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GROUPS OF TYPE (p, p) ACTING ON p-SOLUBLE GROUPS

by

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In this paper we continue with the work of (l) and (2), considering now an operator group A of type (p, p) acting on a p- soluble group G. The aim is to show that if the p length of G is large enough, fixed points appear for such an operator group on a P' section; moreover this is non trivial in the sense that it is not centralized by e very Sylow p subgroup of G. We show that such fixed points must appear if the p length of G/O_p is at least 5 (if p = 2) or 7 (if p - 2). This improves on the bound for odd p given in (l) ; for p = 2 such fixed points do not seem to have been found before. Our result is almost certainly not the best possible; our method is based on a device due to Hartley (3) which probably gives a bound 2k - 1 where k is the best possible.

The search for such fixed points was motivated by a conjecture of J.G. Thompson which states that if a p group A of given order p^k. is contained in only one Sylow p subgroup of the p-soluble group G then the p length of G should be bounded by a linear function of k . An exponential bound was found in (l) for p odd, and the linear bound 2k + 1 for A cyclic and any p . We hope that this paper may suggest a way of getting a linear bound if A is elementary Abelian.

This paper may be regarded as of "Hall Higman type" (see (4)) ; we proceed by showing that if no fixed points of the required type appear in the top 4 (or 6 if p = 2) p sections of the upper p series, then a free module for A must occur in the next one down(Theorem D below gives a linear version of this). In fact it seems likely that such free modules must continue to appear at each successive stage, but we have not been able to establish this. We also give a number of results analogous to Hall-Higman's Theorem B for an operator group of of type (p, p) (see section one) .

The proof uses the same overall method as Shult in (5) where the case that A and G have coprime orders is solved; this case has subsequently been investigated for general A by T.R. Berger (6), (7), (8), but we have not been able to use any of his powerful methods. The main difference between the methods used here and those of (3) is that we have, essentially, to keep track of the whole of G and, unlike (5), we are not able to use induction on G to any significant extent. This, together with the calculations necessary to handle p = 2 and 3, is responsible for the length of the paper.

§1. Statement of Results

<u>Theorem A.</u> Let A be elementary Abelian of order p^2 and let A act on the p-soluble group G. Suppose the p length 1_p of $G/0_p$ satisfies

$$1_p \geq 5 \quad \text{if} \quad p \geq 3$$

$$1_p \geq 7 \quad \text{if} \quad p = 2$$

then A has a fixed point on some A invariant P' section which is acted on fixed point freely by an A invariant p-subgroup of G.

We shall prove this by assuming that A has no such fixed points in the top few members of a p-series and then forcing their existence lower down. It will be convenient to refer to such fixed points as <u>disallowed</u>.

<u>Note.</u> It is not difficult to see that if A is a subgroup of G this Theorem forces A to be contained in at least two distinct Sylow p subgroups, provided the p length satisfies the stated inequality (see (2) "Lemma").

From now on, unless otherwise stated,

A will always denote an elementary Abelian group of order p^2 acting on a p-soluble group G.

In addition, in Theorem B, D and Proposition E, F below we always assume that, writing

$$0_1 = 0_{p\,p} \quad ; \quad 0_2 = 0_{p'p\,p'p} \quad ; \ldots\ldots\ldots\ldots\ldots\ldots$$

we have

(α) $[A, G] \not\subseteq 0_{1p}(G)$;

(ß) A has no fixed points on any disallowed p sections of G.

For all except Bi and C1 V will denote an irreducible module for AG over some finite field k . For all but E and F, V is faithful

Our conclusion will always be the existence of a cyclic A module U, of dimension $\geq (p-1)^2$ - which we call "almost free" - or actually free.

This will be found as a submodule of V_A, except in the case of

B_1 and C_1 where it will be in some A invariant Abelian p <u>subgroup</u> of G. (It is of crucial importance that this is in a <u>subgroup</u> not just in a p section of G). We summarize our results in the following table;

	Condition on		Hypotheses (in addition to α and β)	Conclusion about U
	P	1_p		
B	≥ 5	2	chk . = p	Almost free dim ≥ $(p - 1)^2$.
B*	≥ 5	2	ch k = p ; | 0_p ,(G) | odd.	Free
C	2	4	chk = p	Free
B1	≥ 5	4		Almost free
B1*	≥ 3	4	|G| odd	Free
C1	2	6		Free
D	P≥ 3 p = 2	4 6	chk ≠p	Free
E		1	E4. (Abelian AG p-section in G/0p ,) etc.	p > 2 Almost free; p = 2 Free.
F	≥ 3	2	F4, 5 (two Abelian AG p - sections) etc.	Free ,

U "in" a p subgroup of G

Theorem A follows from B1 and C1 for p ¹ 3 ; from B_1* if p = 3 and G is odd ; and from F if p = 3. Proposition F is necessary because the almost free GF(p) modules provided by E and B, while sufficing for p ≥ *5,* do not suffice for p = 3, due to the simple fact that if the cyclic module U has dimension 4 then

$[U, \underbrace{A, A, A}_{3}] = 1.$ (see 2.21)

= 1 .

We now state these results in detail.

<u>Theorem B.</u> Suppose $p \geq 5$ and

(0) ch k = p ,V is a faithful and irreducible k AG module,

(1) $1_p(G) = 2$.

Then V_A contains a cyclic submodule of dimension $\geq (p - 1)^2$.

<u>Theorem B*</u>. If, in addition to the hypotheses of B, the P' radical of G has odd order, then V_A. contains a free A module.

<u>Theorem G.</u> Suppose p = 2 and

(0) ch k = p ; V is a faithful and irreducible k AG module ;

(1) $1_p (G) = 4$

Then V_A. contains a free A module .

Intermediate between B and C and A we have group theoretic corollaries of B and C (in fact B1* holds for p = 3 as well) :

<u>Theorem B1.</u> Suppose $p \geq 5$ and 1p (G) = 4 •

Then A has a cyclic module of dimension $\geq (p - 1)^2$ on some A invariant Abelian p subgroup of G.

<u>Theorem B1* .</u> Suppose $p \geq 3$, G is odd and $1_p (G) = 4$; then A has a free module on some A invariant Abelian p subgroup of G .

<u>Theorem C1.</u> Let p = 2 and $1_p (G) = 6$.

Then A has a free module on some A invariant Abelian p subgroup of G.

The next result is of particular interest in the case p = 3 (Theorem B.. can only handle the case | G | odd) .

<u>Theorem D.</u> Suppose

(0)V is a faithful and irreducible k AG module and chk $\neq$ p .

(1) 1_p (G) = 4 if $p \geq 3$

1_p(0) =6 if p = 2

Then A has a free module on V.

Theorems B - D will be proved by using a device to Hartley (3) to obtain a section of G having roughly half the p length of G, but rich in useful Abelian p sections, and then proving two rather technical module theoretic results (Propositions E and F below) about the case that the p length is 1 or 2.

Proposition E. Let P be an A invariant S subgroup of G. Suppose

(0) V is an irreducible *k*. AG module where ch k = P •

(1) 1_p (G) = 1 .

(2) [A, G] covers G/0 $_{p'p}$ which is non trivial.

(3) Condition (β) . A has no fixed points on any disallowed *P′* section.

(4.) P contains an Abelian N_{AG} (P) invariant subgroup B not centralised by an $S_{p'}$ subgroup of N_G(P) , and minimal with these properties.

(5) B_V is non trivial.

Then V_A. contains a cyclic module of dimension $\geq$ $(p - 1)^2$; if p = 2 it contains a free module.

Proposition E*. If we add to hypothesis E4, that [B, 0 p']$_V$ is not a 2 group, then V_A contains a free module.

Proposition F. Suppose $p \neq 2$ and that P is an A invariant S_P subgroup of G. Suppose

(0) V is an irreducible *k* AG module where ch *k*= P ,

(1) 1_p(G) = 2 ; G = 0 $_{p\,p'pp'}$; let R = 0 $_{p\,p'}$, etc.

(2) [A, G] covers G/0 $_{p\,p'pp'}$; which is non trivial.

(3) Condition *β* (see E3) .

(4) P/R_o contains an Abel.ian $N_{GA}(P)$ Invariant subgroup B/R_o not centralized by an Sp' of $N_G(p)$ and minima] with this property.

(5) R contains an Abeliar. AG invariant subgroup F not centralized by an $S_{p'}$ subgroup of $[R_1, B]$. F is minimal with this property.

(6) E_v is non trivial.

Then V_A. contains a free submochile.

The major difficulty in proving E and F is in handling the cases o = 2 or p = 3 ; E is essentially to deal with p = 2, while F gives us a free A submodule, which is required to prove A and D for p = 3 . Proposition E is also required to prove Theorems B and C ; it might be possible to prove F for p = 2 as well - thus making B, C and E unnecessary for the proof of Theorem A, but the proof of F is somewhat more involved than that of E, and B and C may have independant interest.

Notation

In addition to notation of (l), we use

$0_1 = 0_{p'p}$, $0_2 = 0_{p'pp'p}$, etc. (0_2 will not be used for the 2 radical). $1_p(G)$ is the p length of G.

$0_{t,p'}$ is defined by $0_{t,p'}/0_t = 0_{p'}/(G/0_t.)$ where t = 1, 2. .. as above.

Thus $0_{1,p'} = 0_{p'pp'}$, $0_{2p'}$ $0_{2,p'} = 0_{p'pp'pp'}$ and so on.

H^G denotes the normal closure. $\cap H^G$ the normal interior of H under G. Thus

$$\cap H^G = \bigcap_{X \in G} H^X$$

S 2 . The main purpose of this section is to deduce Theorem A from Propositions E and F. For $p \geq 5$ the arguments are straightforward; this case is covered in 2.3. The cases $p = 2$ and 3 give rise to some technical difficulty and have to be settled by different arguments. For the rest of this paper we will have two lines of argument, one, via Proposition E, dealing with all $p \neq 3$, and the other, via Proposition F, with $p \neq 2$. It is possible that the latter could be extended to deal with $p = 2$ as well, but since this second method is already somewhat more complicated it seems worthwhile to use both. Both of these will deal with $p \geq 5$ en passant, thus making 2.3 below redundant; however it is so much simpler that it is probably worth retaining.

We start with two elementary lemmas; these form, for $p \neq 3$ the link between the almost free module U of B1 and C1 and the free modules and fixed points of A and D. The argument is thoroughly familiar (it forms the basis of (9) for example). Unfortunately it will not be any use if $p = 3$, unless we restrict the 2 - part of G.

2.11 Lemma. Let V be a cyclic module for $A = E_p 2$ of dimension

$\geq (p - 1)^2$ over GF(p) .

Then provided $p \geq 5$, the p fold commentator

$$U=V(A-1)^p \neq 0 .$$

Proof. Since V is the image of a one generator free A module, it suffices to show that, if V is free, U has dimension

Let and be generators for A and 1 a generator for V. Then

$$(1-\alpha)^r (1-\beta)^s \quad r + s \geq p \ . \quad r, \ s \leq p - 1$$

are linearly independent elements of U ; clearly there are $\geq \dfrac{p (p - 1)}{2}$

such ; for $p \geq 5$ this is greater than $2p - 1$.

2.12 Lemma. Let G have an elementary Abelian A invariant subgroup B such that

$$[B, \underbrace{A, .. A}_{P}] \neq 1.$$

Then if V is a faithful irreducible kAG module where ch $k \neq p$, it follows that V_A contains a free module.

<u>Proof</u>. We note first that we may take k to be a splitting field for all subgroups of AG. For, if not, let k_1 be a finite extension of k which is such a splitting field and consider

$$V_1 = V \otimes k_1 .$$

Let U be an irreducible k_1 AG constituent of this. Then since

$$(V_1)_{kAG} \cong | k_1 : k | \, V ,$$

U_{kAG} is a multiple of V ; thus U is faithful and we may apply our result to deduce that U_A contains a free module. But then it follows that V must. Now

$$B_1 = [B, \underbrace{A .. A}_{P}] \neq 1 .$$

Let V_1 be an irreducible AB constituent of V on which B_1 acts non trivially ; we apply Clifford's theorem (Huppert (10) page 565) to B on V_1 . Suppose W_1 is a homogeneous component, that

$$L = \ker B \text{ on } W_1$$

$$A = \text{stab}_A \ (B \text{ on } W_1) ,$$

and that A_1 is non trivial. Then, since $B_1 \lhd AB$ it is clear that

$$B_1 \not\subseteq L .$$

On the other hand, since k is a splitting field for B, the stabilizer A_1 centralises B/L and so also does

$$[A_1, B] \leq \bigcap L^A = K \quad \text{say.}$$

But now R/K is a cyclic module for A/A_1 and so K must contain B_1. This contradiction shows that $A_1 \neq 1$, so that A has a free module on V_1 an required. It follows that V_A contains a free summand.

2.13 Corollary. If $p \neq 3$, Theorem D follows from B1 and C1.

Proof. Immediate from 2.11 and 2.12.

We next deduce Theorem A, for $p \neq 3$, from B1 and C1. We need an elementary lemma which will be useful later.

2.21. Lemma. Let $G/0_{pp'}$ have an Abelian A invariant p subgroup B such that

$$[B, \underbrace{A.....A}_{p}] \neq 1 .$$

Then A has a fixed point on some disallowed section of $0_{pp'}/0_p$.

Proof. Clearly we may assume that $0 = 1$. Let P be an A invariant p subproun such that $P0_{p'}/_{p'} = B$, and Q an AP invariant Sylow q subgroup (for a suitable prime q) on which

$$P_1 = [P, \underbrace{A.....A}_{p}]$$

acts non trivially. Let V be an irreducible AP constituent of Q on which P_1 acts non trivially; then by 2.12 we have a free module, and hence a fixed point for A on V. Since $V = [V, P_1]$ it is clear that this is disallowed.

2.22. If $p \neq 3$, Theorem A follows from Theorems B1 and C1.

Proof. Firstly it is clear that, on factoring out by a suitable term in the upner p series for $G/0_p$, we may assume

$$0_p(G) = 1 \text{ and } 1_p(G) = \begin{cases} 5 & p \neq 2 \\ ? & p = 2 \end{cases} .$$

Write $1 - 1_p$ (0) and let. $0_{1-1,p'}/0_{1-1}$ denote the P' radical of $G/0_{1-1}$ (recall that $0 = 0_{p'p}$, $0_2 = 0_{p'p'pp}$ etc.). Then it is clear that we may assume

$$[A, 0_{1-1\,p'}] \not\subseteq 0_{1-1}.$$

Now put

$$G_1 = [A, 0_{1-1,p'}]\ 0_{1-1}$$

Applying B1 ($p \geq 5$) or C1 ($p = 2.$) to $G_1/0_1$ we deduce the existence of the subgroup B required by 2.21. This lemma now completes the proof.

We include here a similar lemma which will be used later to handle the case $p = 3$ (in fact $p \geq 3$) .

2·23 Lemma. Let $G/0_{p\,p'}$ satisfy F1 - 5 . Then, assuming Proposition F, there is a disallowed fixed point for A in $0_{p\,p'} / 0_p$

Proof of 2.23.. Clearly we may take $0_p = 1$.

Let P_o be an A invariant p subgroup of G such that $P_o\ 0_{p'}/0_{p'} = F$.

Let P be an A invariant S_p subgroup of G containing P , and L an AP invariant Sylow s subgroup (for suitable prime s) of $0_{p'}$ on which P_o acta trivially. Let $N = N_G(L)$ and V be an AN/L constituent of L on which P_o acts non trivially. Then, since N covers $G/0_{p'}$, A, N. V satisfy the hypotheses of Proposition F and 2.23 follows.

We now use Harley's method ((3) section ?) to handle the case $p \geq 5$. This is a simple version of the first main line of argument.

2.31. Theorem B1 follows from Theorem B.

Proof. Let

$$R = C_{p'pp'p}\ (G)$$

and P be an A invariant S_p subgroup of R ; we shall use Hartley's method to find a suitable Abelian subgroup B of P acted on by $N = N_G(P)$ - which covers G/R - and to apply Theorem B to the action of AN on B .

Let

$$X = 0_4(G)\ [A, G]$$

$$N = N_x(P)$$

$$F = 0^{P\,P\,'P\,P'\,'P}{}_{(N)}$$

$$Y = 0^{p\,p'\,p\,p\,'p}(X)_R \quad .$$

Then Y/R is non trivial and covered by F. Let Q be an S_p subgroup of F; then Q covers Y/R .

Finally, let

$$T = [Q, P].$$

We assert that T <u>has a characteristic Abelian subgroup not centralized by Q</u>.

Before proving this assertion we show that it suffices to prove

2.31. Let B be an AN invariant subgroup of T minimal subject to being Abelian and not centralized by Q. Then since $[B, P] \neq B$ this is centralized by Q ; thus Q acts non trivially on B/[B, P] , Moreover, AN acts on this section ; let V be an irreducible AN constituent of this,subject to non trivial action by Q. Let $G_1 = N_v$. and apply Theorem B to A, G_1, V ; the hypotheses of this are clear except perhaps that A might not be faithful. However if some non trivial subgroup of A centralizes G and V then Theorem B of (l) gives a fixed point for A on some disallowed p'section of G_1 .

Finally we prove the assertion. We note first that it suffices to prove that T does not have class two (see for example (l0) Theorem 13.6 pare 352) Let

$$P_1 = P \cap 0_{p'p}$$

Then, since $Q \not\subseteq R$, $T \not\leq P_1$. Thus, since $p \geq 5$ Hall Higman' s Theorem B shows that

$$[P_1, T, T, T] \neq 1 .$$

Since $[P_1, T] \leq T$ this shows that T is not of class two as required.

2.32. Theorem B (B*) follows from Proposition E (E*) .

Proof. Let $R = 0_{p' p p' p}$, $X = R [A, G]$, $N = N_x(P)$ and $F = [N, A] P$. Then if Q is an $S_{p'}$ subgroup of F, Q covers X/R ; let

$$T = [Q, P] .$$

Now exactly as in 2.31 we may show that T does not have class two, set

$$G_1 = Q \ T \ 0_{p'}$$

and let V_1 be an irreducible AG_1 constituent of V on which $[B, 0_{p'}]$ acts non trivially. Then A, G_1, V_1 satisfies the hypotheses of E(E* in the case that $|0_{p'}(G)|$ is odd) . For these are all clear except perhaps for E 2. But $Q \leq S [Q, A]$ and covers the non trivial group X/R ; thus E 2 holds. The proof of 2.32 is now clear.

For the rest of this section we are primarily concerned with the cases $p = 2$ and $p = 3$. The main difficulty is with repeating the argument of 2.31 for $p = 2$; this is done, using three steps in the p series where two were used before, in two main stages ; The first and most technical is in 2.41 where we show that the subgroup T is not of class two provided T is, essentially, not contained in 0_p. We then have to handle what is really the easy case,that T is Abelian modulo 0_p by, essentially, using Theorem A in $G/0_p$ to obtain a disallowed fixed point. This second stage is also necessary for the case $p = 3$; it is messy in detail but straightforward conceptually.

The first and most technical lemma is an extension of Lemma 3.1 of (1) to the case that A is non cyclic. We recall that $0_1 = 0_{p'p}$; $0_2 = 0_{p'pp'p'}$, . etc.

2.41. Suppose that either

(a) $p = 2$ and $1_p(G) = 3 = 1$ or

(b) $p \geq 3$ and $1_p(G) = 2 = 1$.

Suppose $G \neq 0_1(G) = R$; that P is an S_p subgroup of G and Q a P' subgroup of $N_G(P)$ covering G/R . Let $T = [P, Q]$. Then, provided

$$T' \nsubseteq 0_{1-1}(G)$$

T is not of class two.

<u>Proof.</u> We deal with case (b) first as this is immediate from Hall Higman. For, if $P_1 = P \cap 0_{p'p}$ then (see (l), Lemma 2.3 for example)

$$[P_1, T', T'] \neq 1 .$$

Since $[P_1, T'] \subseteq T' \lhd P$, the conclusion is clear.

Thus suppose (a) holds ; $p = 2$, $1 = 3$. We note first that, setting

$$R_o = 0_{p'pp'pp'} , \ R_1 = 0_{p'pp'pp'} , \ \ldots \ R_3 = 0_{p'p}$$

we may assume that

(0) $[T', R_0]$ and $[T', R_2]$ are both, modulo R_1 and R_3 respectively, non Abelian 3 groups. For if not,then 3.1 of (l) shows that T does not have class two as required. Now let M be an $S_{p'}$ subgroup of G containing Q and let $M_o = M \cap R_o$. Then if

$$\tilde{N} = N_G(M_o)$$

we have that

(1) $L = \tilde{N} \cap P$ is an S_p subgroup of $\tilde{N}$. For, since $M_0P = R$ we have that

$$\tilde{N} \cap R = M_0 (\tilde{N} \cap P) .$$

But now M covers $\tilde{N}/\tilde{N} \cap R$. Thus L is an S_p subgroup of $\breve{N}$. In particular

(2) L covers R/R_0 .

Now let

$$\check{T} = [\, L, \; Q \,] \; .$$

Then since $\check{T} \geq T$ it suffices to show that T does not have class two ; also since both Q and L normalize M_o we have

(3) $\check{T}$ normalizes M_o .

Now $\check{T}$ covers T R_o/R_o (by (2)) ; thus

(4) $\Gamma = [\tilde{T}' , R_o]$ R_1/R_1 is a non Abelian 3 group.1

Now let M_1 be a T' invariant S_3 subgroup of M_o (by (3)) and

$$\tilde{M} = [\, M_1, \; \tilde{T}' \,] .$$

We have

(5) $\tilde{M}$ covers Γ ; in particular $\tilde{M}' \not\subseteq R_1$.

The main step is to prove

(6) $\tilde{M}$ R_3/R_3 is not of class two. This will show that T is not of class two. For then some T invariant Abelian subgroup B/R_3 will not be centralized by $\tilde{T}'$ and we may apply 3.1 of (1) to deduce that

$$[R_3, \; \check{T}, \; \tilde{T}, \; \tilde{T} \,] \not\equiv 1 \quad \text{modulo } 0_{p'} \; .$$

Since $[P, \check{T}] \leq T$ it is clear that T can not have class two.

We now come to the proof of (6). We wish to apply our usual argument, with M in place of T to an $S_{\{2,\,3\}}$ subgroup of

$$\Delta = R_o / R_3 \; ;$$

we must first ensure that this is sufficiently large, which follows from the abservation that

(7) $\Delta_1 = [\, \Delta , \; T \,]$ is soluble. For certainly this is soluble modulo R_2 ; factoring out its 3-radical R_2 is centralized by T' . Thus

$$(\Delta_1 \cap R_2/R_3)$$

is a 3 group and (7) is proved.

Now let $\tilde{\Delta}$ be an $S_{\{2,3\}}$ subgroup of Δ_1 containing $\tilde{M}$ modulo R_3 .

Since Δ_1 is soluble this is a Hall subgroup and, making the natural identifications,

(8) $\tilde{M}'$ is not contained in the 3-radlcal of $\tilde{\Delta}$. For if it is then

$$\tilde{M}' \subseteq 0_{p'}(\Delta = R_2/R_3 .$$

But this contradicts (5) .

Now applying Hall Higman (see for example (1) Lemma 2.3) to the group $\tilde{\Delta}$ we find that (6) holds. As we remarked above this completes the proof of 2.41.

We now apply 2.41 to make deductions from E and F.

2.42. Suppose either

(a) $p = 2$ and $t = 3$ or

(b) $p \geq 3$ and $t = 2$; $0_p(G) = 1$: in either case

that $G = 0_{t+1,p'} \neq 0_{t+1}$ and that $[A, G]$ covers $G/0_{t+1}$.

Suppose further that $0_{t+1}/0_{t,p'}$ has an AG invariant Abelian subgroup $B/0_{t,p'}$ not centralized by any $S_{p'}$ subgroup of G. Then, provided A has no fixed points on any disallowed p' section of G (condition β), and Propositions E and F hold, it follows that p = 2 and G has an elementary Abelian A invariant p subgroup which affords a free module for A,

Proof. Replacing B by a subgroup if necessary we may assume that B is minimal: now let $R = 0_{t,p'}$ and P_o be an A invariant S_p subgroup of B. Next let $P = P_o \cap R$, and $N = N_{G}.(P)$. Let

$$F = [N_R(P) , P_O]$$

and Q be an $S_{p'}$ subgroup of F. Then $T = [P, Q]$ is not special. For we note first that

$$T' \nleq 0_{t-1}(G) = R_o .$$

To see this suppose $T' \leq R$; consider first case (a) $p = 2$, $t = 3$. Consider the action of AN on T/T' ; let V be a faithful irreducible AN constituent of $T/T \cap R_o$. We assert that A, N, V satisfies the hypotheses of Proposition E (after taking B_1 minimal in B as required by E4). Thus A has a free module on V and by Lemma 2.21 we find a disallowed fixed point for A in $0_{t-1,p'}$.

Case (b) $p \geq 3$, $t = 2$. Here we arrive at the same conclusion, using F in place of E. We assert that $G/0_{p'}$ satisfies the hypotheses of 2.23 ; equivalently $G/0_{1,p'}$, satisfies F1 - 5. But, since N covers this quotient, it is clear that F1 - 4 hold and, taking F minimal in T subject to non trivial action by Q, that F5 holds as well. Now 2.23 ensures that A has a disallowed fixed point as asserted.

Thus we have that T is non Abelian modulo R_o, and we may apply 2.41 to $F0_t$ to deduce that T is not of class two. Thus T has a characteristic Abelian subgroup D not centralized by Q (see for example (10) Theorem 111 13.6 page 352). Suppose now that $p \geq 3$. We assert that AN $0_{p'}$ satisfies the hypotheses of 2.23 ; equivalently (since $0_p(G) = 1$) that N satisfies F1 - 5 modulo $0_{p'}$. Since N covers $G/0_{1,p'}$ it is clear that F1 - 4 hold ; finally, taking $F \subseteq D$ minimal subject to non trivial action by Q, that F5 holds as well. Thus A has, by 2.23 a disallowed fixed point in $0_{p'}$. Thus $p = 2$.

Now take V an irreducible GF(p)AN constituent of D^+ on which Q acts non trivially ; then A,N,V satisfies the hypotheses of E and A has a free module on V as required. This completes the proof of 2.42.

2.51 Theorem C1 follows from Proposition E.

Proof. Let P be an A invariant S subgroup of G, $N = N_G(P)$ and

F = [N, A] P. Then let Q be an $S_{p'}$ subgroup of F and

$$T=[P, Q] .$$

We assert that T is not of class two modulo 0_3.

This follows from 2.41 provided

$$T' \nleq 0_5.$$

But if $T' \leq 0_5$, then we consider $G_1 = G/0_2$; we assert that this satisfies the hypotheses of 2.42. But then 2.21 ensures that A has a disallowed fixed pointj thus $T' \leq 0_5$ as required, and T is not of class two modulo 0_3 · We now apply 2.41 again; this shows that A on $F0_3$ satisfies the hypotheses of 2.42. This completes the proof of 2.51.

We do not give full details of the following.

2.52 Theorem B1* follows from E*.

Proof:. this follows from 2.41 and 2.42 - suitably amended to use E* in place of E - exactly as 2.51 does.

2.61. For $p \geq 3$, Theorem A follows from Proposition F.

Proof. This follows from 2.41 and 2.42 exactly as 2.51 does.

2.62. For $p \geq 3$ Theorem D follows from Proposition F.

Proofs, this follows using a suitably amended version of 2.42, from 2.41 and 2 just as 2.51 does.

Finally we do not give full details of the proof of C as this is not needed.

2.71. Theorem C follows from Proposition E. We proceed as in 2.32 using Lemma 2.41 to ensure that T does not have class two.

§ 3. In this section we tighten up the structure of the groups considered in Proposition E and F. We first show that in proving E we may take G to satisfy

<u>Hypothesis I</u>A is elementary abelian of order p^2 and G is a soluble group acted on by A; P is an A invariant Sylow p subgroup of G.

(0) V is a faithful irreducible kAG module where k is a field of characteristic p.

(1) $G = 0_{p'pp'}(G)$; write $R_1 = 0_p$, $R_2 = 0_{p'p}$. p'p

(2) $A_1 = C_A(G/R_2) \neq 1$; $A_1 \neq A$;

[A,G] covers G/R_2 ; G/R_2 is a q group for some prime $q \neq p$.

(3) $E = [R_1, P]$ is anr group for some prime $r \neq p$; A has no fixed points on any section of E not centralized by an S_p

(4) P contains an abelian $N_{AG}(P)$ invariant subgroup B not centralized by any Sylow q subgroup Q of $N_G(P)$; B is minimal with these properties.

(5) P is minimal subject to satisfying both

(i) P contains B satisfying (4)

(ii) There exists a Sylow q subgroup Q of $N_G(P)$ such that AQP is a group.

(6) R has a normal Sylow r subgroup R and if Q is a Sylow q subgroup of $N_G(P)$ then G = QPR; Q is minimal such that QPR is A invariant and satisfies (2) and (4); $Q/\phi(Q)$ is an irreducible A module, modulo P; if p = 2, Q is cyclic.

<u>Hypothesis I</u> * In addition to Hypothesis I, $r \neq 2$.

<u>Proposition 3</u>. 1. In proving Proposition E we may assume that G satisfies Hypothesis I. In proving Proposition E* we may assume I*.

<u>Proof</u>. We show that if A, G, V satisfy the hypothesis of Proposition E, (E*) then there are sections G_1, V_1 of G and V respectively such that A, G_1, V_1 satisfies Hypothesis I (I*). We prove this by induction on $|G|$, assuming that A, G, V provides a counter example with least possible $|G|$.

Write $R_1 = 0_{p'}(G)$, $R_2 = 0_{p'p}(G)$.

(1) G is faithful on V. For if not, consider $G^* = G_V$; we show that A, G*, V satisfies the conditions of Proposition E (E*), and so, by induction, $G^* = G$.

These are clear except for E4, 5 (4*, 5*). Let $B^* = B_V$, etc. We assert that, if Q is an $S_{p'}$ subgroup of $N_G(P)$,

(i) $[B^*, Q] \neq 1$

(ii) B* is minimal as required.

Let S be the kernel of G on V, and let $S_1 = S \cap B$. Then we are given that $S_1 < B$. By the minimality of B, it is clear that S_1 is centralized by Q, so that

$$[Q, B] \not\subseteq S_1 .$$

Thus (i) holds. Now suppose $B_1{}^* \subseteq B^*$ satisfies E(4.). Then consider

$$B_1 \cap BS = (B_1 \cap B) S .$$

Then $B_1 \cap B$ cannot be centralized by Q, and so, by the minimality of B is B itself.

Thus (ii) is proved. It is now clear that E_4, 5 (4*, 5*) hold for A, G*, V so that (1) is proved.

(2) G satisfies 1.2. Choose a prime q and an S_q subgroup Q of $N_G(P)$ such that

(i) A normalizes QP

(ii) $[Q, B] \neq 1$.

Let Q_1 be an S_q subgroup of [Q, A] ; then by E2, Q_1 satisfies (i) and (ii).

Let $r = Q_1 P / C_{Q1}(B) P$

Then Γ is generated by the centralizers of non trivial elements of A((1.1) Theorem 3.16, page 188), and so at least one of these,

$C_{\Gamma,}$ (A) say, must fail to centralize A. Let

$$C_\Gamma (A_1) = Q_2/C_{Q1} \quad (B) \quad (\text{modulo } P)$$

and Q_3 be an S_q subgroup of $[Q_2, A]$. Finally let $G_1 = A Q_3 P R_1$ and $B_1 \subseteq B$ B be minimal subject to E4. Finally let V_1 be an irreducible AG_1 constituent of V on which $[B_1, Q_3]$ acts non trivially. Then A, G_1, V_1 satisfies conditions E(E*) and so by induction must be A, G, V. But it also satisfies 1.2. Thus (2.) is proved.

(3) G satisfies 1.5. For if not choose $M \not\subseteq P$ satisfying 1.5. Then M contains an abelian normal subgroup B_1 satisfying I.4, and for some S_q subgroup Q_1 of N_G (M), AQ_1 M is a subgroup. Let $G_1 = AQ_1 MR_1$ and V_1 be an irreducible AG_1 constituent on which $[B_1, Q_1]$ acts non trivially. Then A, G_1, V_1 satisfies conditions E (E*) , and since $|G_7| < |G|$ we may apply induction.

(4) G satisfies 1.3. Let R be an AP invariant S_r subgroup of R_1, for suitable r so that B acts non trivially on R (if E* holds we take $r \neq 2$). Let $N = N_G$ (R) and Q be an S_q of N_N (P) . Finally let G_1 = QPR and V_1 be an irreducible AG_1 constituent of V_1 on which [B, RJ acts non trivially. Then A, G_1, V_1 satisfies conditions E(E*) and also I.3.

(5) G satisfies 1.6. We already have in (4) above that G = QPR; now choose $Q_1 \subseteq Q$ minimal such that $G_1 = Q_1PR$ is A invariant and satisfies 1.2 and 1.4. Then take V_1 as an irreducible AG_1 constituent of V. Clearly E(E*) holds so that we may take $Q_1 = Q$, $V_1 = V$. Now if $D = C_Q$ (B) it is clear that Q/D affords an irreducible A module (modulo P). Thus $D \supseteq \phi$ (Q) , and if

$$\overline{Q} = \overline{Q}_1 \oplus \overline{D}$$

where $\overline{Q}_1$ is, modulo P, an A module,then unless $Q_1 = Q$, Q_1 gives a contradiction to the minimality of Q. Thus D = f (Q) .

Finally, if p = 2 it is clear that Q is cyclic. Thus (5) is proved.

We next tighten up Proposition F in a similar manner.

Hypothesis II. A, G, P as in preamble to I above; p # 2 .

(0) as 1.0 except that ch k - s $\neq$ p .

(1) G = $0_{p p' p p'}$; R_o = 0_p , R_1 = $0_{p p'}$, , R_2 = · · .

(2) I.2.

(3) - (6) I.3 - 6 for G/R_o .

(7) R contains an abelian AG invariant subgroup F such that F is not centralized by any S_r subgroup of $[R_1, B]$.

F is minimal with this property.

<u>Proposition 3</u>.2. In proving Proposition F we may assume that Hypothesis II holds.

<u>Proof</u>. The proof is very similar to that of 3.1; we carry out only the first two steps in detail as these involve all the necessary modifications to the arguments of 3.1.

<u>Proof of 3.2</u>

(1) G is faithful on V . As before let B* denote B_V , G* denote G_V and so on. Then as in 3.1 we assert

(i) F* is not centralized by any $S_{p'}$ subgroup of [R , B]

(ii) $[B^*, Q] \nsubseteq 0_p(G)$ for Q an $S_{p'}$ subgroup of $N_G(P)$.

(iii) B* is minimal in $G^*/0_p(G^*)$ as required by F4.

To see this we let S be the kernel of G on V . Let $F_1 = F \cap S$; then since F_1 is properly contained in F it is centralized by an S_p, subgroup T of $[R_1 , B]$; clearly $F/F_1 = F^*$ cannot be centralized by T. Thus (i) holds. Now let $\tilde{R} \supseteq S$ and

$$0_p (G^*) = \tilde{R}^*.$$

Suppose that

$$[B^*, Q] \subseteq \tilde{R}^* \quad .$$

Let $$B_2 = [B, Q] \ .$$

Then $$B_2 \subseteq \tilde{R}$$

and by the minimality of B we deduce that

$$B \subseteq \tilde{R} \ .$$

Thus

$$[B, R_1] \subseteq [\tilde{R}, R_1] \subseteq \tilde{R} \cap R_1$$

But

$$\tilde{R} \cap R_1 \subseteq R \quad (R_1 \cap S) \quad .$$

Thus

$$[B, R_1] \subseteq (R_1 \cap S) R_o \quad .$$

But now any $S_{p'}$ subgroup of $[B, R_1]$ is contained in S, which contradicts (i) . Thus we have proved (ii) .

Next suppose $B_3{}^* \subseteq B^*$ and has the required properties for B (F4) . Then, as before, consider

$$B_3 \cap BS = (B_3 \cap B) S \quad .$$

Now $B_3 \cap B \subseteq B$ and is not centralized by an $S_{p'}$ subgroup of $N_G(P)$ modulo R_o . But this means that $B_3 \cap B = B$ so that $B_3{}^* = B_1{}^*$.

Finally let $F_1 \leq F^*$ be minimal subject to F5 ; then as in 3.1 we have demonstrated (1) .

(2) G satisfies II.2. Chose a prime q and an S_q subgroup Q of N_G (P) such that

(i) A normalizes QP

(ii) $[B, Q] \not\subseteq R_o$.

Let Q_1 be an S_q subgroup of [Q, A] and $D = C_{Q_1}(B/R_o)$. Consider

$$\Gamma = Q_1 P/DP$$

Then Γ is generated by the centralizers of non trivial elements of A((11) Theorem 3.16 page 188) and so at least one of these, $C_\Gamma\ (A_1)$ say, must fail to centralize A.

Let

$$C_\Gamma (A\ ) = Q_2/D \qquad \text{modulo } P$$

and Q_3 be an S_q subgroup of $[\,Q_2\,,\ A\,]$.

Finally let

$$G_1 = AQ_3\ PR_1$$

and

$$B_1 / R_o \subseteq B/R_o$$

be minimal subject to F4. Then if

$$\Delta = R_1/C_{R1}\ (F)\ R_o$$

we have

$$R_o \subseteq C_B\ (\Delta) \subsetneq B$$

and so $C_B(\Delta)/R_o$ is centralized by Q. Thus B_1 does not centralize Δ . Now take F_1 minimal in P subject to F5 (with G_1 and B_1 in place of G and B). Finally let V_1 be an irreducible AG_1 constituent of V on which F_1 acts non trivially. Then as in 3.1, we have proved (2).

Finally steps (3), (4) and (5) of 3.1 carry over in a similar manner; the crucial fact is that $C_B(\Delta)/R_o$ is centralized by Q, so that we may decrease P,B and Q without causing B to act trivially on Δ .

§ 4. A special case. In this section we carry out a simple version of some of the basic arguments and deal with a technical difficulty in the case that $p = 2$. Basically we show that, in proving the special case of Proposition E and F to which section three has reduced up. we may assume that V_{R_1} is <u>not</u> homogeneous - or, for $p = 2$, somewhat more than this (see 4.3 below). For Proposition E, R_1 is just $0_{p'}$ and the argument is a straight forward version of Shult' s (5) ; for F however, R_1 is $0_{pp'}$ and we have to introduce some new methods - which will be important in § 5.

A is, as always, elementary of order p^2 and acts on the p soluble group G,

4.. 11. Let V be a faithful irreducible AG module where G is an r group for some prime $r \neq p$ and ch$k = p$.

Then V_A contain s a cyclic module of dimension at least $(p - 1)^2$; if $p = 2$, V_A contains a free module.

4.11*. If $\neq 2$ then V_A contains a free module.

Proof. We note first that, as in 2.12, we may assume that k is a splitting field for subgroups of G.

We use induction on $|G| + \dim V$. Let A, G, V be a counter example with $|G| + \dim V$ as small as possible. Then, following Shult (5) .

(l) V_G is homogeneous. For suppose

$$V_G = W1 \oplus \;..\; \oplus W_t$$

where the W_1 are Wedderburn (homogeneous) components. Then if the stabilizer, I, of G on W_1 is proper it must be A_1G for some non trivial A_1 (by Clifford's theorem). Thus, by Hall Higman

$$[A_1, G]_{w1} = 1 .$$

But since A permutes the W. transitively we deduce that

$$[A_1, G] = 1 ,$$

which is not the case since AG has trivial p radical - since ch $k = p$.

(2) $[G, A] = G$. For if not $C_G(A)$ covers $G/[G, A]$ and if

$$V_A[A,G] = W_1 \oplus \ldots \oplus W_t$$

the homogeneous components W_1 .. W_t are permuted transitively by

$C_G(A)$. By induction A is not faithful on

$$[A, G]\, W_1 .$$

But now if A is the kernel of A on this, we deduce that $[A_1, G]_V = 1$.

But this is not the case since AG has trivial p radical.

(3) Let $M \subset G$ be a maximal proper AG invariant subgroup. Then V_M

is homogeneous. For if not let I be the stabilizer of a component.

Since V_G is homogeneous, we may assume that

$$I = AM$$

(For I covers AG/G ; by Sylow's theorem it may be assumed to contain A ;

since G/M is an irreducible A module, $I \cap G = M$) .

Since G is nilpotent, G/M is Abelian and so the kernel, A_1 say, of A

on G/M is non trivial. Since

$$M\, C_G(A_1) = G$$

the components of M on V are permuted transitively by $C_G(A_1)$. Thus,

since A does not centralize G/M ((2) above), for some $x \in C_R(A_1)$ we

have

$$A \cap I^x = A .$$

By Hall Higman's Theorem B, and Clifford's Theorem we deduce that

$$[A_1^{x-1}, M]\, W_1 = \mathbf{1} .$$

Now, as in (1), it follows that

$$[A_1, G] = 1 ,$$

a contradiction.

(4) For each AG invariant subgroup T of G, V_T is homogeneous. If T is Abelian it is cyclic and central in AG. This follows immediately from (3). for suppose V_T were not homogeneous ; let $I \geq A$ be the stabilizer and $I \cap G = I_1$. Then, since G is nilpctert, I_1 i contained in some maximal AG invariant subgroup M of G, But since V_M r p ; is homogeneous, I_1 covers G/M .

(5) Every characteristic Abelian subgroup of G is cyclic. Thus G is extraspecial ($r \neq 2$) or, if r = 2, it is the central product of extraspecial, dihedral, generalized quaternion and quasidihedral groups (see (10) page 357 Satz 13.10) .

By induction G is the direct sum of two irreducible A modules. Let the kernels of these be A_1 and A_2 . Then

$$G = [G, A_1] \; [G, A_2] = G_1 \; G_2 \quad \text{say} .$$

Now if p = 2, each of G_1 and G_2 is Abelian - which contradicts (4) ; moreover, in general, since A centralizes G we have

$$G_i = C_G \; (A.) \qquad j \neq i \quad .$$

The "three subgroups lemma" applied to A_i , G_j. and G for $i \neq j$ then shows that

$$[G_i , G_j] = 1 .$$

Thus, since neither G. can be Abelian (4), AG is the central product of A_i G_i and V is an "outer" tensor product

$$V = u_1 \oplus k \; U_2$$

where each U_i is an irreducible A_i G_i module.

By Hall Higman we have the required result (note that for 4.11*, if $r \neq 2$ we cannot have an "exceptional" case) .

We now come to the result needed for the proof of Proposition F.

We are essentially, looking at the p - P' radical of a group G satisfying Hypothesis II.

4.12. Let V be a faithful irreducible k.AG module where

(i) ch$k$.= p ;

(ii) G $=0_{p,r}$ (G) ;

(iii) $0_p(G) = R_o$ contains an Abelian AG invariant subgroup F such that A acts faithfully on $G/R_oC_G(F)$;

(iv) V is homogeneous for G.

Then V_A. contains a free submodule.

<u>Proof.</u> As for 4.11 we note first that we may take k to be a splitting field for subgroups of AG, and proceed by induction on | G | + dim V, considering a counter example in which this is as small as possible.

(1) V_{Ro} is homogeneous. If not let $M \supseteq R$ be a maximal proper AG invariant subgroup and assume that V_M is not homogeneous. Since G/R_o is nilpotent, G/M is Abelian and

$$A_1 = C_A(G/M)$$

is non trivial. Now if we let

$$D/R_o = C_{G/Ro}(A_1)$$

we find that D supplements M. Let I be the stabilizer of M on a Wedderburn constituent W say ; as before we may take

$$I = AM .$$

Thus I is supplemented by D . For x ∈ D\M we have

$$A \cap I^x = A_1 .$$

Now considering A_1M on W_1 x , letting Q_1 be an S_r subgroup of $[A_1, G]$, we find that

$$[Q_1, F]_{W_1x} = 1.$$

For, let L - Ker F on W_1 and suppose $[Q_1, F]$ is not contained in L^x.

Then

$$[A_1, F] \not\subseteq L^x .$$

So by a well known argument, applying Clifford's theorem as in 2.12, we find a free module for A_1 on W_1 x and, by Clifford again, one for A on

$$(W_1 x)_{A1} \; A .$$

Now let Q be an S_r subgroup of G containing Q_1. Then Q_1 is normal in Q and since Q covers G/M we find

$$[Q_1, F] \subseteq \bigcap_{X \in G\backslash M} L^X .$$

Thus $[Q_1, F] = 1$. But now we have a contradiction to (iii).Thus

(1) is proved.

(2) $0^P(G) = G$. Let $\hat{G} = 0^P(G)\ [G, A]$.

Then if $0^P \neq G$, it follows that $\hat{G} \neq G$.

Thus (2) will follow if we can show that $V_{\hat{G}A.}$ is homogeneous. Suppose not, and let

$$V_{\hat{G}A} = w_1 \oplus \ldots \oplus w_t .$$

Then, by induction, for some non trivial subgroup A_1 of A we have that, for any S_r subgroup Q_1 of $[G, A_1]$

$$[Q_1, F] \leq L = \ker \text{ on } W_1 .$$

But now $C_{R_o}(Q_1)$ supplements $\hat{G}$ and so must permute the homogeneous components transitively ; clearly this implies, as before, that $[Q_1, F] = 1$. But this contradicts (iii). Thus (2) is proved.

To complete the proof we show that V_F is homogeneous, a contradiction. To achieve this we consider M a maximal proper AG invariant subgroup of R_o containing F. Suppose V_M is not homogeneous. Let I be the stabilizer in AG of a Wedderburn component W_1.

Then

(3) No conjugate of I contains A. For suppose I A. Consider first

Case (i) A is faithful on R_o/M . Then by 4.11 applied to the action of AG on R /M, we obtain a cyclic A module, of dimension at least $(p - 1)^2$, contained in $(R_o/M)^+$. Taking x as a generator of this we find that C_A (x) = 1 ; now

$$A \cap I^x = (A^{x-1} \cap {}^{I})^x \leq (AR_o \cap I)^x = (AM)^X .$$

Thus

$$A \cap I^X \leq A \cap A^x \text{ modulo } M .$$

But since $C_A(x) = 1$ we find that

$$A \cap I^x = 1 .$$

Thus we are done by Clifford's theorem.

Case (ii) The kernel A_1 of A on R_o/M is non trivial. Let Q_1 be an S_r subgroup of I A_1, G] contained in I. Then

$$D = C_{R0} (Q_1)$$

covers R_o/M . Now by (2) above A does not centralize R_o/M ; thus we may take

$$x \in D \setminus C_D (AH/M) .$$

Then

$$A \cap I^x = A_1$$

and thus,

$$[A_1, F]_{w_1 x} = 1 .$$

But since the components are permuted by D, we may deduce that

$$[Q_1, F] = 1 .$$

This contradicts our hypothesis (iii). Thus (3) is proved.

We deduce that

(4) $| I^x \cap A | = p$ for all x in G .

To complete the proof we must use more sophisticated versions of the two arguments given in (3) above. First we establish

(5) A is not faithful on R_o/M . This is similar to (3) case (i) above. Let $x \in R_o$ generate a cyclic A module of dimension at least $(p - 1)^2$ modulo M. Now, as before, if

$$T = AR_o \cap I ,$$

we have, for $y \in G$,

$$I^y \cap A = T^y \cap A .$$

Now let

$$T \cap A = < \alpha > ;$$

$$T^x \cap A = < \beta > .$$

Since x cannot centralize α we have, working modulo M ,

$$A = < \alpha, \beta > ;$$
$$T = < \alpha , \beta^{x-1} > .$$

Consider $T^{x2} \cap A$; we may take this to be $\alpha\beta$; but now we have

$$\alpha\beta \in <\alpha^{x2} , \beta^x > .$$

Thus

$$\alpha\beta = \alpha^{x2} \beta^x ,$$

or

$$[\alpha, x^2] = [\beta^{-1}, x] .$$

It follows that

$$[A, < x >]$$

has dimension at most p - 1, so that $<x>$ has dimension at most p which is not the case.

Finally, let $A_1 = < \beta >$ be the kernel of A on R_o/M ; then taking Q_1 as an S subgroup of [G, A] contained in I , the centralizer K of Q_1 in R_o covers R_o/M .

case (i) $I \cap A = A_1$. Then clearly $I^x \cap A = A_1$ for all x in K_2,

so that

$$[Q_1 , F]_{w_1 x} = 1 \qquad \text{for } x \in K .$$

But, since K centralizes Q_1 , vis find, that $[Q_1, F]_V = 1$ which is not the case.

Case (ii) $I \cap A = A_2 \neq A_1$, Let $A_2 = < \propto >$; L = ker F on W_1, and, if Q in an S_r subgroup of I containing Q_1 ,

$$D = C_Q (F/L) .$$

Then $$D \supseteq [A_2 , Q]$$

Also, as in case (i), since K covers R_o /M, we have

$$\bigcap_{X \in K} L^X = 1 .$$

Thus D $\not\supseteq$ Q_1 ; in fact D does not cover [A, Q] modulo R_o. But, by (2)

$$[A_2 , R_o] \nleq M .$$

Thus we may take x Î K not centralizing A_2 modulo M ; then

$$A \cap I^x = A_3 \neq A_2 .$$

We deduce that

$$[A_3 , Q^X] \leq D^X$$

or $$[A_3^{x-1} , Q] \subseteq D .$$

Since A_3 and. A_2 are distinct module R_o , we deduce that D contains Q_1 .

a. contradiction.

This completes the proof of 4.12 . It is now a simple matter to deduce an important corollary, E denotes the subgroup introduced in section three (Hynothesis I) or $R_o [R_1 , B]$ (for II.)

4.2 Corollary, in proving Proposition E and F we may assume that A, G, V satisfy Hypothesis 1, II respectively and provided A acts faithfully on

$$\Delta = E \quad \text{or} \quad E/C_E(F) \; R_o$$

respectively, V_R is not homogeneous.

Finally we strengthen 4.2 in the case p = 2.

4.3 In proving Proposition E for p = 2 we may assume that if A, G, V satisfy Hypothesis I and C_A (B) = 1 , then V_{BR} is not homogeneous.

Proof. Suppose A, G, V satisfy Hypothesis I, that C_A (B) = 1 and p = 2. Let V_1 be an irreducible ABR constituent of V. Then if V_{BR} is homogeneous we have that $(V_1)_{BR}$ is also, and is faithful for BR. Thus V_1 is faithful for ABR. We first establish

0) [B, A, A] = 1 . For if not, choose an irreducible AB constituent U of E on which [B, A, A] acts non trivially. Let I be the stabilizer of B on a component of U; as in 2.12, I = B and we have a fixed point for A in U, by Clifford. But this contradicts I.

(2) $(V_1)A[B,A]_R$ is not homogeneous. For let V_2 be an A [B, A] R constituent of V_1 ; then let

$$(V2)_{AR} = W_1 \oplus \,.\,.\, \oplus \; W_t \,.$$

by 4.1, $[A_1 , R] \leq L = \ker R$ on W_1 for some non trivial subgroup A_1 of A ; also

$$\bigcap_{y \in [B,\ A]} L^y = \ker\ R \text{ on } V_2 \ ,$$

But, by (1) above, [B, A] centralizes A.

Thus we have

$$[A_1 \;\; R]\; V_2 = 1 \;\; .$$

Now B acts faithfully on R since R contains E = $[R_1, B]$ and G has trivial p - radical.

Thus R is not faithful on V_2 and (2) follows.

Next let T ≥ A[B, A] be maximal in AB, chosen so that $(V_1)T$ is not homogeneous.

Let $b \in AB \setminus T$ be an involution ($b^2 = 1$) .

Then

$$(V_1)_T - V_{11} \oplus V_{11} \otimes b \quad .$$

Let $L = \ker E$ on V_{11};

$$S = C_T \ (E/L) \quad .$$

By 4.1,

$$S \cap A \neq 1 \quad .$$

Now since A has no fixed points on $\overline{E}$, we have

$$S \cap A = A_1 \neq A .$$

Let $S^b \cap A = A_2$.

Then $A_2 \neq A$.

Now $S \wedge S^b$ centralizes E and so

$$S \cap S^b \leq k = C_{AB} \ (E) \quad .$$

Also, since K AB and $K \cap B = 1$, we have

$$K \leq C_{AB} \ (B) = B \ ,$$

since $C_A \ (B) = 1$. Thus K = 1, and

$$S \cap S^b = 1 \quad .$$

In particular, $A_2 \neq A_1$. Let $A_1 = <\propto>$, $A_2 = <\beta>$;

then

$$S \geq <\propto , \beta^b > \quad .$$

Now since $\propto$ centralizes E/L , β can have no fixed points on

$$E/L \ ø \ (E) = \Gamma \quad ;$$

sitnilarly $\propto^b$ has none. Thus $\alpha^b \beta$ must centralize Γ, so that

$$\propto^b \beta \in S$$

But $\alpha \beta^b \in S$; thus

$$\propto \beta^b \in S \cap S^b \quad .$$

But we have shown that $S \cap S^b = 1$; since $\propto \beta^b \neq 1$ we have a contradiction. This completes the proof of 4.3.

§5. Proof of Propositions E, E* and F. In this section we complete the proof of the special case of Propositions E, E* and F to which 3.1 and 3.2 have reduced us. Many of the methods have already appeared in section four; thus we follow Shalt's approach, moving down the upper p - series of G showing that the restriction of V to successive terms of this series is homogeneous. Using the results of section four it is, loosely speaking, only necessary to come down as far as $0_{pp'} = R_1$. It is perhaps worth pointing out that the proof of E for $p \geq 5$ occupies only a small proportion of this section; it is mainly proving F and dealing with the case p = 2 where, as in 4.3, a lot of detailed special pleading is required. Since the proof of E* involves only trivial modifications to that of E - the only difference being that exceptional Hall Higman situations are excluded at a number of points - we shall leave these to the reader.

To avoid repetition we carry out the proofs of Propositions E and F simultaneously; we assume that A, G, V satisfies either Hypothesis I or II as the case may be. Several frequently used arguments involving the application of Clifford's Theorem have appeared in 4.1 ; we shall not always give full details of these.

We note first that, as in 2.12, we may take k to be a splitting field for subgroups of AG.

We recall that if Hypothesis I holds $E = [R_1, B]$; if Hypothesis II holds let

$$E = R_o[R_1, B] .$$

Then let

$$\Delta = E \text{ if I holds ;}$$

$$\Delta = E/R_o\, C_E(F) \text{ if II holds .}$$

(1) A acts faithfully on Δ . For, let U be a non trivial irreducible AG constituent of $\overline{\Delta}$; then B_U is Abelian and centralized by the kernel of A on U, but not by A. For if it were centralized by A then it would be centralized by Q ; but the minimality of B ensures that no proper AG invariant quotient of B can be centralized by Q. Thus the argument of 2.12 shows that A/A_1 has a free module on U, giving a disallowed fixed point for A.

(2) Let $\breve{G} = G A_1$. Then V_G is homogeneous. For suppose not, let

$$V_{\breve{G}} = W_1 \oplus \ldots$$

be the Wedderburn decomposition. Then

$$A = \mathrm{Stab}_A (\breve{G} \text{ on } W_1) .$$

Now suppose first that Hypothesis I holds ; we deduce, by Hall Higman, that

$$[A_1, R_1]_{w_1} = 1 .$$

But since $[A, A_1] = 1$ we deduce that A_1 centralizes R_1 and so Δ , a contradiction to d). Suppose then that II holds. Here we deduce that

$$[A_1 , F]_{w_1} = 1$$

and hence that A_1 centralizes F ; but then A_1 centralizes Δ and we contradict (l) again.

(3) V_G is homogeneous. For if not the stabilizer in A must be A_2 , distinct from A_1.. But then it is clear that A_2 does not centralize R_1. (in the case of I) or F(in case II) so we may proceed as in (2) ,

(4) Let $\breve{R}_2 = R_2 A_1$. Then $V_{\breve{R}_2}$ is homogeneous. Let M be the unique maximal proper AG invariant subgroup of $\breve{G}$ containing $\tilde{R}_2$. We show that V_M is homogeneous - from which (4) follows as in (3) of 4.11. Suppose that V_M is not homogeneous. Then let I be the stabilizer in AG of M on

A homogeneous constituent W_1 say. Since, by (3), I covers AG/G we may assume (as in 4.12 (1)) that

$$I = AM$$

Thus

$$A_1 = I^x \cap A \qquad \text{for } x \in \tilde{G} \backslash M .$$

Now let $L = \ker E$ on W_1 ; in case I we find

$$L \supseteq 2\ [E, A_1{}^x] \qquad \text{for } x \in \tilde{G} \backslash M ;$$

while for case II ,

$$L \supseteq [F, A_1{}^x] \qquad \text{for } x \in \tilde{G} \backslash M .$$

We next deduce that if

$$H = < A_1{}^x \mid x \in Q\backslash M >$$

then, writing $\tilde{P} = A_1 P$,

(5) H has a $\tilde{P}$ invariant fixed set on some irreducible AG constituent U of $\bar{\Delta}$. Suppose first that Hypothesis I holds. Then, since

$$\bigcap_{X \in \tilde{G}} L^X = 1 .$$

we know that L does not cover $\bar{E} = \bar{\Delta}$; (5) then follows immediately (note that L is $\tilde{P}$ invariant and contains [H, E]) .

Now suppose Hypothesis II holds. Let $L \cap F = F_1$. Then

$$F_1 \lhd E\,PA$$

and

$$F_1 \neq F ;$$

consider the centralizer D of F/F_1 in E. We assert that this does not cover E/R_o . For if it did it would contain the characteristic subgroup generated by all p ' elements of E, which would then centralize F/F_1 . But since R_o is a p group we would then have

$$[F, E] \neq F .$$

This clearly contradicts the minimality of F. But now

$$[A_1^x, E] \leq D \qquad \text{for } x \in \tilde{G} \setminus M \; .$$

Thus if

$$L_1 = DR_o$$

we have

$$L_1 \supseteq C_E(F)\, R_o \quad ;$$

$$L_1 \not\supseteq E \; .$$

Thus L does not covert $\bar{\Delta}$ and we have, writing $\bar{L}_1$ for L_1 modulo $\phi(\Delta)$,

$$[H, \bar{\Delta}] \subseteq L_1 \neq \Delta$$

Now $H \leq \tilde{P}$ which is completely reducible on $\bar{\Delta}$; since L_1 is $\tilde{P}$ invariant we have a $\tilde{P}$ invariant fixed set X for H in $\bar{\Delta}$. But now (5) follows.

The following lemma will allow us to deduce a contradiction to (5), thus proving (4).

5.1 Lemma. Let G = PQ where P is the p-radical and Q is an S_q subgroup. Suppose P has an AG invariant Abelian subgroup B, on which Q acts non trivially, and minimal with this property. Suppose

$$A_1 = \ker A \text{ on } G/P$$

is non trivial, [A, Q] covers *G/P* , and A_1G has a unique maximal AG invariant subgroup containing A_1P, M say . Let U be an irreducible AG module on which B acts non trivially. Then if

$$H = \langle A_1^x \mid x \in Q \setminus M \rangle$$

has a non trivial PA_1 fixed set, it follows that A has a fixed point on U.

Proof. Write $\breve{G} = GA_1$, $\breve{P} = PA_1$. Then

(A) $U_{\breve{G}}$ is homogeneous. For if not let

$$U_{\breve{G}} = U_1 \oplus . .$$

Then since H is completely reducible on U we may assume that H has a fixed point on U . But then, since

$$A1 \leq H^x \qquad \text{for } x \in Q \setminus M \; ,$$

we find that A_1 has a fixed point on U_1 . Clearly it follows that A has a fixed point on U.

(B) $U_{\tilde{P}}$ is homogeneous. For consider

$$U_M = U_1 \oplus \ldots$$

$$I = Stab_{AG} (M \text{ on } U_1)$$

Then if I AG we have, as in (4), that

$$I = AM \quad ;$$

$$I^X \cap A = A_1 \qquad \text{for } x \in \tilde{G} \setminus M \quad .$$

<u>Case</u> (a). H has a fixed point, u say, on U_1 .

Choose

$$x \in \tilde{G} \setminus M \quad .$$

We assert that u x is a fixed point for A., and, if $\propto \in A \setminus A_1$, that

$$u \otimes x \ (1 + \propto + \qquad + \propto^{p-1})$$

is a non trivial fixed point for A.

<u>Case</u> (b). $[H, U_1] = U_1$. Then H has a fixed point u Ä x say on $U_1 \otimes x$ for some $x \in \tilde{G} \setminus M$. Then, choosing

$$y \in \tilde{G} \setminus M \quad ; \qquad y \not\equiv x \quad (M)$$

(for $|\breve{G}/M| \neq 2$) we find that, if $A_1 = <\beta>$,

$$u \otimes y \ \beta = u \otimes x \, \beta^{y^{-1}x} \, x^{-1} y = u \otimes y$$

since $y^{-1} x \notin M$. Thus we may proceed as in (a) above.

It now follows that U_P is homogeneous as asserted. But now, since H has a $\tilde{P}$ invariant fixed set, H must in fact act trivially on U. But then, since

$$A_1 \leq H^G$$

we have that A_1 acts trivially. But since [A, B] does not, we find a

fixed point for A as in (l) above. This completes the proof of 5.1.
(6) V_{R2} is homogeneous. For let I be the stabilizer of a Wedderburn constituent. Then by (3) we may assume I contains A ; by (4) that I covers $AG/\breve{R}_2$; thus I = G as required.

(7) V_{R_1} is homogeneous provided either

(i) $\{p, q\} \neq \{2, 3\}$; <u>or</u>

(ii) p = 3 and for every maximal AG invariant subgroup M of $\breve{R}_2$ containing B we have V_M homogeneous. The proof of (7) will involve a substantial amount- of argument. We take M a maximal AG invariant subgroup of $\tilde{R}_2$ containing R_1 and show that, under (i) or (ii), V_M is homogeneous. Assume then that (i) or (ii) holds but that V_M is inhomogeneous. Then by (6) above M is not R_2 ; let I be the stabilizer of M on a homogeneous component W_1 say. By (4) above I covers AG/R_2 and so contains an S subgroup of G, and further

$$I \cap \tilde{R}_2 = M .$$

The proof divides into three cases.

Case . (a). Some conjugate of I contains A. Here we may assume that I actually contains A. We note first that

$$M \not\geq B .$$

For if M contains B, since I contains an S subgroup of G we have a contradiction to 1.5 (the minimality of P) .

We deduce that [B, Q] covers $\tilde{R}_2/M$; moreover $B \cap M$ is AG invariant and so must be centralized by Q (and hence also by P = [p, Q]) . Thus

$$[B, Q] \text{ complements } M \text{ in } \breve{R}_2 .$$

Now since $A \leq I$ we must have $A_1 \leq M$; thus if

$$X = \{ [B, Q] \setminus C_G (AM/M) \}$$

we have

$$A_1 \;=\; I \cap A^x \qquad \text{for } x \in X \quad .$$

Let $\qquad H \;=\; \langle\, A_1^{\,x} \mid x \in X \,\rangle \quad .$

We shall prove that

(λ) $\quad H \supseteq A_1{}^{[B,\ Q]}$,

and, letting

$$L \;=\; \ker M \text{ on } W_1 \quad ;$$

$$L_o \;=\; C_E\ (F)\ R_o \quad \text{(under Hyp. II)} \quad :$$

$$L_1 \;=\; L \cap E \qquad \text{(under Hyp.I)}$$

$$=\; C_E\ (F/F \cap L)\ R_O/L_O \quad \text{(under Hyp.II)} \quad ,$$

that

(μ) $\quad [H,\ \Delta] \;\subseteq\; L_1 \quad$ and $\quad \cap\, L_1{}^{[B, Q]} \;=\; 1$

(where, as above, $\Delta \;=\; E$ or E/L_o under I and II respectively).

These two statements show that A_1 acts trivially on Δ, contradicting (1).

We first prove (λ). Since A does not centralize $\widetilde{R}_2/M$ we know that X is non empty. Let

$$1 \neq x \in X \quad .$$

Then if $y \in [B,\ Q] \setminus X$ we have

$$x\,y \in X \quad .$$

Thus, since B is Abelian

$$\beta^{xy} \;=\; \beta\,[\beta,\ x]\,[\beta,\ y] \;=\; \beta^{X}[\,\beta\ ,\ y] \;\in\; H \;,\ \text{and } \langle\beta\rangle \;=\; A_1 \quad ,$$

and hence

$$[\beta, y] \in H \,.$$

Also

$$\beta^{x2} \;=\; \beta^{x}\,[\beta\,, x] \qquad \text{and, } \underline{\text{if } p \neq 2}$$

we deduce that

$$[\beta\ ,\ x] \;\in\; H\,.$$

Alternatively, <u>if p = 2</u>, since $q \neq 3$, we know that $|\,X\,| > 2$ and we may pick $x_1 \in X$ with $x\,x_1 \quad X$. Then

$$\beta^{x\,x_1} \;=\; \beta^{x_1}\ [\,\beta\ , x] \quad \in \quad H \quad ,$$

and we again have $[\beta, x] \in H$.

It now follows that $\beta \in H$, and hence that $\beta^y \in H$ for all $y \in B \setminus X$.

We have thus proved (λ) .

We now prove (μ) . Under Hypothesis I this is clear ; for by Hall Higman

$$[A_1^X, E]_{w_1} = 1 \qquad \text{for } x \in X .$$

Thus

$$[A_1^x, E] \subseteq L = L_1$$

and since [B, Q] complements I ,

$$\cap L_1^{[B,Q]} = 1 .$$

Now suppose Hypothesis II holds. Then we have

$$[A_1^x, F] \subseteq L \qquad \text{for } x \in x$$

Thus $[H, \Delta] \leq L_1$; it only remains to show that

$$\cap L_1 [B,Q] = 1 .$$

Let

$$\tilde{L}_1/L_o = L_1 ,$$

and

$$\cap \tilde{L}_1[B,Q] = T ;$$

$$D = C_T(F/F \cap L) .$$

Then $D \lhd T$ and D covers T/R_o . Thus D contains the characteristic subgroup generated by all P' elements of T ; it follows that

$$D_1 = \cap D^{[B,Q]}$$

covers T/R_o . But D_1 centralizes F . Thus $T \leq L_o$ and (μ) is proved.

In the next case we complete the case that (i) holds.

<u>Case (b).</u> No conjugate of I contains A and either $\{p, q\} \neq \{2, 3\}$ or p = 3 and AQ does not act as $SL_2(3)$ on $\tilde{R}_2/M$. We remark first that

$$I \not\geq A_1 .$$

For if $I \supseteq A_1$ it follows that $A_1 \subseteq M$, so that

$$A_1 = I^X \cap A \quad \text{for } x \in \tilde{R}_2 .$$

But then, as in case (a) above, we find that A_1 centralizes Δ , contradicting (1) .

Thus

$$I \cap A = A_o \neq A_1$$

and

$$| I \cap A^x | = p \quad \text{for } x \in \tilde{R}_2 .$$

We show that, provided $\{ p, q \} \neq \{2, 3\}$, this cannot happen; our method is to show that

$$[\tilde{R}_2 , A] \ M/M = A_1 \ M/M \qquad (*) .$$

But this implies that the irreducible GF(p) AQ module $\tilde{R}_2/M$ has dimension two (by Hall Higman, for example) which cannot happen unless $\{p,q\} = \{2, 3\}$ and AQ acts as Σ_3 or SL_2 (3) .

We must prove (*). First we note: that

$$I \cap A^x \leq I \cap AR_2 = A_oM$$

since A_oM/M is a Sylow p subgroup of I/M contained in AR_2/M. Thus, modulo M,

$$A_o = I \cap A^X \quad \text{for } x \in \tilde{R}_2 .$$

Now let $A_o = < \propto >$ and take x in $\tilde{R}_2$.

Then, for some j ,

$$\propto = (\propto \beta^j)^x = \propto^x \beta^j \quad \text{(modulo M)}$$

or $$[\propto , x] \in A_1M .$$

This proves (*) and so (7) is proved in case (b) .

<u>Note.</u> Apart from a simple argument (see (11) below) we have now proved E and F for $p \geq 5$.

To complete the proof of (7) we must deal with condition (ii) .

Case (c). No conjugate of I contains A ; p = 3, M does not contain B and AQ acts as $SL_2(3)$ on $\tilde{R}_2/M$ (in the natural way) .

We note first that, as in case (b)

(A) $A_1 \not\subseteq M$. A_1^Q covers R_2/M .

It follows that we may assume

(B) $I = A_o QM$ for some $A_o \leq A$, $A_o \neq A_1$;

$$I/M \cong SL_2(3)$$

and $\tilde{R}_2/M$ is the natural module for this.

As in case (a) we find

(C) $[A_0{}^I, \Delta] \subseteq L_1$; $\bigcap L_1^{\langle A_1{}^Q \rangle} = 1$.

We will complete this case by a detailed analysis of the action of I on Δ ; the key to this is

(D) $[M \cap P, \Delta] = 1$. We show first that

$$[M, Q] \cap P = M \cap P .$$

Now $MB = R_2$; by the minimality of B , $[B, Q]$ complements M ; thus

$$[M, Q][B, Q] \cap M = [M, Q] .$$

But P is contained in $[MB, Q]$; thus

$$P \cap M \leq [M, Q]$$

as required.

Thus, by (C) above, writing $M = M \cap P$,

$$[M_o, \Delta] \leq L_1 ;$$

$$\bigcap [M_0, \Delta]^{\langle A_1{}^Q \rangle} = 1 .$$

But M_o is $A_1{}^Q$ invariant ; thus (D) follows.

To complete the proof we consider an irreducible AG subraodule U of $\overline{\Delta}$.

Clearly we have (by (C))

(E) $[U, I] \neq U$; $(AG)_U$ is the split extension of B_U by $St_2(3)$ acting in the natural way.

Now consider the Wedderburn decomposition

$$U_B = U1 \oplus \dots \oplus U_t \quad .$$

Let K be the kernel of B on U_1 . Without loss of generality A_o centralizes B/K so that A_o stabilizes B on U_1 . In this case

$$K_{U1} = (A_1)\, U_1 = 1$$

so that, since A has no fixed points on U_1 we must have

$$[A_0, U_1] = U_1 .$$

But clearly this implies that

$$[A_o Q, U] = U$$

which contradicts (E) . This completes the proof of (7).

(8) If $p = 3$, V_{R1} is homogeneous; if $p = 2$, V_{R1B} is homogeneous. Let M be a maximal AG invariant subgroup of R_2 containing R_1 and not equal to R_2 ; then in view of (7) it suffices to prove that if M contains B then V_M is homogeneous. Thus we assume that V_M is not homogeneous; let I be the stabilizer of a Wedderburn constituent W_1 . Then, as in 7 case (a), 1.5 or II.5 ensures that no conjugate of I contains A. Moreover, as before we have

$$I \cap A = A_1$$

so that

$$A_1 \not\subseteq M$$

and A_1^Q covers $\tilde{R}_2/M$. we note also that, by (4), I may be chosen to contain Q.

Let $B_2 = [B, A_1]^{AQP}$ (modulo R_o in case II)

$\Delta_2 = [\Delta, B_2]$.

We assert that, if $I \cap A = A_o$ then

(9) $[A_o^{\,I}, \Delta_2] \neq \Delta_2$. To see this we proceed as in 7 case (a). Let L

(B) A has fixed points on U . For $[A_1 , B]_U$ is certainly non trivial.

(C) $U_{\breve{G}}$ is homogeneous. For if not let W_1 be the first Wedderburn constituent; without loss of generality A_1 has a fixed point on W_1 , leading to a fixed point for A on U.

(D) U_{R2}. is homogeneous. Let $M_1 \supseteq \tilde{R}_2$ be the maximal proper AG invariant subgroup of $\tilde{G}$. We show that $U_{M1.}$ is homogeneous. Suppose this is not the case; let I_1 be the stabilizer. Then we may take

$$I_1 = A\,M_1$$

and $$I_1{}^X \cap A = A_1 \qquad \text{for } x \in \tilde{G} \setminus M_1 .$$

As before we deduce

$$[A_1{}^x, B] \subseteq s = \ker \tilde{R}_2 \text{ on } W_1 \qquad \text{for } x \in Q \setminus M_1 .$$

But now

$$\cap \quad S_U{}^Q = 1$$

while

$$[A_1{}^x, B] \subseteq [\tilde{P}, B] \neq B .$$

By the minimality of B, the commutator $[\tilde{P}, B]$ is centralized by Q. Thus

$$[A_1 , B]_U = 1$$

which is not the case ; (D) now follows .

(E) U_M is homogeneous, where M is the maximal AG invariant subgroup of $\tilde{R}_2$ considered in (8) above. Note that since A_o has an M invariant fixed subspace in U this implies that A acts trivially on U ; this contradiction then completes the proof.

Now suppose that I_1, the stabilizer of M on a Wedderburn constituent W_1 is proper. Then we may assume that I_1 contains Q ; but then I_1 is the normalizer of Q modulo M and so must coincide with I . Thus

$$I = I_1 .$$

Thus A has no fixed points on W_1 ; suppose then that $W_1 \otimes x$ is fixed elementwise by A_o . A_o clearly stabilizes this component so that

$$A_o = A \cap I^X .$$

be the kernel of E on W_1, and, under Hyp II, as in (1) ,

$$L_o = C_E(F) \quad R_o .$$

Then put

$$L_1 = L \qquad \text{(under Hyp. I)}$$

$$= C_E \; (F/F \cap L) \; R_o / L_o \quad \text{(under Hyp II)} .$$

As before we have

$$[A_o^I , \Delta] \subseteq L_1 .$$

Now since A_o^I contains Q we deduce

$$[B, Q, \Delta] \subseteq L_1 .$$

As in 7 (a) (μ) we have $\langle A_1^Q \rangle$

$$\cap \; L_1 = 1 .$$

Thus

$$[B, Q, A_1] \not\subseteq C_B (\Delta / L_1) .$$

We deduce that Δ_2 is non trivial ; since it is AG invariant it cannot be contained in L_1 . Thus

$$[A_o^I , \Delta_2] \; L \cap \Delta_2 \neq \Delta_2$$

and (9) is proved. In fact it is clear that we may take an irreducible AG constituent U of Δ_2 such that

$$[A_o^I , U] \neq U .$$

<u>Lemma</u> 5.2. In the above circumstances, A has a fixed point on U .

<u>Proof</u>

(A) A_o has an M invariant fixed subspace on U. This is clear since A_o is contained in an M invariant Sylow p subgroup of A_o^I (modulo the kernel of AG on U) - for

$$M_U \cong (M \cap P)_U \times (R_1 \cap Q)_U$$

where $R_1 \cap Q$ centralizes A_o - and U is a GF(r) module where $r \neq p$.

But now we get, by Clifford, a fixed point for A as required.

This completes the proof of (8). Putting together (7) and (8) and using the simple (11) below, we find that Proposition F is proved and that E is also unless $p = 2$, $q = 3$ and Hyp I holds.

From now on we assume that $p = 2$, $q = 3$ and Hypothesis I holds ; then, by I.6, the S_q subgroup Q of N_G (P) is cyclic; if M is a maximal AG invariant subgroup of $\tilde{R}_2$ distinct from $\tilde{R}_2$ then A/A_1 is free on $\tilde{R}_2/M$.

(10) If C_A (B) $\neq$ 1 then V_{R1} is homogeneous. In view of (7) and (8) we may take M not containing B and assume that V_M is not homogeneous.

We note first that, clearly

(A) $A_1 = C_A$ (B) .

We next deduce, as in 7 (a) that

(B) $A_1 \not\subseteq M$; $I = A_o QM$ for suitable $A_o \leq A$ and Q an S_q subgroup of N_G (P) . Since A/A_1 is free on $\tilde{R}_2/M$, complements to R_2 in A R_2 modulo M) are conjugate ; thus for some $A_o \leq A$, the normalizer N_1 of Q contains A_o modulo M. But now we may assume, since I covers $AG/\tilde{R}_2$ that I contains N_1 . It follows that $I = A_o QM$ as asserted. Now suppose A_1 is contained in M . Then I contains A and for

$$x \in B \setminus C_B \ (AM/M)$$

we have

$$A \cap I^x = A_1 .$$

Thus

$$[\, A_1^{x-1} , R_1]_{W1} = 1 ;$$

since A_1 centralizes B it follows that A_1 centralizes R_1 contradicting item (l) .

Next we utilize the fact that $p = 2$ to deduce

(C) We may assume A_o normalizes Q .

This follows from a simple lemma.

5.3 Lemma. Suppose $| A_o | = 2$ acts on $G = PQ$ where P is a normal 2 - subgroup and Q is a q group for some prime $q \neq 2$. Then A_o normalizes some S subgroup Q_1 of G.

Proof. We use induction on $| G |$,

(1) P is elementary Abelian. For, if not let

$$Z = \Omega_1 (Z (P))$$

and consider $G/2$ By induction A_o normalizes Q_1 for some S_q subgroup Q_1.

(2) P^+ is an irreducible $A_o Q$ module. This follows exactly as (1).

(3) A is free on P^+ or trivial. In the first case all complements to P in AP are conjugate, while N(Q) complements P in AG ; the result follows. In the second case Q is normal.

We now make a detailed analysis of the structure of P.

(D) B is central in $\tilde{P}$; $\tilde{P} = B\tilde{T}$ where $\tilde{T} = A_1{}^Q$. For, since A_O normalizes Q, the normal closure $\tilde{T}$ is A_o Q invariant and $AQ\tilde{T}$ is a subgroup. Thus

$$\tilde{T} B = \tilde{P}$$

by minimality of P (1.5). Since A_1 centralizes B it is clear that $\tilde{T}$ does. Thus (D) follows.

(E) Q centralizes M. For, since $I \cap A = A_o$, we have

$$[A_o{}^I , R_1] \subseteq L = \ker R_1 \text{ on } W_1 .$$

Also, since B supplements I, the normal interior

$$\cap \{ A_o{}^I \}^B$$

centralizes R_1. But now $A_o{}^I$ contains [M, Q] which, by (D) above is centralized by B. Thus [M, Q] must be trivial as asserted,

(F) $\tilde{P} \cong B \times H$; B has rank two. For, since B is central in P, [B, Q] is $< A, Q >$ invariant and so, by minimality, is B. Similarly

$[B, Q]$ is irreducible and so has rank 2 (recall that $q = 3$ and, by I.6, the group Q_B must have order 3) . But now $B \cap M$ is trivial so that (F) is established.

To complete the proof we consider $U = \overline{E}$ as an AG module; we have that since

$$[A_o^{\ I}, E] \neq E \quad ,$$

$$[A_O^{\ I}, U] \neq U \quad .$$

Now consider the Wedderburn decomposition

$$U_B = w_1 \oplus W_2 \oplus W_3$$

- since B acts fixed point freely on U we have three distinct irreducibles in B on U. Now A_o Q acts on these ; since B^+ is an irreducible Q module, Q cannot fix any component ; clearly A_o must fix one, say W_1 , and if $Q = \langle \tau \rangle$ we have

$$W_2 = W_1 \tau \quad ; \qquad W_3 = W_1\tau^2 \quad .$$

Thus

$$(w_1)_{A_o} \quad \text{covers} \quad (U/[U,Q])_{A_o}$$

so that A_o has fixed points on w_1 . Let

$$x = C_{w1}(A_o)$$

Then A_1 must have no fixed points on X ; if $\langle \beta \rangle = A_1$ then $\beta = -1$ on X . Now let

$$\beta = \beta_1 \gamma \qquad 1 \neq \beta 1 \in B, \quad \gamma \in M..$$

Now since A_o interchanges W_2 and W_3 , b can have no fixed points on $X\tau$ or $X\tau^2$; thus β^τ and $\beta\tau^2$ are -1 on X . Now

$$\beta^\tau = \beta_1^\tau \gamma$$

$$\beta^{\tau 2} = \beta_1^{\tau 2} \gamma$$

$$\beta = \beta_1 \gamma \ .$$

Thus $\beta 1$, $\beta_1^{\tau 2}$, β_1^τ must all act in the same way on X . Clearly they must centralize it, in which case we find that B centralizes X, which in not possible. This completes the proof of (10) .

(11) Let $p \neq 2$ or $C_A(B) \neq 1$. Then V_R is homogeneous. For if not let T be the stabilizer of a component. We may assume that I contains AP and since

$$AG = N(P)R$$

we have

$$I = AQ_1 PR$$

where $Q_1 = I \wedge Q$ and covers G/R_2 . By the minimality of Q we have that

$$Q_1 = Q \quad .$$

Thus I = G as required.

In exactly the same way we may prove

(12) Let $p = 2$ and $G_A(B) = 1$. Then V_{BR} is homogeneous.

In view of (11) and (12) , Proposition 4.3 and Corollary 4.2 complete the proof of Propositions E and F .

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