Ask not what bilateralist intuitionists can do for Cut, but what Cut can do for bilateralist intuitionism

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On a bilateralist reading, sequents are interpreted as statements to the effect that, given the assertion of the antecedent it is incoherent to deny the succedent. This interpretation goes against its own ecumenical ambitions, endowing Cut with a meaning very close to that of *tertium non datur* and thus rendering it intuitionistically unpalatable. This paper explores a top-down route for arguing that, even intuitionistically, a prohibition to deny is as strong as a license to assert.

1 Preamble

Frege's (1919) doctrine that to deny a statement is to assert its negation led to denial being pushed to the periphery of logical theorising. In the past decades, however, denial made a spectacular comeback (cf. Humberstone, 2000). Thus, for instance, Price (1983) initiates a programme aimed to provide an account of meaning based on both 'assertion and denial conditions' which validates classical logic. Smiley (1996) adds to this generic programme of recovering denial (he calls it 'rejection') for the purposes of logical theorising, arguing, e.g., that 'the equivalence between rejecting [p] and asserting [$\sim p$] does not make rejection redundant' (4).

For all that, denial returned in a changed world, in which challenges to the hegemony of classical logic abound and its usefulness must be assessed also in relation to non-classical logics. Recently, Restall (2005) appealed to denial for a defence of classical logic, albeit from a pluralist perspective that forces us to account for denial in relation to non-classical logics, and in particular with intuitionist logic, as well.[1]

However, it is not immediately clear that denial can successfully be used in this way. This paper presents the problem and proposes a solution to it. For good measure, the solution itself relies on reconsidering a fundamental

[1] For other uses of denial from a non-classical perspective, see, e.g., Priest (2005), chapter 6.

idea about how sequent calculi determine consequence relations, namely that the sequents themselves are a kind of entailments.

2 The problem

Restall (2005) proposes a bilateralist reading of sequents which places denial (or its mental counterpart, rejection) on a par with assertion (or its mental counterpart, acceptance). At its core lies the concept of a *state*, i.e., a pair $X : Y$ of finite collections of statements in which *all* the members of X are asserted and *all* the members of Y are denied. Some states are *coherent*: there is no clash in jointly asserting all the members of X and jointly denying all the members of Y. Others are *incoherent*: the assertion of X clashes with the denial of Y. Hereafter, the colon will serve as a sign for both coherent and incoherent states, disambiguation is achieved by (con)text.

Sequents represent incoherent states: having asserted the antecedent, it is incoherent to deny the succedent. On this reading, the rule of Identity (Id)

$$\frac{}{A : A}\,\text{Id}$$

says that given the assertion of A, it is incoherent to deny A. Weakening in the succedent

$$\frac{X : Y}{X : Y, A}$$

states that if the assertion of X and the denial of Y are jointly incoherent, then the incoherence is preserved if A is added to Y. According to the rules for introducing conjunction in the antecedent,

$$\frac{A_i, X : Y}{A_1 \mathbin{\&} A_2, X : Y}\,\&\text{L}_i\ (i \in \{1, 2\})$$

if the denial of Y is ruled out by the assertion of A_i, X, then its denial is also ruled out by the assertion of $A_1 \mathbin{\&} A_2, X$. The introduction in succedent

$$\frac{X : Y, A_1 \qquad X : Y, A_2}{X : Y, A_1 \mathbin{\&} A_2}\,\&\text{R}$$

states that if the denial of Y, A_1 is ruled out by the assertion of X and the denial of Y, A_2 is also ruled out by the assertion of X, then the denial of $Y, A_1 \mathbin{\&} A_2$ is ruled out by the same assertion.

The main beneficiary of this re-interpretation of the sequent calculus is the classical logician. On a bilateral reading, succedents consisting of more than one formula occurrence are no longer construed as disjunctions – a very contentious and often criticised feature of the classical sequent calculus (Dummett, 1991; Steinberger, 2010). Instead, they read as conjunctions under a *sui generis* force operator: ‘do not deny’.

However, the account has a more ecumenical scope and it is meant to be acceptable, e.g., to the intuitionist. This is problematic because a bilaterally interpreted Cut

$$\frac{X : Y, A \qquad A, X : Y}{X : Y}\ \text{Cut}$$

can be seen as an extensibility condition on coherent states:

Extensibility If $X : Y$ is coherent, then so is one of $A, X : Y$ and $X : Y, A$.[2]

This principle is reminiscent of the law of the excluded middle, coming suspiciously close to saying that every statement is either deniable or assertible (in a context).

Restall goes to some pains to ensure that, despite appearances to the contrary, '[extensibility] does not rule out truth-value gaps and it does not implicitly endorse the law of the excluded middle.'

In the intuitionist case, denial must be taken in the following 'subtle' sense, which

> is not as strong as the intuitionist's assertion of a negation, but not as weak as the intuitionist's mere failure to assert. The requirement is that to deny, in our sense, is to *refuse* to accept. A statement is rejected if any move to accept it would be a change of mind, and not merely a supplementation with new information. (Restall, 2005, 9, fnote 5)

Even if, at least *prima facie*, this successfully allays the intuitionist's misgivings, assessing the precise strength of a *refusal to accept* and, in particular, the precise strength of a *prohibition to refuse to accept* is still a difficult matter. The latter – our *sui generis* undeniability operator – must come very close to a license to assert, at least if the logical consequence relation is to retain the strength usually attached to it. Customarily, sequents are understood as entailments, i.e., claims to the effect that the succedent follows from the antecedent. Yet the statements in the succedent of a bilaterally interpreted sequent are merely placed under an interdiction to refuse to accept them. One cannot take it for granted that one can recover consequence with its usual strength, amounting to a license to assert, from incoherent states – see Rumfitt (2008); Steinberger (2010) for criticism of bilateralism in this spirit.

That this is, nonetheless possible, is suggested by an analysis of Cut inspired by a remark of Girard (1989, 30) proposing that Cut may be used, alongside Id, to assess the strength of formulae in a calculus. Cut states that the right-hand side occurrence of the Cut-formula is stronger than its left-hand side occurrence. Id states that the left-hand side occurrence of the

[2] In the case of the intuitionistic sequent calculus LJ, the right-hand side of the sequents can be at most a singleton, so Y is empty. Cut for sequents – If $X : Y, A$ is incoherent and $A, X : Y$ is incoherent, then $X : Y$ is incoherent – follows from Extensibility by contraposition. Closure under extensibility has been criticised independently of intuitionism, cf. Ripley (2015).

Id-formula is stronger than its right-hand side occurrence. Together they cancel each other out and establish the equal strength of the right and the left side of a sequent.

What if we could use Cut to assess the relative strength of intuitionist assertion and denial? In its applications, the Cut formula appears twice, once in a context where its denial is ruled out and once in a context where it is asserted. This suggests that having a license to accept a statement and not being allowed to refuse to accept it are, in a sense, of equal strength. Can we turn this 'suggestion' into a rigorous argument? And can we do it without generating intuitionistic misgivings?

I believe that both these *desiderata* are within reach. The thought is that the admissibility of Cut is *eo ipso* evidence that even intuitionistically the commitment to assert a sentence in a given context is no stronger than the commitment not to deny (in Restall's sense) that sentence. So let us have a closer look at Cut in the context of a sequent calculus for intuitionist logic (LJ).

3 The ingredients

Standardly, a sequent calculus is taken to determine a consequence relation on formulae via an interpretation of sequents as consequence claims. That is, the consequence relation usually associated to a sequent calculus S is its so-called *internal* consequence relation (Avron, 1991):

Definition 1 (Internal consequence) *A is an internal consequence of X in S iff the sequent $X : A$ is derivable in S.*

Gentzen (1935) proved that LJ and LJ$^-$, i.e., LJ without Cut, have the same derivable sequents. Anything that can be derived with Cut can be derived without it or, equivalently, its addition to LJ$^-$ does not modify the stock of derivable sequents. This means that Cut is an *admissible* rule in LJ$^-$.

However, incoherent states themselves can be the objects of *inferring* and the bilateral interpretation provides a good account of inferring as a process of ampliating incoherent states. In turn, this inferential process generates a reflexive, monotonic, and transitive derivability relation between sequents. In other words, it generates a *Blok-Jónsson* consequence relation on sequents (Blok and Jónsson, 2006):

Definition 2 (Blok-Jónsson consequence) *Let U be a set, $\mathcal{P}(U)$ its powerset and $a \in U$. A Blok-Jónsson consequence relation on U is a relation $\vdash \subset \mathcal{P}(U) \times U$ that is reflexive, monotonic, and transitive.*

Unlike the more familiar *Tarskian* consequence relations (q.v. Tarski, 1956, ch. 3), Blok-Jónsson consequence relations are carried by arbitrary sets, not just sets of formulae or statements. In particular, they can arise between incoherent states or sequents.

The consequence relations between incoherent states determined by LJ and LJ$^-$, $\vdash^S_{\mathrm{LJ}}$ and $\vdash^S_{\mathrm{LJ}^-}$ respectively, are clearly distinct. For instance, whereas $\{p : r, r : s\} \vdash^S_{\mathrm{LJ}} p : s$, $\{p : r, r : s\} \nvdash^S_{\mathrm{LJ}^-} p : s$. In general, LJ$^-$ lacks many sequent-to-sequent inferences (or *metainferences*) that are available in LJ. Thus, the metainference rule

$$\frac{A \,\&\, B, X : C}{A, B, X : C}\ \&\mathrm{L}^{\iota}_{m}$$

is *derivable* in LJ. That is, there is an LJ-derivation of its conclusion sequent from its premiss sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A : A}}{A, B : A}\,\mathrm{WL} \quad \dfrac{\dfrac{}{B : B}}{A, B : B}\,\mathrm{WL}}{A, B : A \,\&\, B}\,\&\mathrm{R} \qquad A \,\&\, B, X : C}{A, B, X : C}\,\mathrm{Cut}$$

Since LJ$^-$ lacks Cut, this derivation fails; nor is there any other way to derive $\&\mathrm{L}^{\iota}_{m}$, as an inverted proof search will show. (But this rule is, nonetheless, admissible in LJ$^-$.)[3]

The fundamental insight of the Blok-Jónsson paradigm is that the objects related by a consequence relation are less important than the reasoning that generates it. One's account of consequence should be neutral with respect to the 'ontologies' upon which it is based. The upshot of their definition of consequence is a theory of equivalence of consequence relations exhibiting this kind of neutrality:

Definition 3 (Equivalence of consequence relations) *Let U_1 and U_2 be two sets and $\vdash_1 \subseteq \mathcal{P}(U_1) \times U_1$ with, respectively, $\vdash_2 \subseteq \mathcal{P}(U_2) \times U_2$ their associated BJ consequence relations. $\vdash_1$ and $\vdash_2$ are equivalent iff there exist mappings $\tau : U_1 \to \mathcal{P}(U_2)$ and $\rho : U_2 \to \mathcal{P}(U_1)$ such that:*

1. *$X \vdash_1 a$ iff $\tau(X) \vdash_2 \tau(a)$*
2. *$b \dashv\vdash_2 \tau(\rho(b))$*

for every $X \cup \{a\} \subseteq U_1$ and every $b \in U_2$.[4]

This allows one to define, for any sequent calculus, an equivalence class of consequence relations induced by the Blok-Jónsson consequence on sequents

[3]This is a familiar situation in the case of classical logic. Gentzen's cut-less classical sequent calculus (LK$^-$) can be used to formalise the logic ST advocated in Cobreros et al. (2013) as a means of retaining classical reasoning in the presence of the paradoxes. The ST theorists claim that it is simply classical logic. There are serious doubts that this claim is cogent (Barrio et al., 2015), and the Blok-Jónsson framework provides a way to argue against ST's pretensions of classicality (Dicher and Paoli, 201x). The intuitionist case has received less attention; see though Thomas (2014).

[4]In Blok and Jónsson (2006), this is called 'similarity'.

associated to it. Take, for instance, the consequence relation on formulae induced internally by LJ, $\vdash^{I}_{\mathrm{LJ}}$, and the sequent-to-sequent consequence relation $\vdash^{S}_{\mathrm{LJ}}$ of LJ. We can define τ and ρ as, respectively,

$$\tau(A) =_{df} \emptyset : A \qquad \rho(X : A) =_{df} (A_1 \,\&\, \ldots \,\&\, A_n) \to A$$

for all $A_i \in X$, $i = 1, \ldots, n$. Under these mappings, $\vdash^{S}_{\mathrm{LJ}}$ and $\vdash^{I}_{\mathrm{LJ}}$ fall in the same equivalence class. So does the following *external* consequence relation, $\vdash^{E}_{\mathrm{LJ}}$, defined not over sequents in general, but over sequents with empty antecedents (Avron, 1991):

Definition 4 (External consequence) *A is an external consequence of X in* LJ *iff the sequent $\emptyset : A$ is derivable in* $\mathrm{LJ} \cup \{\emptyset : B_i | \forall B_i \in X\}$.

The internal and external consequence relations of LJ coincide in the presence of Cut; without it, i.e., in LJ^{-}, they are extensionally distinct. Take, for instance, *modus ponens*. Obviously, $A, A \to B \vdash^{I}_{\mathrm{LJ}^{-}} B$. This is just a matter of applying →L:

$$\frac{X : A \qquad B, X : C}{A \to B, X : C} \to\mathrm{L}$$

to the axioms $A : A$ and $B : B$. Nevertheless, neither $\vdash^{E}_{\mathrm{LJ}^{-}}$ nor, *a fortiori*, $\vdash^{S}_{\mathrm{LJ}^{-}}$ sanction this entailment. Its image under τ is $\{\emptyset : A, \emptyset : A \to B\} \vdash \emptyset : B$, which LJ^{-} cannot deliver in the absence of Cut on formulas.

4 The arguments

Now we can attempt to formulate two arguments showing that Restall's 'prohibition to refuse to deny' indeed has all the properties required of intuitionist denial.

The first argument runs as follows. We start with the fact that LJ^{-} yields intuitionist logic as its internal consequence relation; this is true even if LJ^{-} is interpreted bilaterally. The intuitionist is worried that this interpretation assigns to conclusions a 'lesser' status than that which they deserve. This, however, boils down to Cut not being correct on the bilateral interpretation. Thus, the intuitionist has no viable, independent and non-prejudicial, way of objecting to the rules of LJ^{-}, even if bilaterally interpreted. Thus

P1 The bilaterally interpreted LJ^{-} is a calculus for intuitionist logic.

Grant, for the sake of the argument, that

P2 Cut, bilaterally interpreted, is intuitionistically incorrect.

At the very least, this means that Cut is not admissible in LJ^{-}. But in this case, the external consequence relation of LJ^{-} will not be intuitionist logic. Suppose that we also grant the extra premiss:

P3 It is an essential feature of intuitionist logic that $\vdash^{I}_{\mathrm{LJ}^{-}}$ and $\vdash^{S(E)}_{\mathrm{LJ}^{-}}$ coincide.

I shall not argue for the supplementary premiss, except by default: these relations *do* normally coincide. It follows that

C The bilaterally interpreted LJ^- – i.e., LJ^- without admissible Cut – is not a calculus for intuitionist logic.

But **C** contradicts **P1**. So either **P1** or **P2** must go. Since there is no case against **P1** that could be made independently of **P2**, one must reject the latter: It is not the case that Cut is not correct on the bilateral interpretation. Hence, it is correct.

This argument is not without problems. The most glaring, presumably, is the paucity of evidence offered for **P3**. But in truth, this is the least problem of the argument. Though not canvassed here, such evidence could be provided, e.g., along the lines of Barrio et al. (2018). This aside, one may, for instance, be sceptical about **P1** and its defence. Consider the intuitionist $\neg$R rule:

$$\frac{X, A :}{X : \neg A} \neg\mathrm{R}$$

Read bottom-to-top and as a condition on coherent states, it seems to licence the passage from 'it is coherent to accept X and refuse to accept $\neg A$' to 'it is coherent to accept X, A and refuse to deny some contradiction'. (Recall that the empty succedent is the syntactical mark of the falsity constant, $\bot$.) This seems to be an inferential passage from two iterations of 'negative' attitudes to a 'positive' one, which may be problematic to the extent that it is reminiscent of double-negation elimination.[5]

Moreover, the intuitionist, even having granted **P1-P3** and **C**, may still be suspicious about the final conclusion of the argument. After all, we obtain it by shaving off double-negations and moving from 'it is not the case that (...) is not correct' to '(...) is correct'. Can we be sure that this instance of double negation elimination is correct?

Attempting to fix this argument is beyond the scope of this paper. I take it, however, that the complications affecting it are illustrative of the difficulties inherent in dealing with nonstandard senses of denial. Fortunately, since I believe that there is a safer and more direct route to the same conclusion, its failure, albeit instructive, is not of great significance.

A rather more straightforward argument would proceed from the fact that $\vdash^S_{\mathrm{LJ}}$, $\vdash^E_{\mathrm{LJ}}$, and $\vdash^I_{\mathrm{LJ}}$ fall in the same equivalence class. So take $\vdash^S_{\mathrm{LJ}}$ ($\vdash^E_{\mathrm{LJ}}$ tags along as a particular case) under the bilateral reading of LJ. This is safe: even the intuitionist ought to accept that one can reason intuitionistically about incoherent states. Take also $\vdash^I_{\mathrm{LJ}}$ under the standard construal of LJ. These are two consequence relations over distinct entities which, nonetheless, fall in the same equivalence class via the transformers from Definition 3. But those transformers map undeniable statements to warrantedly assertible statements. This, I submit, is compelling evidence that, *as regimented by the*

[5]I owe this example, though given with a different intent, to an anonymous referee.

sequent calculus, the prohibition to refuse to accept a statement carries the same strength as the licence to assert that statement, again, *as regimented by the sequent calculus.* This argument does not ascribe any kind of priority, conceptual or otherwise, to $\vdash^S_{\mathrm{LJ}}$ over $\vdash^I_{\mathrm{LJ}}$, or the other way round. It simply takes each of these relations, with the interpretation attached to them and to the statements that occur in them, and measures their relative strength. Thus it is perfectly egalitarian.

Just like the preceding argument, this too has its weak points. One may reply that the argument merely shows that LJ is a calculus for intuitionist logic: after all, nothing specific to bilateralism plays any part in it. This is a cogent point as far as it concerns the fact that the transformers would have yielded the same result irrespective of the interpretation attached to the calculi in question. Yet the objection disregards the injunction to consider those consequence relations under a particular interpretation. This ought to be taken seriously in the heuristic of the argument. To do otherwise is to reject a priori the possibility of a top-down argument in favour of bilateralism. So the argument, while defeasible, is not defeated yet.

Furthermore, by dispensing with the egalitarianism professed above, we can use the Blok-Jónsson paradigm to present the argument in a somewhat more enticing light:

In effect, the bilateral interpretation re-conceptualises *inferring* as a passage between incoherent states. In a more familiar and more general jargon, it moves from inferences to *meta*inferences. In doing so, it indicates, perhaps inadvertently perhaps, the *vertical* dimension of a sequent calculus as the dominant and indeed natural vector for the generation of logical consequence. The plight of the bilateralist springs from the fact that, while she offers a novel account of inferring, she nonetheless fails to offer an appropriately updated account of how inference and consequence connect. Instead, she insists on recovering consequence *directly* from sequents, by, as it were, simply swapping the sequent sign with the consequence sign.

The Blok-Jónsson account of consequence allows us to match on the consequence side what the bilateralist did on the inference side. We can take the sequent-to-sequent derivability relation of LJ as a *bona fide* consequence relation (Dicher and Paoli, 2018). Moreover, we can also see it as the main relation that is determined by that calculus: after all, sequent calculi deal with sequents, not formulae. All in all, it is possible to do justice to the bilateralist having made the vertical dimension of a sequent calculus the fulcrum for the determination of a consequence relation.

Then the fact that, in the absence of Cut, LJ^- fails to determine the intuitionist logical consequence (over sequents) receives an entirely new significance. The bilateralist and the more traditional intuitionist can agree that one can reason intuitionistically about both incoherent states and warrantedly assertible statements as regimented in standard natural deduction systems or Hilbert-style calculi, etc. To put it another way, they can agree that one can track, via intuitionistic reasoning, both the propagation of the warrant to assert and the correct extension of incoherent states. From the Blok-Jónsson perspective, via their concept of equivalence of consequence relations, we can see that a warrantedly assertible statement in one case is a statement that one cannot refuse to accept in the other. If this is so, then how can one persist in doubting that there is a sense of denial of the required kind?

5 Epilogue

If these remarks are correct, then we have a viable, albeit highly abstract, argument that Restall's posited sense of intuitionist denial can indeed deliver the goods. There is another way to look at this. It seems to me that the Blok-Jónsson paradigm deflates the whole issue, as far as consequence is concerned. There is no need to worry about what sequents are or about how many formulae they have in the succedent etc. Logical consequence is a relation that can occur between many things. Among other things, it can occur between sequent, which, whatever they are, need not be entailments. This is the consequence relation properly and primarily determined by a sequent calculus. Derivatively, such calculi can also determine consequence relations among other kinds of things. They do this in the sense of Definition 3. How we construe sequents is an important matter, but not one that should affect our understanding of consequence.[6]

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