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LAGRANGIAN FINITE ELEMENT and
FINITE DIFFERENCE METHODS FOR
POISSON PROBLEMS.

BY

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ABSTRACT

The use of Lagrangian finite element methods for
solving a Poisson problem produces systems of linear
equations, the global stiffness equations. The
components of the vectors which are the solutions
of these systems are approximations to the exact
solution of the problem at nodal points in the
region of definition. There is thus associated
with each nodal point an equation which can be
thought of as a difference equation. Difference
equations resulting from the use of polynomial trial
functions of various orders on regular meshes of
square and isosceles right triangular elements are
derived. The rival merits of this technique of
setting up a standard difference equation, as distinct
from the more usual practice with finite elements
of the repeated use of local stiffness matrices, are
considered.

1 .Introduction and Finite Element Method

Two methods for producing numerical approximations to the solutions of elliptic boundary value problems are those of finite elements and finite differences. However, these methods are closely related. In this paper an approach is adopted whereby the linear equation which results at a mesh point from the application of the finite element method is regarded as a difference equation. Thus emphasis is placed on the idea of using the finite element method with a regular grid to produce difference molecules which are used repeatedly over the mesh and which change only when affected by the boundary.

The discussion is restricted to two dimensional Poisson problems with Dirichlet boundary conditions, Let the function u(x,y) be the solution of the problem

$$\left.\begin{array}{ll} - \Delta \, | \, u(x, \ y) \, | = g(x, \ y), & (x, \ y) \ \varepsilon \, \Omega \ , \\[2mm] u(x, \ y) = 0, & (x, \ y) \ \varepsilon \, \partial\Omega, \end{array}\right] \quad (1)$$

where $\Omega \ \varepsilon \ E^2$ is an open bounded domain with polygonal boundary $\partial\Omega$ A is the Laplacian operator and $g \ \varepsilon \ L_2 \ (\Omega \ )$. Under these assumptions the solution u(x,y) of (1) minimizes the energy functional

$$I[V] \ = \ \iint_\Omega \ \left[\left(\frac{\partial V}{\partial X} \right)^2 \ + \ \left(\frac{\partial V}{\partial y} \right)^2 \ - \ 2gv \right] \qquad dxdy \quad (2)$$

over the class of functions $\overset{0}{W}{}^1_2(\Omega,)$, see Zlamal (11). As usual $W^{\frac{1}{2}}_2(\Omega)$ is the Sobolev space of functions which

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together with their first generalized derivatives exist

and are in L_2, and $\overset{0}{W}{}^1_2(\Omega)$ is the subspace of functions

in $W^1_2(\Omega)$ which are identically zero on $\partial\Omega$.

The region Ω is divided into a number of non-overlapping triangular or rectangular elements, and it is assumed that on the partition there are m mesh nodes: in Ω and n nodes on $\partial\Omega$

An approximation U(x,y) to u(x,y) is constructed using the Ritz method by solving

$$\min_{S^h_o}\big(I[V]\big) \tag{3}$$

where S^h_o is an m-dimensional subspace of $\overset{O}{W}{}^1_2(\Omega,)$, and the t: parameter h is a measure of the size of the elements. The finite dimensional space S^{Oh} consists of functions which are piecewise polynomial over Ω, and satisfy the homogeneous boundary conditions on $\partial\Omega$.

In each element for Lagrangian methods the choice of the approximating function is motivated by an interpolating function which takes the values of u at k nodes in the element. Let the interpolant $\tilde{u}$ in each element have the form

$$\tilde{u}(x,y) = \sum_{i=1}^{k} u_i \Phi_i(x,y), \tag{4}$$

where the ϕ_i are the cardinal basis functions (shape functions) of the interpolation with respect to the values u_i of u at the k nodal points. Note that in (4) the numbering of the shape functions is local to the element. The approximating function $U^e(x,y)$ then has

in each element the form

$$U^e(x, y) = \sum_{i=1}^{k} U_i^e \varphi_i(x, y), \qquad (5)$$

where now the U_i^e are the values of U at the same nodal points of the element as those of u in (4). For the whole of Ω. the nodes have a global ordering p = 1,2,..,m, and the collection of all shape functions associated with the pth node defines a function B_p (x,y). This method of construction results in the B_p being nonzero only in elements which contain the pth node. Thus the shape functions Φ of (4) and (5), considered over the total number of elements in Ω, form the linearly independent set

$$\{B_p\}, \quad p = 1,2,...,m,$$

members of which span the space $\overset{o}{S}h$ A function $V \varepsilon \overset{o}{S}h$ can therefore be written as

$$V(x, y) = \sum_{p=1}^{m} U_p B_p(x, y). \qquad (6)$$

In order hat $\overset{oh}{S}$ may be a subspace of $\overset{o1}{W2}$ the interpolating polynomials and nodes in each element

must be chosen so that $V \varepsilon \overset{oh}{S}$ satisfies certain continuity

properties across interelement boundaries. The

condition that $\overset{oh}{S} C W_2^1$ is the <u>conforming</u> <u>condition</u>, and

for problems of type (1)it is that $\overset{oh}{S} \subset \overset{o}{C} \overline{(\Omega)}$ where

$\overline{\Omega} \equiv \Omega \bigcup \partial\Omega.$

Since $\overset{oh}{S} \subset \overset{o1}{W2}(\Omega\Omega) V \varepsilon \overset{Oh}{S}$ is admissible function

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(see | 8 |) so that we may write

$$\| V \| = \sum_e \iint_{\Omega_e} \left[\left(\frac{\partial V}{\partial X} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 - 2gV \right] dxdy$$

$$\cong \sum_e I|U^e|, \tag{7}$$

where u^e is the local trial function in the eth element
and Ω_e is the area of that element . From (5) and (7)
it follows that for the eth element:

$$\| U^e \| = \| \sum_{i=1}^{k} U_i^e \phi_i (x, y) | \quad ,$$

$$\tag{8}$$

$$= |U^e|_{1xk}^r \quad K_{kxk}^e \quad U_{kx1}^e \quad - 2 |U^e|_{1xk}^T \quad z_{kx1} \quad ,$$

where, see Zlamal |11|, Ke $^{is \ the \ elements \ stiffness}$

matrix, U^e vec$(u_{1.} u_2^e, \ldots, u^e_k)$, z is a column
vector the components of which depend on $g(x,y)$. The
summing of I | u^e | to form I | V | as in (7) produces a
quadmatic function of the variables U_1, U_2, . .. ,U_m,
where these U's are simply the total number over all
the elements of the U^o_1 associated with the m nodes in
Ω The vector U vec $(U_1, U_2, \ldots, U_m)$ can be found
by solving, the tinear system

$$\frac{\partial \|V\|}{\partial U_p} = O, \qquad p = 1, 2, \ldots, m. \tag{9}$$

which, after the boundary conditions have been taken
into account, is written

$$KU = Z . \tag{10}$$

The mxm matrix K is the global stiffness matrix. Substitution of u in (6) gives the approximation U(x,y) to the solution u(x,y) of (1).

2. Difference Stars Using Langrangian Interpolation with C^o Approximating Functions

The above procedure may be termed a standard finite element approach for setting up the global stillness equations (10), and requires that the local stiffness matrix K^e be calculated for each element. For a regular mesh, using co-ordinates local to each element, the K^e will all be the same. However, their repeated use will be necessary for the construction of (10).

An alternative method, which *is* useful for regular meshes, is suggested by thinking of the equations (10) as difference equations, so that the equation $\partial I |V|/\partial U_p =0$ is the difference equation at the pth node. Nonzero contributions to the function $\partial I[V]\partial Up$ come only from the set of elements, {Np }, which contain the pth node. Thus the summation in (7) need only be performed over these elements. The technique adopted here is for every node in Ω to form $I| u^e|$, I, and hence $\partial I|.u^e|/\partial Up$, for all elements of {Np } and then to form

$$\sum_{Np} \frac{\partial I[U^e]}{\partial U_p}=0$$

(11)

Equation (11) is exactly the pth equation of (9). As a difference equation it is derived only once. However, the region Ω is covered with a regular mesh, and so equation (11) is used for all nodal points at

6.

which the element contiguration is as in { N $_p$ }. As
before the boundary conditions have to be taken into
account

In Sections 2.1. — 2.3 various interpolating
functions are used to produce difference molecules at
nodes on regular triangular and square meshes. As the
main interest is the form of the difference molecules at
general interior mesh nodes, no account is taken of the
boundary or boundary conditions.

2.1 <u>Standard Triangular Elements</u>

Consider a mesh made up entirely of isosceles
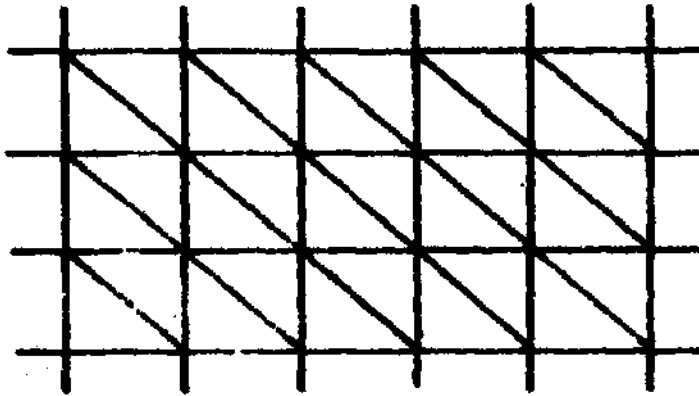right angled triangles as in Figure 1.

Figure 1

The complete Mth order polynomial

$$I_M(x) = \sum_{i+j=0}^{M} \alpha_{ij} \, x^i \, y^j$$

can ho used to interpolate a function at $k = \dfrac{1}{2}(M+1)(M+2)$

symmetrically placed nodes in a triangle. This k is
the same as the upper limit of summation of the series
in (5). Interpolation using different orders of
polynomial is now illustrated.

Linear Interpolation ; M = 1

Here k = 3 and in any triangle of the mesh the nodes are taken at the vertices. Linear interpolation to the values of U at these nodes produces over Ω a global approximating function which is continuous, so that the conforming condition is satisfied.

Use is now made of the standard triangle T, see Birkhoff and Mansfield [1]. In terms of local co-ordinates (X,Y) this has vertices (0,0), 2 (h,0) and 3 ≡ (0,h), as in Figure 2.
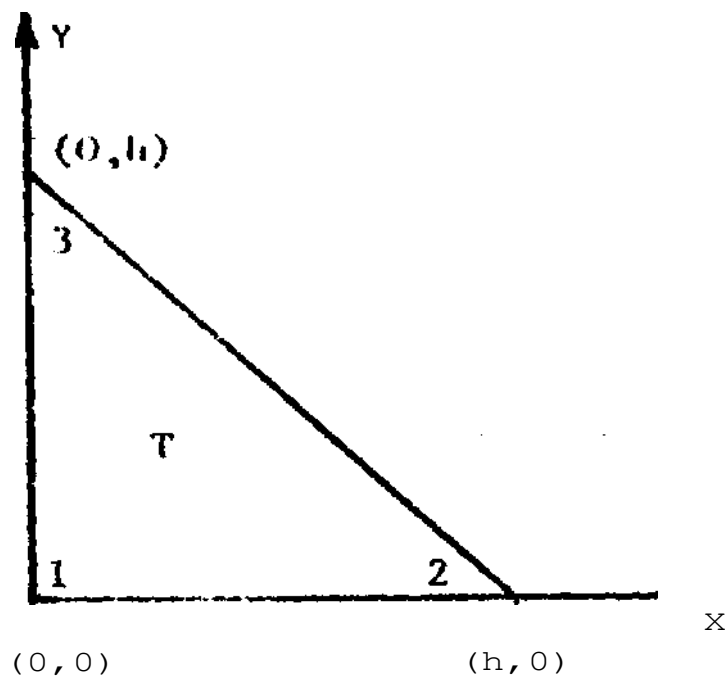

Figure 2

The Function which is linear in X and Y and which interpolates to the values $U_1^e \equiv U^e(0,0)$, U_2^e $U^e(h,0)$, U_3^e U^e (O,h) can *be* written in cardinal basis form as

$$u^e(X,Y) = U_1^e(1-X/h-Y/h) + U_2^e x/h + U_1^e Y/h \ . \tag{12}$$

Thus for the standard triangle, if G(X,Y) g(x,y) ,

$$I \mid U^e \mid = \iint_T \left[\frac{1}{h^2} \{ (U_2^e - U_1^e)^2 + \left(U_3^e - U_1^e \right)^2 \} \right.$$

$$- 2G(x, \ y) \{ U_1^e (1 - \frac{X}{h} - \frac{Y}{h}) + U_2^e \frac{X}{h} + U_3^e \frac{Y}{h} \} \ dXdY \left. \vphantom{\frac{1}{h^2}} \right]$$

$$= \frac{1}{2} \{ (U_2^e - U_1^e)^2 + (U_3^e - U_1^e)^2 \} - 2F^e(U_1^e, U_2^e, U_3^e), \tag{13}$$

where F^e is the appropriate linear function of the

variables $U^e{}_1, \ U^e{}_2, \ U_3$, so that

$$\frac{\partial \ I[U^e]}{\partial \ U_1^e} = 2U_1^e - U_2^e - U_3^e - 2 \frac{\partial \ F^e}{\partial \ U_1^e}, \tag{14}$$

$$\frac{\partial \ I[U^e]}{\partial \ U_2^e} = U_2^e - U_1^e - 2 \frac{\partial \ F^e}{\partial \ U_2^e} \tag{15}$$

and

$$\frac{\partial \ I[U^e]}{\partial \ U_3^e} = U_3^e - U_1^e - 2 \frac{\partial \ F^e}{\partial \ U_3^e}. \tag{16}$$

Any interior point of the mesh is a node for six
elements, the assemblage of these being as in Figure 3.
In particular the pth node of the global ordering may
be taken as the node 1 of the standard triangle T.
Considering only the part of I[U] which is associated
with the Laplacian $-\Delta U$, it follows from (14) that the

contribution to $\quad \dfrac{\partial I[U]}{\partial U_p}$ from the triangle T at the point

$(x,y) \ \varepsilon \Omega$ is

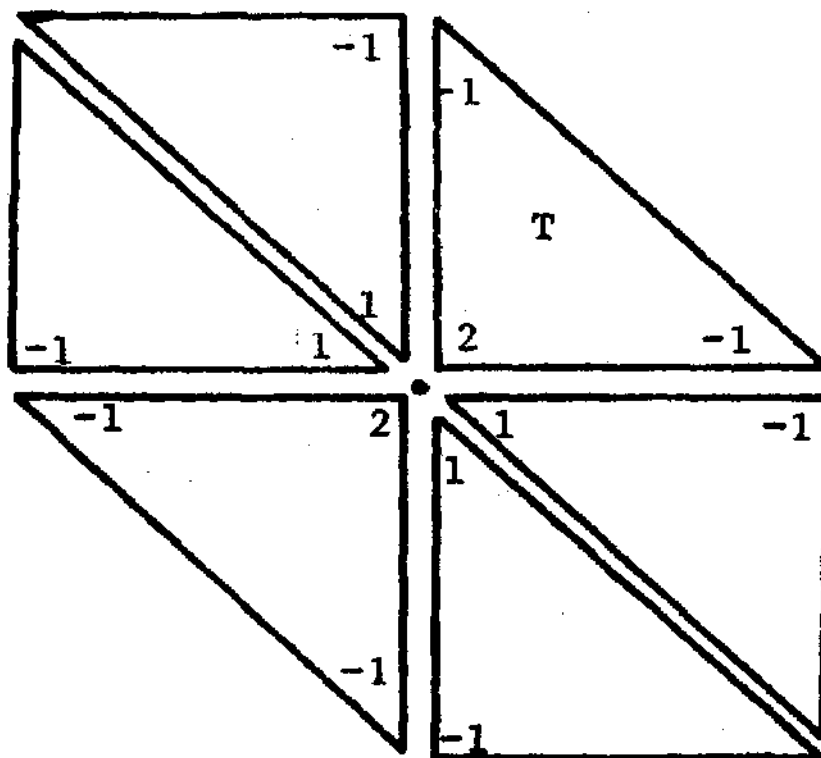$$2 \ U(x,y) - U(x+h,y) - U(x,y+h) \ .$$

Figure 3

The contributions from the other five elements of N_p follow immediately from (14)-(16) using symmetry, and on summation produce at the point (x,y) the familiar five-point finite difference replacement

$$4 \ U(x,y) \ - \ U(x+h,y) \ - \ U(x,y+h) \ - \ U(x-h,y) \ - \ U(x,y-h)$$

(17)

for the Laplacian, which has $0(h^2)$ local discretization error.

The fact that this five point formula can be obtained using linear interpolants is well known, and has been pointed out by Fix and Strang [6], Pian [7] and Whiteman [10].

<u>Quadratic Interpolation</u>; M = 2

Here k = 6 and in any triangle the nodes are taken as the vertices and the centre points of the sides. Interpolation with the complete quadratic in x and y to values at these nodes again produces a continuous global approximating function. For the standard triangle the quadratic interpolant to the function values U^e_i i = 1, 2, ..., 6 at the six nodes 1 ≡ (0,0),

2 = (h,0), 3 ≡ (0,h), 4 ≡ $(\frac{h}{2},0)$, 5 ≡ $(\frac{h}{2},\frac{h}{2})$, 6 ≡ $(O,\frac{h}{2})$
can be written in cardinal basis form as

$$U^e(X, Y) = U^e_1 (1 - \frac{X}{h} - \frac{Y}{h})(1 - \frac{2X}{h} - \frac{2Y}{h}) + U^e_2 \frac{X}{h}(\frac{2X}{h} - 1) + U^e_3 \frac{Y}{h}(\frac{2Y}{h} - 1)$$

$$+ U^e_4 \, 4\frac{X}{h}(1 - \frac{X}{h} - \frac{Y}{h}) + U^e_5 \, 4\frac{X}{h}\frac{Y}{h} + U^e_6 \, 4\frac{Y}{h}(1\frac{X}{h} - \frac{Y}{h}). \qquad (18)$$

In a completely analogous manner to that used with linear interpolation, the interpolant (18) leads at a mesh point with co-ordinates (x,y), which is a vertex of a triangle, to a <u>non-standard</u> nine-point difference replacement for the Laplacian of the form

$$12U(x, \ y) - 4\{U(x \ + \frac{h}{2},y) + U(x, \ y + \frac{h}{2}) + U(x \ - \frac{h}{2},y) + U(x, \ y - \frac{h}{2})\}$$

$$+ U(x \ + h, y) + U(x, \ y + h) + U(x \ - h, y) + U(x, \ y - h). \qquad (19)$$

Formula (19) is clearly a combination of five-point replacements at (x,y) based on mesh lengths h and $\frac{h}{2}$.

However, at a mid-side node (e.g. nodes 4,5 or 6 of the standard triangle) contributions to $\partial I[U]/\partial U_p$

come only from two elements, and in this case the difference replacement is again the familiar five—point formula (17), except that now it is based on a mesh of length h/2.

It is thus clear that the use of this form of quadratic interpolation leads to a double system of difference equations the form of which differs between vertex and mid-side nodes.

Cubic and Higher Order Interpolation

In the previous two cases the difference replacements produced have been either of a familiar form or closely related to this. There is with the increase in order of interpolating polynomial an increase in the number of nodes in each triangle (M = 3, k = 10; M = 4 k = 15;..,). There are also more classes of nodes and for each class the replacement is so complicated as to be worthless in its own right as a difference replacement.

As an illustration, we give in Figure 4 in tabular form the replacement at a mesh point which is an element vertex obtained using cubic interpolants to function values at the vertices, the points of trisection of the sides and the centre of the triangle of the mesh of isosceles right angled triangles.

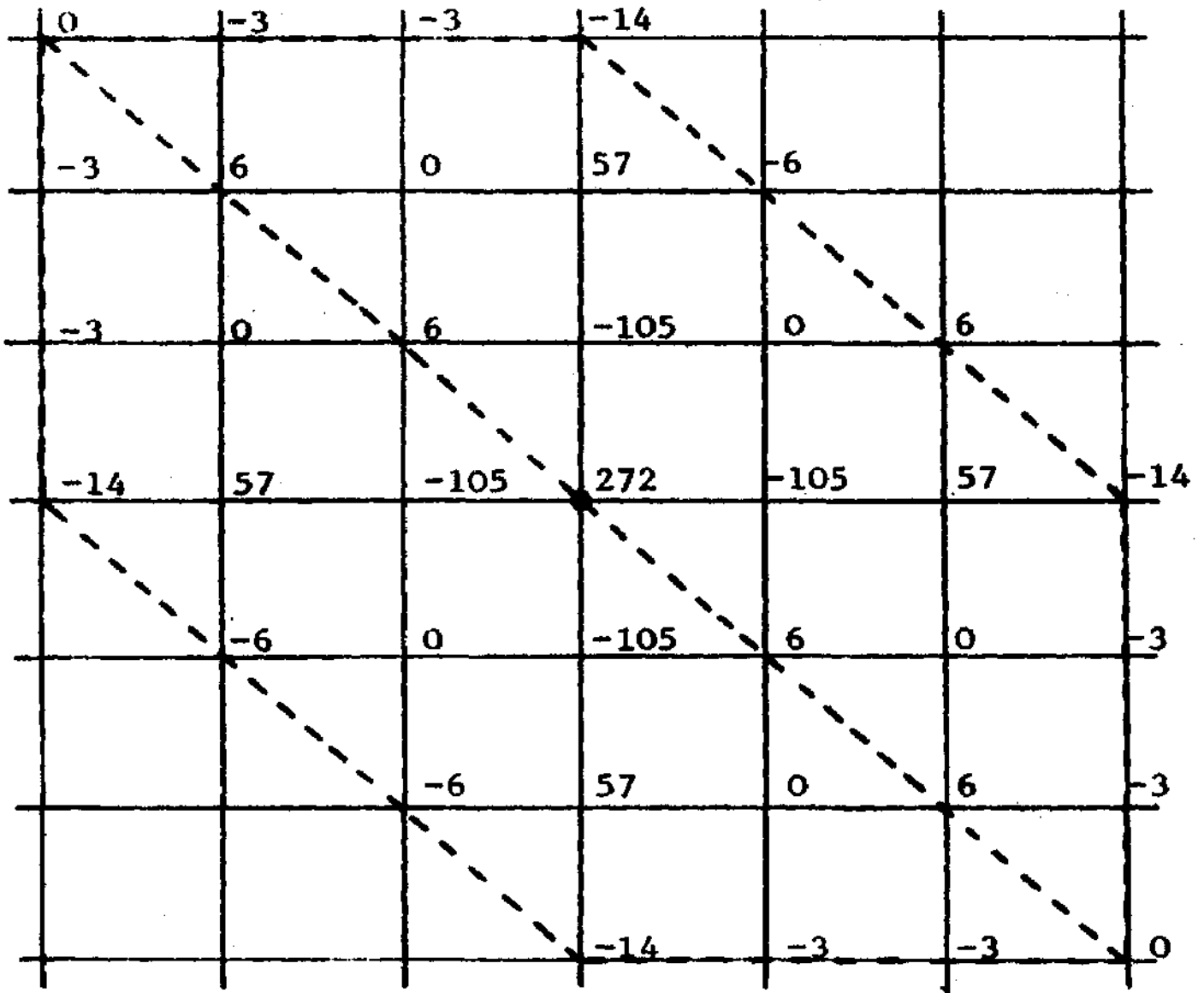| 0	-3	-3	-14			
-3	6	0	57	-6		
-3	0	6	-105	0	6	
-14	57	-105	272	-105	57	-14
	-6	0	-105	6	0	-3
		-6	57	0	6	-3
			-14	-3	-3	0

Figure 4.

2.2 Standard Square Elements

Consider now a mesh made up entirely of squares of side h. The interpolation is performed with polynomials of the form

$$Q_M(x, y) = \sum_{i=0}^{M} \sum_{j=0}^{M} \alpha_{ij} \, x^i \, y^j \, ,$$

where successively M = 1 (bilinear), M = 2 (biquadratic) etc.

Bilinear Interpolation; M = 1

In this case there are four terms in Q (x,y) so that, using square elements, a continuous global approximating function can be obtained if interpolation to function values at the corners of the squares is performed.

The bilinear function which interpolates to the values $U^e_1 \equiv U^e_1(0,0)$, $U^e_2 = U^e(h,0)$, $u^e_3 \equiv U^e(h,h)$, $U^e_4 \equiv U^e(0,h)$ at the corners of the standard square is, in cardinal basis form

$$U^e(X, Y) = U^e_1 (1 - \frac{x}{h})(1 - \frac{Y}{h}) + U^e_2 \frac{X}{h}(1 - \frac{Y}{h}) + U^e_3 \frac{X}{h}\frac{Y}{h} + U^e_4 (1 - \frac{X}{h})\frac{Y}{h}.$$

$$(20)$$

Use of (20) leads at a point (x,y) of the mesh to another non-standard nine-point finite difference replacement for the Laplacian of the form

8U(x,y) − U(x+h,y) − U(x,y+h) − U(x−h.y) − U(x,y−h)

  − U(x+h,y+h) − U(x−h,y+h) − U(x−h,y−h) − U(x+h,y−h)

$$(21)$$

This replacement has $0(h^2)$ local discretization error, and is given by Birkhoff, Schultz and Varga in [2].

Biquadratic Interpolation; M = 2

For biquadratic interpolation nine nodes on the standard square are. used, and these are taken at the

Points $\quad 1 \equiv (0,0), \quad 2 \equiv (\frac{h}{2}, 0), \quad 3 \equiv (h, o), \quad 4 \equiv (o, \frac{h}{2}),$

$5 \equiv (\frac{h}{2}, \frac{h}{2}), \quad 6 \equiv (h, \frac{h}{2}), \quad 7 \equiv (0, h), \quad 8 \equiv (\frac{h}{2}, h), \quad 9 \equiv (h, h).$

The biquadratic interpolant to function values at these nodes is , in cardinal basis form,

$$U^e(X,Y) \quad = \quad \{ U_1^e \; p(X) + U_2^e \; q(X) + U_3^e \; r(X) \} \; p(Y)$$

$$+\{ U_4^e \; p(X) + U_5^e \; q(X) + U_6^e \; r(X) \} \; q(Y)$$

$$+\{ U_7^e \; p(X) + U_8^e \; q(X) + U_9^e \; r(X) \} \; r(Y) , \qquad (22)$$

where $p(t) = (1-t)(1-2t) ,$

$\qquad q(t) = 4t(1-t) ,$

$\qquad r(t) = t(2t-1) .$

Use of (22) leads at a node (x,y) which is a corner of a mesh square to the twenty-five-point replacement of Figure 5.

1	5	3	5	1
5	-16	-18	-16	5
-3	-18	112	-18	-3
5	-16	-18	-16	5
-1	5	-3	5	-1

Figure 5

At nodes which are respectively centre points of the
squares and mid—points of sides of squares the
replacements of Figures 6 and 7 are obtained. The
difference replacements of Figures 5, 6 and 7 all have
$0(h^2)$ local discretization errors.

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-1	-3	-1
-3	16	-3
-1	-3	-1

Figure 6 (Mid-point of square)

5	0	5
-16	-48	-16
-18	176	-18
-16	-48	-16
5	0	5

Figure 7 (Mid-point of side)

2.3 A. Nonconforming Element

If in each element of a triangular mesh a Linear trial function interpolates to function values at the mid-points of the sides of the triangle, then the global approximating function so produced is not in general continuous and the conforming condition is violated.

Suppose that for the standard triangle of Section 2.1 the nodes are now taken at the points $(\frac{h}{2}, 0)$, $(\frac{h}{2}, \frac{h}{2})$ and $(0, \frac{h}{2})$ In cardinal basis form the linear interpolant to the function values at these points can be written as

$$U^e(X, Y) = U(\frac{h}{2}, 0)(1 - 2\frac{Y}{h}) + U(\frac{h}{2}, \frac{h}{2})(2\frac{X}{h} + 2\frac{Y}{h} - 1) + U(0, \frac{h}{2})(1 - 2\frac{X}{h}). \quad (23)$$

Use of (23) leads at a point of type mid-point of the hypotenuse of a triangle, to the difference replacement $(\frac{h}{2}, \frac{h}{2})$ that is the

$$4U(x, y) - U(x + \frac{h}{2}, y) - U(x, y + \frac{h}{2}) - U(x - \frac{h}{2}, y) - U(x, y - \frac{h}{2}). \quad (24)$$

However, at the mid-point of a vertical side (x=constant) of; a triangle, the replacement is

$$2U(x, y) - U(x + \frac{h}{2}, y) - U(x - \frac{h}{2}, y) \quad (25)$$

and at the mid-point of a horizontal side (y = constant) it is

$$2U(x, y) - U(x, y + \frac{h}{2}) - U(x, y - \frac{h}{2}).$$

Both of the difference replacements (25) and (26) are inconsistent.

3. Discussion and Application to Model Problem

It has been shown in Sections 2.1 – 2.3 that
finite element techniques, based on Lagrangian
interpolauts with triangular and square elements,
produce difference replacements for the Laplacian,
and that the systems of equations so derived are
exactly those of (9) and (10). Most of these
replacements are not those with which the users of
finite differences are familiar; see for example
Collatz |4 |, pp.542-543. However, the methods of
Sections 2.1 and 2.2 produce consistent replacements
for which, with the usual techniques based on Taylor's
series, it can be shown that the local discretization
errors are $0(h^2)$. Some of these schemes are
unconventional in that with a single trial function the
finite element technique produces several difference
replacements; the particular replacement which is
appropriate to a mesh point being determined by the
position of this (nodal) point in an element. The
mesh points thus fall into classes with each of which
is associated a particular type of replacement. This
property suggests that it may be possible, by
permuting the rows and columns, to rearrange the global
stiffness matrix K so that it can be partitioned in a
form which facilitates the solution of the linear
system (10).

Many theoretical bounds have been derived for the
error in the finite element approximations to the
solutions of problems of the type (1); see for example
Bramble and Zlamal [3] for triangular elements and

Birkhoff, Schultz and Varga [2] for rectangular
elements. Provided that the conforming condition
$(\overset{O}{S}{}^{h}, \overset{O}{W}{}^{1}_{2}(\Omega))$ is satisfied, and that the solution u of

(1) has specific derivatives bounded throughout $\overline{\Omega}$,
these hounds hold. The difference methods of Sections
2.1 and 2.2, being exactly "conforming finite element
methods", are subject to these same error bounds and so,
under the same conditions on u, convergence with
decreasing mesh size of the "finite difference"
solution to the true solution is assured.

The discussion in this paper has been limited to
finite element methods based on Lagrangian
interpolation. In higher order problems it is usual
for trial functions to interpolate to values of
derivatives at nodal points as well as to function
values. These are finite element methods based on
Hermite interpolation, such as are discussed in (2).
In biharmonic problems, for example, the integrand in
the energy functional which corresponds to (2) contains
second derivatives, and conforming global trial
functions possess continuous first derivatives. The
final system of global stiffness equations in this
case involves derivative values. Clearly such linear
equations involving function and derivative values as
unknowns cannot be thought of as conventional
difference equations. However, in the difference
context such Hermitean methods (mehrstellenverfahren)
have been considered by Collatz [5].

The difference approach based on finite element
techniques does not allow for the full exploitation of

the versatility of the standard f i n i t e element method;
namely that the form of local trial function in each
element may be varied over Ω so as to produce better
numerical approximations. Further, the difference
approach is only really useful when _regular_ meshes are
involved. However, as has been indicated, when the
difference approach is viable, it produces equations
which are identical with the global stiffness
equations.

  The criterion by which to judge the two methods
is the respective computation times taken to produce
the same numerical solutions. The approach adopted
here is to use the standard finite element method to
solve a model problem and to analyse the times taken
for the various parts of the computation. This
computation involves the repeated call of local
stiffness matrices for the assembly of the global
stiffness matrix; a process which will largely be
absent in the difference approach. In the model
problem $\overline{\Omega}$ is the square

$$\overline{\Omega} \quad \{(x,y) : 0 \leq x \leq 1 \;, \; 0 \leq y \leq 1 \}$$

with interior Ω and boundary $\partial\Omega$, and the function
u(x,y) satisfies

$$-\Delta[u(x,y)] = x-x^2+y-y^2 \;, \quad (x,y) \; \varepsilon\Omega \;,$$

$$u(x,y) = 0 \;, \quad (x,y) \; \varepsilon \; \partial\Omega, \; .$$

This problem has exact solution $u(x,y) = \frac{1}{2} xy(1-x)(1-y)$.

The. region Ω *is* partitioned into right triangular elements by subdivision first into equal squares, and then by further subdivision of each square into two triangles along the diagonal parallel to x+y = 1. Numerical approximations U to u are calculated using respectively piecewise linear, quadratic and cubic approximating functions on meshes of the above type. As accuracy is not the main aim here, the partitions for the three cases are chosen so that the total number of nodes is approximately constant. Results at two points in are given in Table 1. Details of the partitions and the global stiffness matrices are given

Trial functions in Each Element	Total Number of Nodes	Values of U at $\left(\frac{1}{2},\frac{1}{2}\right)$	Values of U at $\left(\frac{3}{4},\frac{1}{4}\right)$
Linear	289	.031017	.017496
Quadratic	289	.031254	.017578
Cubic	256	.031243	.017581
Exact solution		.031250	.017578

Table 1

in Table 2, together with a break down of the computation times. It is seen that the generation of the local stiffness matrix, and of the global stiffness equations through its repeated call, in the three cases takes respectively 45, 30 and 35 seconds. When the difference approach is adopted, the generation of the

TABLE 2

PARTITIONS

Trial Functions	Linear	Quadratic	Cubic
Length of short sides of each triangle	1/16	1/8	1/5
No. of nodes in $\overline{\Omega}$	289	289	256
No. of boundary nodes	64	64	60
No, of elements	512	128	50
GLOBAL STIFFNESS MATRICES			
Bandwidth	42	82	96
No. of non-zero elements (i) in half of g.s.m.	709	793	1197
(ii) with fill in using Gaussian elimination	3363	4653	6737
COMPUTATION TIMES IN SECS ON CDC 6500			
Generation of mesh, co-ordinates, data	30	25	20
Generation of standard element stiffness matrix	7	14	24
Generation of global stiffness matrix	38	16	11
Gaussian elimination	8	15	35
Back substitution	3	6	6
TOTAL TIMES	86	76	96

global stiffness equations in each case takes
approximately 5 seconds. As can be seen from Table 2,
this results in savings of the order of 45%, 40% and
30% in the respective computation times.

Clearly this model problem is ideally suited for
the difference approach, so that this is not a totally
Lair comparison. However, it does illustrate the
potential of this alternative way of viewing the global
stiffness equations.

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