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A new perturbative solution to the motion around triangular Lagrangian points in the elliptic restricted three-body problem

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Abstract The equations of motion of planar elliptic restricted three body problem are transformed to four decoupled Hill's equations. By using the Floquet theorem analytic solution to the oscillator equations with time dependent periodic coefficients are presented. We show that the new analytic approach is valid for system parameters $0 < e \leq 0.05$ and $0 < \mu \leq 0.01$ where e denotes the eccentricity of primaries while μ is the mass parameter, respectively. We also clarify the transformation details that provide the applicability of the method.

Keywords ERTBP · Hill's Equation · Floquet theorem

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1 Introduction

In the era of exoplanets and specifically designed space missions, the co-orbital motion in the vicinity of the equilateral points L_4 and L_5 became again the focus of attention. Since the seminal work of Szebehely [Szebehely(1967)] the orbits near the libration points have been discussed extensively by the community. Analytic description of the Trojan-like resonant dynamics in the elliptic case of restricted three-body problem (ERTBP) is based mainly on averaged motion. rdi [Erdi(1977),Erdi(1978)] showed the perturbation effects up to second order in Jupiter's eccentricity, perihelion and ascending node precession by using a three-parameter expansion. Morais [Morais(2001)] considered an averaged disturbing potential to describe the secular variation of the Trojans' orbital elements in case of an oblate primary. Recently, [Robutel et al(2016)Robutel, Niederman, and Pousse] and [Páez et al(2016)Páez, Locatelli, and Efthymiopoulos]

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investigated the co-orbital resonance based on Hamiltonian formalism whereby the fast angles had been averaged out. These latter analytical studies are also capable to locate higher-order resonances as well as very slow secular frequencies.

It has been demonstrated [Tschauner(1971), Erdi(1974), Meire(1981), Matas(1982)] that the coupled equations of the ERTBP can be written in the form of independent ordinary differential equations with variable coefficients. The primary goal of these studies is to explore the stability map of eccentricity–mass parameter dated back to [Danby(1964)]. Interestingly, the analysis given by [Erdi(1977)] [Eq. (24)] and [Robutel et al(2016)Robutel, Niederman, and Pousse] [Eq. (56)] also terminates at a pendulum-like equation, however, they do not attempt to solve it by classical techniques such as Floquet theorem [Lichtenberg and Lieberman(1983) Here we propose a detailed derivation of Hill's equations of ERTBP and make a comprehensive analysis of their applicability which is still out of literature. Furthermore, analytic expressions for the solution of Hill's equations are given in the regime of moderate eccentricities and mass parameter with good agreement of numerical calculations.

2 Basic context

In this paper we mainly follow the notations used in e.g. [Tschauner(1971), Meire(1980), Meire(1982)]. Motion around the L_4 and L_5 Lagrangian points is determined by the coupled differential equations [Szebehely(1967)]

$$x'' - 2y' = rc_1 x, \tag{1}$$
$$y'' + 2x' = rc_2 y, \tag{2}$$

where the notations are

$$r = \frac{1}{1+e\cos(v)}, \quad \mu = \frac{m_2}{m_1+m_2}, \quad g = 3\mu(1-\mu) \quad \text{and} \quad c_i = \frac{3}{2}(1+(-1)^i\sqrt{1-g}) \quad (i=1,2). \tag{3}$$

Here $'$ denotes the derivation with respect to the true anomaly v.

2.1 Hill's equations

We will show, that Eqs. (1)-(2) with a suitable transformation can be rewritten to four decoupled second order differential equations. Let us introduce y_1, y_2, y_1^*, y_2^* with the following transformation

$$\begin{pmatrix} x \\ y \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_1^* \\ y_2^*, \end{pmatrix}, \tag{4}$$

where $\mathbb{1}_2$ is the 2-dimensional identity matrix, and furthermore $\mathbf{P}_1$ and $\mathbf{P}_2$ are introduced as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_1^{*\prime} \\ y_2^{*\prime} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_1^* \\ y_2^* \end{pmatrix} \tag{5}$$

relation stands. Let us use the temporary notations $\tilde{x} = (x, y)$, $\tilde{y}_1 = (y_1, y_2)$ and $\tilde{y}_2 = (y_1^*, y_2^*)$. The elements of $\mathbf{P}_i$ $(i = 1, 2)$ matrices can be gained by using the following identities

$$\begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} \tilde{y}_1 + \tilde{y}_2 \\ \mathbf{P}_1\tilde{y}_1 + \mathbf{P}_2\tilde{y}_2 \end{pmatrix}, \tag{6}$$

from which it simply follows, that

$$\begin{pmatrix} \tilde{x}' \\ \tilde{x}'' \end{pmatrix} = \begin{pmatrix} \tilde{y}_1' + \tilde{y}_2' \\ \mathbf{P}_1'\tilde{y}_1 + \mathbf{P}_1\tilde{y}_1' + \mathbf{P}_2'\tilde{y}_2 + \mathbf{P}_2\tilde{y}_2' \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbb{1}_2 \\ r\mathbf{C} & 2\mathbf{D} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{x}' \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbb{1}_2 \\ r\mathbf{C} & 2\mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbb{1}_2 & \mathbb{1}_2 \\ \mathbf{P}_1 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}, \tag{7}$$

where $\mathbf{C} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 0 & 1 \\ -1 & 0. \end{pmatrix}$. It can be recognized, that

$$\begin{pmatrix} \tilde{y}_1' + \tilde{y}_2' \\ \mathbf{P}_1'\tilde{y}_1 + \mathbf{P}_1\tilde{y}_1' + \mathbf{P}_2'\tilde{y}_2 + \mathbf{P}_2\tilde{y}_2' \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_1' + \mathbf{P}_1^2 & \mathbf{P}_2' + \mathbf{P}_2^2 \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \quad \Rightarrow \quad \mathbf{P}_i' + \mathbf{P}_i^2 = r\mathbf{C} + 2\mathbf{D}\mathbf{P}_i, \tag{8}$$

which is a Riccati-type matrix differential equation. Based on Tschauner's argument [Tschauner(1971)] the following matrix elements satisfy Eq. (8)

$$p_{11}^{(i)} = -\frac{1}{2} re\sin(v)(1 + ke\cos(v)), \tag{9}$$

$$p_{12}^{(i)} = r\left(a_2^{(i)} + e\cos(v) - \frac{1}{4}ke^2\cos(2v)\right), \tag{10}$$

$$p_{21}^{(i)} = -r\left(a_1^{(i)} + e\cos(v) + \frac{1}{4}ke^2\cos(2v)\right), \tag{11}$$

$$p_{22}^{(i)} = -\frac{1}{2} re\sin(v)(1 - ke\cos(v)), \tag{12}$$

where

$$k = \frac{1}{\sqrt{1-g}}, \quad c = \sqrt{1 - 9g + 2e^2 + k^2e^4}, \quad a_1^{(i)} = \frac{1}{4}(2c_1 + 1 + (-1)^i c), \quad a_2^{(i)} = \frac{1}{4}(2c_2 + 1 + (-1)^i c) \tag{13}$$

notations are introduced. As all elements of $\mathbf{P}_i$ have a multiplicative factor of r, therefore we define the following $q = p/r$ quantities as

$$q_{11}^{(i)} = -\frac{1}{2} e\sin(v)(1 + ke\cos(v)) \quad \Rightarrow \quad q_{11}^{(i)\prime} = -\frac{1}{2}(e\cos(v) + ke^2\cos(2v)), \tag{14}$$

$$q_{12}^{(i)} = \left(a_2^{(i)} + e\cos(v) - \frac{1}{4}ke^2\cos(2v)\right) \quad \Rightarrow \quad q_{12}^{(i)\prime} = 2q_{22}^{(i)}, \tag{15}$$

$$q_{21}^{(i)} = -\left(a_1^{(i)} + e\cos(v) + \frac{1}{4}ke^2\cos(2v)\right) \quad \Rightarrow \quad q_{21}^{(i)\prime} = -2q_{11}^{(i)}, \tag{16}$$

$$q_{22}^{(i)} = -\frac{1}{2} e\sin(v)(1 - ke\cos(v)) \quad \Rightarrow \quad q_{22}^{(i)\prime} = -\frac{1}{2}(e\cos(v) - ke^2\cos(2v)). \tag{17}$$

(18)

Let $\det \mathbf{Q}^{(i)}$ be the determinant of the matrix with the above elements. It can be shown

$$\det \mathbf{Q}^{(i)} = \frac{1}{2r}\left((-1)^i c + 1 + 3e\cos(v)\right). \tag{19}$$

According to Eq. (5) $\tilde{y}_i' = \mathbf{P}_i\tilde{y}_i$, from which the $\tilde{y}_i'' = \mathbf{P}_i'\tilde{y}_i + \mathbf{P}_i\tilde{y}_i' = (2\mathbf{DP} + r\mathbf{C})\tilde{y}_i$ relation follows. Consequently,

$$y_1'' = (2p_{21}^{(1)} + rc_1)y_1 + 2p_{22}^{(1)}y_2 = \frac{1}{q_{12}^{(1)}}(q_{12}^{(1)}rc_1 - 2r\det\mathbf{Q}^{(1)})y_1 + \frac{2q_{22}^{(1)}}{q_{12}^{(1)}}y_1', \tag{20}$$

$$y_2'' = (-2p_{12}^{(1)} + rc_2)y_2 - 2p_{11}^{(1)}y_1 = \frac{1}{q_{21}^{(1)}}(q_{21}^{(1)}rc_2 + 2r\det\mathbf{Q}^{(1)})y_2 - \frac{2q_{11}^{(1)}}{q_{21}^{(1)}}y_2', \tag{21}$$

where we used the relations

$$y_1 = \frac{y_2' - p_{22}^{(1)}y_2}{p_{21}^{(1)}} \quad \text{and} \quad y_2 = \frac{y_1' - p_{11}^{(1)}y_1}{p_{12}^{(1)}}. \tag{22}$$

Similar arguments are true for y_1^* and y_2^* with elements of matrix $\mathbf{P}_2$.

Considering the general form of the ordinary differential equation $y'' + a(v)y' + b(v)y = 0$, the transformation $y = \xi(v)\exp(-1/2\int_0^v a(x)\mathrm{d}x)$ eliminates the first order derivative term y'. Applying this conversion to Eqs. (20) and (21) we can introduce the following transformations

$$y_1 = \sqrt{|q_{12}^{(1)}|}\xi_1, \quad y_1' = \frac{q_{22}^{(1)}}{\sqrt{|q_{12}^{(1)}|}}\xi_1 + \sqrt{|q_{12}^{(1)}|}\xi_1', \quad y_1'' = \frac{q_{22}^{(1)\prime}|q_{12}^{(1)}| - q_{22}^{(1)^2}}{|q_{12}^{(1)}|^{3/2}}\xi_1 + \frac{2q_{22}^{(1)}}{\sqrt{|q_{12}^{(1)}|}}\xi_1' + \sqrt{|q_{12}^{(1)}|}\xi_1'', \tag{23}$$

$$y_1^* = \sqrt{|q_{12}^{(2)}|}\xi_2, \quad y_1^{*\prime} = \frac{q_{22}^{(2)}}{\sqrt{|q_{12}^{(2)}|}}\xi_2 + \sqrt{|q_{12}^{(2)}|}\xi_2', \quad y_1^{*\prime\prime} = \frac{q_{22}^{(2)\prime}|q_{12}^{(2)}| - q_{22}^{(2)^2}}{|q_{12}^{(2)}|^{3/2}}\xi_2 + \frac{2q_{22}^{(2)}}{\sqrt{|q_{12}^{(2)}|}}\xi_2' + \sqrt{|q_{12}^{(2)}|}\xi_2''. \tag{24}$$

By using equations above, (20) and (21) become the differential equations of harmonic oscillators with periodic coefficient. These equations are also known as Hill's equation

$$\xi_1'' + J_1(v)\xi_1 = 0, \quad \text{where} \quad J_1(v) = -\left(rc_1 + 2 - \frac{3r\det\mathbf{Q}^{(1)} + c_2}{q_{12}^{(1)}} + \frac{3q_{22}^{(1)^2}}{q_{12}^{(1)^2}}\right), \quad y_1 = \sqrt{|q_{12}^{(1)}|}\xi_1, \tag{25}$$

$$\xi_3'' + J_3(v)\xi_3 = 0, \quad \text{where} \quad J_3(v) = -\left(rc_2 + 2 + \frac{3r\det\mathbf{Q}^{(1)} + c_1}{q_{21}^{(1)}} + \frac{3q_{11}^{(1)^2}}{q_{21}^{(1)^2}}\right), \quad y_2 = \sqrt{|q_{21}^{(1)}|}\xi_3 \tag{26}$$

The corresponding transformations (not presented) can also be carried out for y_1^* and y_2^* yielding the following Hill's equations

$$\xi_2'' + J_2(v)\xi_2 = 0, \quad \text{where} \quad J_2(v) = -\left(rc_1 + 2 - \frac{3r\det\mathbf{Q}^{(2)} + c_2}{q_{12}^{(2)}} + \frac{3q_{22}^{(2)^2}}{q_{12}^{(2)^2}}\right), \quad y_1^* = \sqrt{|q_{12}^{(2)}|}\xi_2, \tag{27}$$

$$\xi_4'' + J_4(v)\xi_4 = 0, \quad \text{where} \quad J_4(v) = -\left(rc_2 + 2 + \frac{3r\det\mathbf{Q}^{(2)} + c_1}{q_{21}^{(2)}} + \frac{3q_{11}^{(2)^2}}{q_{21}^{(2)^2}}\right), \quad y_2^* = \sqrt{|q_{21}^{(2)}|}\xi_4. \tag{28}$$

In Eqs. (25)-(28) J_i ($i = 1 \ldots 4$) are periodic coefficients with period of 2π. Square root of the coefficients gives the frequency of the oscillator. We can obtain the original Cartesian coordinates by using Eq. (4), thus, x, y coordinates can be calculated as $x = y_1 + y_1^*$ and $y = y_2 + y_2^*$.

It is clear from the coefficients J_i that Eqs. (25)-(28) do not have solutions if $q_{jk}^{(l)}(\mu, e) = 0$ ($j = 1$ or 2, $k = 1$ or 2 and $l = 1$ or 2, see Eqs. (15)-(18)). The forbidden parameter pairs (μ, e) as solid lines are depicted in Fig. 1. We note that $q_{12}^{(1)}$ and $q_{12}^{(2)}$ associated to Eqs. (25) and (27) do not take zero value anywhere in the shaded region. However, the black solid line between domain I and II corresponds to those (μ, e) pairs where $q_{21}^{(1)} = 0$. Similarly, $q_{21}^{(2)} = 0$ along the line between the regions II and III.

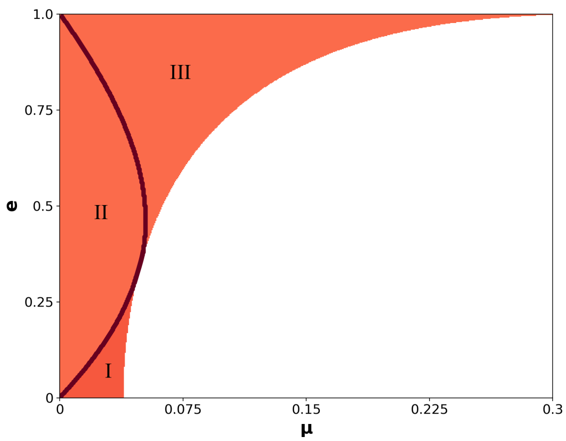

Fig. 1: Shaded region of $\mu - e$ parameter plane describes where Hill's equations might have real solutions. Along the black solid lines between domains I, II, and III. condition $q_{21}^{(i)} = 0$ holds. That is Eqs. (27) and (28) have no solutions. No stable solution exists in white part.

Transformations Eqs. (23)-(24) can be substituted into Eq. (22), from which we get

$$y_2 = \left(\frac{q_{22}^{(1)}}{r|q_{12}^{(1)}|^{3/2}} - \frac{q_{11}^{(1)}}{\sqrt{|q_{12}^{(1)}|}} \right) \xi_1 + \frac{1}{r\sqrt{|q_{12}^{(1)}|}} \xi_1' \quad \text{and} \quad y_2^* = \left(\frac{q_{22}^{(2)}}{r|q_{12}^{(2)}|^{3/2}} - \frac{q_{11}^{(2)}}{\sqrt{|q_{12}^{(2)}|}} \right) \xi_2 + \frac{1}{r\sqrt{|q_{12}^{(2)}|}} \xi_2'. \tag{29}$$

Doing this Eq. (29) reduces the four Hills equations to two. Thus Eqs. (25) and (27) fully describe the problem[1]. In other words, Eqs. (29) allows one to use safely the transformations (23) and (24) to solve Hill's equations. Fig. 2 depicts the trajectory for $e = 0.048$, $\mu = 0.000954$ (the case of Jupiter). The solution of Eqs. (1)-(2) and Eqs. (25)-(27) originating from the appropriate initial conditions perfectly overlap. This means, that the transformations Eqs. (23)-(24) lead to the same result, therefore, Hill's equations can be applied to solve the equations of motion around the L_4 and L_5 points.

[1] We note that any two equations can be selected but for practical reasons the pair of ξ_1 and ξ_2 is the best choice.

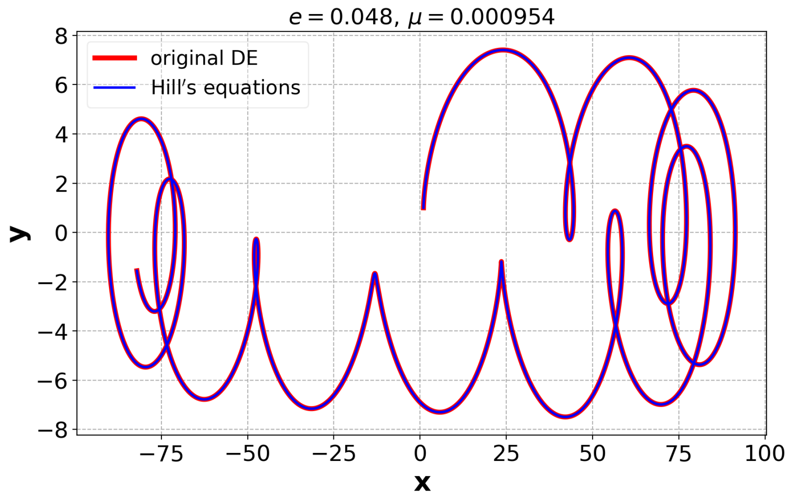

Fig. 2: Numerical solutions around the L_4 and L_5 points. The initial conditions and the parameters are $x_0 = 1$, $y_0 = 1$, $v_{x0} = 0$, $v_{y0} = 0$, $e = 0.048$, $\mu = 0.000954$ (the case of Jupiter), respectively.

3 Perturbative solution

In this section we give the perturbative solution of the differential equations. Hill's equations (Eqs. (25)-(27)), as they are second order differential equations with periodic coefficients, can be solved by Floquet theorem [Hagel(1992)]. We seek the solution in the form of

$$\xi(v) = aw(v)\cos(\psi(v) + b), \tag{30}$$

where $w(v)$ is the so-called Floquet function, which has the same period as $\xi(v)$. Constants a and b are determined by the initial conditions. Since the derivation for both ξ_1 and ξ_2 are the same, we omit the indices in the rest part of the paper. Let us rewrite Eqs. (25)-(27) to $w(v)$ and $\psi(v)$

$$\xi'(v) = aw'(v)\cos(\psi(v)+b) - aw(v)\sin(\psi(v)+b)\psi'(v), \tag{31}$$

$$\begin{aligned}\xi''(v) = {} & aw''(v)\cos(\psi(v)+b) - 2aw'(v)\sin(\psi(v)+b)\psi'(v) - aw(v)\cos(\psi(v)+b)\psi'^2(v) \\ & - aw(v)\sin(\psi(v)+b)\psi''(v).\end{aligned} \tag{32}$$

The differential equations above split into two parts with the coefficients of sin and cos

$$w''(v) - w(v)\psi'^2(v) + J(v)w(v) = 0, \tag{33}$$

$$2w'(v)\psi'(v) + w(v)\psi''(v) = 0. \tag{34}$$

From Eq. (34) we obtain

$$2\frac{w'(v)}{w(v)} = -\frac{\psi''(v)}{\psi'(v)} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}v}\log(w^2(v)) = \frac{\mathrm{d}}{\mathrm{d}v}\log\left(\frac{1}{\psi'(v)}\right) \quad \Rightarrow \quad \psi'(v) = \frac{C}{w^2(v)}, \tag{35}$$

where C is a constant. With this equation Eq. (33) becomes

$$w'' + J(v)w(v) - \frac{C^2}{w^3(v)} = 0. \tag{36}$$

Now we are looking for the solution of $w(v)$ in a third order Taylor series in the eccentricity e

$$w(v) = w^{(0)}(v) + ew^{(1)}(v) + e^2w^{(2)}(v) + e^3w^{(3)}(v) + \mathscr{O}(e^4). \tag{37}$$

3.1 Taylor series of $J_1(v)$ and $J_2(v)$

The periodic coefficients to be solved have complicated forms, therefore, the solution can be obtained by a third order Taylor expansion in the eccentricity. Let us first utilize J_1

$$J_1(v) = -\left\{c_1 r + 2 - \frac{\frac{3}{2}(1-c+3e\cos(v))+c_2}{\frac{1}{4}(2c_2+1-c)+e\cos(v)-\frac{1}{4}ke^2\cos(2v)} + 3\left(\frac{-\frac{1}{2}e\sin(v)(1-ke\cos(v))}{\frac{1}{4}(2c_2+1-c)+e\cos(v)-\frac{1}{4}ke^2\cos(2v)}\right)^2\right\}, \tag{38}$$

by using the earlier introduced notations. Useful expressions will be $\lambda \equiv \sqrt{1-9g}$ and $B \equiv (2c_2+1-\lambda)^{-1}$. Third order Taylor expansion of J_1 is then

$$J_1(v;e,\mu) = \alpha_1 + \beta_1\cos(v)e + (\gamma_1 + \delta_1\cos(2v))e^2 + (\varepsilon_1\cos(v) + \eta_1\cos(3v))e^3 + \mathscr{O}(e^4),$$

where

$$\begin{aligned}
\alpha_1 &= -c_1 - 2 + B(6 - 6\lambda + 4c_2),\\
\beta_1 &= c_1 + 18B + 8B^2(-3 + 3\lambda - 2c_2),\\
\gamma_1 &= \frac{B^2}{\lambda}(6 - 12\lambda + 4c_2) - \frac{6B}{\lambda} - \frac{c_1}{2} - 4B^3(10c_2 - 3 + 3\lambda),\\
\delta_1 &= B^2k\left(6 - 6\lambda + 4c_2 + \frac{6}{k}\right) - \frac{c_1}{2} - 4B^3(10c_2 - 3 + 3\lambda),\\
\varepsilon_1 &= \frac{B^4}{\lambda}\left\{\frac{20c_2 - 6 + 30\lambda + 10c_2\lambda k - 3\lambda k + 3\lambda^2 k}{B} + \frac{6k\lambda}{B^2} + 64c_2^2 + 32c_2 +\right.\\
&\quad + 208c_2\lambda - 32c_2k\lambda - 12k\lambda + 24k - 216gk + 32c_2k - 288kc_2g - 12k\lambda^3 - 16c_2^2k\lambda -\\
&\quad \left. - 72\lambda + 72 - 648g\right\} + \frac{3c_1}{4},\\
\eta_1 &= \frac{B^4}{\lambda}\left\{\frac{10c_2\lambda k - 3\lambda k + 3\lambda^2 k + 24\lambda}{B} + \frac{6k\lambda}{B^2} - 32c_2k\lambda - 12k\lambda + 24k - 216gk +\right.\\
&\quad \left. + 32c_2k - 288c_2gk - 12k\lambda^3 - 16c_2^2k\lambda + 80\lambda c_2 - 24\lambda + 24 - 216g\right\} + \frac{c_1}{4}.
\end{aligned} \tag{39}$$

The expression of the other periodic coefficient J_2 is

$$J_2(v) = -\left\{c_1 r + 2 - \frac{\frac{3}{2}(1+c+3e\cos(v))+c_2}{\frac{1}{4}(2c_2+1+c)+e\cos(v)-\frac{1}{4}ke^2\cos(2v)} + 3\left(\frac{-\frac{1}{2}e\sin(v)(1-ke\cos(v))}{\frac{1}{4}(2c_2+1+c)+e\cos(v)-\frac{1}{4}ke^2\cos(2v)}\right)^2\right\}. \tag{40}$$

To calculate the Taylor series of J_2 (again up to third order in e) we use $D \equiv (2c_2+1+\lambda)^{-1}$. Then J_2 becomes

$$J_2(v;e,\mu) = \alpha_2 + \beta_2\cos(v)e + (\gamma_2 + \delta_2\cos(2v))e^2 + (\varepsilon_2\cos(v) + \eta_2\cos(3v))e^3 + \mathscr{O}(e^4),$$

where

$$
\begin{aligned}
\alpha_2 &= -c_1 - 2 + D(6 + 6\lambda + 4c_2),\\
\beta_2 &= c_1 + 18D - 8D^2(3 + 3\lambda + 2c_2),\\
\gamma_2 &= -\frac{D^2}{\lambda}(6 + 12\lambda + 4c_2) + \frac{6D}{\lambda} - \frac{c_1}{2} + 4D^3(-10c_2 + 3 + 3\lambda),\\
\delta_2 &= D^2 k\left(6 + 6\lambda + 4c_2 + \frac{6}{k}\right) - \frac{c_1}{2} + 4D^3(-10c_2 + 3 + 3\lambda),\\
\varepsilon_2 &= \frac{D^4}{\lambda}\Bigg\{\frac{-20c_2 + 6 + 30\lambda + 10c_2\lambda k - 3\lambda k - 3\lambda^2 k}{D} + \frac{6k\lambda}{D^2} - 64c_2^2 - 32c_2 +\\
&\quad + 208c_2\lambda - 32c_2k\lambda - 12k\lambda - 24k + 216gk - 32c_2k + 288kc_2g - 12k\lambda^3 - 16c_2^2k\lambda -\\
&\quad - 72\lambda - 72 + 648g\Bigg\} + \frac{3c_1}{4},\\
\eta_2 &= \frac{D^4}{\lambda}\Bigg\{\frac{10c_2\lambda k - 3\lambda k - 3\lambda^2 k + 24\lambda}{D} + \frac{6k\lambda}{D^2} - 32c_2k\lambda - 12k\lambda - 24k + 216gk -\\
&\quad - 32c_2k + 288c_2gk - 12k\lambda^3 - 16c_2^2k\lambda + 80\lambda c_2 - 24\lambda - 24 + 216g\Bigg\} + \frac{c_1}{4}.
\end{aligned}
\tag{41}
$$

Let us write back the results of the Taylor expansions into Eq. (36), and use the fact that

$$
\begin{aligned}
&\frac{1}{(w^{(0)}(v)+ew^{(1)}(v)+e^2w^{(2)}(v)+e^3w^{(3)}(v))^3} = \frac{1}{w^{(0)^3}(v)} - \frac{3w^{(1)}(v)}{w^{(0)^4}(v)}e + \frac{6w^{(1)^2}(v) - 3w^{(0)}(v)w^{(2)}(v)}{w^{(0)^5}(v)}e^2 +\\
&+ \frac{-3w^{(0)^2}(v)w^{(3)}(v) + 12w^{(0)}(v)w^{(1)}(v)w^{(2)}(v) - 10w^{(1)^3}(v)}{w^{(0)^6}(v)}e^3 + \mathscr{O}(e^4).
\end{aligned}
\tag{42}
$$

Then we can collect the terms for e^0, e^1, e^2 and e^3, thus 4 new differential equations can be obtained (also for $i = 1,2$) for the terms of $w(v)$:

$$
w^{(0)''}(v) + w^{(0)}(v)\alpha - \frac{C^2}{w^{(0)^3}(v)} = 0, \tag{43}
$$

$$
w^{(1)''}(v) + w^{(0)}(v)\beta\cos(v) + w^{(1)}(v)\alpha + \frac{3C^2w^{(1)}(v)}{w^{(0)^4}(v)} = 0, \tag{44}
$$

$$
w^{(2)''}(v) + w^{(0)}(v)\Big(\gamma + \delta\cos(2v)\Big) + w^{(1)}(v)\beta\cos(v) + w^{(2)}(v)\alpha - \frac{6C^2w^{(1)^2}(v)}{w^{(0)^5}} + \frac{3C^2w^{(2)}(v)}{w^{(0)^4}(v)} = 0, \tag{45}
$$

$$
\begin{aligned}
&w^{(3)''}(v) + w^{(0)}(v)\Big(\varepsilon\cos(v) + \eta\cos(3v)\Big) + w^{(1)}(v)\Big(\gamma + \delta\cos(2v)\Big) + w^{(2)}\beta\cos(v) + w^{(3)}(v)\alpha +\\
&\qquad + \frac{3C^2w^{(3)}(v)}{w^{(0)^4}(v)} - \frac{12C^2w^{(1)}(v)w^{(2)}(v)}{w^{(0)^5}(v)} + \frac{10C^2w^{(1)^3}(v)}{w^{(0)^6}(v)} = 0.
\end{aligned}
\tag{46}
$$

Again we note, that for all cases $w^{(j)}(v) = w^{(j)}(v+2\pi)$, $(j = 0,1,2,3)$, as also $\xi(v) = \xi(v+2\pi)$. It can be easily seen, that the unique solution for Eq. (43) is:

$$w^{(0)}(v) = \frac{C^{1/2}}{\alpha^{1/4}} \equiv w_{0,0}. \tag{47}$$

Differential equations (44)-(45)-(46) are second order linear differential equations, therefore the solution can be written up as the sum of the solution of the homogeneous equation ($w_h^{(j)}(v)$) and a particular solution of the inhomogeneous equation ($w_{ih}^{(j)}(v)$). Homogeneous part of Eq. (44) is

$$w_h^{(1)''}(v) + \left(\alpha + \frac{3C^2}{w_{0,0}^4}\right) w_h^{(1)}(v) = 0, \tag{48}$$

which is a harmonic oscillator with frequency $\left(\alpha + \frac{3C^2}{w_{0,0}^4}\right)^{1/2}$, thus the solution of the equation is

$$w_h^{(1)}(v) = K_1 \sin\left(\sqrt{\alpha + \frac{3C^2}{w_{0,0}^4}}v\right) + K_2 \cos\left(\sqrt{\alpha + \frac{3C^2}{w_{0,0}^4}}v\right), \tag{49}$$

where the constants K_1 and K_2 must be determined from the initial conditions. In order to fulfill the 2π periodicity of $w(v)$, the constants must be $K_1 = K_2 \equiv 0$. For the inhomogeneous solution we use the following trial function

$$w_{ih}^{(1)}(v) = w_{1,1}\cos(v) + w_{1,0}, \tag{50}$$

where $w_{1,1}$ are $w_{1,0}$ constants. By calculating the derivatives from the coefficients we can simply obtain the values of $w_{1,1}$ and $w_{1,0}$, namely

$$w_{1,1} = -\frac{w_{0,0}\beta}{\alpha + \frac{3C^2}{w_{0,0}^4} - 1}, \qquad w_{1,0} = 0. \tag{51}$$

We use the same steps for the solution of Eq. (45). By using trigonometric identities it can be seen, that the differential equation has the following form

$$w^{(2)''}(v) + \left(\alpha + \frac{3C^2}{w_{0,0}^4}\right) w^{(2)}(v) = \left(-w_{0,0}\gamma - \frac{1}{2}w_{1,1}\beta + \frac{3C^2 w_{1,1}^2}{w_{0,0}^5}\right) - \left(w_{0,0}\delta + \frac{1}{2}w_{1,1}\beta - \frac{3C^2 w_{1,1}^2}{w_{0,0}^5}\right)\cos(2v). \tag{52}$$

Like in the previous case the solution of the homogeneous part is

$$w_h^{(2)}(v) = K_1 \sin\left(\sqrt{\alpha + \frac{3C^2}{w_{0,0}^4}}v\right) + K_2 \cos\left(\sqrt{\alpha + \frac{3C^2}{w_{0,0}^4}}v\right), \tag{53}$$

where again K_1 and K_2 must disappear for the 2π periodicity, $K_1 = K_2 \equiv 0$. The trial function of the particular solution of the inhomogeneous equation is:

$$w_{ih}^{(2)}(v) = w_{2,2}\cos(2v) + w_{2,0}. \tag{54}$$

Again by calculating the appropriate derivatives the equality of the coefficients imply:

$$w_{2,2}=\frac{\dfrac{3C^2w_{1,1}^2}{w_{0,0}^5}-w_{0,0}\delta-\dfrac{1}{2}w_{1,1}\beta}{\alpha+\dfrac{3C^2}{w_{0,0}^4}-4},\quad w_{2,0}=\frac{\dfrac{3C^2w_{1,1}^2}{w_{0,0}^5}-w_{0,0}\gamma-\dfrac{1}{2}w_{1,1}\beta}{\alpha+\dfrac{3C^2}{w_{0,0}^4}}. \tag{55}$$

Only the solution of Eq. (46) is left

$$w^{(3)\prime\prime}(v)+w^{(3)}(v)\left(\alpha+\frac{3C^2}{w_{0,0}^4}\right)=-\left(w_{0,0}\varepsilon+w_{1,1}\gamma+\frac{1}{2}w_{1,1}\delta+w_{2,0}\beta+\frac{1}{2}w_{2,2}\beta-\frac{12C^2w_{1,1}w_{2,0}}{w_{0,0}^5}-\right.$$
$$\left.-\frac{6C^2w_{1,1}w_{2,2}}{w_{0,0}^5}+\frac{15C^2w_{1,1}^3}{2w_{0,0}^6}\right)\cos(v)-\left(w_{0,0}\eta+\frac{1}{2}w_{1,1}\delta+\frac{1}{2}w_{2,2}\beta-\frac{6C^2w_{1,1}w_{2,2}}{w_{0,0}^5}+\frac{5C^2w_{1,1}^3}{2w_{0,0}^6}\right)\cos(3v). \tag{56}$$

The homogeneous solution reads

$$w_h^{(3)}(v)=K_1\sin\left(\sqrt{\alpha+\frac{3C^2}{w_{0,0}^4}}v\right)+K_2\cos\left(\sqrt{\alpha+\frac{3C^2}{w_{0,0}^4}}v\right), \tag{57}$$

where again the constants are $K_1=K_2\equiv 0$ due to the periodicity of $w(v)$. The trial function for the particular solution of the inhomogeneous equation

$$w_{ih}^{(3)}(v)=w_{3,1}\cos(v)+w_{3,3}\cos(3v), \tag{58}$$

where the forms for $w_{3,1}$ and $w_{3,3}$ coefficients are

$$w_{3,1}=-\frac{w_{0,0}\varepsilon+w_{1,1}\gamma+\dfrac{1}{2}w_{1,1}\delta+w_{2,0}\beta+\dfrac{1}{2}w_{2,2}\beta-\dfrac{12C^2w_{1,1}w_{2,0}}{w_{0,0}^5}-\dfrac{6C^2w_{1,1}w_{2,2}}{w_{0,0}^5}+\dfrac{15C^2w_{1,1}^3}{2w_{0,0}^6}}{\alpha+\dfrac{3C^2}{w_{0,0}^4}-1},$$
$$w_{3,3}=-\frac{w_{0,0}\eta+\dfrac{1}{2}w_{1,1}\delta+\dfrac{1}{2}w_{2,2}\beta-\dfrac{6C^2w_{1,1}w_{2,2}}{w_{0,0}^5}+\dfrac{5C^2w_{1,1}^3}{2w_{0,0}^6}}{\alpha+\dfrac{3C^2}{w_{0,0}^4}-9}. \tag{59}$$

Then by using the fact that $\psi'(v) = Cw^{-2}(v)$, Eq. (33), $\psi(v)$ can be calculated if we again expand $\psi'(v)$ into Taylor series in e up to third order

$$\frac{1}{C}\psi(v) = \frac{v}{w_{0,0}^2} - 2\frac{w_{1,1}\sin(v)}{w_{0,0}^3}e + \frac{1}{w_{0,0}^4}\left\{3w_{1,1}^2\left(\frac{\sin(2v)}{4}+\frac{v}{2}\right) - w_{0,0}w_{2,0}v - \frac{\sin(2v)w_{0,0}w_{2,2}}{2}\right\}e^2 -$$
$$- \frac{1}{w_{0,0}^5}\left\{\frac{2}{3}w_{1,1}^3\left(\frac{9}{4}\sin(v)+\frac{\sin(3v)}{4}\right) + \sin(v)w_{0,0}^2w_{3,1} + \frac{\sin(3v)w_{0,0}^2w_{3,3}}{3} - 2\sin(v)w_{0,0}w_{1,1}w_{2,0} -\right.$$
$$- 2w_{0,0}w_{1,1}w_{2,2}\left(\frac{\sin(v)}{2}+\frac{\sin(3v)}{6}\right) + \frac{2}{3}w_{1,1}\left(\frac{w_{1,1}^2}{4} - \frac{w_{0,0}w_{2,2}}{2}\right)\sin(3v) +$$
$$\left. + 2w_{1,1}\left(\frac{3}{4}w_{1,1}^2 - \frac{1}{2}w_{0,0}w_{2,2} - w_{0,0}w_{2,0}\right)\sin(v)\right\}e^3 + \mathscr{O}(e^4). \tag{60}$$

Now we have expressions for $w(v)$ and $\psi(v)$, thus $\xi(v) = aw(v)\cos\big(\psi(v)+b\big)$ can be calculated. One can easily see, that all $w_{i,j}$ coefficients and the expression of $\psi(v)$ have a common multiplicative factor $\sqrt{C}$ and C respectively, therefore this factor can be chosen to be $C = 1$. The effects of C will be considered with the initial conditions. It is left to determine constants a and b, which are controlled by the initial conditions $\xi(0) \equiv \xi_0$ and $\xi'(0) \equiv \xi_0'$. As the differential equations are second order linear differential equations with periodic coefficients, the initial conditions can be arbitrary, therefore we use the simple conditions of $x_0 = 1$, $y_0 = 1$, $v_{x0} = 0$, $v_{y0} = 0$, from which $\xi_{1,0}$, $\xi_{1,0}'$, $\xi_{2,0}$ and $\xi_{2,0}'$ can be easily achieved. By using the values ξ_0 and ξ_0'

$$\xi_0 = aw(0)\cos\big(\psi(0)+b\big), \quad \xi_0' = aw'(0)\cos\big(\psi(0)+b\big) - \frac{1}{w(0)}\sin\big(\psi(0)+b\big), \quad \text{therefore}$$
$$a = \sqrt{\big(w'(0)\xi_0 - \xi_0'w(0)\big)^2 + \left(\frac{\xi_0}{w(0)}\right)^2}, \quad b = \arccos\left(\frac{\xi_0}{aw(0)}\right) - \psi(0). \tag{61}$$

At the end the only task is to use the transformations detailed in Eqs. (23)-(24), calculate y_2 and y_2^* with Eq. (29), then turn back to the x, y coordinates as $x = y_1 + y_1^*$ and $y = y_2 + y_2^*$.

4 Illustrations and discussion

The prominent example of co-orbital dynamics is the Sun-Jupiter-Trojan configuration in our own Solar System. We apply the perturbative solution described in Sec. 3.1 to this structure first. Fig. 3 depicts the trajectory around the Sun-Jupiter triangular Lagrangian point. The integration time is 20 periods of Jupiter (ca. 240 years). The analytic and numerical solutions match perfectly, although after some time ($\sim 38-40$ periods) they start to deviate.

Recently, [Lillo-Box et al(2018)Lillo-Box, Leleu, Parviainen, Figueira, Mallonn, Correia, Santos, Robutel, Lendl, Boffin, Faria, Bar studied the physical parameters and dynamical properties of possible exo-Trojans in systems with close-in (orbital period < 5 days) planets. We selected two of them, HAT-P-20b ($e = 0.015$, $\mu = 0.0091$) and WASP-36b ($e = 0.0$, $\mu = 0.0021$), to demonstrate analytic solution in these regimes[2]. The orbits are plotted in

[2] The orbital periods are HAT-P-20b : 2.87 days, WASP-36b : 1.53 days.

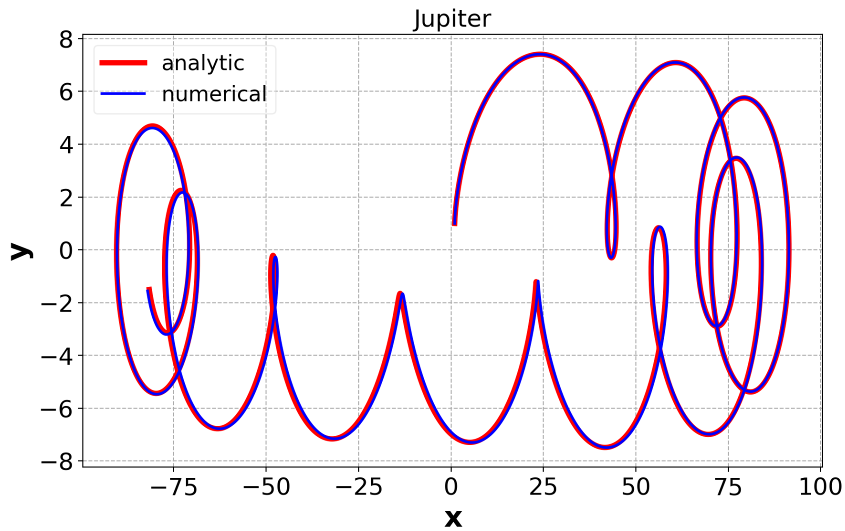

Fig. 3: Analytic and numerical solution in Sun-Jupiter system. Initial conditions are the same as in Fig 2.

Fig. 4a and b, respectively. The panels show the paths for T=20 periods again. Due to the zero eccentricity of the planet, the analytic solution for WASP-36b remains very close to the numerical outcome for much longer times (not shown here).

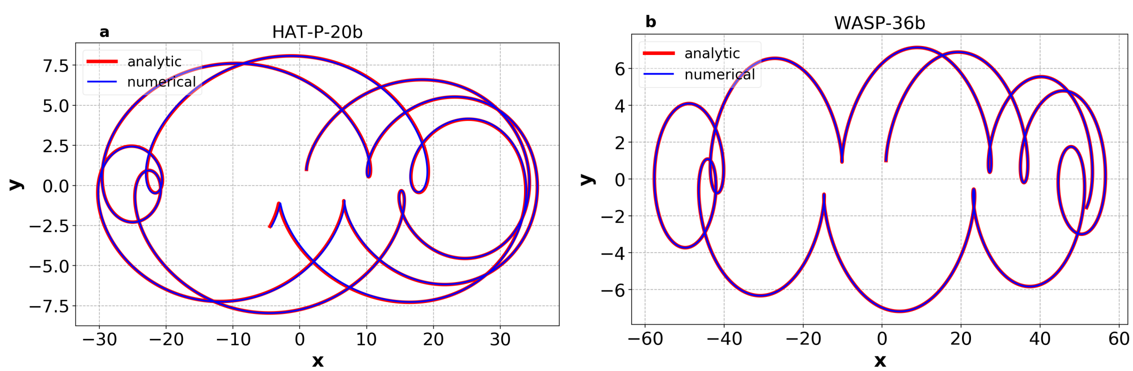

Fig. 4: Perturbative solution for particular exoplanetary systems. Initial conditions are the same as in Fig 2. For parameters see the main text.

Considering the Earth-Moon system with $e = 0.054$ and $\mu = 0.012$ it falls close to the limit of third order solution. The analytic solution diverges after 5-6 revolutions ($\sim 130-150$ days) of the Moon. We have seen that for Sun-Jupiter system the analytic curve traces the numerical method reasonable well while the eccentricity falls into the same range. In addition, we have found that the rather large mass parameter - compared to planetary systems - does not affect the precision of the analytic solution provided the eccentricity is small enough, practically zero. This is, however, not the case for Moon. Consequently, systems with moderate non-zero eccentricity and mass parameter in the same magnitude requires further improvement to the analytic solution, e.g. higher order expansion in mass.

In this work we fully describe the motion around triangular Lagrangian points with Hill's equations. As a perturbative solution, a third order expansion of Floquet function $w(v)$ in eccentricity is presented. This method is capable to follow analytically the orbit of a massless particle around the equilibrium points L_4 and L_5 in ERTBP. Precise trajectory forecast for moderate eccentricity ($e \leq 0.05$) and mass parameter ($\mu \leq 0.005$) is achievable for tens of secondary's orbital period. Furthermore, we note that Eq. 61 can be used to identify periodic orbits around L_4 and L_5 points. This calculation is postponed elsewhere.

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