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THE ZEROS OF PARTIAL SUMS OF A MACLAURIN EXPANSION

by

A. TALBOT

1 . Introduction. A problem in approximation theory on which I have recently worked [2] required for its solution a proof that the zeros of all partial sums of the expansion

$$(1-z)^{-\frac{1}{2}} = 1 + a_1 z + a_2 z^2 + \ldots\ldots\ldots\ldots + a_n z^n + \ldots\ldots , \tag{1}$$

$$a_n = \frac{1.3\ldots.(2n-1)}{2.4\ldots\ldots 2n} = \frac{(2n)!}{2^{2n}(n!)2} \tag{2}$$

$$= \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \ldots\ldots\ldots$$

all lie outside the unit circle $|z| = 1$. It is easily verified that this is indeed true for the first few partial sums, but I have been unable either to devise a function-theory proof for the general case or to find in the literature any general theorems on the zeros of partial sums which could be applied to this expansion.

The proof to be presented here proceeds in the following way. We show that the real part of any partial sum is positive on the unit circle. Being harmonic, it is therefore also positive inside the circle, and hence the partial sum cannot vanish there. Now the real part on the circle is a function of the polar angle θ, and we shall show separately, and by entirely different methods, that it is positive (a) for a range $0 \le \theta \le \theta_2$, and (b) for a range $\theta_1 \le \theta \le \pi$, where $\theta_1 \le \theta_2$. No single method has been found that covers the whole range $\theta_1 \le \theta_2 < \pi$. For (a) we approximate to the trigonometric sum involved by a related integral. For (b) we make use of trigonometric manipulations. There is nothing fixed about θ_1 and θ_2 : many variations of (a) and (b) can be exhibited leading to different values of θ_1 and θ_2. All that is necessary is that we obtain values such that $\theta_1 < \theta_2$, thus making the two ranges of θ overlap.

It is not difficult to convince oneself that the real part of every partial sum is positive on the unit circle: if for example one tabulates the values of

$$1 + a_1 \cos\theta + \dots\dots\dots + a_n \cos n\theta$$

for θ between 0° and 180° at intervals of, say, 10°, and for n = 1, 2, ... up to say 10 or 20, the result stated Is seen to be highly likely. But closer examination of the values suggests that an even stronger result holds, namely that these values are greater than or equal to ½, with equality only for n = 1 and θ = 180°. It turns out that it is almost as easy to prove this, by methods (a) and (b), as to prove the weaker result, and we shall therefore proceed to do this. Thus, letting

$$s_n(\theta) = \tfrac{1}{2} + a_1 \cos\theta + a_2 \cos 2\theta + \dots + a_n \cos n\theta. \qquad (3)$$

we shall show that

$$s_n(\theta) \geq 0, \quad \forall \quad \forall \qquad (4)$$

It is clearly only necessary to consider the range $0 \leq \theta \leq \pi$.

2. Range $0 \leq \theta \leq \theta_2$. The sum $s_n(\theta)$ may be written, for any fixed θ,

$$s_n(\theta) = \sum_{0}^{n}{}' a_k, \cos k\theta$$

$$= \Sigma' \; f(x_k) \;, \qquad (5)$$

where $x_k = k\theta$ and $f(x) = a(x/\theta)\cos x$, a(t) being a continuous function taking the value a_k when t = k, (The prime on Σ denotes that the first term is halved.) It is clear therefore that $s_n(\theta)$ may be roughly regarded as a trapezoidal approximation to the integral

$$\int f(x)\,dx$$

taken between appropriate limits, and the value of the integral gives an estimate for $s_n(\theta)$. Obviously for the estimate to be useful θ should not be too large.

The formula (2) for a_n is too complicated to be practicable here, so we obtain an approximation using Stirling's formula. For our purpose we need both an upper and a lower estimate for a_n, and these are given by the inequalities

$$(1-\frac{1}{8n})\frac{1}{\sqrt{(\pi n)}} < a_n < \frac{1}{\sqrt{(\pi n)}}, \quad \forall n \geq 1. \tag{6}$$

These are proved as follows. By one form of Stirling's formula

$$n! = (\frac{n}{e})^n \sqrt{(2\pi n)} \exp(\frac{1}{12n} - \frac{\lambda'}{360n^3}), \quad 0 < \lambda' < 1,$$

$$(2n)! = (\frac{2n}{e})^{2n} \sqrt{(4\pi n)} \exp(\frac{1}{24n} - \frac{\lambda''}{2880n^3}), \quad , \quad 0 < \lambda'' < 1.$$

Then (2) gives

$$a_n = \frac{1}{\sqrt{(\pi n)}} \exp(-\frac{1}{8n} + \frac{\lambda}{180n^3}), \quad -\frac{1}{16} < \lambda < 1. \tag{7}$$

Now using Tylor's Theorem with Lagrange remainder

$$1 > \exp(\frac{-1}{8n}) = 1 - \frac{1}{8n} + \frac{\exp(-\mu/8n)}{128n^2}, \quad 0 < \mu < 1$$

$$> 1 - \frac{1}{8n} + \frac{7/8}{128n^2},$$

while

$$\exp(\frac{\lambda}{180n^3}) > 1 + \frac{\lambda}{180n^3} > 1 - \frac{1/16}{180n^3}.$$

It follows that

$$\exp(\frac{-1}{8n} + \frac{\lambda}{180n^3}) > \exp(\frac{-1}{8n}) - \frac{0.00035}{n^3} > 1 - \frac{1}{8n},$$

giving the first inequality in (6). The second follows at once from (7).

We now let $f(x) = \frac{\cos x}{\sqrt{x}}$ and $x_k = k\theta$, with θ fixed. Then by (6) the terms of (3) satisfy

$$a_k \cos k\theta \geq r_k \sqrt{(\frac{\theta}{\pi})}\, f(x_k), \quad k \geq 2, \tag{8}$$

where

$$r_k = \frac{15}{16} \text{ if } f(x_k) > 0, \ 1 \text{ if } f(x_k) < 0.$$

If we write

$$n_p = [(2p-1)\pi/2\theta)], \quad p = 1,2, \ldots ,$$

then

$$\cos k\theta \geq 0, \; k \in [0,n_1],[n_2+1, n_3],[n_4+1, n_5], \ldots , \tag{9}$$

$$\cos k\theta \leq 0, \; k \in [n_1+1, n_2],[n_3+1, n_4], \ldots \tag{10}$$

With the help of (8) we shall prove that there exist θ_2, μ and σ, with $\mu \geq \sigma$, such that for $0 \leq \theta \leq \theta_2$,

$$s_{n_2}(\theta) = \sum_{0}^{n_2}{}' a_k \cos k\theta > \mu > o , \tag{11}$$

and

$$s_{n_{2q}}(\theta) - s_{n_2}(\theta) = \sum_{n_2+1}^{n_{2q}} a_k \cos k\theta > -\sigma, \; q = 2,3, \ldots \tag{12}$$

Then clearly $S_{n_{2q}}(\theta) > 0$, $q = 1,2, \ldots$, and it will follow at

once that

$$s(\theta) > 0, \quad \forall n, \; 0 \leq \theta \leq \theta_2 \tag{13}$$

For if

$$n \in [n_{2q}+1, n_{2q+1}], \quad s_n(\theta) \geq s_{n2q}(\theta) \quad \text{by (9)},$$

while if

$$n \in [n_{2q-1}+1, n_{2q}], \quad s_n(\theta) \geq s_{n2q}(\theta) \quad \text{by (10)}.$$

To obtain an inequality of the form (12), let us write for simplicity $S = s_{n_{2q}}(\theta) - s_{n_2}(\theta)$, $Q = n_{2q}$, $P = n_2$. Then

$$2S\sin\tfrac{1}{2}\theta = -a_{p+1}\sin(P+\tfrac{1}{2})\theta + (a_{P+1}-a_{P+2})\sin((P+\tfrac{3}{2}))\theta + \ldots + (a_{Q-1}-a_Q)\sin((Q-\tfrac{1}{2})\theta$$

$$+ a_Q \sin((Q+\tfrac{1}{2})\theta .$$

Now

$$n_2\theta = P\theta = \frac{3\pi}{2} - \delta, \text{ where } 0 \leq \delta < \theta. \tag{14}$$

Thus

$$-\sin(P+\tfrac{1}{2})\theta = \cos(\delta - \tfrac{1}{2}\theta) \geq \cos\tfrac{1}{2}\theta ,$$

and hence

$$2S\sin\tfrac{1}{2}\theta > a_{p+1}\cos\tfrac{1}{2}\theta - (a_{p+1}-a_{p+2}) - \ldots - (a_{Q-1}-a_Q) - a_Q ,$$

i.e.

$$S > -\frac{1}{2} a_{P+1} \tan\frac{\theta}{4} , \qquad \forall\mathbf{0} . \tag{15}$$

Now by (6) and (14), $a_{P+1} < \frac{1}{\pi} \sqrt{(\frac{2\theta}{3})}$. It follows that

$$S > -\frac{1}{\pi} \sqrt{(\frac{\theta}{6})} \tan\frac{\theta}{4} = -\sigma(\theta) \ , \text{ say,}$$

and since 0(8) is an increasing function,

$$\mathbf{s > -\sigma(\theta_2) \text{ if } 0 \le \theta \le \theta_2} , \tag{16}$$

which is (12) if we take $a = \sigma(\theta_2)$.

For (11) we must use an entirely different procedure. We observe first that

$$f''(x) = x^{-3/2}(\sin x- (x\text{-}3/4x) \cos x),$$

which has just one root in $(0, 3\Pi/2)$, namely at $x = \gamma = \frac{3\pi}{2} - \mathbf{0.22760}$,

and is positive to the left of this point. Thus the graph of f(x) is concave upwards in (0,y) and any trapezoidal sum of values of f(x) in (0,y) exceeds the corresponding integral. By (8) we deduce a similar inequality for a sum of terms $a_k \cos k\theta$, where $k\theta < \gamma$. We cannot however use concavity in this way in $(\gamma, 3\pi/2)$, where f(x) is concave downwards. Instead we use the standard result

$$\left|\frac{h}{2}(f(a)+f(a+h)) - \int_a^{a+h} f(x)dx\right| < \frac{1}{12} h^3(\max|f''(x)| \text{in } (a, a+h))$$

for the error in trapezoidal quadrature- Now let

$$\bar{n} = [\gamma/\theta] .$$

Then it is easy to see from the above considerations that

$$s_{n_2}(\theta) > \frac{1}{2} + \frac{1}{2}\cos\theta + \frac{1}{\sqrt{(\pi\theta)}}\left(\int_0^{\pi/2} f(x)dx - \frac{1}{16}\int_0^{\pi/2} f(x)dx\right) +$$

$$+ \frac{1}{\sqrt{\pi}}\left(\frac{15}{12}A - \frac{1}{16}B - \frac{1}{12}C\right) , \tag{17}$$

where $$A = \sqrt{\theta}\theta\left(\frac{1}{2} f(x_2) - \frac{1}{\theta}\int_0^{2\theta} f(x)dx\right) ,$$

$$B = \sqrt{\theta}\left(\frac{1}{2} f(x_{n_1}) - \frac{1}{\theta}\int_{n_1\theta}^{\pi/2} f(x)dx\right) ,$$

$$c = \sqrt{\theta}(\theta(n_2\theta - \bar{n}\theta) + \delta^2)\ M\ ,$$

$$M = \max\ |f''(x)|\ \text{ in } (\bar{n}\theta,\ 3\pi/2)\ .$$

We deal with A, B and C in turn, and shall assume that

$$0 < \theta < \pi/4\ .$$

Then

$$A = \frac{7}{2\sqrt{2}}\cos 2\theta - \frac{2}{\sqrt{\theta}}\int_0^{2\theta} x^{\frac{1}{2}}\sin x\, dx$$

is an increasing function of 0, for its derivative is

$$-\frac{1}{\sqrt{2}}\sin 2\theta + \frac{1}{\theta\sqrt{\theta}}\int_0^{2\theta} x^{\frac{1}{2}}\sin x\, dx > -\sqrt{2\theta} + \frac{1}{\theta\sqrt{\theta}}\int_0^{2\theta}\frac{2}{\pi}x^{3/2}dx$$

$$= \sqrt{2\theta}\left(\frac{16}{5\pi} - 1\right) > 0\ .$$

Thus $\qquad A \geq -\frac{7}{2\sqrt{2}}$

Next, writing $n_1\theta = \frac{\pi}{2} - \delta^1,\quad 0 \leq \delta' < \theta < \frac{\pi}{4}$,

$$B < \sqrt{\theta}\left(\frac{\sin\delta'}{2\sqrt{(\frac{\pi}{2} - \delta')}}\right) - \frac{1}{\theta}\int_{\pi/2-\delta'}^{\pi/2}\sqrt{(\frac{2}{\pi})}\cos x\, dx\ )$$

$$< \sqrt{\theta}\left(\sqrt{\left(\frac{2}{2(\pi - 2\delta')} + \frac{2}{\pi\theta^2}\right)} - \frac{1}{\theta}\sqrt{(\frac{2}{\pi})}\right)$$

$$< \frac{\theta\sqrt{(\pi\theta/2)}}{4(\pi - 2\delta')} < \frac{\pi\sqrt{2}}{32}\ .$$

Again, $\quad \delta = \frac{3\pi}{2} - n_2\theta < \theta\ ,\quad \bar{n}\theta > \gamma - \theta > \gamma - \frac{\pi}{4} > \frac{7\pi}{6}\ ,$

and $\quad M \leq \max\ |f''(x)|\ \text{ in } (\frac{7\pi}{6}, \frac{3\pi}{2})\ .$ Now if we write

$$v(x) = x^{3/2} f''(x) - \sin x - (x - \frac{3}{4x})\cos x\ ,$$

it is easily seen that $v'(x) < 0$ in $(\frac{7\pi}{6}, \frac{3\pi}{2})$. It follows that

$$M < (\frac{7\pi}{6})^{-3/2}\ \max |v(\frac{7\pi}{6})|\ ,\ |v(\frac{3\pi}{2}|)\ , < 0.3559\ .$$

Further, we either have

$$\bar{n}\theta < \gamma < n_2\theta\ ,\quad n_2\theta - \bar{n}\theta < \frac{\pi}{3},\ \text{and}\ \ \delta < -\frac{3\pi}{2} - \gamma < 0.2277,$$

or

$$\bar{n}\theta = n_2\theta \leq \gamma\ ,\quad \text{and}\quad \delta < \theta\ .$$

Thus if $\theta < \pi/4$,

$$c < \frac{\sqrt{\pi}}{2}\left(\frac{\pi^2}{12} + 0.2277^2\right) M < 0.1556\sqrt{\pi} .$$

Finally, collecting results and using the values ([1]. p.329)

$$\frac{1}{\sqrt{\pi}}\int_0^{3\pi/2} f(x)\,dx = \sqrt{2}\,C(\sqrt{3}) > 0.45405 ,$$

$$\frac{1}{\sqrt{\pi}}\int_0^{\pi/2} f(x)\,dx = \sqrt{2}\,C(1) < 1.10294 ,$$

Where

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt$$

we obtain from (17)

$$s_{n_2}(\theta) > \mu(\theta) = \frac{0.385}{\sqrt{\theta}} + \frac{1}{2}\cos\theta - 0'827, \quad 0 < \theta < \frac{\pi}{4} ,$$

and since $\mu(\theta)$ is a decreasing function,

$$s_{n_2}(\theta) > \mu(\theta_2) \quad \text{if} \quad o < \theta < \theta_2 < \frac{\pi}{4}. \tag{18}$$

Comparing (16) and (18) with (12) and (11), the desired result (13) will now follow if we can find $\theta_2 < \pi/4$ such that $\mu(\theta_2) \geq \sigma(\theta_2)$. A suitable value (and one near the maximum possible) is $\theta_2 = 0.69$, for $\mu(0.69) > 0.022$ while $\sigma(0.69) < 0.019$.

3. Range $\theta_l \leq \theta \leq \pi$. By multiplying (3) by $\sin\frac{1}{2}\theta$ it readily follows that if $n \geq 2$,

$$s_n(\theta) = \phi_n(\theta) + \frac{a_n \sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta} \tag{19}$$

where

$$\phi_n(\theta) = \frac{1}{4} + \frac{\sum_2^n b_k \sin\left(n + \frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta} , \tag{20}$$

with

$$b_k = a_{k-1} - a_k = \frac{a_{k-1}}{2k} = \frac{a_k}{2k-1} . \tag{21}$$

Thus the b's, like the a's, form a positive decreasing sequence. They are in fact the coefficients in the expansion

$$(1-z)^{\frac{1}{2}} = (1-z)(1-z)^{-\frac{1}{2}} = 1 - b_1 z - b_2 z^2 - \dots .$$

We need also to rewrite (19) as

$$s_n(\theta) = \Phi_p(\theta) + \psi_{n,p}(\theta), \quad p<n, \tag{22}$$

Where $$\psi_{n,p}(\theta) \frac{\sum_{p+1}^{n} b_k \sin(k-\frac{1}{2})\theta + a_n \sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} \qquad \textbf{(23)}$$

Then $$|\psi_{n,p}(\theta)| < \frac{\sum_{p+1}^{n} b_k + a_n}{2\sin\frac{1}{2}\theta} = \frac{a_p}{2\sin\frac{1}{2}\theta} \qquad \textbf{(24)}$$

by (21). Noting also (19) it follows that

$$s_n(\theta) \geq \Phi_p(\theta) - \frac{a_p}{2\sin\frac{1}{2}\theta}, \quad \forall\ \theta, \ n \geq p, \qquad (25)$$

with equality only if n is odd, p = n and $\theta = \pi$.

Let x denote cos 0 . Then in the range $-1 \leq x \leq 1$,

$$s_1(\theta) = \frac{1}{2}x + \frac{1}{2} \geq 0, \text{ with equality only when } x = -1, \qquad (26)$$

$$s_2(\theta) = \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{8} = \frac{3}{4}(x+\frac{1}{3})^2 + \frac{1}{24} > \mathbf{0}\ . \qquad (27)$$

Now take p = 3 in (25). Since for any k,

$$\frac{\sin(k-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} = U_{k-2}(x) + U_{k-1}(x),$$

where U_m is the Chebyshev polynomial of the sec ond kind , we have

$$\frac{\sin\frac{3}{2}\theta}{\sin\frac{1}{2}\theta} = \mathbf{2}x + \mathbf{1}, \quad \frac{\sin\frac{5}{2}\theta}{\sin\frac{1}{2}\theta} = \mathbf{4}x^2 + \mathbf{2}x + \mathbf{1}.$$

Thus the right hand side of (25) becomes y/32, where

$$y = 4x^2 + 6x + 9 - \frac{5\sqrt{2}}{\sqrt{(1-x)}}.$$

We may write this as

$$y = (2x+1)^2 + u, \quad u = 2x + 8 - \frac{5\sqrt{2}}{\sqrt{(1-x)}},$$

and

$$y = (2x-1)^2 + w, \quad w = 10x + 8 - \frac{5\sqrt{2}}{\sqrt{(1-x)}},$$

and since $u'' < 0$ and $w'' < 0$ in (-1,1), the graphs of u and w are both concave downwards, and therefore lie above any chord. Now $U(-1) = 1$, $u(0) = 8 - 5\sqrt{2} > 0$, whence $u > 0$ in [-1 ,0]. Similarly $w(0) > 0$, $w(0{\cdot}8) = 16 - 5\sqrt{10} > 0$, so $w > 0$ in $[0, 0{\cdot}8]$, It follows that

$$y > 0 \text{ in } [-1, 0{\cdot}8],$$

and since $\cos(0{\cdot}644) < 0{\cdot}8$, we have by (25), (26) and (27)

$$s_n(\theta) \geq 0 \quad, \forall n, \quad \theta_1 = 0.644 \leq \theta \leq \pi, \tag{28}$$

with equality only when n = 1 and 0 = π.

Since we have proved (13) with $\theta_2 = 0.69 > \theta_1$, (4) follows.

Remark 1. It is in fact the case that y> 0 in a wider interval than (-1, 0.8), namely in (-1, ξ) where ξ = 0.818 ... , this being the only root of y in (-1,1). This is not difficult to prove, with the help of Rolle's theorem, and gives a lower value of θ, but the above method is simpler and gives a low enough θ_1 for our purpose. The same applies to several other possible variations on the use of (25).

Remark 2. Suppose it has been proved that for some value of p

$$\phi_p(\theta) \geq \frac{a_p}{2\sin\frac{1}{2}\theta} \quad , \theta \geq \alpha \ . \tag{29}$$

Then by (25) it follows that

$$s_n(\theta) \geq 0, \quad \theta \geq \alpha, \quad n \geq p \ . \tag{30}$$

Now $\dfrac{\sin(k-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$, $k \geq 2$, decreases steadily from the value 2k - 1 at θ = 0 to zero at θ = 2π/(2k-1). Thus if we take

$$m = [\frac{\pi}{\alpha} + \frac{1}{2}] \ , \tag{31}$$

then α 2π/(2m-l) and

$$\phi_m(\theta) > \phi_m(\alpha) \ , \quad 0 \leq \theta < \alpha \ .$$

Now suppose that m > p . (This is not necessarily the case, for if a is arbitrarily increased towards π, m decreases towards 1.

If however a has the smallest value for which (29) holds, i.e. is such that $\phi_p(\alpha) = a_p/2\sin\frac{1}{2}\alpha$, then probably m > p follows, but I have been unable to prove this.) Then $\phi_m(\alpha) > \phi_p(\alpha)$, since all terms of $\phi_m(\alpha)$ are positive and they include the terms of $\phi_p(\alpha)$, while $a_m < a_p$. Thus if we define α' by

$$\sin\tfrac{1}{2}\alpha' = \frac{a_m}{2\phi_m(\alpha)} \ , \tag{32}$$

we have $\sin \frac{1}{2}\alpha' < \frac{a_p}{2\phi(\alpha)} \le \sin \frac{1}{2}\alpha$, by **(29)** i.e. $\alpha' < \alpha$. Moreover ,
if $\alpha' \le \theta < \alpha$, we have

$$\phi_m(\theta) > \frac{a_m}{2\sin\frac{1}{2}\alpha'} \ge \frac{a_m}{2\sin\frac{1}{2}\theta},$$

so that (30) can be improved (as regards θ) to

$$s_n(\theta) > 0, \quad 0 \ge \alpha^1, \quad n \ge m.$$

Further, if $n < m$ and $k \le n+1$, $(k-\frac{1}{2})\alpha \le \pi$, and it follows by (19) that

$$s_n(\theta) > 0, \quad \theta \le \alpha, \quad n < m,$$

and thus finally $s_n(\theta) \ge 0, \quad \theta \ge \alpha', \quad n \ge p,$ (33)

i.e. (30) with a replaced by $\alpha' < \alpha$.

The whole process can now be iterated: putting α' in place of a in (31) gives m', certainly ≥ m, and putting α' and m' in (32) in place of a and m gives α'', certainly < α' (since $\phi_{m'}$,$(\alpha') \ge \phi_m(\alpha') > \phi_m(\alpha)$).
In this way we produce a strictly decreasing sequence a, α', α'', ...
If this sequence could be proved to converge to zero, we would have a proof that $s_n(\theta) \ge 0$, Vθ, n≥p. In fact however, taking p = 1, 2 or 3, the sequence does not appear to converge to zero, so we cannot use this method as a means of avoiding the use of the method of section 2.
This convergence behaviour is associated with the slow rate of convergence to zero of the sequence a , and indeed this is at the root of failure of several other methods that have been tried for solving the present problem.

Remark 3. (24), and hence (25), depends on the b_k's and a_n being all positive, i.e. the a_k's forming a positive decreasing sequence, and the same method could be used to prove $s_n(\theta) \ge 0$ for $\theta \ge \theta_r$ whenever the coefficients have this property. Similarly (15) also holds in all such cases, and the method of section 2 can be attempted whenever bounds on a_n as in (6) are known.

4. Conclusion. We have proved that the trigonometric series (3) has the simply property (4) when the a_n are the coefficients in the Maclaurin series (1), and hence, as a corollary, that all partial sums of (1) have all their zeros outside the unit circle. Inspite of its complicated nature, the present proof is put forward in the hope of stimulating the search for a better one, for surely such exists !

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