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SYMMETRIC SMOOTH INTERPOLATION ON TRIANGLES

by

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ABSTRACT

Rational and polynomial 'smooth' interpolation schemes are derived which interpolate a function and its derivatives along the boundary of a triangle. The interpolation schemes are symmetric and affine invariant on the triangle and can be used to construct a piecewise defined function which is $C^N(\Omega)$ over a triangular subdivision of a polygonal region Ω.

1. INTRODUCTION

Smooth or blending function Interpolation on a triangle involves the construction of a function which matches data defined on the boundary of the triangle. Examples of such interpolants are described in Barnhill, Birkhoff, and Gordon [1], where rational functions which interpolate the boundary data are given. In particular, a symmetric rational interpolant is described which requires certain compatibility of the boundary data derivatives, as is shown in Barnhill and Mansfield [4]. The problem of smooth polynomial interpolation on triangles is considered in Barnhill and Gregory [3] but the polynomial interpolants are not symmetric on all the sides of the triangle.

This paper presents a method of deriving symmetric functions which, for any positive integer N, interpolate a function $F \in C^N(\overline{T})$, and its derivatives of order N and less, on the boundary ∂T of the triangle T. The interpolation functions are invariant under affine transformation and have the symmetry that each side of the triangle is treated in the same way. Two types of interpolation function are introduced. One is a rational function which, other than $F \in C^N(\overline{T})$, does not require compatibility of the boundary data derivatives. The other is a polynomial function which does require certain compatibility of the boundary data derivatives, although these conditions can be removed by the addition of rational terms.

The interpolation functions can be used to define a piecewise approximation function which is $C^N(\Omega)$ over a triangular subdivision of a polygonal region Ω Such approximation functions have applications to finite element analysis and computer aided geometric design. The functions can be used to blend together given space curves or finite dimensional schemes can be derived by suitable choice of the triangle boundary data. Further, interpolants can be constructed on triangles adjacent to the boundary of Ω which completely satisfy boundary data on Ω, whilst being compatible with finite dimensional schemes on triangles in the interior of Ω. Another application of the smooth interpolation schemes is that they can be used to define transformations of a triangle T onto a region with curved boundaries.

2. THE GENERAL SMOOTH INTERPOLATION SCHEME

By affine invariance, it is sufficient to consider the triangle T with vertices at $V_1 = (1,0)$, $V_2 = (0,1)$, and $V_3 = (0,0)$. The side opposite the vertex V_k. is denoted by E_K, and thus E_1 is the side $x = 0$, E_2. is the side $y = 0$, and E_3 is the side $z = 0$, where $z = 1-x-y$.

The smooth interpolation scheme makes use of the following finite dimensional interpolant: Let L be the affine invariant polynomial interpolation Projector (idempotent linear operator) over the $3(N+1)^2$ dimensional set of polynomials which are of degree $2N+1$ along parallels to the three sides of T, where

$$(2.1) \qquad L\ F = \sum_{i,j\le N} \alpha_{i,j}(x,y)\left([-\frac{\partial}{\partial x}]^i [-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}]^j F \right)(1,0)$$

$$+ \sum_{i,j\le N} \beta_{i,j}(x,y)\left([\frac{\partial}{\partial x} - \frac{\partial}{\partial y}]^i [-\frac{\partial}{\partial y}]^j F \right)(0,1)$$

$$+ \sum_{i,j\le N} \gamma_{i,j}(x,y)\left(\frac{\partial^{i+j}F}{\partial x^i\ \partial y^i} \right)(0,0)$$

and $\alpha_{i,j}$, $\beta_{i,j}$, and $\gamma_{i,j}$ are the cardinal basis functions. The case N = 1 is the tricubic polynomial interpolant of Birkhoff [5]. An explicit representation of the interpolant (2.1) is

$$(2.2) \qquad L\ F = x^{N+1} \sum_{i,j\le N} z^{(i)} y^{(j)} \left([-\frac{\partial}{\partial x}]^i [-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}]^j \frac{F(x,y)}{x^{N+1}} \right)(1,0)$$

$$+ y^{N+1} \sum_{i,j\le N} x^{(i)} z^{(j)} \left([\frac{\partial}{\partial x} + \frac{\partial}{\partial y}]^i [-\frac{\partial}{\partial y}]^j \frac{F(x,y)}{y^{N+1}} \right)(1,0)$$

$$+ z^{N+1} \sum_{i,j\le N} x^{(i)} y^{(j)} \left(\frac{\partial^{i+j}}{\partial x^i \partial y^j} \frac{F(x,y)}{z^{N+1}} \right)(0,0) \ ,$$

Where $x^{(i)} = x^i/i!$ etc., see Gregory [6]. (Barnhill and Mansfield [4] show that L is the product operator of the three Hermite projectors defined in Section 3 of this paper.) The first three cardinal functions of (2.1) have the properties that

$$(2.3) \qquad \alpha_{0,0}\ (x,y) + \beta_{0,0}\ (x,y) + \gamma_{0,0}\ (x,y) = 1 \ ,$$

$$(2.4) \qquad (D^{\upsilon} \alpha_{0,0})\ (E_1) = (D^{\upsilon}\beta_{0,0})(E_2) = (D^{\upsilon} \gamma_{0,0})(E_3)\ C\ ,\ |\ \upsilon\ |\ \le\ N,$$

where $\upsilon = (m,n)$, $D^{\upsilon} = \partial^{m+n}/\partial x^m \partial y^n$, and $(D^{\upsilon} \alpha_{0,0})(E_1)$ denotes $D^{\upsilon} \alpha_{0.0}$ (x,y) evaluated on the side E_1, etc. The property (2.3) follows from the precision of (2.1) for F = 1 and (2.4) follows since the basis functions have factors x^{N+1}, y^{N+1}, and z^{N+1}respectively, see (2.2).

<u>Examples</u>

(i) For N = 0

$$(2.5)\qquad \begin{cases} \alpha_{0,0}(x, y) = x , \\ \beta_{0,0}(x, y) = y , \\ \gamma_{0,0}(x, y) = z , \end{cases}$$

(ii) For N = 1 ,

$$(2.6)\qquad \begin{cases} \alpha_{0,0}(x, y) = x^2 (3 - 2x + 6yz) , \\ \beta_{0,0}(x, y) = y^2 (3 - 2z + 6xy) , \\ \gamma_{0,0}(x, y) = z^2 (3 - 2z + 6xy) , \end{cases}$$

The smooth interpolation scheme is defined in the following theorem.

<u>Theorem 2. 1.</u> Let P_k, k = 1 ,2,3, be linear operaters such that P_kF is a function which interpolates $F \in C^N(\overline{T})$, and its derivatives of order N and less, on the sides E_i. and E_j of the triangle T adjacent to the vertex V_k Then the function

$$(2.7)\qquad PF = \alpha_{o,o}(x,y)\, P_1F + \beta_{0,0}(x,y)\, P_2F + \gamma_{0,0}(x,y)\, P_3F$$

interpolates F, and its derivatives of order N and lees, on the boundary ∂T of the triangle T, where P is a linear operator.

<u>Proof</u> It is sufficient to consider the side E_1, where, from (2.4),

$$(2.8)\qquad (D^{\upsilon}PF)(E_1) = (D^{\upsilon}[\beta_{0,0}\, P_2F + \gamma_{0,0}P_3F])\,(E_1),\ |\,\upsilon\,| \leqq N.$$

Now

$$(D^{\upsilon}P_2F)(E_1) = (D^{v}P_3F)(E_1) = (D^{\upsilon}F)(E_1), \quad |\nu| \leqq N ,$$

and (2.3) and (2.4) give that

$$(\beta_{0,0} + \gamma_{0,0})(E_1) = 1 \text{ and })D^{\upsilon}[\beta_{0,0} + \gamma_{0,0}]) \quad (E_1) = 0, \quad |\nu| \leq N$$

Thus application of Leibnitt's rule in (2.8) gives the required result that

$$(D^{\upsilon}PF)(E_1) = (D^{\upsilon}F)(E_1), |v| \leqq N.$$

<u>Remark 1</u>, Let τ be the intersection of the sets of polynomials for which each of the operators P_k k = 1,2,3, ia exact. Then P is exact for at least the set τ (the precision set). The proof of this result follows immediately from (2.3) and (2.7). The precision set is important in that it indicates the accuracy of the interpolation function.

<u>Remark 2.</u> If the operators P_k, k = 1,2,3, are affine invariant and symmetric on the sides of the triangle, then the operator P is affine invariant and symmetric.

<u>Remark 3.</u> If the P. are linear operators such that $P_k(PF) = P_kF$, k = 1,2,3, then it follows that P is a projector, i.e. P is a linear operator and P(PF) = PF. This property holds for the schemes discussed in Sections 3 and 4.

3. <u>A RATIONAL SMOOTH INTERPOLANT</u>

The Hermite two point Taylor projectors P_k, k = 1 ,2,3, along parallels to the sides E_k of the triangle T, are appropriate operators for the interpolation scheme (2.7). Explicitly these projectors are defined by

$$(3.1)\qquad P_1F = \sum_{i\le N} \varphi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{o,1}(x,o) + \sum_{i\le N} \psi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{o,i}(x,1-x) ,$$

$$(3.2)\qquad P_2F = \sum_{i\le N} \varphi_i\left(\frac{x}{1-y}\right)(1-x)^i F_{o,1}(o,y) + \sum_{i\le N} \psi_i\left(\frac{x}{1-y}\right)(1-x)^i F_{o,i}(1-y,y)$$

$$(3.3)\qquad P_3F = \sum_{i\le N} \varphi_i\left(\frac{x}{x+y}\right)(x+y)^i\left(\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right]^i F\right)(o,x+y)$$
$$+ \sum_{i\le N} \varphi_i\left(\frac{x}{x+y}\right)(x+y)^i\left(\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right]^i F\right)(x+y,o) ,$$

where the $\Phi_i(t)$ and $\Psi_1(t) = (-1)^i\Phi_i(1-t)$ are the cardinal basis functions for Hermite two point Taylor interpolation on [0,1].

Excluding the apparent singularity at the vertex V_k, the functions P_kF, k = 1,2,3, are N times continuously differentiable on T, provided that the boundary data in P. F is If times continuously differentiable with respect to its single variables on the sides of T. Also, Barnhill and Gregory [2] show that a sufficient condition that P_k. F is N times continuously differentiable as the limit to the vertex V_k is approached from T, is that the functions F in P_kF be N+1 times continuously differentiable. These are sufficient conditions that (2.7) can defins a piecewise interpolation function which is $C^N(\Omega)$ over a triangular r subdivision of a polygonal region Ω. These conditions are satisfied if the boundary data on the triangle is polynomial,

From Remark 1 , Section 2, and the definition of the projectors P_k, it follows that the interpolation scheme (2.7) is exact for τ_{2N+1} the set of polynomials which are of degree 2N+1 along parallels to the three sides of T.

Example

(i) For N = 0 ,

$$(3.4)\qquad PF = x[\frac{z}{1-x}F(x,o) + \frac{y}{1-x}F(x,i-x)]$$

$$+ y[\frac{z}{1-y}F(o,y) + \frac{x}{1-y}F(1-y,y)]$$

$$+ z[\frac{z}{1-z}F(o,x+y) + \frac{x}{1-z}F(x+y,0)]$$

with precision set $\mathcal{J}_1$

(ii) For N = 1

$$(3.5)\qquad PF = z^2(3-2z+6xy)[\varphi_0(\frac{x}{x+y})F(o,x+y) + \varphi_1(\frac{x}{x+y})(x+y)([\frac{\partial}{\partial x}-\frac{\partial}{\partial y}]F)(o,x.y)$$

$$+ \psi_o(\frac{x}{x+y})F(x+y,0) + \psi_1(\frac{x}{x+y})(x+y)([\frac{\partial}{\partial x}-\frac{\partial}{\partial y}]F)(x+y,o)]$$

+ dual terms in P_1F and P_2F,

where

$$(3.6)\qquad \begin{cases} \varphi_o(t) = (t=1)^2(2t+1), & \varphi_1(t) = (t-1)^2 t, \\ \psi_o(t) = t^2(-2t+3), & \psi_1(t) = t^2(t-1). \end{cases}$$

The precision set is τ_3.

4. A POLYNOMIAL SMOOTH INTERPOIANT

The polynomial Taylor projectors T_i^j on the sides E_j of the triangle, along parallels to the sides E_i, are defined by

$$
(4.1)\quad \begin{cases}
T_1^2 F = \sum_{i\le N} y^{(i)} F_{o,i}\,(x,o)\ , & T_1^3 F = \sum_{i\le N} (x+y-1)^{(i)}\ F_{o,1}(x,1-x) \\
T_2^1 F = \sum_{i\le N} x^{(i)} F_{i,o}\,(o,y)\ , & T_2^3 F = \sum_{i\le N} (x+y-1)^{(i)}\ F_{o,1}(1-y-y) \\
T_3^1 F = \sum_{i\le N} x^{(i)}([\frac{\partial}{\partial x}-\frac{\partial}{\partial y}]^{i}\ F)\,(o,x,y)\ , = & T_3^2 F = \sum_{i\le N} y^{(i)}\,([\frac{\partial}{\partial x}-\frac{\partial}{\partial y}]^{i}\ F)\,(x+y,o)
\end{cases}
$$

Where $x^{(i)} = x^i/i!$ etc. Polynomial projectors P., k = 1,2,3, appropriate for the interpolation scheme (2.7), are defined in the following theorem.

<u>Theorem 4.1.</u> Let $F \in C^N(\overline{T})$ Satisfy the compatibility condition

$$
(4.2)\qquad (\frac{\partial^{m+n}}{\partial n_i^m\,\partial m_j^n})\,(V_k) = (\frac{\partial^{n+m}F}{\partial n_j\, n\partial m_i^m})\,(V_k)\quad ,\quad m,n \le N;\qquad m+n > N\quad ,
$$

at the vertex V_k, with adjacent sides E. and E_j, where $\partial/\partial s\ell$ denotes differentiation along the side E.. Then the polynomial Boolean sum function

$$
(4.3)\qquad P_{kF} = (T^j{}_i \oplus T_j{}^i)F
$$

$$
= (T^j{}_i + T^i{}_j - T^J{}_I\ T^i{}_j)F
$$

interpolates $F \in C^N(\overline{T})$, and its derivatives of order N and less, on the sides E_i. and E_j of the triangle T.

<u>Proof</u> By affine transformation it is sufficient to consider the case

$$
(4.4)\qquad P_3F = (T_i^j \ \oplus T_1^2)F
$$

$$
= \sum_{i\le N} x^{(i)} F_{i,0}\,(0,y) + \sum_{j\le N} y^{(j)} F_{0,j}\,(x,0)
$$

$$
+ \sum_{i,j\le N} x^{(i)} y^{(j)} (\frac{\partial^{i+j}F}{\partial x^i\,\partial y^j})\,(0,0)
$$

Since $D^{\upsilon}(I - T^1{}_2)$ is null on E_1 for all $|\upsilon| \leq N$, it follows that

$$D^{\upsilon}F - D^{\upsilon}P_3F = D^{\upsilon}(I - T^1{}_2)(I - T^2{}_1)F$$

is zero on E_1. Also, the compatibility condition (4.2), and $F \in C^N(\overline{T})$ imply that $T^1{}_2$ and $T^2{}_1$ are commutative and he nce the dual result holds on E_2.

The compatibility condition can he removed by the addition of a suitable rational term to the projector, as is shown in [3]. Let T_k, k = 1,2,3, be the rational Hermits projectors defined by equations (3. 1) - (3.3) respectively. Then

$$T_k[F - (T^j_i \oplus T^i_j)F]$$

is a rational function which Interpolates the remainder function

$F - (T^i_j \oplus T^i_j)F$ on E_i and E_j Hence

$$(4.5) \quad P_kF = (T^j_i \oplus T^i_j)F + T_k[F-(T^j_i \oplus T^i_j)F]$$

Interpolates $F \in C^N(\overline{T})$ on the sides E_i and E_j of the triangle T adjacent to the vertex V_k Further, the symmetry of the interpolation scheme (2.7) is retained if the projectors P_k are defined by the average

$$(4.6) \quad P_kF = \tfrac{1}{2}(T^i_j \oplus T^i_j)F + \tfrac{1}{2}(T^i_j \oplus T^j_i F$$
$$+ T_k[F - \tfrac{1}{2}(T^j_i \oplus T^i_j)F - \tfrac{1}{2}(T^i_j \oplus T^j_i)F].$$

If F satisfies (4.2) at each vertex, the rational terms are zero and the projector reduces to (4.3).

The projector P_k is precise for the union of the precision sets of T^j_i and T^i_j, namely

$$(4.7) \qquad \xi_i^m \, \xi_j^n \qquad \begin{cases} o \le m \le N \text{ for all } n , \\ o \le n \le N \text{ for all } m , \end{cases}$$

where $\xi_1 = x$, $\xi_2 = y$, and $\xi_3 = z$. Thus, from Remark 1, Section 2, the interpolation scheme (2.7) is exact for P_{2N+1}, the set of polynomials which are of degree 2N+1 or less.

Examples

(i) For N = 0 ,

$$(4.8) \qquad PF = x[F(1-y,y) + F(x+y,0) - F(1,0)]$$

$$+ y[F(x,1-x) + F(0,x+y) - F(0,1)]$$

$$+ z[F(0,y) + F(x,0) - F(0,0)] ,$$

with precision set P_1.

(ii) For N = 1 ,

$$(4.9) \qquad PF = z^2(3-2z+6xy)[F(0,y) + xF_{1,0}(0,y) + F(X,0) + F_{0,1}(X,0) - F(0,0)$$

$$- xF_{1,0}(0,0) - yF_{0,1}(0,0) - \frac{xy}{2}\left\{(\frac{\partial^2 F}{\partial x \partial y})(0,0) + (\frac{\partial^2 F}{\partial y \partial x})(0,0)\right\}$$

$$+ \frac{xy(x-y)}{2(x+y)}\left\{(\frac{\partial^2 F}{\partial x \partial y})(0,0) - (\frac{\partial^2 F}{\partial y \partial x})((0,0)\right\}]$$

+ dual terms in P_1F and P_2F ,

where the rational term is the compatibility correction. The precision set is P_3 .

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