - H_2)$ is unbounded and 0leinik's original method breaks down. We therefore introduce a function, strictly monotone in both H and (ϕ, defined by

$$F(\phi, H) = H + \gamma\phi \qquad (3.2)$$

where γ is any positive constant. A second relaxation of the conditions imposed by Oleinik is that we require the source term Q to be a smooth function of the enthalpy, rather than of the temperature. When $L > 0$ this permits the source term to be discontinuous across the phase boundary.

(3.1) may be rewritten as

$$\int_0^T \int_\Omega (F_1 - F_2)\left\{A\frac{\partial \Psi}{\partial t} + B\nabla^2\Psi + AC\,\Psi\right\} d\Omega\, d\, t = 0 \qquad (3.3)$$

Where

$$\left.\begin{aligned} A(x,\ t) &= \frac{H_1 - H_2}{F_1 - F_2} \\ B(x,\ t) &= \frac{\phi_1 - \phi_2}{F_1 - F_2} \\ c(x,\ t) &= \frac{Q_1 - Q_2}{H_1 - H_2} \end{aligned}\right\} \tag{3.4}$$

and A, B, C are defined to be zero when $\phi_1 = \phi_2$ and $F_1 = F_2$ From the definition of F it follows that

$$0 \leq A \leq 1,$$

and

$$0 \leq B \leq 1/\gamma, \tag{3.5}$$

while since Q is required to be a smooth function of H, we have

$$|C\ (x,t)| < M \tag{3.6}$$

for some constant M. Thus the coefficients of $\frac{\partial \Psi}{\partial t}, \nabla^2 \Psi$, ψ in (3.3) are all bounded and A and B are non-negative.

We note that the introduction of the function F, with $\gamma > O$, is necessitated by the possible unboundedness of the ratio B/A, due to the vanishing of the specific heat. This prevents our following exactly Oleinik's proof, which corresponds to taking γ zero in (3.2) and for which the restriction c > 0 implies B/A is bounded.

The object is now to show that

$$\int_0^T \int_0^\Omega (F_1 - F_2)\ A\ G\ d\Omega\, dt = 0$$

for an arbitrary smooth bounded function G(x,t), using procedures analogous to those of Oleinik. Let $\{A_n\}$, $\{B_m\}$, $\{C_l\}$ be sequences of smooth bounded functions such that

$$\begin{aligned} &\|A - A_n\| < \alpha/n, && 1/n < A_n \le 1 \\ &\|B - B_m\| < \beta/m, && 1/m < B_m \le 1/\gamma \\ &\|C - C_1\| < 1/\ell && |C_1| < M \end{aligned} \tag{3.7}$$

where α and β are constants.

Now consider the equation

$$\frac{\partial \xi_{mnl}}{\partial t} + \frac{B_m}{A_n} \nabla^2 \xi_{mnl} + C_1 \xi_{mnl} = G(x, t) \tag{3.8a}$$

with the boundary consitions

$$\xi_{mnl} = 0 \quad \text{on} \quad t = T, \quad \xi mn1 \,|_{\partial\Omega} = 0, \; t \in [0, T] \tag{3.8b}$$

for an arbitrary smooth bounded function G such that

$$G\,|_{\partial\Omega} = 0.$$

With the change of variable $t = T - \tau$ this becomes

$$\begin{aligned} &\frac{-\partial \xi_{mnl}}{\partial \tau} + \frac{B_m}{A_n} \nabla^2 \xi_{mnl} + C_1 \xi_{mnl} = G(x, \tau) \\ &\xi_{mnl} = 0 \text{ on } \tau = 0, \; \xi_{mnl} \,|_{\partial\Omega} = 0 \quad \tau \in [0, T] \end{aligned} \tag{3.9}$$

The coefficient $\dfrac{B_m}{A_n}$ is continuous, positive and bounded below away from zero. The differential operator in (3.9) is therefore uniformly parabolic, and by the maximum principle (see for example Friedman (1964)) ξ_{mnl} is bounded independent of m, n, 1. Thus, being continuous with continuous derivatives $\nabla \xi_{mnl}$, $\nabla^2 \xi_{mnl}$ $\partial\xi_{mnl}/\partial t$, ξ_{mnl} belongs to the set of test functions $\{\psi\}$.

Writing (3.3) with ξ_{mnl} in place of ψ and substituting for $\frac{\partial \xi_{mnl}}{\partial t}$

from (3,8) yields

$$\int_0^T \int_\Omega (F_1 - F_2)\, A\, G\; d\Omega\Omega d = \int_0^T \int_\Omega (F_1 - F_2)\left[\frac{B_m}{A_n}(A - A_n) + (B_m - B)\right]\nabla^2 \xi_{mnl}\, d\Omega dt$$

$$+\int_0^T \int_\Omega (F_1 - F_2)(AC - A_n C_1)\; \xi_{mnl}\, d\Omega dt \qquad (3.10)$$

The remainder of the proof is then to demonstrate the right-hand side of (3,10) is arbitrarily small as m, n, 1 $\rightarrow$ ∞. This will be shown in the next section. Taking this as given, we thus obtain

$$\int_0^T \int_\Omega (F_1 - F_2) A\; G\; d\Omega dt = 0 \qquad (3.11)$$

for an arbitrary smooth bounded function F(x,t) such that $G\,|\partial\Omega = 0$.

Therefore we have

$$A(F_1 - F_2) = 0 \quad \text{a.e. in } \Omega \times [0,T]$$

which, substituting from (3.4) yields

$$H_1 = H_2 \quad \text{almost everywhere}$$

Returning now to (3,1), we have $Q_1 = Q_2$ a.e. since Q is required to be a smooth continuous function of the generalised enthalpy H, and thus (3.1) reduces to

$$\int_0^T \int_\Omega (\phi_1 - \phi_2)\nabla^2 \Psi d\Omega dt = 0$$

where $\nabla^2\psi$ is an arbitrary smooth bounded function. Therefore $\phi_1 = \phi_2$ almost everywhere in Ω x [0,T],and so the weak solution $\{H, \phi\}$ is indeed unique.

The proof given here is readily extended to cover more general boundary conditions on the fixed surface. If (2.6) is replaced by

$$\frac{\partial \varphi}{\partial \nu} + w\phi = 0, \ \ w \geq 0, \text{ on } \partial\Omega, \ t \in [0, T]$$

where v denotes the outward normal, then (2.8) is changed correspondingly, so that (3.8) becomes

$$\frac{\partial \xi_{mnl}}{\partial \nu} + w\xi_{mn\,l} = 0, \ \ w \geq 0, \ \text{ on } \ \partial\Omega, \ t \in [0, T].$$

In this case we appeal to comparison theorems (Friedman (1964)) to show that ξ_{mnl} is bounded independent of 1, m, n. The rest of the proof then proceeds as outlined above, with minor modifications to some estimates in section 4.

§4. Some bounds

We seek now to estimate the terms on the right-hand side of (3.10) in order to justify (3.11). (3.10) is

$$\int_0^T \int_0^T (F_1 - F_2) \ AG \ \ d\Omega \, dt = I_1 + I_2 + I_3 \tag{4.1}$$

where

$$\begin{aligned} I_1 &= \int_0^T \int_\Omega (F_1 - F_2)(A - A_n)\frac{B_m}{A_n}\nabla^2\xi_{mn\,1}\, d\Omega\, dt, \\ I_2 &= \int_0^T \int_\Omega (F_1 - F_2)(B - B_m)\nabla^2\xi_{mnl}\ \ d\Omega\, dt, \\ I_3 &= \int_0^T \int_\Omega (F_1 - F_2)(AC - A_n C_1)\xi_{mnl}\ \ d\Omega\, dt. \end{aligned} \tag{4.2}$$

It is first necessary to obtain some bound for $\int_\Omega \nabla^2 \xi_{m\,n\,l}$

Multiplying (3.8) by $e^{\theta t}\nabla^2\xi_{mnl}$, where θ is arbitrary ,and integrating over Ω x[0,T] yields

$$\int_0^T \int_\Omega \frac{\partial \xi mnl}{\partial t}\nabla^2\xi_{mnl}e^{\theta t} + \frac{B_m}{A_n}e^{\theta t}(\nabla^2\xi_{mnl})^2 + C_1\xi_{mnl}e^{\theta t}\nabla^2\xi_{mnl}d\Omega dt$$

$$= \int_0^T \int_\Omega G(x,t)\nabla^2\xi_{mnl}e^{\theta t}d\Omega dt \tag{4.3}$$

On integrating all terms except the second on the left hand side by parts, using the conditions $\xi_{mnl}|_{\partial\Omega} = 0$, $G|_{\partial\Omega} = 0$, $\xi_{mnl} = 0$ on $t=T$ and rearranging (4.3) becomes

$$\int_0^T \int_\Omega \left\{ \frac{B_m}{A_n}e^{\theta t}(\nabla^2\xi_{mnl})^2 + (\frac{1}{2}\theta - C_1)e^{\theta t}(\nabla\xi_{mnl})^2 \right\} d\Omega dt$$

$$= \int_0^T \int_\Omega e^{\theta t}\xi_{mnl}\nabla^2 G + e^{\theta t}\xi_{mnl}\nabla\xi_{mnl}\nabla c \quad d\Omega dt$$

and hence

$$\int_0^T \int_\Omega \left\{ \frac{B_m}{A_n}e^{\theta t}(\nabla^2\xi_{mnl})^2 + (\frac{1}{2}\theta - C \quad - \frac{\mu}{2})e^{\theta t}(\nabla\xi_{mnl})^2 \right\} d\Omega dt$$

$$= \int_0^T \int_\Omega e^{\theta t}\xi_{mnl}\nabla^2 G + \frac{e^{\theta t}}{2\mu}\xi_{mnl}\nabla\xi_{mnl}{}^2(\nabla c_1)^2 \quad d\Omega dt$$

for any μ, $\theta > 0$. Thus, choosing $\theta > 2M + \mu$ (M as in (3.6)) we have

$$\int_0^T \int_\Omega \frac{B_m}{A_n}(\nabla^2\xi_{mnl})^2 d\Omega dt, \; \int_0^T \int_\Omega (\nabla\xi_{mnl})^2 d\Omega dt$$

bounded above independent of m, n, by K_1^2, K_2^2 say.

Returning to (4.2) and using the Cauchy-Schwarz inequality

$$I_1 \le k_1 \left[\int_0^T \int_\Omega (F_1 - F_2)^2 \frac{B_m}{A_n} (A - A_n)^2 \, d\Omega dt \right]^{\frac{1}{2}}$$

and, if $K_3 = \mathrm{Max}_{\Omega x}[0, T](F_1 - F_2)$,

$$\le \frac{2K_1 K_3}{\gamma^{\frac{1}{2}}} \cdot \left[\int_0^T \int_\Omega \frac{(A - A_n)^2}{A_n} \, d\Omega \, dt \right]^{\frac{1}{2}}$$

Let E denote the set in $\Omega \times [0,T]$ on which $|A_n| < \sigma_1$. Then

$$\int_{(\Omega - E) \times [0, T]} \frac{(A - An)^2}{A_n} \, d\Omega\Omega d \le \frac{1}{\sigma_1} \| A - A_n \|^2 \le \frac{\alpha^2}{\sigma_1 n^2} \text{ by (3.7)}$$

$$\text{and} \int_E \frac{(A - A_n)^2}{A_n} \, d\Omega \, dt \le \sigma_1 \int_E \frac{(A - A_n)^2}{A_n^2} \, d\Omega \, dt \le \alpha^2 \sigma.$$

Thus

$$I_1 \le \frac{2k_1 k_3 \alpha}{\gamma^{\frac{1}{2}}} \left(\sigma_1 + \frac{1}{\sigma_1 n^2} \right)^{\frac{1}{2}}. \tag{4.4}$$

The estimation of the second integral is similar, taking

$$I_2 \le \left[\iint (F_1 - F_2)^2 \frac{(B - B_m)^2}{B_m} \, d\Omega dt \right]^{\frac{1}{2}} \left[\iint B_M (\nabla^2 \xi_{mnl})^2 \, d\Omega dt \right]^{\frac{1}{2}}$$

and using

$$\frac{B_m}{A_n} (\nabla^2 \xi_{mnl})^2 > B_m (\nabla^2 \xi_{mnl})^2 \qquad \text{(from (3.7))}$$

to obtain

$$I_2 \le 2K_1 K_3 \beta \left(\sigma_2 + \frac{1}{\sigma_2 m^2} \right)^{\frac{1}{2}}. \tag{4.5}$$

Lastly

$$I_3 = \int_0^T \int_\Omega (F_1 - F_2)\,[A(c - c) + c\ \ (A - A_n)]\,\xi_{mnl} d\Omega dt$$

$$\leq k_3 \quad \left\{\int_0^T \int_\Omega (c - c)^2 d\Omega dt\right\}^{\frac{1}{2}} + K_3 M \left\{\int_0^T \int_\Omega (A - A_n)^2 d\Omega dt\right\}^2 \quad (4.6)$$

$$\leq K_3 \ \left(\frac{1}{1} + \frac{\alpha M}{n}\right).$$

Thus we have

$I_1 < \frac{\in}{3}, I_2 < \frac{\in}{3}, I_3 < \frac{\in}{3}$, for sufficiently large l,m,n, and hence

in the limits as l, m, n,, $\to \infty$ we obtain, as stated in (3.11),

$$\int_0^T \int_\Omega (F_1 - F_2) AG \quad d\Omega\, d\, t \ = 0.$$

§5. Some numerical examples

Here we describe the numerical solution of some physical problems of practical importance. The weak formulation of these problems is developed, and equations of the form (2.4) on a fixed domain, with appropriate boundary conditions, are then solved using a finite difference scheme. The computational advantage of this approach is that the moving boundary condition is automatically satisfied in the weak formulation, and so need not be explicitly applied. Thus the moving boundary need not be tracked, but simply appears as the surface separating the region where $\phi > 0$ from that where $\phi < 0$.

In the proof of the uniqueness of the weak solution, we have specifically considered the situation when the generalised temperature ϕ is not a single-valued function of the generalised enthalpy H.

Therefore the finite difference scheme used must be of the form

$$H(\underline{x},\ t + \delta t) - H(\underline{x},\ t)$$
$$= \delta t \{\theta \nabla^2 \phi\ (\underline{x}, t + \delta t) + (1-\theta)\ \nabla^2 \phi\ (\underline{x}, t)\} \qquad (5.1)$$

for $0 < \theta \leq 1$, with $\theta = 0$ which gives an explicit scheme being excluded. An implicit scheme of the type (5,1) calculates a linear combination of H ϕ and of the form of $F(\phi)$ in (3.2) at each step, and since this function is strictly monotone in both H and ϕ, the values of these functions are uniquely determined.

From numerical experiments it is known that the finite difference scheme is convergent, although this has not yet been shown analytically. The vanishing of the specific heat, which excludes the use of explicit finite difference schemes, also causes the breakdown of the convergence proof for implicit schemes described by Elliott (1976).

1. Electro-chemical machining

In the electro-chemical machining process the metal piece to be shaped is used as the anode in an electrolytic cell. The passage of current causes metal to be dissolved from the surface of the anode, which is therefore a moving boundary. This is basically a one-phase problem, since the differential equation is trivially satisfied in the other phase. The configuration is shown in Figure1. In the quasi-steady model derived by McGeough and Rasmussen (1974) the differential equation in the electrolyte is

$$\nabla^2 \phi = 0, \qquad (5.1)$$

where ϕ is the electric potential ,with the boundary conditions at the anode

$$\phi = 0$$

$$M \frac{\partial s}{\partial t} = -\nabla \phi . \nabla s \text{ on } s(\underline{r}, t) = 0, M > 0, \tag{5.2}$$

and

$$\phi = - \mathbf{v} < 0$$

on the cathode $r = a$.

Thus we have a one-phase problem with an elliptic governing equation and a Stefan boundary condition with non-zero latent heat. In order to cast this problem into the form (2.4) we introduce an 'enthalpy' function, simply a step function of ϕ, given by

$$H = \begin{cases} 0 & \phi < 0 \\ \in [0, \quad M] & \phi = 0 \\ M & \phi > 0. \end{cases} \tag{5.3}$$

To formulate the equations on a fixed domain, we regard the anode as being a region where $\phi = 0$, $H \equiv M$. The problem thus becomes

$$\frac{\partial H}{\partial t} = \nabla^2 \phi \tag{5.4}$$

with the boundary conditions $\phi = -v$, $H = 0$ on $r = a$, and initially

$$\phi = -v, \; H = 0 \;, \quad t = 0, \quad \text{outside } s(\underline{r},t) = 0$$

$$\phi = 0, \; H = M \;, \quad t = 0 \quad \text{within } s(\underline{r},t) = 0$$

The conservation form of (5.4) across a surface of discontinuity $s(\underline{r}, t) - 0$ is

$$[H] \frac{\partial s}{\partial t} = M \frac{\partial s}{\partial t} = [\nabla \phi . \nabla s]$$

where [ ] denotes the jump between the side where $H \leq 0$ and that where $H \geq M$. Therefore since $\nabla\phi \equiv 0$ inside the anode, (5.2) is satisfied by a solution of (5.4).

Two calculations were carried out for the electrochemical machining problem. In the first the case where the initial anode surface is given by $r = 9.25 + 0.25 \sin 29$ and the cathode by r»10 is solved numerically, and the results compared with those obtained, both numerically and using perturbation methods by Christiansen and Rasmussen (1976). Since the amplitude of the perturbations is small, this example is not particularly suitable for numerical solution. However the average radius of the anode is accurately predicted, while the amplitude of the oscillations is in agreement until it is of the same order as the mesh spacing. A comparison of the results calculated using the weak formulation with those of Christiansen and Rasmussen is given in Table 1.

The second solution calculated for this problem is the rounding of an anode, which is initially a square of half-diagonal 9.25, when it is placed within a circular cathode of radius $r = 10$. The position of the anode surface at later times is shown in Figure 2. In both examples the cathode potential is taken as $\phi = 1$.

2. <u>Saturated/unsaturated flow in porous medium</u>

In the flow of incompressible fluid in a porous medium, governed by Darcy's law two distinct regimes occur. In the first, the flow is unsaturated, that is the medium is only partially filled with fluid, and the fluid content may change. In the second regime the medium is saturated, and no further fluid may be added.

The differential equation governing the flow is

$$\frac{\partial H}{\partial t} = -\nabla \, (k \, \nabla \, p)$$

where H is a measure of the air content of the medium, K is the hydraulic conductivity and p is the pressure head. Introducing the velocity potential ϕ , we obtain

$$\frac{\partial H}{\partial t} = \nabla^2 \phi. \tag{5.6}$$

We consider the case, chosen by Horung (1977), where with a suitable choice of origin for ϕ the relationship between H and ϕ is given by

$$H = \begin{cases} \frac{1}{2}\phi^2 & \phi > 0 \\ 0 & \phi < 0. \end{cases} \tag{5.7}$$

Since H is not discontinuous at $\phi = 0$ the condition at the moving boundary between the two flow regimes requires the continuity of $\nabla\phi$.

A sample problem of the flow resulting when an initially unsaturated medium, with $H = 0.5$, $\phi = 1$, is filled by a constant flux of fluid $\nabla\phi = \pm 1$ through the walls $x = + 1$ was solved using the weak formulation (5.6), (5.7). It should be noted that the solution of this problem ceases to exist when the medium becomes saturated everywhere, since the problem then consists of an elliptic equation in a fixed domain, with Neumann boundary conditions Figure 3 shows the profiles of ϕ and H at various times, from which the time when the medium becomes totally saturated is seen to be $t = 0.5$.

It may be seen that this prediction is correct from the following conservation argument: integrating (5.6) over (-1,1) x (0,t) and using the initial condition yields

$$\int_{-1}^{1} H(x, t)\, dx - \int_0^T \phi_x(1, t) - \phi_x(-1, t)\, dt = 1.$$

When the medium becomes totally saturated, at $t = T$, say, $H(x,T) = 0$, and using the boundary conditions on $x = \pm 1$, we obtain $2T = 1$,
The non-existence of a solution to (5.6)(5.7) for $t > 0.5$ is indicated numerically by the failure of the iterative scheme for solving the finite difference equations to converge at the next time step.

From these examples, it may be seen that the weak formulation, which we have shown to possess a unique solution, provides a useful fixed domain method for the numerical solution of moving boundary problems, and furthermore the results are in agreement with those obtained by other methods.

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TABLE 1

Comparison of results of weak solution approach with those obtained by Christiansen & Rasmussen

t	Average radius			Amplitude of oscillation		
	wk. soln.	C & R Num.	C & R Pertn.	wk. soln.	C & R Num.	C & R Pertn.
0	9.25	9.25	9.25	0.25	0.25	0.25
0.5	8.719	8.71	8.727	0.156	0.15	0.151
1	8.320	8.34	8.356	0.117	0.11	0.118
1.5	8.031	8.03	8.048	0.094	0.10	0.100
2	7.781	7.76	7.778	0.094	0.09	0.089
2.5	7.531	7.52	7.533	0.094	0.08	0.080

CAPTIONS

Figure 1. Sketch showing the configuration of the electrolytic cell in the electrochemical machining problem.

Figure 2. The position of the moving boundary at various times in the electrochemical machining of an initially square anode.

Figure 3. The profiles of Φ and H at various times for the saturated/unsaturated flow problem.

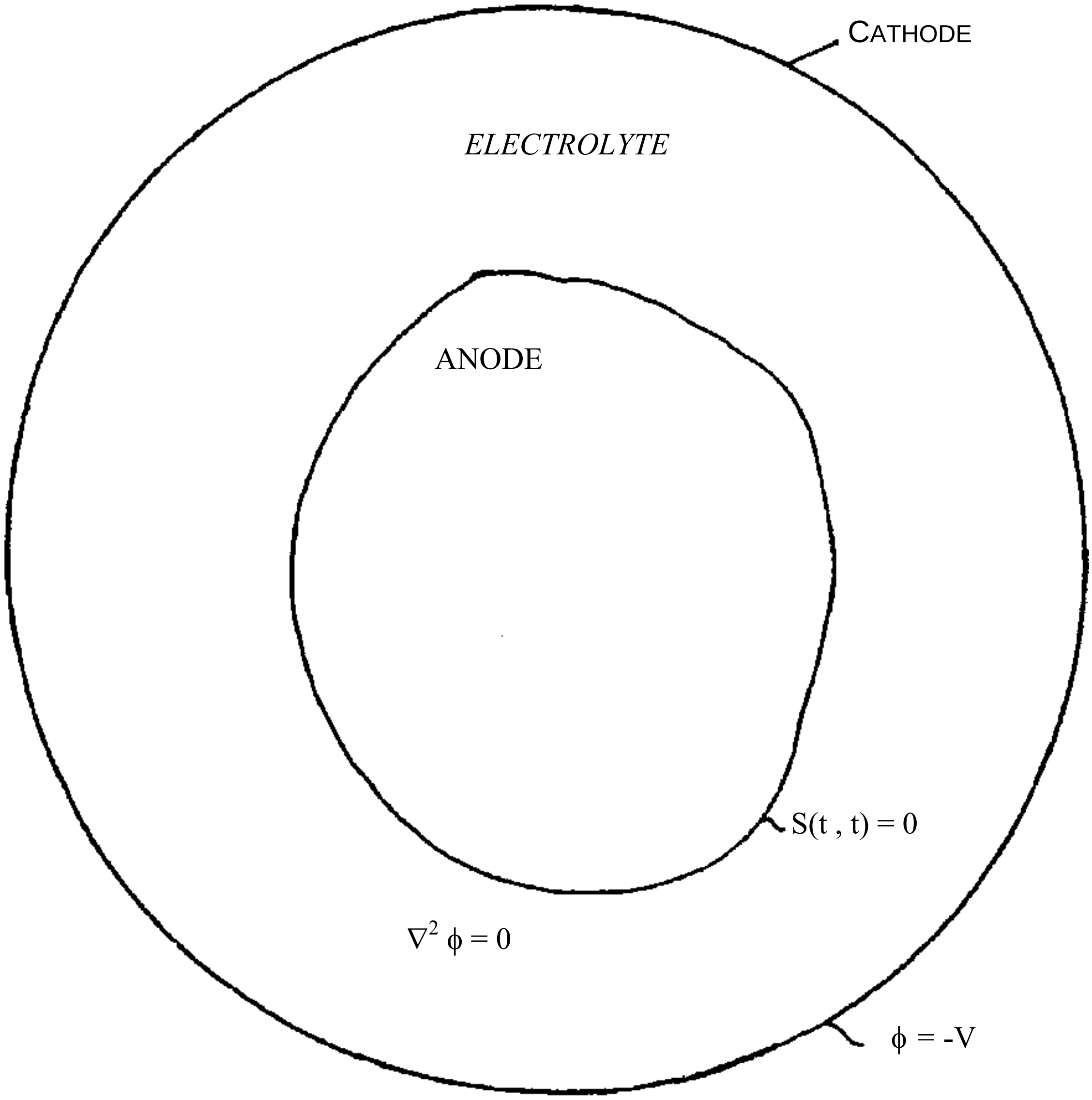

Fig. 1.

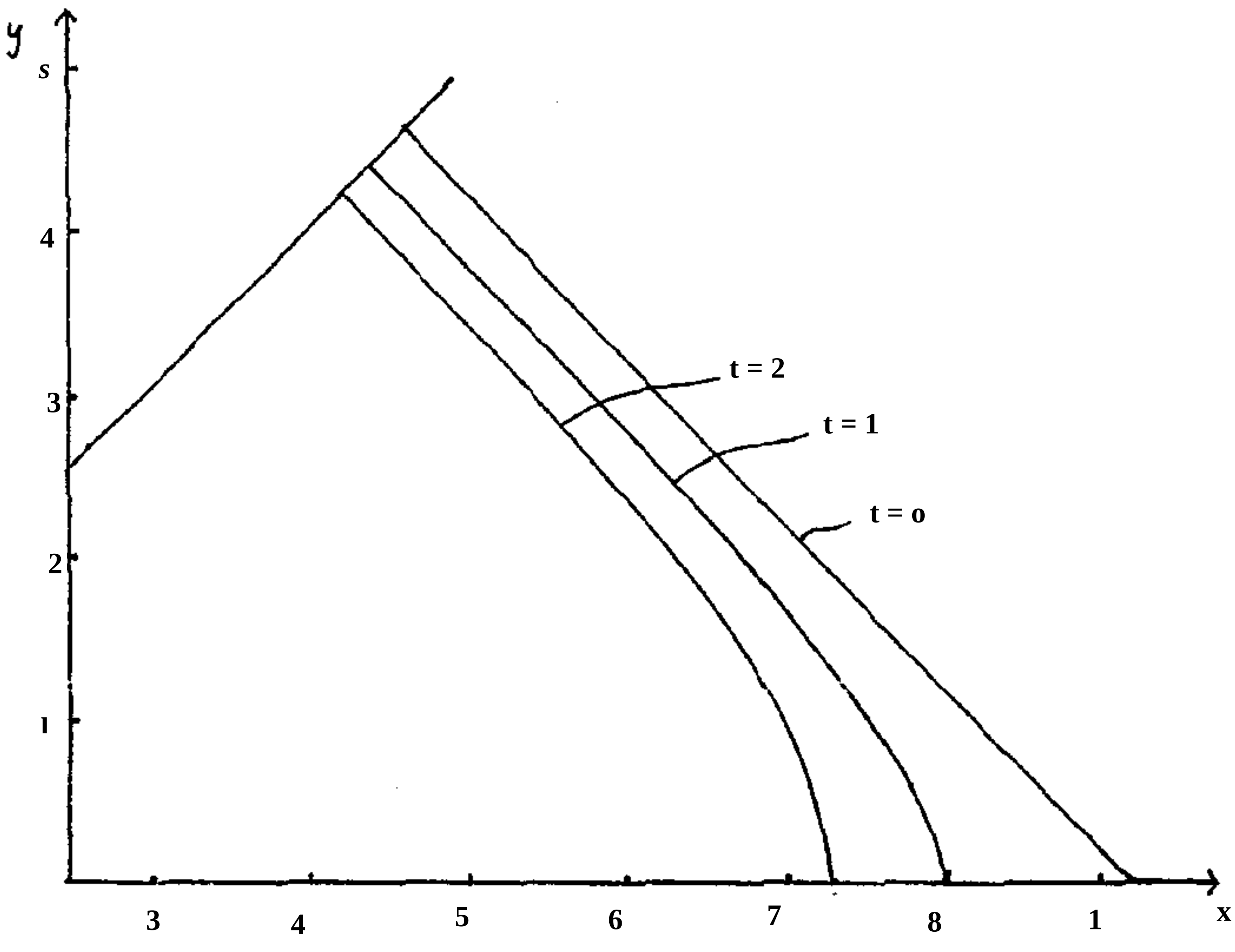

Fig. 2.

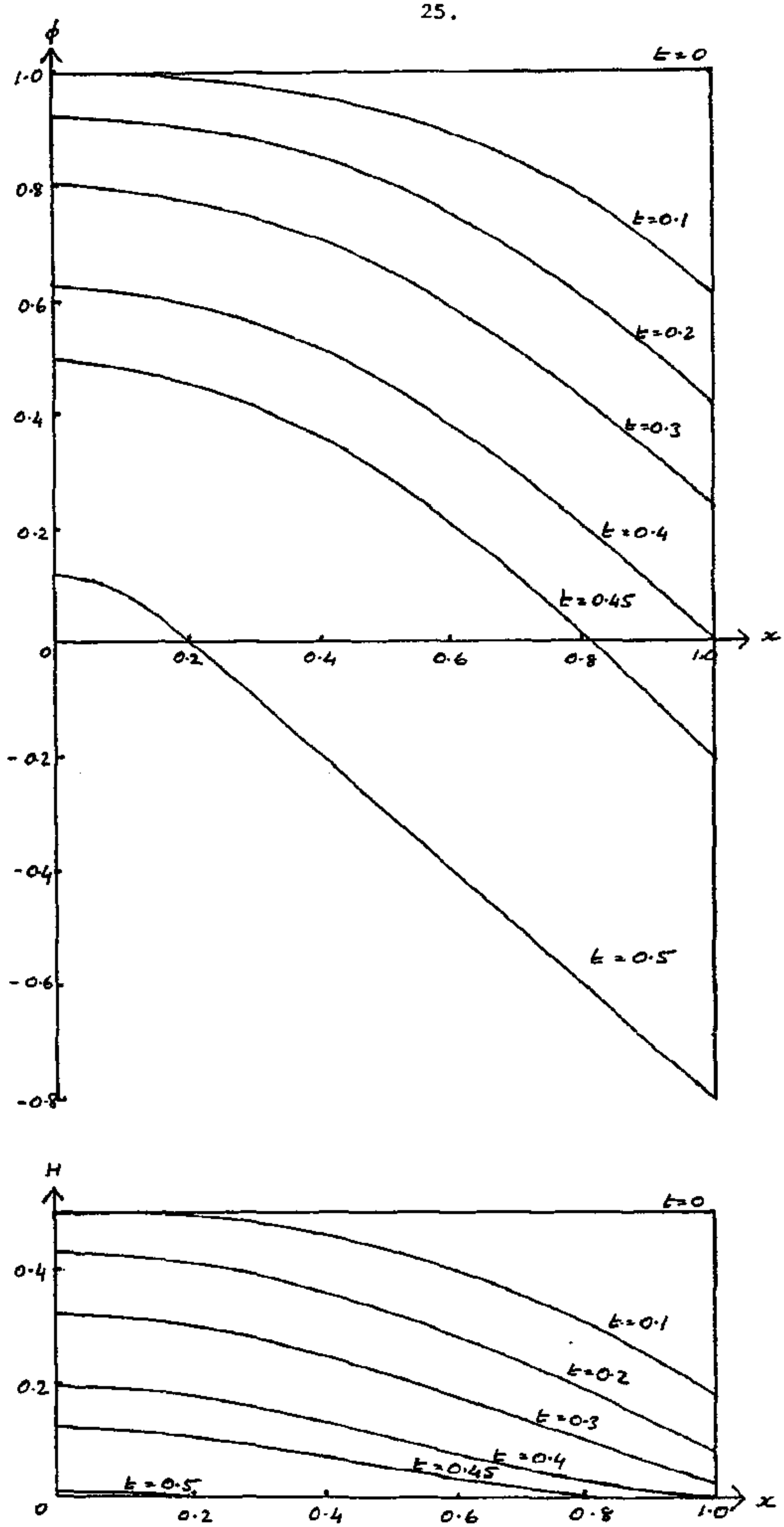
φ
1·0
0·8
0·6
0·4
0·2
0
-0·2
-0·4
-0·6
-0·8
0·2
0·4
0·6
0·8
1·0
x
t=0
t=0·1
t=0·2
t=0·3
t=0·4
t=0·45
t=0·5
H
0·4
0·2
0
0·2
0·4
0·6
0·8
1·0
x
t=0
t=0·1
t=0·2
t=0·3
t=0·4
t=0·45
t=0·5

Fig. 3