

Edge-ordered Ramsey numbers*

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Abstract

We introduce and study a variant of Ramsey numbers for *edge-ordered graphs*, that is, graphs with linearly ordered sets of edges. The *edge-ordered Ramsey number* $\overline{R}_e(\mathfrak{G})$ of an edge-ordered graph \mathfrak{G} is the minimum positive integer N such that there exists an edge-ordered complete graph \mathfrak{K}_N on N vertices such that every 2-coloring of the edges of \mathfrak{K}_N contains a monochromatic copy of \mathfrak{G} as an edge-ordered subgraph of \mathfrak{K}_N .

We prove that the edge-ordered Ramsey number $\overline{R}_e(\mathfrak{G})$ is finite for every edge-ordered graph \mathfrak{G} and we obtain better estimates for special classes of edge-ordered graphs. In particular, we prove $\overline{R}_e(\mathfrak{G}) \leq 2^{O(n^3 \log n)}$ for every bipartite edge-ordered graph \mathfrak{G} on n vertices. We also introduce a natural class of edge-orderings, called *lexicographic edge-orderings*, for which we can prove much better upper bounds on the corresponding edge-ordered Ramsey numbers.

1 Introduction

An *edge-ordered graph* $\mathfrak{G} = (G, \prec)$ consists of a graph $G = (V, E)$ and a linear ordering \prec of the set of edges E . We sometimes use the term *edge-ordering of G* for the ordering \prec and

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also for \mathfrak{G} . An edge-ordered graph (G, \prec_1) is an *edge-ordered subgraph* of an edge-ordered graph (H, \prec_2) if G is a subgraph of H and \prec_1 is a suborder of \prec_2 . We say that (G, \prec_1) and (H, \prec_2) are *isomorphic* if there is a graph isomorphism between G and H that also preserves the edge-orderings \prec_1 and \prec_2 .

For a positive integer k , a k -*coloring* of the edges of a graph G is any function that assigns one of the k colors to each edge of G . The *edge-ordered Ramsey number* $\overline{R}_e(\mathfrak{G})$ of an edge-ordered graph \mathfrak{G} is the minimum positive integer N such that there exists an edge-ordering \mathfrak{K}_N of the complete graph K_N on N vertices such that every 2-coloring of the edges of \mathfrak{K}_N contains a monochromatic copy of \mathfrak{G} as an edge-ordered subgraph of \mathfrak{K}_N .

More generally, for two edge-ordered graphs \mathfrak{G} and \mathfrak{H} , we use $\overline{R}_e(\mathfrak{G}, \mathfrak{H})$ to denote the minimum positive integer N such that there exists an edge-ordering \mathfrak{K}_N of K_N such that every 2-coloring of the edges of \mathfrak{K}_N with colors red and blue contains a red copy of \mathfrak{G} or a blue copy of \mathfrak{H} as an edge-ordered subgraph of \mathfrak{K}_N . We call the number $\overline{R}_e(\mathfrak{G}, \mathfrak{H})$ the *non-diagonal edge-ordered Ramsey number*.

To our knowledge, Ramsey numbers of edge-ordered graphs were not considered in the literature. On the other hand, Ramsey numbers of graphs with ordered vertex sets have been quite extensively studied recently; for example, see [1, 3, 10]. For questions concerning extremal problems about vertex-ordered graphs consult the recent surveys [23, 24]. A *vertex-ordered graph* $\mathcal{G} = (G, \prec)$ (or simply an *ordered graph*) is a graph G with a fixed linear ordering \prec of its vertices. We use the term *vertex-ordering of G* to denote the ordering \prec as well as the ordered graph \mathcal{G} . An ordered graph (G, \prec_1) is a *vertex-ordered subgraph* of an ordered graph (H, \prec_2) if G is a subgraph of H and \prec_1 is a suborder of \prec_2 . We say that (G, \prec_1) and (H, \prec_2) are *isomorphic* if there is a graph isomorphism between G and H that also preserves the vertex-orderings \prec_1 and \prec_2 . Unlike in the case of edge-ordered graphs, there is a unique vertex-ordering \mathcal{K}_N of K_N up to isomorphism. The *ordered Ramsey number* $\overline{R}(\mathcal{G})$ of an ordered graph \mathcal{G} is the minimum positive integer N such that every 2-coloring of the edges of \mathcal{K}_N contains a monochromatic copy of \mathcal{G} as a vertex-ordered subgraph of \mathcal{K}_N .

For an n -vertex graph G , let $R(G)$ be the Ramsey number of G . It is easy to see that $R(G) \leq \overline{R}(\mathcal{G})$ and $R(G) \leq \overline{R}_e(\mathfrak{G})$ for each vertex-ordering \mathcal{G} of G and edge-ordering \mathfrak{G} of G . We also have $\overline{R}(G) \leq \overline{R}(\mathcal{K}_n) = R(K_n)$ and thus ordered Ramsey numbers are always finite. Proving that $\overline{R}_e(\mathfrak{G})$ is always finite seems to be more challenging; see Theorem 1.

The Turán numbers of edge-ordered graphs were recently introduced in [14], motivated by a lemma in [15, Lemma 23]. The authors of [14] proved, for example, a variant of the Erdős–Stone–Simonovits Theorem for edge-ordered graphs, and also investigated the Turán numbers of small edge-ordered paths, star forests, and 4-cycles; see also the last section of [24].

Another related problem is to determine the maximum length of a monotone increasing path that must appear in any edge-ordered complete graph on n vertices. The Chvátal–Komlós conjecture [9] says that this quantity is linear in n and the authors of [5] could prove an almost linear lower bound.

For $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1, \dots, n\}$. We omit floor and ceiling signs whenever they are not crucial. All logarithms in this paper are base 2.

2 Our results

We study the growth rate of edge-ordered Ramsey numbers with respect to the number of vertices for various classes of edge-ordered graphs. As our first result, we show that edge-ordered Ramsey numbers are always finite and thus well-defined.

Theorem 1. *For every edge-ordered graph \mathfrak{G} , the edge-ordered Ramsey number $\overline{R}_e(\mathfrak{G})$ is finite.*

Theorem 1 also follows from a recent deep result of Hubička and Nešetřil [17, Theorem 4.33] about Ramsey numbers of general relational structures. In comparison, our proof of Theorem 1 is less general, but it is much simpler and produces better and more explicit bound on $\overline{R}_e(\mathfrak{G})$. It is a modification of the proof of Theorem 12.13 [21, Page 138], which is based on the Graham–Rothschild Theorem [16]. In fact, the proof of Theorem 1 yields a stronger induced-type statement where additionally the ordering of the vertex set is fixed; see Theorem 8. Theorem 1 can also be extended to k -colorings with $k > 2$.

Due to the use of the Graham–Rothschild Theorem, the bound on the edge-ordered Ramsey numbers obtained in the proof of Theorem 1 is still enormous. It follows from a result of Shelah [22, Theorem 2.2] that this bound on $\overline{R}_e(\mathfrak{G})$ is primitive recursive, but it grows faster than, for example, a tower function of any fixed height. Thus we aim to prove more reasonable estimates on edge-ordered Ramsey numbers, at least for some classes of edge-ordered graphs.

As our second main result, we show that one can obtain a much better upper bound on non-diagonal edge-ordered Ramsey numbers of two edge-ordered graphs, provided that one of them is bipartite. For $d \in \mathbb{N}$, we say that a graph G is d -degenerate if every subgraph of G has a vertex of degree at most d .

Theorem 2. *Let \mathfrak{H} be a d -degenerate edge-ordered graph on n' vertices and let \mathfrak{G} be a bipartite edge-ordered graph with m edges and with both parts containing n vertices. If $d \leq n$ and $n' \leq t^{d+1}$ for $t = 3n^{10}m!$, then*

$$\overline{R}_e(\mathfrak{H}, \mathfrak{G}) \leq (n')^{2t^{d+1}}.$$

In particular, if \mathfrak{G} is a bipartite edge-ordered graph on n vertices, then $\overline{R}_e(\mathfrak{G}) \leq 2^{O(n^3 \log n)}$. We believe that the bound can be improved. In fact, it is possible that $\overline{R}_e(\mathfrak{G})$ is at most exponential in the number of vertices of \mathfrak{G} for every edge-ordered graph \mathfrak{G} ; see Section 6 for more open problems. We note that, for every graph G and its vertex-ordering \mathcal{G} , both the standard Ramsey number $R(G)$ and the ordered Ramsey number $\overline{R}(\mathcal{G})$ grow at most exponentially in the number of vertices of G .

In general, the difference between edge-ordered Ramsey numbers and ordered Ramsey numbers with the same underlying graph can be very large. Let M_n be a *matching* on n vertices, that is, a graph formed by a collection of $n/2$ disjoint edges. There are ordered matchings $\mathcal{M}_n = (M_n, <)$ with super-polynomial ordered Ramsey numbers $\overline{R}(\mathcal{M}_n)$ in n [1, 10]. In fact this is true for almost all ordered matchings on n vertices [10]. On

the other hand, all edge-orderings of M_n are isomorphic as edge-ordered graphs and thus $\overline{R}_e(\mathfrak{M}_n) = R(M_n) \leq O(n)$ for every edge-ordering \mathfrak{M}_n of M_n .

In Section 5, we consider a special class of edge-orderings, which we call *lexicographic edge-orderings*, for which we can prove much better upper bounds on their edge-ordered Ramsey numbers and which seem to be quite natural.

An ordering \prec of edges of a graph $G = (V, E)$ is *lexicographic* if there is a one-to-one correspondence $f: V \rightarrow \{1, \dots, |V|\}$ such that any two edges $\{u, v\}$ and $\{w, t\}$ of G with $f(u) < f(v)$ and $f(w) < f(t)$ satisfy $\{u, v\} \prec \{w, t\}$ if either $f(u) < f(w)$ or if $(f(u) = f(w) \ \& \ f(v) < f(t))$. We say that such mapping f is *consistent* with \prec . Note that, for every vertex u , the edges $\{u, v\}$ with $f(u) < f(v)$ form an interval in \prec . Also observe that there is a unique (up to isomorphism) lexicographic edge-ordering $\mathfrak{K}_n^{\text{lex}}$ of K_n . Setting $\{u, v\} \prec' \{w, t\}$ if either $f(u) < f(w)$ or if $(f(u) = f(w) \ \& \ f(v) > f(t))$ we obtain the *max-lexicographic* edge-ordering \prec' of G . Observe that in the max-lexicographic ordering, for every vertex u , the edges $\{u, v\}$ with $f(u) < f(v)$ again form an interval in \prec' . When compared to the lexicographic edge-ordering, each of these intervals is reversed, but the ordering of the intervals is kept the same.

For a linear ordering $<$ on some set X , we use $<^{-1}$ to denote the *inverse ordering* of $<$, that is, for all $x, y \in X$, we have $x <^{-1} y$ if and only if $y < x$.

The lexicographic and max-lexicographic edge-orderings are natural, as Nešetřil and Rödl [20] showed that these orderings are canonical in the following sense.

Theorem 3 ([20]). *For every $n \in \mathbb{N}$, there is a positive integer $T(n)$ such that every edge-ordered complete graph on $T(n)$ vertices contains a copy of K_n such that the edges of this copy induce one of the following four edge-orderings: lexicographic edge-ordering \prec , max-lexicographic edge-ordering \prec' , \prec^{-1} , or $(\prec')^{-1}$.*

Theorem 3 is also an unpublished result of Leeb; see [19]. It is thus natural to consider the following variant of edge-ordered Ramsey numbers, which turns out to be more tractable than general edge-ordered Ramsey numbers. The *lexicographic edge-ordered Ramsey number* $\overline{R}_{\text{lex}}(\mathfrak{G})$ of a lexicographically edge-ordered graph \mathfrak{G} is the minimum N such that every 2-coloring of the edges of the lexicographically edge-ordered complete graph $\mathfrak{K}_N^{\text{lex}}$ on N vertices contains a monochromatic copy of \mathfrak{G} as an edge-ordered subgraph of $\mathfrak{K}_N^{\text{lex}}$. Observe that $\overline{R}_e(\mathfrak{G}) \leq \overline{R}_{\text{lex}}(\mathfrak{G})$ for every lexicographically edge-ordered graph \mathfrak{G} .

For every lexicographically edge-ordered graph $\mathfrak{G} = (G, \prec)$, the lexicographic edge-ordered Ramsey number $\overline{R}_{\text{lex}}(\mathfrak{G})$ can be estimated from above with the ordered Ramsey number of some vertex-ordering of G . More specifically, we have the following result.

Lemma 4. *Every lexicographically edge-ordered graph $\mathfrak{G} = (G, \prec)$ satisfies*

$$\overline{R}_{\text{lex}}(\mathfrak{G}) \leq \min_f \overline{R}(\mathcal{G}_f),$$

where the minimum is taken over all one-to-one correspondences $f: V \rightarrow \{1, \dots, |V|\}$ that are consistent with the lexicographic edge-ordering \mathfrak{G} and \mathcal{G}_f is the vertex-ordering of G determined by f .

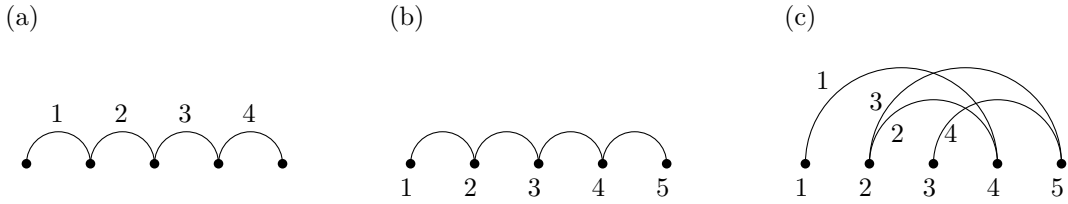


Figure 1: (a) The edge-monotone path on 5 vertices. (b) The monotone path on 5 vertices. (c) A different ordered path on 5 vertices and the corresponding lexicographic edge-ordering. The label of each edge and vertex denotes the position in the edge- and vertex-ordering, respectively.

We prove Lemma 4 in Section 5. Since $\overline{R}(\mathcal{K}_n) = R(K_n)$, it follows from Lemma 4 and from the well-known bound $R(K_n) \leq 2^{2n}$ by Erdős and Szekeres [11] that the numbers $\overline{R}_{lex}(\mathfrak{G})$ are always at most exponential in the number of vertices of G . In fact, we have $\overline{R}_{lex}(\mathfrak{K}_n^{lex}) = \overline{R}(\mathcal{K}_n) = R(K_n)$ for every n . The equality is achieved in the statement of Lemma 4, for example, for graphs with a unique vertex-ordering determined by the lexicographic edge-ordering. Such graphs include graphs where each edge is contained in a triangle. Additionally, combining Lemma 4 with a result of Conlon et al. [10, Theorem 3.6] gives the estimate

$$\overline{R}_{lex}(\mathfrak{G}) \leq 2^{O(d \log^2(2n/d))}$$

for every d -degenerate lexicographically edge-ordered graph \mathfrak{G} on n vertices. In particular, $\overline{R}_{lex}(\mathfrak{G})$ is at most quasi-polynomial in n if d is fixed.

We note that the bound in Lemma 4 is not always tight. For example, $R(K_{1,n}) = \overline{R}_{lex}(\mathfrak{K}_{1,n})$ for every edge-ordering $\mathfrak{K}_{1,n}$ of $K_{1,n}$, as any two edge-ordered stars $K_{1,n}$ are isomorphic as edge-ordered graphs. However, the Ramsey number $R(K_{1,n})$ is known to be strictly smaller than $\overline{R}(\mathcal{K}_{1,n})$ for n even and for any vertex-ordering $\mathcal{K}_{1,n}$ of $K_{1,n}$; see [6] and [2, Observation 11 and Theorem 12].

As an application of Lemma 4 we obtain asymptotically tight estimate on the following lexicographic edge-ordered Ramsey numbers of paths. The *edge-monotone path* $\mathfrak{P}_n = (P_n, \prec)$ is the edge-ordered path on n vertices v_1, \dots, v_n , where $\{v_1, v_2\} \prec \dots \prec \{v_{n-1}, v_n\}$; see part (a) of Figure 1.

Proposition 5. *For every integer $n > 2$, we have*

$$\overline{R}_{lex}(\mathfrak{P}_n) \leq 2n - 3 + \sqrt{2n^2 - 8n + 11}.$$

The proof of Proposition 5 uses the fact that the one-to-one correspondence f consistent with the lexicographic edge-ordering of P_n is not determined uniquely. Indeed, we can choose the mapping f so that it determines the vertex-ordering \mathcal{P}_n of P_n where edges are between consecutive pairs of vertices. Such vertex-ordering \mathcal{P}_n is called *monotone path*; see part (b) of Figure 1. However, it is known that $\overline{R}(\mathcal{P}_n) = (n - 1)^2 + 1$ [7] and thus we cannot apply Lemma 4 to this ordering to obtain a linear bound on $\overline{R}_{lex}(\mathfrak{P}_n)$. Instead we

choose a different mapping f that determines a vertex-ordering of P_n with linear ordered Ramsey number; see part (c) of Figure 1.

As our last result, we show an upper bound on edge-ordered Ramsey numbers of two graphs, where one of them is bipartite and suitably lexicographically edge-ordered. This result uses a stronger assumption about \mathfrak{G} than Theorem 2, but gives much better estimate. For $m, n \in \mathbb{N}$, let $\mathfrak{K}_{m,n}^{lex}$ be the lexicographic edge-ordering of $K_{m,n}$ that induces a vertex-ordering, in which both parts of $K_{m,n}$ form an interval.

Theorem 6. *Let \mathfrak{H} be a d -degenerate edge-ordered graph on n' vertices and let \mathfrak{G} be an edge-ordered subgraph of $\mathfrak{K}_{n,n}^{lex}$. Then*

$$\overline{R}_e(\mathfrak{H}, \mathfrak{G}) \leq (n')^2 n^{d+1}.$$

The proof of Theorem 1 is presented in Section 3. Both Theorem 2 and Theorem 6 are proved in Section 4. Section 5 contains the proofs of Lemma 4 and Proposition 5. Finally, we mention some open problems in Section 6.

3 Proof of Theorem 1

In this section, we prove Theorem 1 by showing that edge-ordered Ramsey numbers are always finite. The proof is carried out using the Graham–Rothschild Theorem [16]. To state this result, we need to introduce some definitions first. We follow the notation from [21].

Let N and t be nonnegative integers with $t \leq N$ and let A be a finite set of symbols not containing the symbols $\lambda_1, \dots, \lambda_t$. Then the set $[A] \binom{N}{t}$ of t -parameter words of length N over A is the set of mappings $f: [N] \rightarrow A \cup \{\lambda_1, \dots, \lambda_t\}$ such that for every j with $1 \leq j \leq t$ there exists $i \in [N]$ such that $f(i) = \lambda_j$ and $\min(f^{-1}(\lambda_i)) < \min(f^{-1}(\lambda_j))$ for all i and j with $1 \leq i < j \leq t$. The composition $f \cdot g \in [A] \binom{N}{r}$ of $f \in [A] \binom{N}{t}$ and $g \in [A] \binom{t}{r}$ is defined by

$$(f \cdot g)(i) = \begin{cases} f(i), & \text{if } f(i) \in A \text{ and} \\ g(j), & \text{if } f(i) = \lambda_j. \end{cases}$$

The following result, called the Graham–Rothschild Theorem, is a strengthening of the famous Hales–Jewett Theorem.

Theorem 7 (The Graham–Rothschild Theorem [16, 21]). *Let A be a finite alphabet and let $r, t \in \mathbb{N}_0$ and $k \in \mathbb{N}$ be integers such that $r \leq t$. Then there is a positive integer $N = GR(|A|, k, r, t)$ such that for every k -coloring χ' of $[A] \binom{N}{r}$ there exists a monochromatic $f \in [A] \binom{N}{t}$, that is, f satisfies*

$$\chi'(f \cdot g) = \chi'(f \cdot h)$$

for all $g, h \in [A] \binom{t}{r}$.

Applying the Graham–Rothschild Theorem similarly as in [21, Page 138], we can derive the following result, which is actually a stronger statement than Theorem 1.

Theorem 8. *Let (F, \prec_v, \prec_e) be a graph with linear orderings \prec_v and \prec_e on its vertices and edges, respectively. Then, for every $k \in \mathbb{N}$, there exists a graph G with orderings \leq_v and \leq_e of its vertices and edges, respectively, such that for every k -coloring of the edges of G there is a monochromatic induced copy of (F, \prec_v, \prec_e) in (G, \leq_v, \leq_e) .*

Note that we can fix the vertex-ordering of the monochromatic copy as well as the edge-ordering. Moreover, the obtained monochromatic copy of F is contained in the large graph G as an induced subgraph. Unfortunately, as we discussed in Section 2, the obtained bound on the number of vertices of G is enormous.

Proof of Theorem 8. We use n and m to denote the number of vertices and edges of F , respectively. Let G be defined as follows. Choose $N \in \mathbb{N}$ such that $N \geq GR(1, k, 3, n + m)$. Let $2^{[N]}$ be the vertex set of G and let $\{X, Y\}$ with $X, Y \subseteq [N]$ be an edge of G if $X \cap Y \neq \emptyset$. For any two sets $X, Y \subseteq [N]$ with $\min(X) < \min(Y)$, we set $X \prec'_v Y$. Note that \prec'_v is a partial ordering on the vertices of G . We let \leq_v be an arbitrary linear extension of \prec'_v . For two edges $\{X, Y\}$ and $\{U, V\}$ of G with $\min(X \cap Y) < \min(U \cap V)$, we set $\{X, Y\} \prec'_e \{U, V\}$. Observe that \prec'_e defines a partial ordering of the edges of G . We let \leq_e be an arbitrary linear extension of \prec'_e .

We show that (G, \leq_v, \leq_e) satisfies the statement of the theorem. We use $v_1 \prec_v \dots \prec_v v_n$ and $f_1 \prec_e \dots \prec_e f_m$ to denote the vertices and edges of F , respectively. Let χ be a k -coloring of the edges of G . We use χ to define a k -coloring χ' of 3-parameter words of length N over the single-letter alphabet $\{0\}$. Given such a word $w \in \{0\}^{\binom{N}{3}}$, we set S_i , $i \in \{1, 2, 3\}$, to be the set of positions from $[N]$ on which w contains the i th variable symbol λ_i . Note that the sets S_1, S_2, S_3 are pairwise disjoint, non-empty and, since the first occurrence of λ_i precedes the first occurrence of λ_j in w for all $i < j$, we also have $\min(S_1) < \min(S_2) < \min(S_3)$. Setting $X = S_1 \cup S_3$ and $Y = S_2 \cup S_3$, we have two vertices of G that form an edge of G and that satisfy $X \prec_v Y$. We let $\chi'(w) = \chi(\{X, Y\})$.

By the Graham–Rothschild Theorem (Theorem 7) applied with $t = n + m$ and $r = 3$, there is an $(n + m)$ -parameter word $w \in \{0\}^{\binom{N}{n+m}}$ such that $\chi'(w \cdot v) = \chi'(w \cdot v')$ for all $v, v' \in \{0\}^{\binom{n+m}{3}}$. Let b be the common color of the words $w \cdot v$ in χ' . Similarly as before, for every $i \in [n + m]$, we let $S_i \subseteq [N]$ be the set of positions on which w contains the i th variable symbol λ_i . Again, observe that the sets S_i are pairwise disjoint and satisfy $\min(S_1) < \dots < \min(S_{n+m})$. For every $i \in [n]$, we let

$$F_i = S_i \cup \bigcup_{\substack{j: j \in [m], \\ v_i \in f_j}} S_{n+j}.$$

The sets F_1, \dots, F_n then induce a vertex-ordered and edge-ordered graph (F^*, \prec_v, \prec_e) in G . We show that F^* is a copy of (F, \prec_v, \prec_e) in (G, \leq_v, \leq_e) . Indeed, since $\min(S_1) < \dots < \min(S_{n+m})$, we have $\min(F_i) = \min(S_i) < \min(S_j) = \min(F_j)$ if $i < j$ and thus $F_i \prec_v F_j$ for all $v_i \prec_v v_j$. Moreover, since the sets S_1, \dots, S_{n+m} are pairwise disjoint, $\{F_i, F_j\}$ is an edge of F^* if and only if there is an edge f_l of F with $f_l = \{v_i, v_j\}$, which gives F as an induced subgraph of G . This is because we have $F_i \cap F_j = S_{n+l}$. Let $\{F_i, F_j\}$ and $\{F_{i'}, F_{j'}\}$

be two edges of F^* . Since $\{F_i, F_j\}$ and $\{F_{i'}, F_{j'}\}$ are edges of G , the sets $S_{n+l} = F_i \cap F_j$ and $S_{n+l'} = F_{i'} \cap F_{j'}$ correspond to the edges $f_l = \{v_i, v_j\}$ and $f_{l'} = \{v_{i'}, v_{j'}\}$ of F , respectively, by the definition of F_1, \dots, F_n . Assume $f_l \prec_e f_{l'}$. Then λ_{n+l} precedes $\lambda_{n+l'}$, as $l < l'$, and thus $\min(S_{n+l}) < \min(S_{n+l'})$. It follows from the fact $S_{n+l} = F_i \cap F_j$ and $S_{n+l'} = F_{i'} \cap F_{j'}$ and from the definition of \prec_e that $\{F_i, F_j\} \prec_e \{F_{i'}, F_{j'}\}$ if and only if $f_l \prec_e f_{l'}$.

It remains to show that all edges of F^* are monochromatic in χ . Let $\{F_i, F_j\}$ be an edge of F^* with $i < j$ and let $S_{n+l} = F_i \cap F_j$. Note that the sets $F_i \setminus S_{n+l}$, $F_j \setminus S_{n+l}$, and S_{n+l} are nonempty, pairwise disjoint, and satisfy $\min(F_i \setminus S_{n+l}) < \min(F_j \setminus S_{n+l}) < \min(S_{n+l})$. We let $v \in \{0, \lambda_1, \lambda_2, \lambda_3\}^{n+m}$ be the word with symbols $\lambda_1, \lambda_2, \lambda_3$ on positions from sets $\{i\} \cup \{n+s : v_i \in f_s \neq f_l\}$, $\{j\} \cup \{n+s : v_j \in f_s \neq f_l\}$, and $\{n+l\}$ in w , respectively. Then $v \in [\{0\}]_3^{(n+m)}$ and $w \cdot v \in [\{0\}]_3^{(N)}$ is the 3-parameter word with variable symbols $\lambda_1, \lambda_2, \lambda_3$ on positions from $F_i \setminus S_{n+l}$, $F_j \setminus S_{n+l}$, and S_{n+l} , respectively. By the choice of w , we have $b = \chi'(w \cdot v) = \chi(\{F_i, F_j\})$. Thus all edges of F^* have the color b in χ . \square

Now, we obtain Theorem 1 as a corollary of Theorem 8.

Proof of Theorem 1. For a given edge-ordered graph $\mathfrak{G} = (G, \prec)$, let $<$ be an arbitrary ordering of the vertices of G . By Theorem 8, there is a graph H and orderings $<'$ and \prec' of its vertices and edges, respectively, such that every 2-coloring of the edges of H contains a monochromatic induced copy of $(G, <, \prec)$. It thus suffices to consider edge-ordered \mathfrak{K}_N that contains (H, \prec') as an edge-ordered subgraph. Then every 2-coloring of the edges of \mathfrak{K}_N contains a monochromatic copy of \mathfrak{G} as an edge-ordered subgraph (not necessarily induced). \square

4 Proofs of Theorems 2 and 6

In this section, we prove both Theorems 2 and 6. That is, we derive a super-exponential upper bound on the edge-ordered Ramsey numbers $\overline{R}_e(\mathfrak{H}, \mathfrak{G})$ of two edge-ordered graphs \mathfrak{H} and \mathfrak{G} , where \mathfrak{G} is bipartite. We then improve this bound under the additional assumption that $\mathfrak{G} \subseteq \mathfrak{K}_{n,n}^{lex}$.

As a first step, we prove the following lemma, which is used in proofs of both Theorem 2 and 6. The proof of this lemma is inspired by a similar “greedy-embedding” approach used, for example, in [2].

Lemma 9. *Let H be a d -degenerate graph on n' vertices and let $v_1 \prec \dots \prec v_{n'}$ be a vertex-ordering of H such that each v_j has at most d neighbors v_i with $i < j$. Then, for every $t \in \mathbb{N}$, there is K_N with $N = (n')^{2t^{d+1}}$ and with the vertex set partitioned into n' sets $I_1, \dots, I_{n'}$ of the same size such that the following statement holds. In every red-blue coloring of the edges of K_N , there is a blue copy of H in K_N with a copy of each v_i in I_i or a red copy of $K_{t,t}$ in K_N with each part contained in a different set I_i .*

Proof. Let $v_1 \prec \dots \prec v_{n'}$ be a vertex-ordering of H such that each v_j has at most d neighbors v_i with $i < j$. Such an ordering exists, as H is d -degenerate. For $N = (n')^{2t^{d+1}}$ and for

every vertex v_i of H , let I_i be a set of vertices such that $|I_i| = M = n't^{d+1}$ and let the disjoint union $I_1 \cup \dots \cup I_{n'}$ be the vertex set of K_N .

Let χ be a red-blue coloring of the edges of K_N . We now try to greedily embed a blue copy of H on vertices $h(v_1), \dots, h(v_{n'})$ in χ such that $h(v_i) \in I_i$ for every $i \in [n']$. We proceed so that if the embedding fails at some step, we obtain a red copy of $K_{t,t}$ in χ with each part contained in a different set I_i .

For each $i \in [n']$, let C_i be a set of candidates for the vertex $h(v_i)$. Initially, we set $C_i = I_i$. We then proceed in steps $i = 1, \dots, n'$, assuming that we have already determined the vertices $h(v_1), \dots, h(v_{i-1})$ in steps $1, \dots, i-1$, respectively.

In step i , assume that, for every neighbor v_j of v_i in H with $i < j$, all but at most $t-1$ vertices from C_i have at least $|C_j|/t$ blue neighbors in C_j . In such a case, if $|C_i| \geq n't$, then there is a vertex of C_i that has at least $|C_j|/t$ blue neighbors in each C_j such that v_j is a neighbor of v_i in H with $i < j$. This is because v_i has at most $n' - 1$ neighbors v_j in H with $i < j$ and

$$|C_i| - (n' - 1)(t - 1) \geq n't - (n' - 1)(t - 1) > 0.$$

We let $h(v_i)$ be an arbitrary such vertex from C_i and we update C_j to be the blue neighborhood of $h(v_i)$ in C_j . Thus the size of C_j decreases at most by a multiplicative factor of t during each update. We update the set C_j so that $|C_j|$ is a multiple of t . Note that each set C_j is updated at most d times, as we update each C_j for every neighbor v_i of v_j in H with $i < j$ and there are at most d such neighbors of v_j , as H is d -degenerate. Since $|I_i| = n't^{d+1}$, we indeed get $|C_i| \geq n't$ after all updates.

If we manage to find $h(v_{n'})$, then the vertices $h(v_1), \dots, h(v_{n'})$ induce a graph that contains a blue copy of H in χ , as $h(v_i)$ is connected to every $h(v_j)$ with a blue edge for every $\{v_i, v_j\} \in E(H)$. Note that $h(v_i) \in C_i \subseteq I_i$ for every $i \in [n']$.

Thus it suffices to consider the case when we cannot find the vertex $h(v_i)$ in some step i . That is, there is a neighbor v_j of v_i in H with $i < j$ such that C_i contains a set W of t vertices, each having at most $|C_j|/t - 1$ blue neighbors in C_j . Then, for each $w \in W$, we remove all blue neighbors of w from C_j . The total number of vertices that stay in C_j after the removal is at least $|C_j| - t \cdot (|C_j|/t - 1) = t$. Together with W , this t -tuple of vertices induces a red copy of $K_{t,t}$ in χ between $C_i \subseteq I_i$ and $C_j \subseteq I_j$. \square

We now proceed with the proof of Theorem 2. The proof is based on a probabilistic argument, which uses the following Chernoff-type inequality.

Theorem 10 (Chernoff bound [18]). *Let $X = \sum_{i=1}^k X_i$ be a random variable, where $\Pr[X_i = 1] = p_i$ and $\Pr[X_i = 0] = 1 - p_i$ for every $i \in [k]$ and all X_i are independent. Let $\mu = \mathbb{E}(X) = \sum_{i=1}^k p_i$. Then, for every $\delta \in (0, 1)$,*

$$\Pr[X \leq (1 - \delta)\mu] \leq e^{-\mu\delta^2/2}.$$

We now state and prove the last auxiliary result needed in the proof of Theorem 2.

Lemma 11. *Let \mathfrak{G} be a bipartite edge-ordered graph with m edges and with both parts having n vertices. For positive integers t and M that satisfy*

$$\binom{M}{t}^2 \cdot e^{-t^2/(3n^2m!)} < 1,$$

there is an edge-ordering $<$ of $K_{M,M}$ such that every copy of $(K_{t,t}, <)$ in $(K_{M,M}, <)$ contains a copy of \mathfrak{G} as an edge-ordered subgraph.

Proof. Let $<$ be the ordering of the edges of $K_{M,M}$ chosen independently and uniformly from the set of all edge-orderings of $K_{M,M}$. The probability that a copy of $(K_{n,n}, <)$ in $(K_{M,M}, <)$ contains a copy of \mathfrak{G} is at least $1/m!$. For a copy of $K_{t,t}$ in $K_{M,M}$, fix a decomposition of $K_{t,t}$ into copies B_1, \dots, B_k of $K_{n,n}$ where any two of them share at most a single edge. Note that $k \geq (t/n)^2$, as we can partition each part of $K_{t,t}$ into t/n sets of size n and then consider copies of $K_{n,n}$ induced by these parts. For $i \in [k]$, let X_i be the random variable such that $X_i = 1$ if $(B_i, <)$ contains a copy of \mathfrak{G} and let $X_i = 0$ otherwise. Then $\Pr[X_i = 1] \geq 1/m!$. Since any two copies B_i and B_j share at most a single edge, all the variables X_i are independent. Let $X = \sum_{i=1}^k X_i$ and note that

$$\mathbb{E}(X) = \sum_{i=1}^k \Pr[X_i = 1] \geq k/m! \geq \frac{t^2}{n^2m!}.$$

Clearly, the probability that a copy of $(K_{t,t}, <)$ in $(K_{M,M}, <)$ does not contain a copy of \mathfrak{G} is at most $\Pr[X = 0]$. By Theorem 10,

$$\Pr[X = 0] \leq e^{-\mathbb{E}(X)/3} \leq e^{-t^2/(3n^2m!)}.$$

The number of copies of $K_{t,t}$ in $K_{M,M}$ is $\binom{M}{t}^2$. The expected number of copies of $(K_{t,t}, <)$ in $(K_{M,M}, <)$ that do not contain a copy of \mathfrak{G} is thus at most

$$\binom{M}{t}^2 \cdot e^{-t^2/(3n^2m!)},$$

which is less than 1 according to our assumptions. Thus there is an edge-ordering $<$ of $K_{M,M}$ such that every copy of $(K_{t,t}, <)$ in $(K_{M,M}, <)$ contains a copy of \mathfrak{G} . \square

We now combine Lemmas 9 and 11 to prove Theorem 2.

Proof of Theorem 2. Let $\mathfrak{H} = (H, \prec_1)$ be a d -degenerate edge-ordered graph on n' vertices and let $\mathfrak{G} = (G, \prec_2)$ be a bipartite edge-ordered graph with m edges and with both parts having n vertices. We set $t = 3n^{10}m!$, $N = (n')^2t^{d+1}$, and $M = N/n'$. Assuming $d \leq n$ and $n' \leq t^{d+1}$, we show

$$\overline{R}_e(\mathfrak{H}, \mathfrak{G}) \leq N.$$

We construct an edge-ordered complete graph $\mathfrak{K}_N = (K_N, <)$ such that every red-blue coloring of the edges of \mathfrak{K}_N contains either a blue copy of \mathfrak{H} or a red copy of \mathfrak{G} . Let

$v_1, \dots, v_{n'}$ be a vertex-ordering of H such that each v_j has at most d neighbors v_i with $i < j$. Such an ordering exists, as H is d -degenerate. For every vertex v_i of H , let I_i be a set of vertices such that $|I_i| = M = n't^{d+1}$ and let the disjoint union $I_1 \cup \dots \cup I_{n'}$ be the vertex set of \mathfrak{K}_N .

By Lemma 11, if $\binom{M}{t}^2 \cdot e^{-t^2/(3n^2m!)} < 1$, then there is an edge-ordering $<'$ of $K_{M,M}$ such that every copy of $(K_{t,t}, <')$ in $(K_{M,M}, <')$ contains a copy of \mathfrak{G} . We have

$$\binom{M}{t}^2 \leq M^{2t} = (n')^{2t} t^{2t(d+1)},$$

thus it suffices to show that the expression

$$(n')^{2t} t^{2t(d+1)} \cdot e^{-t^2/(3n^2m!)} \leq e^{2t \log n' + 2t(d+1) \log t - t^2/(3n^2m!)}$$

is less than 1. From the choice of t and the fact $m \leq n^2$, we have $\log t \leq 5n^2 \log n$ for $n \geq 2$. Also, since $\log n' \leq (d+1) \log t$ and $d \leq n$, we can bound the exponent in the above expression from above by

$$4t(d+1) \log t - t^2/(3n^2m!) \leq t(20(d+1)n^2 \log n - n^8) < 0$$

for $n \geq 2$. Thus there is an edge-ordering $<'$ of $K_{M,M}$ such that every copy of $(K_{t,t}, <')$ in $(K_{M,M}, <')$ contains a copy of \mathfrak{G} .

We now define the edge-ordering $<$ of K_N . Let \prec'_1 be an arbitrary linear ordering of the edges of $K_{n'}$ with the same vertex set as H such that \prec'_1 contains \prec_1 . For two edges $e = \{u, v\}$ and $f = \{x, y\}$ with $u \in I_i, v \in I_j$ and $x \in I_k, y \in I_l$, where $i \neq j, k \neq l$ and $|\{i, j\} \cap \{k, l\}| \leq 1$, we set $e < f$ if and only if $\{v_i, v_j\} \prec'_1 \{v_k, v_l\}$. That is, $<$ is a *blow-up* of \prec'_1 on $I_1, \dots, I_{n'}$. For all i and j with $1 \leq i < j \leq n'$, we let $<$ be the ordering $<'$ on the complete bipartite graph induced by $I_i \cup I_j$. Finally, we order the rest of the edges of K_N arbitrarily, obtaining the edge-ordered graph $\mathfrak{K}_N = (K_N, <)$.

Let χ be a red-blue coloring of the edges of \mathfrak{K}_N . By Lemma 9, there is a blue copy of H in χ with an image of each v_i in I_i or a red copy of $K_{t,t}$ in χ with each part contained in a different set I_i . In the first case, since copy of each v_i lies in I_i and $<$ is a blow-up of \prec'_1 on $I_1, \dots, I_{n'}$, this copy of H has edge-ordering isomorphic to \prec_1 and we obtain a blue copy of \mathfrak{H} in χ .

In the second case, we have a red copy of $K_{t,t}$ between I_i and I_j . Since $<$ corresponds to the ordering $<'$ on the edges between I_i and I_j and, by the choice of $<'$, all copies of $(K_{t,t}, <')$ between I_i and I_j contain a copy of \mathfrak{G} , we obtain a red copy of \mathfrak{G} in χ . \square

The proof of Theorem 6 is also carried out using Lemma 9. However, since we are working with lexicographically edge-ordered graph, we can order edges between two sets I_i and I_j lexicographically and use the fact that \mathfrak{G} is an edge-ordered subgraph of $\mathfrak{K}_{m,n}^{lex}$. The rest of the proof of Theorem 6 is then analogous to the proof of Theorem 2.

Proof of Theorem 6. Let \mathfrak{H} be a d -degenerate edge-ordered graph on n' vertices and let \mathfrak{G} be an edge-ordered subgraph of $\mathfrak{K}_{n,n}^{lex}$. We set $N = (n')^2 n^{d+1}$ and show that $\bar{R}_e(\mathfrak{G}, \mathfrak{H}) \leq N$

by constructing an edge-ordered complete graph $\mathfrak{K}_N = (K_N, <)$ such that every red-blue coloring of the edges of \mathfrak{K}_N contains either a blue copy of \mathfrak{H} or a red copy of \mathfrak{G} .

Letting $v_1, \dots, v_{n'}$ be a vertex-ordering of H such that each v_j has at most d neighbors v_i with $i < j$, for every vertex v_i of H , we let I_i be a set of vertices such that $|I_i| = N/n'$. We again let the disjoint union $I_1 \cup \dots \cup I_{n'}$ be the vertex set of \mathfrak{K}_N . Let $<'_1$ be an arbitrary linear ordering of the edges of $K_{n'}$ with the same vertex set such that $<'_1$ contains $<_1$. We let $<$ be the edge-ordering of K_N that is a blow-up of $<'_1$ on $I_1, \dots, I_{n'}$ and we order edges between two sets I_i and I_j so that they determine a copy of $\mathfrak{K}_{|I_i|, |I_j|}^{lex}$.

Again, for every red-blue coloring χ of the edges of \mathfrak{K}_N , Lemma 9 gives a blue copy of H in χ with an image of each v_i in I_i or a red copy of $K_{n,n}$ in χ with each part contained in a different set I_i . In the first case, we obtain a blue copy of \mathfrak{H} as before. In the second case, since the edges between any two sets $J_i \subseteq I_i$ and $J_j \subseteq I_j$ determine a copy of $\mathfrak{K}_{|J_i|, |J_j|}^{lex}$, we obtain a red copy of \mathfrak{G} , as \mathfrak{G} is an edge-ordered subgraph of $\mathfrak{K}_{n,n}^{lex}$. \square

5 Lexicographic edge-ordered Ramsey numbers

Here, we include proofs of all statements about lexicographically edge-ordered graphs from Section 2. We start with a simple proof of Lemma 4.

Proof of Lemma 4. For a lexicographically edge-ordered graph $\mathfrak{G} = (G, <)$ with the vertex set V , let $f: V \rightarrow \{1, \dots, |V|\}$ be any one-to-one correspondence consistent with \mathfrak{G} and let \mathcal{G}_f be the vertex-ordering of G determined by f . More specifically, the vertex-ordering $\mathcal{G}_f = (G, <')$ is chosen such that $u <' v$ if and only if $f(u) < f(v)$. Without loss of generality, the edge-ordered graph \mathfrak{K}_N^{lex} has the vertex set $[N]$.

We show that every copy of \mathcal{G}_f in \mathcal{K}_N on $[N]$ determines a copy of \mathfrak{G} in \mathfrak{K}_N^{lex} . Let $i: V \rightarrow [N]$ be an inclusion witnessing that \mathcal{G}_f is an ordered subgraph of \mathcal{K}_N . That is, $i(u) < i(v)$ if and only if $f(u) < f(v)$ for all $u, v \in V$. Then \mathfrak{G} is an edge-ordered subgraph of \mathfrak{K}_N^{lex} , since, for edges $\{u, v\}$ and $\{w, t\}$ of G with $f(u) < f(v)$ and $f(w) < f(t)$, we have $\{u, v\} < \{w, t\}$ if and only if $f(u) < f(w)$ or $((f(u) = f(w) \ \& \ f(v) < f(t))$. This is true if and only if $i(u) < i(w)$ or $((i(u) = i(w) \ \& \ i(v) < i(t))$, which corresponds to $\{i(u), i(v)\}$ preceding $\{i(w), i(t)\}$ in \mathfrak{K}_N^{lex} .

Thus, given a 2-coloring χ of the edges of \mathfrak{K}_N^{lex} with $N = \overline{R}(\mathcal{G}_f)$, if we consider χ as a 2-coloring of the edges of \mathcal{K}_N , we obtain a monochromatic copy of \mathcal{G}_f in χ . Since every copy of \mathcal{G}_f in \mathcal{K}_N on $[N]$ determines a copy of \mathfrak{G} in \mathfrak{K}_N^{lex} , we obtain a monochromatic copy of \mathfrak{G} in χ . Since f is an arbitrary mapping consistent with \mathfrak{G} , it follows that $\overline{R}_{lex}(\mathfrak{G}) \leq \overline{R}(\mathcal{G}_f)$ and finishes the proof of Lemma 4. \square

Using Lemma 4, we now present a proof of Proposition 5. That is, we show that $\overline{R}_{lex}(\mathfrak{P}_n) \leq 2n - 3 + \sqrt{2n^2 - 8n + 11}$ for every $n > 2$, where \mathfrak{P}_n is the edge-monotone path on n vertices.

Proof of Proposition 5. The proof is based on the same idea used in [2, Proposition 15]. Let N be a positive integer and assume that there is a 2-coloring of the edges of \mathfrak{K}_N^{lex} with

no monochromatic copy of \mathfrak{P}_n . Let $[N]$ be the vertex set of \mathfrak{K}_N^{lex} and assume that the vertex-order $<$ is determined by the lexicographic edge-ordering of K_N . Without loss of generality, we assume that at least half of the edges with one vertex from $\left[\left\lceil \frac{N}{2} \right\rceil\right]$ and the other one in $\left\{\left\lceil \frac{N}{2} \right\rceil + 1, \dots, N\right\}$ are colored red. Let M be an $\left\lceil \frac{n}{2} \right\rceil \times \left\lfloor \frac{n}{2} \right\rfloor$ matrix with entries from $\{0, 1\}$ such that it contains 1-entries on positions (i, i) and $(i + 1, i)$ and 0-entries otherwise. Note that M naturally corresponds to an ordered path $\mathcal{P} = (P, <)$ on vertices $1 < \dots < n$ by connecting i and $\lceil n/2 \rceil + j$ with an edge if and only if the (i, j) entry of M is 1. Also the identity on the vertex set $[n]$ of \mathcal{P} is consistent with the lexicographic ordering of the edges of \mathcal{P} ; see Figure 1 for an example. Thus the matrix M represents the edge-monotone path \mathfrak{P}_n .

Let A be an $\left\lceil \frac{N}{2} \right\rceil \times \left\lfloor \frac{N}{2} \right\rfloor$ matrix with entries from $\{0, 1\}$ such that A contains 1-entries on positions (i, j) , where $\{i, \lceil \frac{N}{2} \rceil + j\}$ is a red edge of \mathfrak{K}_N^{lex} , and 0-entries otherwise. We say that A *contains* the matrix M if A contains a submatrix that has 1-entries at all the positions where M does. Observe that, since M represents the edge-monotone path \mathfrak{P}_n , the matrix A cannot contain M as otherwise \mathfrak{K}_N^{lex} contains a red copy of \mathfrak{P}_n .

It follows from a result of Füredi and Hajnal [13] (see also Lemma 16 in [2]) that A contains at most

$$\begin{aligned} \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \left\lceil \frac{N}{2} \right\rceil + \left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \left\lfloor \frac{N}{2} \right\rfloor - \left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \\ \leq \frac{2nN + 4n - 4N - 3 - n^2}{4} \end{aligned}$$

1-entries, as M is so-called *minimalist* matrix. On the other hand, since red was the major color among edges between $\left[\left\lceil \frac{N}{2} \right\rceil\right]$ and $\left\{\left\lceil \frac{N}{2} \right\rceil + 1, \dots, N\right\}$, we have at least $\frac{1}{2} \left\lceil \frac{N}{2} \right\rceil \left\lfloor \frac{N}{2} \right\rfloor \geq (N^2 - 1)/8$ such red edges and thus also at least that many 1-entries in A . Altogether, we have the inequality $(2nN + 4n - 4N - 3 - n^2)/4 < (N^2 - 1)/8$, which gives $N \leq 2n - 4 + \sqrt{2n^2 - 8n + 11}$. \square

6 Open problems

Many questions about edge-ordered Ramsey numbers remain open, for example proving a better upper bound on edge-ordered Ramsey numbers than the one obtained in the proof of Theorem 1. Since edge-ordered Ramsey numbers do not increase by removing edges of a given graph, it suffices to focus on edge-ordered complete graphs. It is possible that edge-ordered Ramsey numbers of edge-ordered complete graphs do not grow significantly faster than the standard Ramsey numbers.

Problem 12. *Is there a constant C such that, for every $n \in \mathbb{N}$ and every edge-ordered complete graph \mathfrak{K}_n on n vertices, we have $\overline{R}_e(\mathfrak{K}_n) \leq 2^{Cn}$?*

We note that, very recently, Fox and Li [12] showed that $\overline{R}_e(\mathfrak{H}) \leq 2^{100n^2 \log^2 n}$ for every edge-ordered graph \mathfrak{H} on n vertices.

It might also be interesting to consider sparser graphs, for example graphs with maximum degree bounded by a fixed constant, and try to prove better upper bounds on their edge-ordered Ramsey numbers.

We have no non-trivial results about lower bounds on edge-ordered Ramsey numbers and even lexicographically edge-ordered Ramsey numbers. Proving lower bounds for the latter might be simpler, as one has to consider only the lexicographic edge-ordering of the large complete graph. Is there a class of graphs with maximum degree bounded by a fixed constant such that their corresponding edge-ordered Ramsey numbers grow superlinearly in the number of vertices? If so, then such a result would contrast with the famous result of Chvátal, Rödl, Szemerédi, and Trotter [8], which says that Ramsey numbers of bounded-degree graphs grow at most linearly in the number of vertices.

Another interesting open problem is to determine the growth rate of the function $T(n)$ from Theorem 3. The current upper bound on $T(n)$ is quite large as the proof of Nešetřil and Rödl [20] uses Ramsey's theorem for quadruples and $6! = 720$ colors.

Finally, we showed that the inequality in Lemma 4 is not always tight using examples with stars, where both sides of the inequality differ by 1. It is a natural question to ask how wide this gap can be. In particular, is there a class of graphs for which the ratio between both sides of the inequality in Lemma 4 is arbitrarily large?

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