

Generalized Turán problems for even cycles

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December 18, 2018

Abstract

Given a graph H and a set of graphs \mathcal{F} , let $ex(n, H, \mathcal{F})$ denote the maximum possible number of copies of H in an \mathcal{F} -free graph on n vertices. We investigate the function $ex(n, H, \mathcal{F})$, when H and members of \mathcal{F} are cycles. Let C_k denote the cycle of length k and let $\mathcal{C}_k = \{C_3, C_4, \dots, C_k\}$. We highlight the main results below.

- (i) We show that $ex(n, C_{2l}, C_{2k}) = \Theta(n^l)$ for any $l, k \geq 2$. Moreover, in some cases we determine it asymptotically: We show that $ex(n, C_4, C_{2k}) = (1 + o(1)) \frac{(k-1)(k-2)}{4} n^2$ and that the maximum possible number of C_6 's in a C_8 -free bipartite graph is $n^3 + O(n^{5/2})$.
- (ii) Erdős's Girth Conjecture states that for any positive integer k , there exist a constant $c > 0$ depending only on k , and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ with girth more than $2k$.

Solyosi and Wong [38] proved that if this conjecture holds, then for any $l \geq 3$ we have $ex(n, C_{2l}, \mathcal{C}_{2l-1}) = \Theta(n^{2l/(l-1)})$. We prove that their result is sharp in the sense that forbidding any other even cycle decreases the number of C_{2l} 's significantly: For any $k > l$, we have $ex(n, C_{2l}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^2)$. More generally, we show that for any $k > l$ and $m \geq 2$ such that $2k \neq ml$, we have $ex(n, C_{ml}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m)$.

- (iii) We prove $ex(n, C_{2l+1}, \mathcal{C}_{2l}) = \Theta(n^{2+1/l})$, provided a stronger version of Erdős's Girth Conjecture holds (which is known to be true when $l = 2, 3, 5$). This result is also sharp in the sense that forbidding one more cycle decreases the number of C_{2l+1} 's significantly: More precisely, we have $ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k}\}) = O(n^{2-\frac{1}{l+1}})$, and $ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k+1}\}) = O(n^2)$ for $l > k \geq 2$.
- (iv) We also study the maximum number of paths of given length in a C_k -free graph, and prove asymptotically sharp bounds in some cases.

Keywords: Turán numbers, Erdős Girth Conjecture, Cycles, Extremal graph theory

AMS Subj. Class. (2010): 05C35, 05C38

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1 Introduction

The Turán problem for a set of graphs \mathcal{F} asks the following. What is the maximum number $ex(n, \mathcal{F})$ of edges that a graph on n vertices can have without containing any $F \in \mathcal{F}$ as a subgraph? When \mathcal{F} contains a single graph F , we simply write $ex(n, F)$. This function has been intensively studied, starting with Mantel [31] and Turán [40] who determined $ex(n, K_r)$ where K_r denotes the complete graph on r vertices with $r \geq 3$. See [15, 37] for surveys on this topic.

Let C_k denote a cycle on k vertices and let P_k denote a path on k vertices. *Length* of a path P_k is $k - 1$, the number of edges in it and length of a cycle C_k is k . A theorem of Simonovits [36] implies that for odd cycles, we have $ex(n, C_{2k+1}) = \lfloor n^2/4 \rfloor$ for any $k \geq 1$ and n large enough. For even cycles C_{2k} , Bondy and Simonovits [6] proved the following upper bound.

Theorem 1 (Bondy, Simonovits [6]). *For $k \geq 2$ we have*

$$ex(n, C_{2k}) = O(n^{1+1/k}).$$

The order of magnitude in the above theorem is known to be sharp only for $k = 2, 3, 5$. The case when all cycles longer than a given length are forbidden, was considered by Erdős and Gallai [10].

Theorem 2 (Erdős, Gallai [10]). *If a graph does not contain any cycle of length more than k , then it has at most $\frac{(k-1)n}{2}$ edges.*

On the other hand, if all the short cycles are forbidden, Alon, Hoory and Linial [1] proved the following. To state their result let us introduce the following notation: let A be a set of integers, each at least 3. Then let the set of cycles $\mathcal{C}_A = \{C_a : a \in A\}$. If $A = \{3, 4, \dots, k\}$ for some integer k , then we denote the corresponding set of cycles by \mathcal{C}_k .

Theorem 3 (Alon, Hoory, Linial [1]). *For any $k \geq 2$ we have*

- (i) $ex(n, \mathcal{C}_{2k}) < \frac{1}{2}n^{1+1/k} + \frac{1}{2}n$,
- (ii) $ex(n, \mathcal{C}_{2k+1}) < \frac{1}{2^{1+1/k}}n^{1+1/k} + \frac{1}{2}n$.

For more information on Turán number of cycles one can consult the survey [41].

Notation and definitions

The girth of a graph is the length of a shortest cycle in it. We say a graph has even girth if its girth is of even length, otherwise we say it has odd girth. Now we introduce basic notation that we will use throughout the paper.

- We will denote by $v_1v_2 \dots v_{k-1}v_kv_1$ a cycle C_k with vertices v_1, v_2, \dots, v_k and edges v_iv_{i+1} ($i = 1, \dots, k - 1$) and v_kv_1 . Similarly $v_1v_2 \dots v_{k-1}v_k$ denotes a path P_k with vertices v_1, v_2, \dots, v_k and edges v_iv_{i+1} ($i = 1, \dots, k - 1$).
- For two graphs H and G , let $\mathcal{N}(H, G)$ denote the number of copies of H in G .
- For a vertex v in G , let $N_i(v)$ denote the set of vertices at distance exactly i from v .
- For any two positive integers n and l , let $(n)_l$ denote the product

$$n(n-1)(n-2) \dots (n-(l-1)).$$

1.1 Generalized Turán problems

Given a graph H and a set of graphs \mathcal{F} , let

$$ex(n, H, \mathcal{F}) = \max_G \{ \mathcal{N}(H, G) : G \text{ is an } \mathcal{F}\text{-free graph on } n \text{ vertices.} \}$$

If $\mathcal{F} = \{F\}$, we simply denote it by $ex(n, H, F)$. This problem was initiated by Erdős [9], who determined $ex(n, K_s, K_t)$ exactly. Concerning cycles, Bollobás and Győri [4] proved that

$$(1 + o(1)) \frac{1}{3\sqrt{3}} n^{3/2} \leq ex(n, C_3, C_5) \leq (1 + o(1)) \frac{5}{4} n^{3/2}$$

and this result was extended by Győri and Li [25] showing that

$$ex(n, C_3, C_{2k+1}) \leq \frac{(2k-1)(16k-2)}{3} \cdot ex(n, C_{2k})$$

for $k \geq 2$. This was later improved by Füredi and Özkahya [14] by a factor of $\Omega(k)$.

The systematic study of the function $ex(n, H, F)$ was initiated by Alon and Shikhelman in [2], where they improved the result of Bollobás and Győri by showing that $ex(n, C_3, C_5) \leq (1 + o(1)) \frac{\sqrt{3}}{2} n^{3/2}$. This bound was further improved in [12] and then very recently in [13] by Ergemlidze and Methuku who showed that $ex(n, C_3, C_5) < (1 + o(1)) 0.232 n^{3/2}$. Another notable result is the exact computation of $ex(n, C_5, C_3)$ by Hatami, Hladký, Král, Norine, and Razborov [28] and independently by Grzesik [21], where they showed that it is equal to $(\frac{n}{5})^5$. Very recently, the asymptotic value of $ex(n, C_k, C_{k-2})$ was determined for every odd k by Grzesik and Kielak in [22]. Concerning paths, Győri, Salia, Tompkins and Zamora [26] determined $ex(n, P_l, P_k)$ asymptotically.

In [2], Alon and Shikhelman characterized the graphs F with $ex(n, C_3, F) = O(n)$ and more recently, Gerbner and Palmer [18] showed that for every $l \geq 4$ and every graph F we have either $ex(n, C_l, F) = \Omega(n^2)$ or $ex(n, C_l, F) = O(n)$, and characterized the graphs F for which the latter bound holds. They also showed

Theorem 4 (Gerbner, Palmer [18]). *For $t \geq 2$ and $l \geq 4$ we have*

$$ex(n, C_l, K_{2,t}) = \frac{1}{2l} (t-1)^{l/2} n^{l/2}, \quad ex(n, P_l, K_{2,t}) = \frac{1}{2} (t-1)^{(l-1)/2} n^{(l+1)/2}.$$

Note that the case $t = 2$ was proved independently by Gishboliner and Shapira [20].

In this paper, we mainly focus on the case when H is an even cycle of given length and \mathcal{F} is a family of cycles.

The function $ex(n, H, F)$ is closely related to the area of Berge hypergraphs, see e.g. [17, 19]. Let $k \geq 2$ be an integer. A *Berge cycle* of length k is an alternating sequence of distinct vertices and hyperedges of the form $v_1, h_1, v_2, h_2, \dots, v_k, h_k, v_1$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \dots, k-1\}$ and $v_k, v_1 \in h_k$ and is denoted by *Berge- C_k* . Győri and Lemons [24] proved the following two theorems.

Theorem 5 (Győri, Lemons [24]). *Let $r \geq 3$ be a positive integer. If \mathcal{H} is an r -uniform Berge- C_{2k} -free hypergraph on n vertices, then it has at most $O(n^{1+\frac{1}{k}})$ hyperedges.*

Theorem 6 (Győri, Lemons [24]). *If \mathcal{H} is a Berge- C_{2k} -free hypergraph on n vertices, such that*

$|e| \geq 4k^2$ for every hyperedge e , then we have:

$$\sum_{e \in E(\mathcal{H})} |e| = O(n^{1+\frac{1}{k}}).$$

The previous two theorems easily imply the following corollary that we will use later.

Corollary 7. *If \mathcal{H} is a Berge- C_4 -free hypergraph on n vertices, then we have*

$$\sum_{e \in E(\mathcal{H})} |e| = O(n^{1.5}).$$

1.2 Forbidding a set of cycles

The famous Girth Conjecture of Erdős [8] asserts the following.

Conjecture 8 (Erdős's Girth Conjecture [8] for k). *For any positive integer k , there exist a constant $c > 0$ depending only on k , and a family of graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq cn^{1+1/k}$ and the girth of G_n is more than $2k$.*

This conjecture has been verified for $k = 2, 3, 5$, see [3, 5, 35, 42]. For a general k , Sudakov and Verstraëte [39] showed that if such graphs exist, then they contain a C_{2l} for any l with $k < l \leq Cn$, for some constant $C > 0$. More recently, Solymosi and Wong [38] proved that if such graphs exist, then in fact, they contain many C_{2l} 's for any fixed $l > k$. More precisely they proved:

Theorem 9 (Solymosi, Wong [38]). *If Erdős's Girth Conjecture holds for k , then for every $l > k$ we have*

$$ex(n, C_{2l}, \mathcal{C}_{2k}) = \Omega(n^{2l/k}).$$

The following remark shows that in many cases this bound is sharp.

Remark 1. If $k + 1$ divides $2l$, then

$$ex(n, C_{2l}, \mathcal{C}_{2k}) = O(n^{2l/k}).$$

Indeed, let us associate to each C_{2l} , one fixed ordered list of $2l/(k+1)$ edges $(e_1, e_{k+1}, e_{2k+1}, \dots)$, where e_1 appears as the first edge (chosen arbitrarily) on the C_{2l} , e_{k+1} as the $(k+1)$ -th edge, e_{2k+1} as the $(2k+1)$ -th edge and so on. Note that at most one C_{2l} is associated to an ordered tuple $(e_1, e_{k+1}, e_{2k+1}, \dots)$, because there is at most one path of length $k-1$ connecting the endpoints of any two edges (as all the short cycles are forbidden). Since there are at most $O(n^{1+1/k})$ ways to select each edge, this shows the number of C_{2l} 's is at most $O((n^{1+1/k})^{2l/(k+1)}) = O(n^{2l/k})$, showing that the bound in Theorem 9 is sharp when $k+1$ divides $2l$.

It is worth mentioning that Gerbner, Keszegh, Palmer and Patkós [16] considered a similar problem, where a finite list of allowed cycle lengths is given (thus the list of forbidden cycle lengths is infinite). Another main difference is that in [16], all cycles of allowed lengths are counted, as opposed to only counting the number of cycles of a given length like in this paper.

Constructions

Before mentioning our results in the next section, we present typical constructions of graphs (with many copies of a cycle) that we will refer to, in the rest of the paper.

- For $l, t \geq 1$ the (l, t) -theta-graph with endpoints x and y is the graph obtained by joining two vertices x and y , by t internally disjoint paths of length l .
- For a simple graph F and $n, l \geq 1$ the θ - (n, F, l) graph is a graph on n vertices obtained by replacing every edge xy of F by an (l, t) -theta-graph with endpoints x and y , where t is chosen as large as possible, with some isolated vertices if needed. More precisely let $t = \lfloor \frac{n - |V(F)|}{|E(F)|(l-1)} \rfloor$, and we add $n - (t|E(F)|(l-1) + |V(F)|)$ isolated vertices.

2 Our results

2.1 Forbidding a cycle of given length

We determine the order of magnitude of $ex(n, C_{2l}, C_{2k})$ below.

Theorem 10. *For any $l \geq 3$ and $k \geq 2$ we have*

$$ex(n, C_{2l}, C_{2k}) \leq (1 + o(1)) \frac{2^{l-2}(k-1)^l}{2l} n^l.$$

For any $k > l \geq 2$ we have

$$ex(n, C_{2l}, C_{2k}) \geq (1 + o(1)) \frac{(k-1)_l}{2l} n^l.$$

For any $l > k \geq 3$ we have

$$ex(n, C_{2l}, C_{2k}) \geq (1 + o(1)) \frac{1}{l} n^l.$$

Theorem 10 and Theorem 11 (stated below) show that $ex(n, C_{2l}, C_{2k}) = \Theta(n^l)$ for any $k, l \geq 2$, except for the lower bound in the case $k = 2$, which can be easily shown by counting cycles in the well-known C_4 -free graph constructed by Erdős and Rényi [11] (See Theorem 4 and [20]).

This theorem has recently been proven independently by Gishboliner and Shapira [20]. Our proof is different from theirs, and it gives a better bound if k is fixed (moreover, if l is fixed, then their bound and our bound are both tight). They study odd cycles as well, determining the order of magnitude of $ex(n, C_l, C_k)$ for every $l > 3$ and k , and also provide interesting applications of these results in the study of the graph removal lemma and graph property testing.

Solymsi and Wong [38] asked whether a similar lower bound (to that of Theorem 9) on the number of C_{2l} 's holds, if just C_{2k} is forbidden instead of forbidding \mathcal{C}_{2k} . Theorem 10 answers this question in the negative.

Asymptotic results

If we go beyond determining the order of magnitude we can ask the asymptotics of these functions. In many cases it is a much harder question than the order of magnitude question [2, 4, 12, 30].

We determine $ex(n, C_4, C_{2k})$ asymptotically.

Theorem 11. *For $k \geq 2$ we have*

$$ex(n, C_4, C_{2k}) = (1 + o(1)) \frac{(k-1)(k-2)}{4} n^2.$$

Since most constructions are bipartite, it is natural to consider the bipartite version of the generalized Turán function: Let $ex_{bip}(n, C_{2l}, C_{2k})$ denote the maximum number of copies of a C_{2l} in a bipartite C_{2k} -free graph on n vertices. Our methods give sharper bounds for $ex_{bip}(n, C_{2l}, C_{2k})$ compared to the bounds in Theorem 10 (see Remark 2). In the case $l = 3$, $k = 4$ we can determine the asymptotics.

Theorem 12. *We have*

$$ex_{bip}(n, C_6, C_8) = n^3 + O(n^{5/2}).$$

We prove this theorem in Subsection 4.2. Interestingly, the proof makes use of Corollary 7 concerning Berge- C_4 -free hypergraphs. We leave open the question of determining the asymptotics of $ex(n, C_6, C_8)$, which we believe to be the same as that of $ex_{bip}(n, C_6, C_8)$.

2.2 Forbidding a set of cycles

Theorem 9 implies that if Erdős's Girth Conjecture is true (recall that it is known to be true for $l = 2, 3, 5$), then $ex(n, C_{2l}, \mathcal{C}_{2l-2}) = \Omega(n^{2l/(l-1)})$ for any $l \geq 3$. On the other hand, by Remark 1, this number is at most $O(n^{2l/(l-1)})$. This implies $ex(n, C_{2l}, \mathcal{C}_{2l-2}) = \Theta(n^{2l/(l-1)})$. By Lemma 25 (which is straightforward to prove), we know that when counting copies of an even cycle, forbidding an odd cycle does not change the order of magnitude. Therefore, we have

Corollary 13. *Suppose $l \geq 3$ and Erdős's Girth Conjecture is true for $l - 1$. Then we have*

$$ex(n, C_{2l}, \mathcal{C}_{2l-1}) = \Theta(n^{2l/(l-1)}).$$

In other words, the maximum number of C_{2l} 's in a graph of girth $2l$ is $\Theta(n^{2l/(l-1)})$. We prove that the previous theorem is sharp in the sense that forbidding one more even cycle decreases the order of magnitude significantly: The maximum number of C_{2l} 's in a C_{2k} -free graph with girth $2l$ is $\Theta(n^2)$. That is,

$$ex(n, C_{2l}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^2).$$

More generally, we show the following.

Theorem 14. *For any $k > l$ and $m \geq 2$ such that $2k \neq ml$ we have*

$$ex(n, C_{ml}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

Observe that forbidding even more cycles does not decrease the order of magnitude, as long as we do not forbid C_{2l} itself, as shown by $(l, \lfloor n/l \rfloor)$ -theta graph and some isolated vertices (i.e. the theta- (n, K_2, l) graph). On the other hand it is easy to see that if we forbid every cycle of length other than $2l + 1$, then there are $O(n)$ copies of C_{2l+1} .

Corollary 13 determines the order of magnitude of maximum number of C_{2l} 's in a graph of girth $2l$. It is then very natural to consider the analogous question for odd cycles: What is the maximum number of C_{2k+1} 's in a graph of girth $2k + 1$? Before answering this question, we state a strong form of Erdős's Girth Conjecture that is known to be true for small values of k .

A graph G on n vertices, with average degree d , is called *almost-regular* if the degree of every vertex of G is $d + O(1)$.

Conjecture 15 (Strong form of Erdős's Girth Conjecture). *For any positive integer k , there exist a family of almost-regular graphs $\{G_n\}$ such that $|V(G_n)| = n$, $|E(G_n)| \geq \frac{n^{1+1/k}}{2}$ and G_n is $\{C_4, C_6, \dots, C_{2k}\}$ -free.*

Lazebnik, Ustimenko and Woldar [29] showed Conjecture 15 is true when $k \in \{2, 3, 5\}$ using the existence of polarities of generalized polygons. We show the following that can be seen as the 'odd cycle analogue' of Theorem 9.

Theorem 16. *Suppose $k \geq 2$ and Strong form of Erdős's Girth Conjecture is true for k . Then we have*

$$ex(n, C_{2k+1}, \mathcal{C}_{2k}) = (1 + o(1)) \frac{n^{2+\frac{1}{k}}}{4k+2}.$$

To show that Theorem 16 is sharp and to give an analogue of Theorem 14 (in the case of $m = 2$) for odd cycles, we prove that if we forbid one more even cycle, then the order of magnitude goes down significantly:

Theorem 17. *For any integers $k > l \geq 2$, we have*

$$\Omega(n^{1+\frac{1}{2k+1}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k}\}) = O(n^{1+\frac{l}{l+1}}).$$

However, if the additional forbidden cycle is of odd length, we can only prove a quadratic upper bound. We conjecture that the truth is also sub-quadratic here (see Section 8, Theorem 35).

Theorem 18. *For any integers $k > l \geq 2$, we have*

$$\Omega(n^{1+\frac{1}{2k+2}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k+1}\}) = O(n^2).$$

Concerning forbidding a set of cycles we also determine the asymptotics of $ex(n, C_4, \mathcal{C}_A)$ for every possible set A . Let A_e be the set of even numbers in A and A_o be the set of odd numbers in A .

Theorem 19. *For any $k \geq 3$, we have*

$$ex(n, C_4, \mathcal{C}_A) = \begin{cases} 0 & \text{if } 4 \in A \\ (1 + o(1)) \frac{(k-1)(k-2)}{4} n^2 & \text{if } 4 \notin A \text{ and } 2k \text{ is the smallest element of } A_e \\ (1 + o(1)) \frac{1}{64} n^4 & \text{if } A_e = \emptyset. \end{cases}$$

We also determine the order of magnitude of $ex(n, C_6, \mathcal{C}_A)$ by proving

Theorem 20.

$$ex(n, C_6, \mathcal{C}_A) = \begin{cases} 0 & \text{if } 6 \in A, \\ \Theta(n^2) & \text{if } 6 \notin A, 4 \in A \text{ and } |A_e| \geq 2, \\ \Theta(n^3) & \text{if } 4, 6 \notin A \text{ and } A_e \neq \emptyset, \text{ or if } A_e = \{4\}, \\ \Theta(n^6) & \text{if } A_e = \emptyset. \end{cases}$$

2.3 Maximum number of P_l 's in a graph avoiding a cycle of given length

We study the maximum possible number of paths in a C_{2k} -free graph and prove the following results.

Theorem 21. *For $l, k \geq 2$, we have*

$$ex(n, P_l, C_{2k}) \leq (1 + o(1)) \frac{1}{2} (k-1)^{\frac{l-1}{2}} n^{\frac{l+1}{2}}.$$

Theorem 22. *If $2 \leq l < 2k$, then*

$$ex(n, P_l, C_{2k}) \geq (1 + o(1)) \frac{1}{2} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil}.$$

If $l \geq 2k$, then

$$ex(n, P_l, C_{2k}) \geq (1 + o(1)) \max \left\{ \left(\frac{n}{\lfloor l/2 \rfloor} \right)^{\lceil l/2 \rceil}, \left(\frac{(k-1)}{4(k-2)^{k+2}} \right)^{\lceil \frac{l}{2} \rceil} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil} \right\}.$$

Note that if $l < 2k$ and l is odd, Theorem 21 and Theorem 22 show that $ex(n, P_l, C_{2k})$ is equal to $(1 + o(1)) \frac{1}{2} (k-1)^{\frac{l-1}{2}} n^{\frac{l+1}{2}}$ as k and n tend to infinity.

Finally, we determine the maximum number of copies of P_l in a C_{2k+1} -free graph, asymptotically.

Theorem 23. *For $k \geq 1$ and $l \geq 2$, we have*

$$ex(n, P_l, C_{2k+1}) = (1 + o(1)) \left(\frac{n}{2} \right)^l.$$

Structure of the paper: In Section 3 we prove Theorem 10, determining the order of magnitude of $ex(n, C_{2l}, C_{2k})$

In Section 4 we determine the asymptotics of $ex(n, C_4, C_{2k})$ and $ex_{bip}(n, C_6, C_8)$ (Theorem 11 and Theorem 12).

In Section 5 we prove some basic lemmas for the case when a set of cycles are forbidden and prove Theorem 14 concerning graphs of even girth, along with results about $ex(n, C_4, \mathcal{C}_A)$ and $ex(n, C_6, \mathcal{C}_A)$ for every possible set A (i.e., Theorem 19 and Theorem 20).

In Section 6 we prove the theorems concerning graphs of odd girth: Theorem 16, Theorem 17 and Theorem 18.

In Section 7 we count number of copies of P_l in a C_k -free graph and prove Theorem 21, Theorem 22 and Theorem 23.

Finally in Section 8, we make some remarks and pose questions.

3 Maximum number of C_{2l} 's in a C_{2k} -free graph

Below we prove Theorem 10. Note that the case $l = 2$ of Theorem 14 gives back Theorem 10, and the proof here is also a special case of the proof of that more general statement. We decided to include it here separately for two reasons. On the one hand, this is an important special case. On

the other hand, it serves as an introduction to the similar, but more complicated proof of Theorem 14.

Proof of Theorem 10. Let us start with the lower bound and assume first that $2 \leq l < k$. Then $K_{k-1, n-k+1}$ is C_{2k} -free and it contains

$$\frac{1}{2l} \binom{k-1}{l} \binom{n-k+1}{l} l! l! = (1 + o(1)) \frac{(k-1)(k-2) \dots (k-l)}{2l} n^l = (1 + o(1)) \frac{(k-1)_l}{2l} n^l$$

copies of C_{2l} .

Let us now assume $l > k \geq 3$. Consider a copy of C_{2l} and replace every second vertex u by $\lfloor n/l - 1 \rfloor$ or $\lceil n/l - 1 \rceil$ copies of it, each connected to the two neighbors of u in the C_{2l} . The resulting graph only contains cycles of length 4 and $2l$, thus it is C_{2k} -free and it contains

$$(1 + o(1)) \frac{1}{l} n^l$$

copies of C_{2l} .

Let us continue with the upper bound. Consider a C_{2k} -free graph G . First we introduce the following notation. For two distinct vertices $a, b \in V(G)$, let

$$f(a, b) := \text{number of common neighbors of } a \text{ and } b.$$

Then we have

$$\frac{1}{2} \sum_{a \neq b, a, b \in V(G)} \binom{f(a, b)}{2} \leq (1 + o(1)) \frac{(k-1)(k-2)}{4} n^2, \quad (1)$$

by Theorem 11, since the left-hand-side is equal to the number of C_4 's in G .

Claim 1. *For every $a \in V(G)$ we have*

$$\sum_{b \in V(G) \setminus \{a\}} f(a, b) \leq (2k-2)n.$$

Note that the left-hand side of the above inequality is the number of P_3 's starting at a .

Proof. Recall that $N_1(a)$ is the set of vertices adjacent to a and $N_2(a)$ is the set of vertices at distance exactly 2 from a . Let E_1 be the set of edges induced by $N_1(a)$ and E_2 be the set of edges uv with $u \in N_1(a)$ and $v \in N_2(a)$. It is easy to see that $\sum_{b \in V(G) \setminus \{a\}} f(a, b) = 2|E_1| + |E_2|$.

We claim that there is no cycle of length longer than $2k-2$ in $E_1 \cup E_2$.

First suppose by contradiction that there is a cycle C of length $2k-1$ in $E_1 \cup E_2$. Since the cycle is of odd length it must contain an edge $uv \in E_1$. The subpath of length $2k-2$ between the vertices u and v in C together with the edges ua and va forms a C_{2k} in G , a contradiction.

Now suppose that there is a cycle C of length at least $2k$ in $E_1 \cup E_2$. Observe first that a subpath of length $2k-2$ of C starting from a vertex in $N_1(a)$ cannot have its other endpoint in $N_1(a)$, as that would form a C_{2k} together with the vertex a . Thus there has to be an edge $v_1 v_2$ of E_1 in C . Consider the subpath $v_1, v_2, \dots, v_{2k-1} v_{2k}$ of C . The vertices v_{2k-1} and v_{2k} are both in $N_2(a)$ because they are endpoints of paths of length $2k-2$ starting in v_1 and v_2 respectively. But then, the edge $v_{2k-1} v_{2k} \in E(C)$ is not in $E_1 \cup E_2$, a contradiction again.

Then by Theorem 2 we have $|E_1|+|E_2|=|E_1 \cup E_2| \leq (k-1)n$, which implies the claim. \square

The above claim implies that we have

$$\sum_{a \neq b, a, b \in V(G)} f(a, b) \leq (k-1)n^2. \quad (2)$$

Let us fix vertices v_1, v_2, \dots, v_l and let $g(v_1, v_2, \dots, v_l)$ be the number of C_{2l} 's in G where v_i is the $2i$ -th vertex ($i \leq l$). Clearly $g(v_1, v_2, \dots, v_l) \leq \prod_{j=1}^l f(v_j, v_{j+1})$ (where $v_{l+1} = v_1$ in the product). If we add up $g(v_1, v_2, \dots, v_l)$ for all possible l -tuples (v_1, v_2, \dots, v_l) of l distinct vertices in $V(G)$, we count every C_{2l} exactly $4l$ times. It means the number of C_{2l} 's is at most

$$\frac{1}{4l} \sum_{(v_1, v_2, \dots, v_l)} \prod_{j=1}^l f(v_j, v_{j+1}) \leq \frac{1}{4l} \sum_{(v_1, v_2, \dots, v_l)} \frac{f^2(v_1, v_2) + f^2(v_2, v_3)}{2} \prod_{j=3}^l f(v_j, v_{j+1}). \quad (3)$$

Fix two vertices $u, v \in V(G)$ and let us examine what factor $f^2(u, v)$ is multiplied with in (3). It is easy to see that $f^2(u, v)$ appears in (3) whenever $u = v_1, v = v_2$ or $u = v_2, v = v_1$ or $u = v_2, v = v_3$ or $u = v_3, v = v_2$. Let us start with the case $u = v_1$ and $v = v_2$. Then $f^2(u, v)$ is multiplied with

$$\frac{1}{8l} \left(\prod_{j=3}^{l-1} f(v_j, v_{j+1}) \right) f(v_l, u) = \frac{1}{8l} f(u, v_l) \prod_{j=3}^{l-1} f(v_j, v_{j+1})$$

for all the choices of tuples (v_3, v_4, \dots, v_l) (where these are all different vertices). We claim that

$$\sum_{(v_3, v_4, \dots, v_l)} \frac{1}{8l} f(u, v_l) \prod_{j=3}^{l-1} f(v_j, v_{j+1}) \leq \frac{(2k-2)^{l-2} n^{l-2}}{8l}.$$

Indeed, we can rewrite the left-hand side as

$$\frac{1}{8l} \sum_{v_l \in V(G)} f(u, v_l) \sum_{v_{l-1} \in V(G)} f(v_l, v_{l-1}) \dots \sum_{v_3 \in V(G)} f(v_4, v_3),$$

and each factor is at most $(2k-2)n$ by Claim 15.

Similar calculation in the other three cases gives the same upper bound, so adding up all four cases, we get an additional factor of 4, showing that the number of copies of C_{2l} is at most

$$\frac{1}{2l} \sum_{u \neq v, u, v \in V} f^2(u, v) (2k-2)^{l-2} n^{l-2}.$$

Finally observe that,

$$\sum_{u \neq v, u, v \in V} f^2(u, v) = 2 \sum_{u \neq v, u, v \in V} \binom{f(u, v)}{2} + \sum_{u \neq v, u, v \in V} f(u, v) \leq (1+o(1))(k-1)(k-2)n^2 + (k-1)n^2,$$

which is at most $(1+o(1))(k-1)^2 n^2$. Note that the above inequality follows from (1) and (2). This finishes the proof. \square

Remark 2. Note that if G is bipartite, or even just triangle-free, then in Claim 15, E_1 is empty. Therefore the same proof gives the better upper bound $\sum_{b \in V(G) \setminus \{a\}} f(a, b) \leq (k-1)n$. So we get

$$\sum_{a \neq b, a, b \in V(G)} f(a, b) \leq \frac{k-1}{2} n^2$$

instead of (5). Hence we can obtain

$$ex_{bip}(n, C_{2l}, C_{2k}) \leq ex(n, C_{2l}, \{C_3, C_{2k}\}) \leq (1 + o(1)) \frac{(k - \frac{3}{2})(k-1)^{l-1}}{2l} n^l.$$

Observe that the construction given in Theorem 10 is bipartite. Thus the ratio of the upper and lower bounds of $ex_{bip}(n, C_{2l}, C_{2k})$ is

$$\frac{(k - \frac{3}{2})(k-1)^{l-2}}{(k-1)_l},$$

which goes to 1 as k increases.

4 Asymptotic results

We will first prove the following simple result and use it in the proof of Theorem 19.

Theorem 24. *For any k, l , we have*

$$ex(n, C_{2l}, C_{2k+1}) = (1 + o(1)) \frac{1}{2l} \left(\frac{n^2}{4} \right)^l.$$

Proof. The lower bound is given by the complete bipartite graph $K_{n/2, n/2}$.

Let G be a graph which is C_{2k+1} -free. By a theorem of Győri and Li [25] there are at most $O(n^{1+1/k})$ triangles in G , so let us delete an edge from each of them and call the resulting triangle-free graph G' . This way we delete at most $O(n^{1+1/k})n^{2l-2} = o(n^{2l})$ copies of C_{2l} . So it suffices to estimate the number of C_{2l} 's in G' .

First we count the number of ordered tuples of l independent edges $M_l = (e_1, e_2, \dots, e_l)$ in G' . As the maximum number of edges in a triangle-free graph is at most $\lfloor n^2/4 \rfloor$ by Mantel's theorem, we can pick an edge $e_1 = uv$ of G in at most $\lfloor n^2/4 \rfloor$ ways. Then we can pick the edge e_2 disjoint from e_1 in at most $\lfloor (n-2)^2/4 \rfloor$ ways as the subgraph of G induced by $V(G) \setminus \{u, v\}$ is also triangle-free. We can pick e_3 in at most $\lfloor (n-4)^2/4 \rfloor$ ways, e_4 at most $\lfloor (n-6)^2/4 \rfloor$ ways and so on. Thus G' contains at most

$$\lfloor \frac{n^2}{4} \rfloor \lfloor \frac{(n-2)^2}{4} \rfloor \lfloor \frac{(n-4)^2}{4} \rfloor \dots \lfloor \frac{(n-2l+2)^2}{4} \rfloor = (1 + o(1)) \left(\frac{n^2}{4} \right)^l$$

copies of M_l .

Now we count the number of C_{2l} 's containing a fixed copy of $M_l = (e_1, e_2, \dots, e_l)$, where $e_i = u_i v_i$. To obtain a cycle C_{2l} from M_l , we decide for every i , whether u_i follows v_i or v_i follows u_i in a clock-wise ordering. However, for any given i , after deciding the order for u_i and v_i , we claim that the order for u_{i+1}, v_{i+1} is determined. Indeed, suppose w.l.o.g that v_i follows u_i . Then

v_i can be adjacent to at most one of the vertices u_{i+1}, v_{i+1} because G' is triangle-free. Thus the order of u_{i+1}, v_{i+1} is determined. So once the order of u_1 and v_1 is fixed (in two ways) the cycle C_{2l} is determined. Thus the number of C_{2l} 's obtained from a fixed copy of M_l is at most 2. Note that in this way each copy of C_{2l} in G' is obtained exactly $4l$ times. So the total number of C_{2l} 's in G' is at most

$$(1 + o(1)) \left(\frac{n^2}{4}\right)^l \cdot \frac{2}{4l} = (1 + o(1)) \frac{1}{2l} \left(\frac{n^2}{4}\right)^l$$

as required. □

4.1 Proof of Theorem 11: Maximum number of C_4 's in a C_{2k} -free graph

We restate Theorem 11 below for convenience.

Theorem. *For $k \geq 2$ we have:*

$$ex(n, C_4, C_{2k}) = (1 + o(1)) \frac{(k-1)(k-2)}{4} n^2.$$

Proof. For the lower bound consider the complete bipartite graph $K_{k-1, n-k+1}$.

For the upper bound consider a C_{2k} -free graph G . We call a pair of vertices *fat* if they have at least k common neighbors, otherwise it is called non-fat. We call a C_4 *fat* if both pairs of opposite vertices in that C_4 are fat. First we claim that the number of non-fat C_4 's is at most $\binom{k-1}{2} \binom{n}{2}$. Indeed, there are at most $\binom{n}{2}$ non-fat pairs, and each of them is contained in at most $\binom{k-1}{2}$ C_4 's as an opposite pair.

In the remaining part of the proof we will prove that the number of fat C_4 's is $O(n^{1+1/k})$, by using an argument inspired by the reduction lemma of Győri and Lemons [23]. We go through the fat C_4 's in an arbitrary order, one by one, and pick exactly one edge (from the four edges of the C_4); we always pick the edge which was picked the smallest number of times before (in case there is more than one such edge, then we pick one of them arbitrarily).

After this procedure, every edge e is picked a certain number of times. Let us denote this number by $m(e)$, and we call it the *multiplicity* of e . Note that $\sum_{e \in E(G)} m(e)$ is equal to the number of fat C_4 's in G .

If

$$m(e) < 2(k-2)k^2 \binom{2k}{k}$$

for each edge e , then the number of fat C_4 's in G is at most

$$2(k-2)k^2 \binom{2k}{k} |E(G)| = O(n^{1+1/k})$$

by Theorem 1, as desired.

Hence we can assume there is an edge e with $m(e) \geq 2(k-2)k^2 \binom{2k}{k}$. In this case we will find a C_{2k} in G , which will lead to a contradiction, finishing the proof. More precisely, we are going to prove the following statement:

Claim 2. For every $2 \leq l \leq k$ there is a C_{2l} in G , that contains an edge e_l with

$$m(e_l) \geq 2(k-l)k^2 \binom{2k}{k}.$$

Proof. We prove it by induction on l . For the base case $l = 2$, consider any C_4 containing $e = e_2$. Let us assume now we have found a cycle C of length $2l$ in G and one of its edges $e_l = uv$ has

$$m(e_l) \geq 2(k-l)k^2 \binom{2k}{k}.$$

For any $i \leq 2(k-l)k^2 \binom{2k}{k}$, when e_l was picked for the i th time, the corresponding fat C_4 contained four edges each of which had been picked earlier at least $i-1$ times, thus they have multiplicity at least $i-1$. Let \mathcal{F}_l be the set of those fat C_4 's where e_l was picked for the last $2k^2 \binom{2k}{k} - 1$ times. All the three other edges of each of these fat C_4 's have multiplicity at least

$$2(k-l)k^2 \binom{2k}{k} - 2k^2 \binom{2k}{k} = 2(k-l-1)k^2 \binom{2k}{k}.$$

At most $\binom{2l-2}{2}$ of the C_4 's in \mathcal{F}_l have all four of their vertices in C (note that they all contain the edge e_l).

Observe that G is $K_{k,k}$ -free, as C_{2k} is a subgraph of $K_{k,k}$. This means that any k vertices in C have at most $k-1$ common neighbors. We claim that there are at most $(k-1)\binom{2l}{k}$ vertices in $V(G) \setminus V(C)$ that are connected to at least k vertices in C . Indeed, otherwise by pigeon hole principle, there are k vertices in C such that each of them is connected to the same k vertices in $V(G) \setminus V(C)$, a contradiction. Therefore, at most $(2l-2)(k-1)\binom{2l}{k}$ C_4 's have a vertex in C and a vertex w outside C such that w is connected to at least k vertices in C .

Thus, there are at least

$$(2k^2 \binom{2k}{k} - 1) - \binom{2l-2}{2} - (2l-2)(k-1)\binom{2l}{k} \geq 1$$

four-cycle(s) in \mathcal{F}_l such that one of the following two cases hold. Let $uvxyu$ be one such four-cycle (recall that $e_l = uv$).

Case 1. $x, y \in V(G) \setminus V(C)$.

We replace the edge e_l of C by the path consisting of the edges vx, xy, yu , thus obtaining a cycle of length $2l+2$. The edges vx, xy, yu have multiplicity at least $2(k-l-1)k^2 \binom{2k}{k}$, which finishes the proof in this case.

Case 2. $x \in V(C), y \notin V(C)$ and y has less than k neighbors in C .

Note that in this case $\{y, v\}$ is a fat pair, thus y and v have at least k common neighbors. At least one of those, say w , is not in C . Let us replace the edge e_l of C by the path consisting of the edges uy, yw, wv . This way we obtain a cycle of length $2l+2$, and one of its edges uy has $m(uy) \geq 2(k-l-1)k^2 \binom{2k}{k}$, which finishes the proof of the claim and the theorem. \square

\square

4.2 Proof of Theorem 12: Maximum number of C_6 's in a bipartite C_8 -free graph

Let us recall that Theorem 12 states $ex_{bip}(n, C_6, C_8) = n^3 + O(n^{5/2})$.

Let G be a C_8 -free bipartite graph with classes A and B . Let us define a pair of vertices u, v from the same class *fat*, if there are four different vertices w_1, w_2, w_3, w_4 from the other class such that $uw_1, uw_2, uw_3, uw_4, vw_1, vw_2, vw_3$ and vw_4 are edges of the graph.

Claim 3. *Suppose $v_1v_2v_3v_4v_5v_6v_1$ is a 6-cycle and v_1, v_3 is a fat pair. Then neither v_2, v_4 nor v_2, v_6 is a fat pair.*

Proof. We prove it by contradiction. Let w be a neighbor of both v_1 and v_3 with $w \notin \{v_2, v_4, v_6\}$ and let u be a neighbor of both vertices in the other fat pair (either v_2 and v_4 , or v_2 and v_6) with $u \notin \{v_1, v_3, v_5\}$. We can find such w and u because of the definition of fatness. Then $v_1wv_3v_2uv_4v_5v_6v_1$ or $v_1v_2uv_6v_5v_4v_3wv_1$ is a C_8 , a contradiction. \square

So by Claim 3 we can suppose that if $v_1v_2v_3v_4v_5v_6v_1$ is a 6-cycle and v_1, v_3 is a fat pair, then there can be only one fat pair among v_2, v_4, v_6 and it is v_4, v_6 . This means that if there are two fat pairs among the vertices of a 6-cycle in different classes then (up to permutation of vertices) they should be v_1, v_3 and v_4, v_6 . Let us call a 6-cycle *fat* if it contains fat edges from both classes A and B . First we are going to prove that there are $O(n^{2.5})$ fat 6-cycles in G .

Let $v_1v_2v_3v_4v_5v_6v_1$ and $v_1v'_2v'_3v'_4v'_5v'_6v_1$ be two different fat 6-cycles (i.e., their fat pairs coincide in one of the classes). Claim 3 implies that v_4, v_6 and v'_4, v'_6 are the other fat pairs in these cycles.

Claim 4. *We have $\{v_4, v_6\} \cap \{v'_4, v'_6\} \neq \emptyset$.*

Proof. Let us suppose by contradiction that $\{v_4, v_6\} \cap \{v'_4, v'_6\} = \emptyset$. Then by the definition of fatness we can choose $u \notin \{v_1, v_3\}$ and $w \notin \{v_1, v_3, u\}$ such that u is connected to both v_4 and v_6 , and w is connected to both v'_4 and v'_6 . Then $v_1v_6uv_4v_3v'_4wv'_6v_1$ is a C_8 , a contradiction. \square

Let $N(u, v) := \{w : uw, vw \in E(G)\}$. We prove that for distinct fat pairs these sets of common neighborhoods are almost disjoint.

Claim 5. *Let $v_1, v_3, v'_1, v'_3 \in A$ with $\{v_1, v_3\} \neq \{v'_1, v'_3\}$ such that $\{v_1, v_3\}$ and $\{v'_1, v'_3\}$ are fat pairs of two fat 6-cycles. Then we have*

$$|N(v_1, v_3) \cap N(v'_1, v'_3)| \leq 1.$$

Proof. We prove it by contradiction, let us suppose that we have different vertices $x, y \in N(v_1, v_3) \cap N(v'_1, v'_3)$. By our assumption we can suppose that $v'_1 \notin \{v_1, v_3\}$. Let $v_1v_2v_3v_4v_5v_6v_1$ be a fat cycle.

Assume first $\{x, y\} \cap \{v_4, v_6\} = \emptyset$. As the pair v_4, v_6 is fat, we can find $u \notin \{v'_1, v_1, v_3\}$ that is connected to both v_4 and v_6 . Then $xv'_1yv_3v_4uv_6v_1$ is a C_8 , a contradiction.

Hence we can assume $x = v_4$. Consider now the case $v_6 \neq y$. By the fatness of v_1, v_3 there is a $u \notin \{x, y, v_6\}$ connected to both v_1 and v_3 . By the fatness of v_4, v_6 there is $w \notin \{v_1, v_3, v'_1\}$ connected to both v_4 and v_6 . Then $v_1uv_3yv'_1v_4wv_6v_1$ is a C_8 , a contradiction.

Thus we have $x = v_4, y = v_6$. Assume first $v'_3 \notin \{v_1, v_3\}$. By the fatness of v'_1, v'_3 we have $w \notin \{x, y, v_2\}$ connected to both v'_1 and v'_3 . Then $v_1v_2v_3v_4v'_1wv'_3v_6v_1$ is a C_8 , a contradiction.

Finally, if $v'_3 \in \{v_1, v_3\}$, then the 6-cycle $v_1uv_3v_4v'_1v_6v_1$ contains two fat pairs in one class and a fat pair in the other class, contradicting Claim 3. \square

Let us fix a fat pair v_1, v_3 in one of the parts, A and consider the union of the neighborhoods of the corresponding fat pairs in the other part, B . Let $g(v_1, v_3)$ denote its cardinality, i.e.

$$g(v_1, v_3) := \sum_{\substack{v_4, v_6 \text{ is a fat pair in } B \\ v_1, v_3, v_4, v_6 \text{ are contained in a fat } C_6}} |N(v_4, v_6)|.$$

Claim 6. *For any fat pair v_1, v_3 we have*

$$g(v_1, v_3) \leq 4n.$$

Proof. By Claim 4 we know that the fat pairs v_4, v_6 from B that are contained in a fat 6-cycle $v_1v_2v_3v_4v_5v_6v_1$ must pairwise intersect, so the auxiliary graph G_0 containing these fat pairs as edges is either a star or a triangle.

- If G_0 is a triangle, we are done by using that $|N(u, v)| \leq n$ for any u, v .

- If G_0 is a star with center x , then for every fat pair v_4, v_6 either $x = v_4$ or $x = v_6$. Let G_1 be the graph consisting of those edges where $x = v_4$ and G_2 be the graph consisting of those edges where $x = v_6$.

Observation 1. Suppose that v_1, v_3 is a fat pair in class A , while v_4, v_6 and v_4, v'_6 are distinct fat pairs from class B such that $v_1v_2v_3v_4v_5v_6v_1$ and $v_1v'_2v_3v_4v'_5v'_6v_1$ are fat 6-cycles. Then we have

$$(N(v_4, v_6) \cap N(v'_4, v'_6)) \setminus \{v_1, v_3\} = \emptyset.$$

Proof. Let us suppose by contradiction that there is $x \in (N(v_4, v_6) \cap N(v'_4, v'_6)) \setminus \{v_1, v_3\}$. By the fatness of the pair v_1, v_3 there is $z \in N(v_1, v_3) \setminus \{v_4, v_6, v'_6\}$ and by the fatness of the pair v_4, v_6 , we can find $y \in N(v_6, v_4) \setminus \{v_1, v_3, x\}$. Then $v_1zv_3v_4yv_6xv'_6v_1$ is a C_8 , a contradiction. \square

This implies that for the fat pairs in G_1 we have

$$\sum_{\substack{v_4, v_6 \text{ is a fat pair in } G_1 \\ v_1, v_3, v_4, v_6 \text{ are contained in a fat } C_6}} |N(v_4, v_6) \setminus \{v_1, v_3\}| \leq n,$$

as we add up the cardinalities of disjoint sets. The same statement is true for G_2 . This implies

$$\sum_{\substack{v_4, v_6 \text{ is a fat pair in } G_0 \\ v_1, v_3, v_4, v_6 \text{ are contained in a fat } C_6}} |N(v_4, v_6) \setminus \{v_1, v_3\}| \leq 2n.$$

Observe that there are less than n edges in the star G_0 , thus we subtract the two elements v_1 and v_3 less than n times altogether. This finishes the proof. \square

Now consider the hypergraph \mathcal{H} whose vertex set is B and its edge set is

$$\{N(v_1, v_3) : \{v_1, v_3\} \subset A \text{ is a fat pair that is contained in at least one fat } C_6\}.$$

Recall that a Berge- C_4 in a hypergraph is an alternating sequence of distinct vertices and hyperedges of the form $v_1, h_1, v_2, h_2, v_3, h_3, v_4, h_4, v_1$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, 3\}$ and $v_4, v_1 \in h_4$.

Claim 7. \mathcal{H} is Berge- C_4 -free.

Proof. We prove it by contradiction. Let us suppose that we have a Berge- C_4 in \mathcal{H} that is a sequence

$$x, N(a(x, y), b(x, y)), y, N(a(y, z), b(y, z)), z, N(a(z, w), b(z, w)), w, N(a(w, x), b(w, x)),$$

where we have:

- (1) $x, y, z, w \in B$ are distinct vertices,
- (2) $a(x, y), b(x, y), a(y, z), b(y, z), a(z, w), b(z, w), a(w, x), b(w, x) \in A$, and
- (3) $xa(x, y)y, xb(x, y)y, ya(y, z)z, yb(y, z)z, za(z, w)w, zb(z, w)w, wa(w, x)x, wb(w, x)x$ are all cherries.

Let us consider the subgraph G' of our original graph G that is spanned by the vertices $x, y, z, w, a(x, y), b(x, y), a(y, z), b(y, z), a(z, w), b(z, w), a(w, x), b(w, x)$. If one can find a matching in this bipartite graph that covers x, y, z and w , that would immediately give a C_8 , which is a contradiction. So to get the desired contradiction we apply Hall's theorem [27] and check Hall's condition holds for the set of vertices $\{x, y, z, w\}$.

Let us consider a subset $X \subseteq \{x, y, z, w\}$. Each vertex in $\{x, y, z, w\}$ is connected to at least two vertices in G' , so we are done if $|X| \leq 2$. If $|X| = 3$, we can assume without loss of generality that $X = \{x, y, z\}$. Then x, y and z are connected to the same 2 vertices. This implies $\{a(x, y), b(x, y)\} = \{a(y, z), b(y, z)\}$, thus two hyperedges of the Berge- C_4 coincide, a contradiction. Finally, if $|X| = 4$, then there are at most three vertices in the other part of G' , thus there are at most 3 different hyperedges in the Berge- C_4 , a contradiction. \square

The above claim together with Corollary 7 implies that

$$\sum_{\substack{v_1, v_3 \text{ is a fat pair in } A \text{ that is} \\ \text{contained in at least one fat } C_6}} |N(v_1, v_3)| = O(n^{1.5}). \quad (4)$$

Claim 8. There are $O(n^{2.5})$ fat 6-cycles in G .

Proof. Observe first that we obtain an upper bound on the number of fat 6-cycles if we pick the fat pairs v_1, v_3 and v_4, v_6 , then multiply the number of vertices that can be v_2 with the number of vertices that can be v_5 . These later quantities we can upper bound by $|N(v_1, v_3)|$ and $|N(v_4, v_6)|$, as v_2 is connected to both v_1 and v_3 , while v_5 is connected to both v_4 and v_6 . Thus the number of fat 6-cycles is at most

$$\sum_{\substack{v_1, v_3 \text{ and } v_4, v_6 \text{ are fat} \\ v_1, v_3, v_4, v_6 \text{ are contained in a fat } C_6}} |N(v_1, v_3)| |N(v_4, v_6)| =$$

$$\sum_{v_1, v_3 \text{ is a fat pair in } A} |N(v_1, v_3)| \cdot \sum_{\substack{v_4, v_6 \text{ is a fat pair} \\ v_1, v_3, v_4, v_6 \text{ are contained in a fat } C_6}} |N(v_4, v_6)| \leq$$

$$4n \cdot \sum_{\substack{v_1, v_3 \text{ is a fat pair in } A \text{ that is} \\ \text{contained in at least one fat } C_6}} |N(v_1, v_3)| = 4n \cdot O(n^{1.5}),$$

where we use (4) in the last inequality. □

Now we start to analyze those 6-cycles that contain fat pairs just from one side, say B . We show that the number of these 6-cycles is at most $6\binom{|A|}{3} + O(|A|^{2.5})$. By symmetry it is enough to finish the proof of Theorem 12, since $|A| + |B| \leq n$ implies $6\binom{|A|}{3} + 6\binom{|B|}{3} \leq 6\binom{n}{3} = n^3 + O(n^2)$.

For $x, y, z \in A$ let $h(x, y, z)$ denote the number of 6-cycles containing them in G . We call a set $\{x, y, z\} \subset A$ *good* if $h(x, y, z) \leq 6$, and *bad* otherwise. Note that if there are no fat pairs among x, y, z , then $h(x, y, z) \leq 27$. We say that a 4-set $F \subset A$ is *nice* if the four vertices have three common neighbors in B , forming a copy of $K_{4,3}$ in G . It is easy to see that two vertices from a nice set having another common neighbor would create a C_8 . This implies that 3-subsets of a nice set are good.

We say a pair $x, y \in A$ is *marked*, if they have exactly three common neighbors.

Claim 9. *Let us assume there is a six-cycle $xv_1yv_2zv_3x$ that contains a marked pair x, y , such that x and y are contained in a nice set S , but z is not contained in S . Then $\{x, y, z\}$ is a good triple.*

Proof. Let us pick $s \in S \setminus \{x, y\}$. By the definition of the nice set, there is a 3-set $S' \subset B$ consisting of the common neighbors of the vertices in S . If $v_3 \notin S'$, then $yv_3zv_2xtst'y$ is a C_8 , where t and t' are arbitrary elements of S' , different from v_2 . This is a contradiction. Similarly we can obtain a contradiction if $v_2 \notin S'$ by symmetry. If $v_3 \in S'$ but $v_1 \notin S'$, then $v_3zv_2xv_1ytsv_3$ is a C_8 , where $t \in S' \setminus \{v_2, v_3\}$.

Thus we have $S' = \{v_1, v_2, v_3\}$. Let us assume there is $w \in B \setminus S'$ that is connected to both z and x . Then $zwxv_1sv_2yv_3z$ is a C_8 , a contradiction. Similarly z and y cannot have another common neighbor, just like x and y . Thus all the 6-cycles containing x, y and z have to contain v_1, v_2, v_3 , so there are at most 6 of them. □

Claim 10. *The number of marked pairs that are not in a nice set is $O(|A|^{1.5})$.*

Proof. We prove this by showing that there is no C_4 in the auxiliary graph consisting of marked edges that are not in nice sets. Indeed, assume there is a C_4 , $xyzwx$. Let us consider the auxiliary bipartite graph consisting of the marked pairs xy, yz, zw, wx on one side and the common neighbors of these pairs on the other side, where a marked edge is connected to the three common neighbors defining it. A matching covering xy, yz, zw, wx would correspond to a C_8 $xv_yv_2zv_3wv_4$ in G , where the v_i s are the other vertices in the matching. If there is no such matching, then by Hall's condition the four vertices x, y, z, w have the same three common neighbors, thus they form a nice set, a contradiction. □

This implies that there are $O(|A|^{2.5})$ copies of C_6 containing any marked pairs not in a nice set. Hence from now on we consider only non-marked pairs. Let us consider a triple in A not containing any fat or marked pair.

Claim 11. *There are $O(|A|^2)$ bad triples containing neither a fat, nor a marked pair.*

Proof. Let $\{x, y, z\}$ be such a triple. In every 6-cycle containing x, y, z , there is a vertex connected to both x and y , another connected to both y and z and a third one connected to both z and x . As none of these pairs is marked, there are at most two candidates for each position. If any of them would coincide, the number of 6-cycles containing x, y, z would be at most 4. Thus there are 6 vertices in B $a_1, a_2, b_1, b_2, c_1, c_2$ such that a_1 and a_2 are connected to x and y , b_1 and b_2 are connected to y and z , while c_1 and c_2 are connected to z and x .

Let us consider the auxiliary 3-uniform hypergraph \mathcal{H}_0 of bad triples containing no marked pair. We claim it is a linear hypergraph, finishing the proof. Indeed, otherwise without loss of generality there is another bad triple $\{x', y, z\}$ with the corresponding vertices $a'_1, a'_2, b_1, b_2, c'_1, c'_2$ in B . We can assume that $a_1 \neq a'_1$ and $c_1 \neq c'_1$, and we have $a_1 \neq c'_1$ (similarly $a'_1 \neq c_1$), as otherwise x, y and z (or x', y and z) would have a common neighbor. Then $xa_1ya'_1x'c'_1zc_1x$ is a C_8 in G , a contradiction. \square

The above claims together imply that there are $O(|A|^{2.5})$ bad triples. Indeed, there are $O(|A|^2)$ containing no marked pair by Claim 11. There are two kinds of marked pairs. There are $O(|A|^{1.5})$ of those that are not in a nice set by Claim 10, thus they are contained in at most $O(|A|^{2.5})$ triples. Finally, if a marked pair is contained in a nice set, then it is not contained in a bad triple.

The total number of 6-cycles that do not contain a fat pair from part A is

$$\begin{aligned} \sum_{x,y,z \in A} h(x,y,z) &= \sum_{x,y,z \in A \text{ is a good triple}} h(x,y,z) + \sum_{x,y,z \in A \text{ is a bad triple}} h(x,y,z) \leq \\ & \sum_{x,y,z \in A \text{ is a good triple}} 6 + \sum_{x,y,z \in A \text{ is a bad triple}} 27 \leq 6 \binom{|A|}{3} + 27 \cdot O(|A|^{2.5}). \end{aligned}$$

This finishes the proof of the theorem.

5 Forbidding a set of cycles

In this section we study the case when multiple cycles are forbidden. Recall that if A is a set of integers, such that each integer is at least 3, then the set of cycles $\mathcal{C}_A = \{C_a : a \in A\}$, A_e is the set of even numbers in A and A_o is the set of odd numbers in A .

5.1 Basic Lemmas

The following simple lemma shows that if we count copies of an even cycle of given length, then forbidding odd cycles does not change the order of magnitude.

Lemma 25. *If $2k \notin A$, then*

$$ex(n, C_{2k}, \mathcal{C}_A) = \Theta(ex(n, C_{2k}, \mathcal{C}_{A_e})).$$

Proof. It is obvious that $ex(n, C_{2k}, \mathcal{C}_A) \leq ex(n, C_{2k}, \mathcal{C}_{A_e})$, as a \mathcal{C}_A -free graph is also \mathcal{C}_{A_e} -free. Let G be a \mathcal{C}_{A_e} -free graph. We are going to show that it has a \mathcal{C}_A -free subgraph G' that contains at least $1/2^{2k-1}$ fraction of the $2k$ -cycles of G , finishing the proof.

Let us consider a random 2-coloring of the vertices of G , where every vertex becomes blue with probability $1/2$, and red otherwise. Let us delete the edges inside the color classes, and let G' be the resulting graph. As G' is bipartite, it does not contain any cycle in \mathcal{C}_{A_o} . The probability that a $2k$ -cycle of G is also in G' is $1/2^{2k-1}$, as the first vertex can be of any color, but then the color of all the other vertices is determined. Thus the expected number of $2k$ -cycles in G' is $1/2^{2k-1}$ fraction of the $2k$ -cycles in G , hence there is a 2-coloring with at least that many $2k$ -cycles. \square

Next we show that if we count copies of an odd cycle of given length, then forbidding shorter odd cycles does not change the order of magnitude.

Lemma 26. *Let O_k be the set of odd integers less than $2k + 1$. Then*

$$ex(n, C_{2k+1}, \mathcal{C}_A) = \Theta(ex(n, C_{2k+1}, \mathcal{C}_{A \setminus O_k})).$$

Proof. The proof goes similarly to that of the previous lemma, one of the directions is again trivial. Let G be a $\mathcal{C}_{A \setminus O_k}$ -free graph. We are going to show that it has a \mathcal{C}_A -free subgraph G' that contains at least a constant fraction of the $2k$ -cycles of G .

Let us consider a random partition of the vertices of G into $2k + 1$ classes V_1, \dots, V_{2k+1} , where each vertex goes to each class with the same probability $1/(2k + 1)$. We keep the edges between V_i and V_{i+1} for $i \leq 2k$, and the edges between V_{2k+1} and V_1 . We delete all the other edges and let G' be the resulting graph. It is easy to see that if we delete V_i from G' , we obtain a bipartite graph, hence an odd cycle has to contain a vertex from V_i , for every $i \leq 2k + 1$. This means every odd cycle has length at least $2k + 1$.

It is left to prove that G' contains many $(2k + 1)$ -cycles. An arbitrary cycle in G is a cycle in G' with probability $1/(2k + 1)^{2k}$, finishing the proof. \square

Lemma 27. *Let m and s be fixed positive integers, and let G be a graph on n vertices. Suppose there is a partition of $V(G)$ into sets V_1, \dots, V_s satisfying the following properties:*

- (i) *there is no P_3 with both endpoints in V_i for $i < s$,*
- (ii) *there is no P_{m+1} with endpoints in V_i and V_j if $i \neq j$, and*
- (iii) *V_s is an independent set.*

Then $|E(G)| = O(n)$.

Proof. Let us first delete every vertex with degree at most m . Then repeat this procedure until we obtain a graph with minimum degree greater than m , or a graph with no vertices. We will show that the resulting graph has no vertices, which would imply that G has at most $mn = O(n)$ edges, since obviously at most mn edges were deleted.

If the resulting graph contains a vertex, it has to contain a vertex $x_1 \in V_j$ with $j \neq s$ (because V_s is an independent set). Starting from x_1 , we build a path P_m greedily. First we pick a neighbor x_2 of x_1 . For each $2 \leq i \leq m - 1$, after picking x_i , we pick a neighbor x_{i+1} of it that has not appeared in the path; it forbids at most $i - 1 < m$ neighbors, thus we can pick such a neighbor. After picking x_m , we add one more condition: the neighbor of x_m we pick as x_{m+1} should not be in V_j . As x_m has at most one neighbor in V_j by (i), this is at most one more forbidden neighbor, so we can still find one satisfying all these conditions. This way we obtain a P_{m+1} with endpoints in different parts, a contradiction with (ii). \square

5.2 Proof of Theorem 14: Forbidding an additional cycle in a graph of given even girth

Now we prove Theorem 14. We restate it here for convenience.

Theorem. *For any $k > l$ and $m \geq 2$ such that $2k \neq ml$ we have*

$$ex(n, C_{ml}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m)$$

Proof. By Lemma 26 and 25 it is enough to prove

$$ex(n, C_{ml}, \mathcal{C}_{2l-2} \cup \{C_{2k}\}) = \Theta(n^m).$$

The lower bound is given by the theta- (n, C_m, l) graph. It contains $\Omega(n^m)$ cycles of length ml , and additionally contains only cycles of length $2l$.

First we prove the upper bound for $m = 2$. We consider a graph G on n vertices that does not contain any of the forbidden cycles. We can assume it is bipartite by Lemma 25. Then G has girth at least $2l$, thus it is easy to see that for any vertices u, v and any length $i \leq l - 1$, there is at most one path of length i between u and v .

Claim 12. *If C and C' are $2l$ -cycles in G sharing a path of length $l - 1$, then their intersection is exactly a path of length $l - 1$ or l .*

Proof. Observe that if we can find a closed walk of length less than $2l$ which is not contained in a tree, then it implies the existence of a cycle of length less than $2l$, a contradiction.

Let us consider the longest path $Q = u_1 \dots u_i$ shared by C and C' . If Q has length more than l , then there are paths of length less than l in both C and C' with endpoints u_1 and u_i , and these two paths cannot be the same. Thus they form a closed walk of length less than $2l$.

Let $v \in (V(C) \cap V(C')) \setminus V(Q)$ be the vertex that is the closest to u_1 in C . Let x (resp. x') be the distance between v and u_1 in C (resp. C'). Suppose first that $x \neq x'$. Without loss of generality, we may suppose $x < x'$. Then the subpath of length x between u_1 and v in C , the subpath of length $2l - (i - 1) - x'$ between v and u_i in C' , and the path Q of length $i - 1$ between u_i and u_1 form a closed walk of length less than $2l$.

Hence we can assume $x = x'$. If $x < l$, then the paths of length x between u_1 and v in C and C' form a closed walk of length less than $2l$. If $x \geq l$, then either $v \in V(Q)$ or is adjacent to u_l , contradicting either our assumption that $v \in (V(C) \cap V(C')) \setminus V(Q)$ or our assumption that Q was the longest shared path. \square

The proof of Theorem 14 goes similarly to the proof of Theorem 11 from here. We call a pair of vertices *fat* if there are at least $4l^2$ paths, each of length l (i.e., l edges) between them. We call a copy of C_{2l} *fat* if all the l pairs of opposite vertices are fat.

Claim 13. *Let $\{u, v\}$ be a fat pair and X be a set of at most $4l$ vertices. Then there is a path P of length l between u and v such that $(V(P) \setminus \{u, v\}) \cap X = \emptyset$.*

Proof. For every $i \leq l - 1$, there are at most $4l$ paths $uu_1 \dots u_i \dots u_{l-1}v$ such that u_i is in X . Indeed, there are at most $4l$ ways to choose u_i from X , and after that there is only one choice for the remaining vertices, because there is at most one path of length i from u to u_i , and at most one

path of length $l - i$ from u_i to v . Since there are $l - 1$ ways to choose i , altogether there are at most $4l(l - 1)$ paths intersecting X , finishing the proof. \square

Observe now that the number of non-fat C_{2l} 's is at most $\binom{4l^2-1}{2} \binom{n}{2}$, as there are at most $\binom{n}{2}$ non-fat pairs and each of them is contained in at most $\binom{4l^2-1}{2}$ C_{2l} 's as an opposite pair. This way we count every non-fat C_{2l} at least once.

Let us only consider fat C_{2l} 's from now on. We go through them in an arbitrary order, one by one, and pick exactly one path $u_1u_2 \dots u_l$ of length $l - 1$ from each of them (from the $2l$ paths of length $l - 1$ in the C_{2l}); we always pick the path which was picked the smallest number of times before (in case there is more than one such path, then we pick one of them arbitrarily).

After this procedure, every path Q of length $l - 1$ is picked a certain number of times. Let us denote this number by $m(Q)$, and we call it the *multiplicity* of Q . Note that adding up $m(Q)$ for all the paths Q of length $l - 1$ gives the number of fat C_{2l} 's in G . Assume first that at the end of this algorithm $m(Q) \leq \frac{2k(k-l)}{l-2}$ for every path Q of length $l - 1$. Then the number of fat C_{2l} 's is at most $\frac{2k(k-l)}{l-2}$ times the number of the paths of length $l - 1$, which is at most $\frac{2k(k-l)}{l-2} \binom{n}{2}$, as there is at most one path of length $l - 1$ between any two vertices.

Hence we can assume there is a path Q of length $l - 1$ with multiplicity greater than $\frac{2k(k-l)}{l-2}$. We claim that in this case there is a copy of C_{2k} in G , which leads to a contradiction, finishing the proof. More precisely, we are going to prove the following statement:

Claim 14. *For every $l \leq r \leq k$ there is a C_{2r} in G , that contains a path Q_r of length $l - 1$ with*

$$m(Q_r) \geq \frac{2k(k-r)}{l-2}.$$

Proof. We prove it by induction on r . More precisely, are going to assume that the statement is true for r and show that it is true for $r + l - 1$. Therefore, we need to start with the base cases $l \leq r \leq 2l - 2$. For the base case $r = l$, consider any C_{2l} containing $Q = Q_l$. For the other base cases consider $Q = u_1u_2 \dots u_l$ with $m(Q) \geq \frac{2k(k-l)}{l-2}$. We have two fat C_{2l} 's, say C and C' , containing Q such that every subpath of length $l - 1$ in each of them has multiplicity at least $\frac{2k(k-l)}{l-2} - 2$. By Claim 12 the intersection of C and C' is a path Q' of length l or $l - 1$. It means either $Q = Q'$ or Q' consists of Q plus an additional vertex adjacent to either u_1 or u_l .

Note that for any pair u, v of vertices that are opposite in either C or C' , there is a path $P(u, v)$ of length l between u and v such that $V(P) \setminus \{u, v\} \cap (V(C) \cup V(C')) = \emptyset$, by Claim 13, since u, v is a fat pair.

Let us assume first that $Q' = Q$. Let $1 \leq i \leq l - 2$ and let w be the vertex opposite to u_i in C' . Consider the subpath of Q from u_1 to u_i , the path $P(u_i, w)$ of length l from u_i to w , the subpath of length i from w to u_l in C' , and the path of length $l + 1$ from u_l to u_1 in C . They form a cycle of length $2l + 2i$, that contains a subpath of C of multiplicity at least

$$\frac{2k(k-l)}{l-2} - 1 \geq \frac{2k(k-l-i)}{l-2}.$$

If $Q' \neq Q$, we can assume without loss of generality that $Q' = u_1u_2 \dots u_lu_{l+1}$. Let $1 \leq i \leq l - 3$ and let w' be the vertex opposite to u_{i+1} in C' . Consider the subpath of Q' from u_1 to u_{i+1} , the path $P(u_{i+1}, w')$ of length l from u_{i+1} to w' , the subpath of length i from w' to u_{l+1} in C' , and the

path of length l (different from Q') from u_{l+1} to u_1 in C . They form a cycle of length $2l + 2i$, that contains a subpath of C of multiplicity at least

$$\frac{2k(k-l)}{l-2} - 1 \geq \frac{2k(k-l-i)}{l-2},$$

finishing the base cases.

Let us continue with the induction step. Assume we are given a cycle C of length $2r$ that contains a path $Q_r = u_1 u_2 \dots u_l$ with $m(Q) \geq \frac{2k(k-r)}{l-2}$, and we are going to find a cycle of length $2r + 2l - 2$ that contains a path of length $l - 1$ with multiplicity at least $\frac{2k(k-r-l+2)}{l-2}$. For any $i \leq \frac{2k(k-r)}{l-2}$, when Q_r was picked for the i th time, the corresponding fat C_{2l} contained $2l$ paths of length $l - 1$ each of which had been picked earlier at least $i - 1$ times, thus they have multiplicity at least $i - 1$. Let \mathcal{F}_r be the set of those fat C_{2l} 's where Q_r was picked for the last

$$\frac{2k(k-r)}{l-2} - \frac{2k(k-r-l+2)}{l-2} = 2k$$

times, so $|\mathcal{F}_r| \geq 2k$. All the other paths of length $l - 1$ in each of these fat C_{2l} 's have multiplicity at least

$$\frac{2k(k-r)}{l-2} - 2k = \frac{2k(k-r-l+2)}{l-2}.$$

First observe that for every vertex $w \in V(C) \setminus V(Q_r)$ there is at most one cycle in \mathcal{F}_r which contains w such that w is neither next to u_1 , nor to u_l in that cycle. Indeed, two such cycles would contradict Claim 12. As there are less than $2k$ choices for w , either there exists a cycle $C' = u_1 u_2 \dots u_l v_1 \dots v_l u_1 \in \mathcal{F}_r$ such that all of the vertices v_1, \dots, v_l are not in C or there exists a cycle $C' = u_1 u_2 \dots u_l v_1 \dots v_l u_1 \in \mathcal{F}_r$ such that only v_l or v_1 is in C among v_1, \dots, v_l . Without loss of generality, suppose v_1 is in C .

Consider a path P of length l from u_2 to v_2 that avoids $(V(C) \cup V(C')) \setminus \{u_2, v_2\}$ (such a path exists because u_2, v_2 is an opposite pair in C' , thus it is a fat pair and we can apply Claim 13). Let P' be a subpath of length $l - 1$ in C' from v_2 to u_1 . We replace the edge $u_1 u_2$ in C by the union of paths P and P' . Note that we replaced an edge with a path of length $2l - 1$, so the resulting cycle has length $2r + 2l - 2$ and it contains the subpath $v_2 \dots v_l u_1$, which is a subpath of C' , thus it has multiplicity at least

$$\frac{2k(k-r-l+2)}{l-2},$$

as required. □

We are done with the case $m = 2$, now we consider the case m is larger. Let G again be a graph that does not contain $C_3, C_4, \dots, C_{2l-2}, C_{2k}$. From here we follow the proof of Theorem 10. First we introduce the following notation. For two distinct vertices $a, b \in V(G)$, let

$$f_l(a, b) := \text{number of paths of } l \text{ edges between } a \text{ and } b.$$

In particular $f_2(a, b) = f(a, b)$. Then we have

$$\frac{1}{2} \sum_{a \neq b, a, b \in V(G)} \binom{f_l(a, b)}{2} = O(n^2), \quad (5)$$

by the case $m = 2$, since the left-hand side is equal to the number of C_{2l} 's in G .

Claim 15. *For every $a \in V(G)$ we have*

$$\sum_{b \in V(G) \setminus \{a\}} f_l(a, b) = O(n).$$

Proof. Notice that for any $i < l$, the set $N_i(a)$ does not contain any edges. It is easy to see that $\sum_{b \in V(G) \setminus \{a\}} f_l(a, b)$ is equal to the number of edges between $N_{l-1}(a)$ and $N_l(a)$. We will show this number is $O(n)$ by using Lemma 27 to the bipartite graph G' with vertex set $N_{l-1}(a) \cup N_l(a)$ and the edge set being the set of edges of G between $N_{l-1}(a)$ and $N_l(a)$.

Let w_1, \dots, w_{s-1} be the neighbors of v , and let $V_i = N_{l-1}(a) \cap N_{l-2}(w_i)$ for $1 \leq i \leq s-1$. Let $V_s = N_l(a)$. It is easy to see that V_1, V_2, \dots, V_s partition $V(G')$. Observe that a P_3 in G' with both endpoints inside V_i (for $i < s$) would create a cycle of length at most $2l-2$. This implies (i) of Lemma 27 is satisfied. A path $P_{2k-2l+3}$ in G' with endpoints in V_i and V_j with $i \neq j$ would create a cycle of length $2k$ in G , which shows that (ii) of Lemma 27 is satisfied. Since G' is bipartite, clearly V_s is independent in G' , thus we can apply Lemma 27, finishing the proof of the claim. \square

Let us fix vertices v_1, v_2, \dots, v_m and let $g(v_1, v_2, \dots, v_m)$ be the number of C_{ml} 's in G where v_i is li 'th vertex ($i \leq m$). Clearly $g(v_1, v_2, \dots, v_m) \leq \prod_{j=1}^m f_l(v_j, v_{j+1})$ (where $v_{l+1} = v_1$ in the product).

If we add up $g(v_1, v_2, \dots, v_m)$ for all possible m -tuples (v_1, v_2, \dots, v_m) of l distinct vertices in $V(G)$, we count every C_{ml} exactly $4m$ times. It means the number of C_{ml} 's is at most

$$\frac{1}{4m} \sum_{(v_1, v_2, \dots, v_m)} \prod_{j=1}^m f_l(v_j, v_{j+1}) \leq \frac{1}{4m} \sum_{(v_1, v_2, \dots, v_m)} \frac{f_l^2(v_1, v_2) + f_l^2(v_2, v_3)}{2} \prod_{j=3}^m f_l(v_j, v_{j+1}). \quad (6)$$

Fix two vertices $u, v \in V(G)$ and let us examine what factor $f_l^2(u, v)$ is multiplied with in (6). It is easy to see that $f_l^2(u, v)$ appears in (6) whenever $u = v_1, v = v_2$ or $u = v_2, v = v_1$ or $u = v_2, v = v_3$ or $u = v_3, v = v_2$. Let us start with the case $u = v_1$ and $v = v_2$. Then $f_l^2(u, v)$ is multiplied with

$$\frac{1}{8m} \left(\prod_{j=3}^{m-1} f_l(v_j, v_{j+1}) \right) f(v_m, u) = \frac{1}{8m} f_l(u, v_m) \prod_{j=3}^{m-1} f_l(v_j, v_{j+1})$$

for all the choices of tuples (v_3, v_4, \dots, v_m) (where these are all different vertices). We claim that

$$\sum_{(v_3, v_4, \dots, v_m)} \frac{1}{8m} f_l(u, v_m) \prod_{j=3}^{m-1} f_l(v_j, v_{j+1}) \leq \frac{(2k-2)^{m-2} n^{m-2}}{8m}.$$

Indeed, we can rewrite the left-hand side as

$$\frac{1}{8m} \sum_{v_l \in V(G)} f_l(u, v_m) \sum_{v_{m-1} \in V(G)} f_l(v_m, v_{m-1}) \cdots \sum_{v_3 \in V(G)} f_l(v_4, v_3),$$

and each factor is at most $O(n)$ by Claim 15.

Similar calculation in the other three cases gives the same upper bound, so adding up all four cases, we get an additional factor of 4, showing that the number of copies of C_{ml} is at most

$$\frac{1}{ml} \sum_{u \neq v, u, v \in V} f_l^2(u, v) O(n^{m-2}).$$

Finally observe that

$$\sum_{u \neq v, u, v \in V} f_l^2(u, v) = O(n^2),$$

by (5) and Claim 15. This finishes the proof. \square

5.3 Proofs of Theorem 19 and Theorem 20: Counting C_4 's or C_6 's when a set of cycles is forbidden

Below we determine the asymptotics of $ex(n, C_4, \mathcal{C}_A)$ and the order of magnitude of $ex(n, C_6, \mathcal{C}_A)$. For the convenience of the reader, we restate Theorem 19 and Theorem 20.

Theorem. *For any $k \geq 3$ we have*

$$ex(n, C_4, \mathcal{C}_A) = \begin{cases} 0 & \text{if } 4 \in A \\ (1 + o(1)) \frac{(k-1)(k-2)}{4} n^2 & \text{if } 4 \notin A \text{ and } 2k \text{ is the smallest element of } A_e \\ (1 + o(1)) \frac{1}{64} n^4 & \text{if } A_e = \emptyset. \end{cases}$$

Proof. The first line is obvious. For the second line, the upper bound follows from Theorem 11 as C_{2k} is forbidden, while the lower bound is given by the complete bipartite graph $K_{k-1, n-k+1}$. For the third line, the lower bound is given by the graph $K_{n/2, n/2}$, while the upper bound follows from Theorem 24. \square

Theorem.

$$ex(n, C_6, \mathcal{C}_A) = \begin{cases} 0 & \text{if } 6 \in A, \\ \Theta(n^2) & \text{if } 6 \notin A, 4 \in A \text{ and } |A_e| \geq 2, \\ \Theta(n^3) & \text{if } 4, 6 \notin A \text{ and } A_e \neq \emptyset, \text{ or if } A_e = \{4\} \\ \Theta(n^6) & \text{if } A_e = \emptyset. \end{cases}$$

Proof. The first line is obvious. For the second line, Theorem 14 with $m = 2$ and $l = 3$ gives the upper bound and the lower bound. For the third line, the upper bound follows from Theorem 10 and if $4, 6 \notin A$, then the lower bound is given by the graph $K_{3, n-3}$.

If $A_e = \{4\}$, the upper bound is given by Theorem 10. For the lower bound let us consider the C_4 -free graph G given by [18, 20], which contains $\Theta(n^3)$ C_6 's. G might contain some forbidden odd

cycles. However, Lemma 25 shows they do not change the order of magnitude.

For the fourth line, the lower bound is given by the graph $K_{n/2, n/2}$, while the upper bound is obvious. \square

6 Counting cycles in graphs with given odd girth

6.1 Proof of Theorem 16: Maximum number of C_{2k+1} 's in a graph of girth $2k+1$

In this subsection we prove Theorem 16.

Let G be an almost-regular, $\{C_4, C_6, \dots, C_{2k}\}$ -free graph on n vertices with $n^{1+1/k}/2$ edges given by Conjecture 15. It follows that the degree of each vertex of G is $n^{1/k} + O(1)$. We will show that G contains at least

$$(1 - o(1)) \frac{1}{2k+1} \frac{n^{2+1/k}}{2}$$

copies of C_{2k+1} .

Consider an arbitrary vertex $v \in V(G)$. Recall that $N_i(v)$ denotes the set of vertices at distance i from v . (Note that $N_1(v)$ is simply the neighborhood of v .) First we show the following.

Claim 16. *Let $2 \leq i \leq k$. Each vertex $u \in N_{i-1}(v)$ has at least $n^{1/k} + O(1)$ neighbors in $N_i(v)$. Moreover, no two vertices of $N_{i-1}(v)$ have a common neighbor in $N_i(v)$.*

Proof. Consider an arbitrary vertex $u \in N_{i-1}(v)$. If there are two edges ux, uy with $x, y \in N_{i-2}(v)$, let w be the first common ancestor of x and y . Then the length of the cycle formed by the two paths from w to x and from w to y and the two edges ux, uy is at most $2k$ and is even, a contradiction. So there is at most one edge from u to the set $N_{i-2}(v)$. Now if there are two edges ux, uy with $x, y \in N_{i-1}(v)$, then again consider the first common ancestor w , of x and y and we can find an even cycle of length at most $2k$ similarly. So the degree of u in $G[N_{i-2}(v)]$ is at most one. Therefore, each vertex $u \in N_{i-1}(v)$ has at least $n^{1/k} + O(1)$ neighbors in $N_i(v)$ (recall the degree of u is $n^{1/k} + O(1)$), proving the first part of the claim.

Suppose for a contradiction that there are two vertices $u, u' \in N_{i-1}(v)$, which have a common neighbor in $N_i(v)$. Then consider the first common ancestor of u and u' , and we can again find an even cycle of length at most $2k$. This completes the proof of the claim. \square

The above claim implies the following.

Claim 17. *For all $1 \leq i \leq k$, we have $|N_i(v)| = (1 + o(1))n^{i/k}$.*

Proof. Notice that Claim 16 implies that there are at least $|N_{i-1}(v)| (n^{1/k} + O(1))$ vertices in $N_i(v)$ for each $2 \leq i \leq k$. Since $N_1(v) = n^{1/k} + O(1)$, this proves the claim. \square

Now we claim that there are $(1 - o(1)) \frac{n^{1+1/k}}{2}$ edges of G in $N_k(v)$ (i.e., basically all the edges of G are in $N_k(v)$). Indeed, notice that

$$|N_k(v)| = (1 + o(1))n^{k/k} = (1 + o(1))n$$

by Claim 17, so $|V(G) \setminus N_k(v)| = o(n)$. Therefore the sum of degrees of the vertices in $V(G) \setminus N_k(v)$ is

$$o(n) \cdot (n^{1/k} + O(1)) = o(n^{1+1/k}),$$

showing that the number of edges incident to vertices outside $N_k(v)$ are negligible. This shows that there are $(1 - o(1))\frac{n^{1+1/k}}{2}$ edges in $G[N_k(v)]$, proving the claim.

Now we color each edge ab of $G[N_k(v)]$ in the following manner: If the first common ancestor of a and b is not v , then ab is colored with the color red, but if the only common ancestor of a and b is v then it is colored blue. We want to show that most of the edges in $G[N_k(v)]$ are of color blue. To this end, let us upper bound the number of edges in $G[N_k(v)]$ of color red.

Consider an arbitrary vertex $w \in N_1(v)$. Applying Claim 16 repeatedly, one can obtain that w has

$$(n^{1/k} + O(1))^{k-1} = (1 + o(1))n^{(k-1)/k}$$

descendants in $N_k(v)$. By Theorem 1, in the subgraph of G induced by this set of descendants, there are at most

$$O(n^{(k-1)/k})^{1+1/k} = (1 + o(1))O(n^{(k^2-1)/k^2})$$

edges.

On the other hand, the end vertices of each red edge must have an ancestor $w \in N_1(v)$, so the total number of red edges is at most

$$(1 + o(1)) |N_1(v)| O(n^{(k^2-1)/k^2}) = (1 + o(1))n^{1/k} O(n^{(k^2-1)/k^2}) = o(n^{1+1/k}).$$

This shows that there are $\frac{n^{1+1/k}}{2}(1 - o(1))$ blue edges in $G[N_k(v)]$. Notice that any blue edge ab , together with the two paths joining a and b to v , forms a C_{2k+1} in G containing v . This shows that there are $\frac{n^{1+1/k}}{2}(1 - o(1))$ copies of C_{2k+1} in G containing v . As v was arbitrary, summing up for all the vertices of G , we get that there are at least

$$(1 - o(1))\frac{1}{2k+1} \frac{n^{2+1/k}}{2}$$

copies of C_{2k+1} in G .

Now it only remains to upper bound the number of C_{2k+1} 's in a graph H of girth $2k+1$. For a pair (v, xy) where $v \in V(H)$, and $xy \in E(H)$, there is at most one C_{2k+1} in H such that xy is the edge in the C_{2k+1} opposite to v . Indeed, there is at most one path of length k in H joining v and x , and at most one path of length k in H joining v and y , as H has no cycles of length at most $2k$. On the other hand, a fixed C_{2k+1} consists of $2k+1$ pairs (v, xy) such that v is opposite to an edge xy of the cycle. Therefore, the number of C_{2k+1} 's in H is at most

$$\frac{n}{2k+1} |E(H)| \leq (1 + o(1))\frac{n}{2k+1} \frac{n^{1+1/k}}{2},$$

by Theorem 3. This completes the proof.

6.2 Proofs of Theorem 17 and Theorem 18: Forbidding an additional cycle in a graph of odd girth

In this subsection, we study the maximum number of C_{2l+1} 's in a $\mathcal{C}_{2l} \cup \{C_{2k}\}$ -free graph and the maximum number of C_{2l+1} 's in a $\mathcal{C}_{2l} \cup \{C_{2k+1}\}$ -free graph, and prove Theorem 17 and Theorem 18.

If $l = 1$ we count triangles in a C_{2k} -free graph or a C_{2k+1} -free graph. The second question

was studied by Györi and Li [25] and Alon and Shikhelman [2]. The first question was studied by Gishboliner and Shapira [20]. They showed the following.

Theorem 28 (Györi-Li, Alon-Shikhelman and Gishboliner-Shapira). *For any $k \geq 2$ we have*

- (i) $\Omega(ex(n, \mathcal{C}_{2k})) \leq ex(n, C_3, C_{2k}) \leq O_k(ex(n, C_{2k}))$.
- (ii) $\Omega(ex(n, \mathcal{C}_{2k})) \leq ex(n, C_3, C_{2k+1}) \leq O(k \cdot ex(n, C_{2k}))$.

The above lower and upper bounds are known to be of the same order of magnitude, $\Theta(n^{1+1/k})$ when $k \in \{2, 3, 5\}$ (see [3, 42]).

In the rest of the section we study the case $l \geq 2$. For the lower bounds, we will use the following result of Nešetřil and Rödl [32]. Girth of a hypergraph H is defined as the length of a shortest Berge cycle in it. More formally, it is the smallest integer k such that H contains a Berge- C_k .

Theorem 29 (Nešetřil, Rödl). *For any positive integers $r \geq 2$ and $s \geq 3$, there exists an integer n_0 such that for all $n \geq n_0$, there is an r -uniform hypergraph on n vertices with girth at least s and having at least $n^{1+1/s}$ edges.*

Here we restate Theorem 17.

Theorem. *For any $k \geq l + 1$, we have*

$$\Omega(n^{1+\frac{1}{2k+1}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k}\}) = O(n^{1+\frac{1}{l+1}}).$$

Proof. For the lower bound, consider a $(2l+1)$ -uniform hypergraph of girth $2k+1$ with $n^{1+1/(2k+1)}$ hyperedges (guaranteed by Theorem 29) and then replace each hyperedge by a copy of C_{2l+1} . It is easy to check that the resulting graph does not contain any cycle of length at most $2k$ except $2l+1$.

Now we prove the upper bound. Consider a $\mathcal{C}_{2l} \cup \{C_{2k}\}$ -free graph G . Since all the cycles of length at most $2l$ are forbidden, for any vertex v in G , there are no edges inside $N_i(v)$ for $i < l$, and the number of cycles of length $2l+1$ containing v is equal to the number of edges in $N_l(v)$. For a neighbor w of v let $Q(v, w) = N_l(v) \cap N_{l-1}(w)$.

Claim 18. *For any vertex $v \in V(G)$, there exists a constant $c = c(k, l) \geq 2$ such that there are at most*

- (i) $c|N_l(v)|$ edges inside $N_l(v)$, and
- (ii) $c(|N_l(v)| + |N_{l+1}(v)|)$ edges between $N_l(v)$ and $N_{l+1}(v)$.

Proof. For (i) let w_1, \dots, w_{s-1} be the neighbors of v , and let $V_i = Q(v, w_i)$ for $1 \leq i \leq s-1$. This gives a partition of $N_l(v)$. Observe that an edge inside V_i would create a cycle of length at most $2l-1$, as both its vertices are connected to w_i with a path of length $l-1$. Similarly a P_3 with both endpoints inside V_i would create a cycle of length at most $2l$. Finally, a $P_{2k-2l+1}$ with endpoints in V_i and V_j would create a cycle of length $2k$ together with the two (internally disjoint) paths of l connecting its endpoints to v . Thus we can apply Lemma 27 to finish the proof.

For (ii) we add $V_s = N_{l+1}(v)$ to the family of sets V_i , $1 \leq i \leq s-1$ defined before, and delete the edges inside V_s , as well as the edges inside $N_l(v)$. Observe that if a $P_{2k-2l+1}$ has endpoints in different parts V_i and V_j , then $i \neq s \neq j$ because of the parity of the length of the path. Thus we can apply Lemma 27 to finish the proof. \square

Now we delete every vertex which is contained in at most $4c^{l+1}n^{\frac{l}{l+1}}$ copies of C_{2l+1} from G , and we repeat this procedure until we obtain a graph G' where every vertex is contained in more than $4c^{l+1}n^{\frac{l}{l+1}}$ copies of C_{2l+1} . We claim that G' has at most $O(n^{1+\frac{l}{l+1}})$ copies of C_{2l+1} . As we deleted at most $O(n^{\frac{l}{l+1}})$ C_{2l+1} 's with every vertex, this will finish the proof.

Assume G' contains more than $cn^{1+\frac{l}{l+1}}$ copies of C_{2l+1} . First we show that the maximum degree in G' is at least $cn^{\frac{l}{l+1}}$. Indeed, otherwise $N_i(v)$ contains at most $c^i n^{\frac{i}{l+1}}$ vertices for every $1 \leq i \leq l$ (here we use that G' is \mathcal{C}_{2l} -free), thus there are at most $c^l n^{\frac{l}{l+1}}$ vertices in $N_l(v)$, hence there are at most $c^{l+1} n^{\frac{l}{l+1}}$ edges inside $N_l(v)$ by **(i)** of Claim 18, which means v is contained in at most $c^{l+1} n^{\frac{l}{l+1}}$ copies of C_{2l+1} , so it should have been deleted, a contradiction.

Thus we can assume there is a vertex v of degree at least $cn^{\frac{l}{l+1}}$. We will show that either v or one of its neighbors is contained in at most $c^l n^{\frac{l}{l+1}}$ copies of C_{2l+1} . For a neighbor w of v let $S_0(w) = N_l(w) \cap N_{l-1}(v)$, $S_1(w) = N_l(w) \cap N_l(v)$ and $S_2(w) = N_l(w) \cap N_{l+1}(v)$. Notice that $N_l(w) = S_0(w) \cup S_1(w) \cup S_2(w)$.

Let us sum up the number of edges pq with $p \in Q(v, w)$ and $q \in N_l(v) \cup N_{l+1}(v)$, over all the neighbors w of v . This way we counted every edge inside $N_l(v)$ or between $N_l(v)$ and $N_{l+1}(v)$ at most twice; moreover the number of such edges is at most $2cn$ by Claim 18. Therefore, the total sum is at most $4cn$. As $d(v) \geq cn^{\frac{l}{l+1}}$, v has a neighbor w such that there are at most $4n^{\frac{l}{l+1}}$ edges between vertices in $Q(v, w)$ and vertices in $N_l(v) \cup N_{l+1}(v)$. This also means $|S_1(w) \cup S_2(w)| \leq 4n^{\frac{l}{l+1}}$.

We claim that there are at most $(c+1)n^{\frac{l}{l+1}}$ edges inside $N_l(w) = S_0(w) \cup S_1(w) \cup S_2(w)$. There is no edge inside $S_0(w)$ as there is no edge inside $N_{l-1}(v)$. There is no edge between $S_0(w)$ and $S_2(w)$, since otherwise its endpoint in $S_2(w)$ would have to be in $N_l(v)$. A vertex $u \in S_1(w)$ is connected to at most one vertex in $S_0(w)$, otherwise we would obtain two distinct paths of length l between u and v , giving us a cycle of length at most $2l$. Hence the number of edges inside $N_l(w)$ incident to elements of $S_0(w)$ is at most $|S_1(w)| \leq 4n^{\frac{l}{l+1}}$.

Let us now partition $N_l(w)$ into sets $Q(w, w') = N_l(w) \cap N_{l-1}(w')$ for each neighbor w' of w . Observe that $Q(w, v) = S_0(w)$. For the remaining $d(w) - 1$ parts we want to apply Lemma 27 similarly to **(i)** of Claim 18. In fact, by deleting $S_0(w)$ we obtain another graph G'' where the same cycles are forbidden and the l -th neighborhood of w is $S_1(w) \cup S_2(w)$. Thus applying Claim 18 we obtain that there are at most $c(|S_1(w) \cup S_2(w)|) \leq 4cn^{\frac{l}{l+1}}$ edges inside $S_1 \cup S_2$.

Thus altogether there are at most $4(c+1)n^{\frac{l}{l+1}} < 4c^{l+1}n^{\frac{l}{l+1}}$ edges inside $N_l(w)$ (where $c > 0$ is a constant chosen so that the previous inequality is satisfied), hence w should have been deleted, a contradiction. □

Here we restate Theorem 18 and prove it.

Theorem. *For $k > l \geq 2$ we have*

$$\Omega(n^{1+\frac{1}{2k+2}}) = ex(n, C_{2l+1}, \mathcal{C}_{2l} \cup \{C_{2k+1}\}) = O(n^2).$$

Proof. For the lower bound, consider a $(2l+1)$ -uniform hypergraph of girth $2k+2$ with $n^{1+1/(2k+2)}$ hyperedges and then replace each hyperedge by a copy of C_{2l+1} .

Now we prove the upper bound. Let v be an arbitrary vertex in a $\mathcal{C}_{2l} \cup \{C_{2k+1}\}$ -free graph G . We will upper bound the number of C_{2l+1} 's containing v . There are no edges inside $N_i(v)$ for

each $i < l$. Indeed, if there is an edge then we can find a forbidden short odd cycle containing that edge (because the end points of that edge have a common ancestor). This shows that every C_{2l+1} containing v must use an (actually exactly one) edge from $N_l(v)$. So the number of C_{2l+1} 's containing v is upper bounded by the number of edges in $N_l(v)$. We claim the following.

Claim 19. *The number of edges in $N_l(v)$ is $O(|N_l(v)|) = O(n)$.*

Proof of Claim. Color each vertex in $N_l(v)$ by its (unique) ancestor in $N_1(v)$. Then the resulting color classes A_1, A_2, \dots, A_t partition $N_l(v)$. There are no edges inside the color classes, because such an edge would be contained in a forbidden short odd cycle.

One can partition the color classes into two parts $\{A_i \mid i \in I\}$ and $\{A_i \mid i \in J\}$ (with $I \cup J = \{1, 2, \dots, t\}$ and $I \cap J = \emptyset$), so that at least half of all the edges in $N_l(v)$ are between the vertices of the two parts. Now as C_{2k+1} is forbidden, there is no path of length $2k + 1 - 2l$ between the two parts, as such a path would have its end vertices in different classes. (Note that here we use that the parity of the path length $2k + 1 - 2l$ is odd.) Thus by Erdős-Gallai theorem there are only at most $O(|N_l(v)|) = O(n)$ edges between the two parts. This implies that the total number of edges in $N_l(v)$ is at most twice as many, completing the proof of the claim. \square

So using Claim 19, the number of C_{2l+1} 's containing any fixed vertex v is $O(n)$. Thus the total number of C_{2l+1} 's in G is at most $O(n^2)$, as desired. This completes the proof. \square

We conjecture that even if the additional forbidden cycle has odd length, we can only have a sub-quadratic number of C_{2l+1} 's. See Conjecture 34 stated in the last section.

7 Number of copies of P_l in a graph avoiding a cycle of given length

In the first subsection, we will consider the case when an even cycle is forbidden and prove Theorem 21 and Theorem 22, and in the next subsection, we will deal with the case when an odd cycle is forbidden and prove Theorem 23.

7.1 Bounds on $ex(n, P_l, C_{2k})$

For the upper bound, we use a spectral method similar to the one used in [26].

The spectral radius of a finite graph is defined to be the spectral radius of its adjacency matrix. Given a graph G , the spectral radius of G is denoted by $\mu(G)$. Given a matrix A , its spectral radius is denoted by $\mu(A)$. If A is the adjacency matrix of G , then of course, $\mu(G) = \mu(A)$.

Nikiforov [33] showed the following.

Theorem 30 (Nikiforov). *Let G be a C_{2k} -free graph on n vertices. Then for any $k \geq 1$, we have*

$$\mu(G) \leq \frac{k-1}{2} + \sqrt{(k-1)n} + o(n).$$

Note that in the case $k = 2$, a sharper bound is known: The maximum spectral radius of a C_4 -free graph on n vertices is $\frac{1}{2} + \sqrt{n - 3/4} + O(1/n)$, where for odd n the $O(1/n)$ term is zero. Now we prove Theorem 21, restated below.

Theorem. *We have*

$$ex(n, P_l, C_{2k}) \leq (1 + o(1)) \frac{1}{2} (k-1)^{\frac{l-1}{2}} n^{\frac{l+1}{2}}.$$

Proof. Let A be the adjacency matrix of a C_{2k} -free graph G . Recall that $\mathcal{N}(P_l, G)$ denotes the number of copies of P_l in G . Let $\mathcal{N}(W_l, G)$ denote the number of walks consisting of l vertices in G . Note that $2\mathcal{N}(P_l, G) \leq \mathcal{N}(W_l, G)$, since every path corresponds to two walks.

Then we have,

$$\frac{2\mathcal{N}(P_l, G)}{n} \leq \frac{\mathcal{N}(W_l, G)}{n} = \frac{\mathbf{1}^t A^{l-1} \mathbf{1}}{\mathbf{1}^t \mathbf{1}} \quad (7)$$

Note that $\mathbf{1}$ is the column vector with all entries being 1. The right-hand-side of (7) is at most $\mu(A^{l-1})$ because the spectral radius of any Hermitian matrix M is the supremum of the quotient $\frac{x^* M x}{x^* x}$, where x ranges over $\mathbb{C}^n \setminus \{0\}$. Moreover, using Theorem 30, we have

$$\mu(A^{l-1}) = (\mu(A))^{l-1} = (\mu(G))^{l-1} \leq \left(\frac{k-1}{2} + \sqrt{(k-1)n + o(n)} \right)^{l-1} \leq (1 + o(1)) ((k-1)n)^{\frac{l-1}{2}},$$

completing the proof. \square

Now we provide some lower bounds on $ex(n, P_l, C_{2k})$.

Constructing C_{2k} -free graphs with many copies of P_l

We prove Theorem 22. Note that the behavior of the extremal function seems to be very different in the cases $l < 2k$ and $l \geq 2k$.

Theorem. *If $2 \leq l < 2k$, then*

$$ex(n, P_l, C_{2k}) \geq (1 + o(1)) \frac{1}{2} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil}.$$

If $l \geq 2k$, then

$$ex(n, P_l, C_{2k}) \geq (1 + o(1)) \max \left\{ \left(\frac{n}{\lfloor l/2 \rfloor} \right)^{\lceil l/2 \rceil}, \left(\frac{(k-1)}{4(k-2)^{k+2}} \right)^{\lceil \frac{l}{2} \rceil} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil} \right\}.$$

Proof. In the case $l < 2k$, we take a complete bipartite graph B with parts of size $k-1$ and $n - (k-1)$. Clearly, B is C_{2k} -free and the number of copies of P_l in B is at least

$$\frac{1}{2} (k-1)_{\lfloor \frac{l}{2} \rfloor} (n - (k-1))_{\lceil \frac{l}{2} \rceil} = \frac{1}{2} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil} (1 + o(1)).$$

Now we consider the case $l \geq 2k$. First we give a simple construction. Consider a path $v_1 v_2 \dots v_l$ and for each odd i , replace the vertex v_i by b vertices $v_i^1, v_i^2, \dots, v_i^b$ where each of them is adjacent to the same vertices that v_i was adjacent to. Choose $b = \frac{n - \lfloor l/2 \rfloor}{\lfloor l/2 \rfloor} = (1 + o(1)) \frac{n}{\lfloor l/2 \rfloor}$. The resulting graph only contains cycles of length 4, so it is C_{2k} -free as long as $k \neq 2$. Moreover, it contains at least

$$(1 + o(1)) b^{\lceil l/2 \rceil} = (1 + o(1)) \left(\frac{n}{\lfloor l/2 \rfloor} \right)^{\lceil l/2 \rceil}$$

copies of P_l . The case $k = 2$ is dealt with in Theorem 4

Now we give a different construction which gives a better lower bound when l is large compared to k . We will use the following theorem of Ellis and Linial [7].

Theorem 31 (Ellis, Linial [7]). *Let r, d and g be integers with $d \geq 2$ and $r, g \geq 3$. Then there exists an r -uniform, d -regular hypergraph \mathcal{H} with girth at least g , and at most*

$$n_r(g, d) := (r - 1) \left(1 + d(r - 1) \frac{(d - 1)^g (r - 1)^g - 1}{(d - 1)(r - 1) - 1} \right) < 4((d - 1)(r - 1))^{g+1}$$

vertices.

Let $g = k + 1$ and $r = k - 1$. Consider the hypergraph \mathcal{H} given by Theorem 31 with

$$|V(\mathcal{H})| \leq n_r(g, d) = n_{k-1}(k + 1, d). \quad (8)$$

Notice that the number of hyperedges in \mathcal{H} is

$$|E(\mathcal{H})| \leq \frac{d \cdot n_r(g, d)}{r} = \frac{d \cdot n_{k-1}(k + 1, d)}{k - 1}. \quad (9)$$

Let $E(\mathcal{H}) = \{h_1, h_2, \dots, h_m\}$. To each hyperedge $h_i \in E(\mathcal{H})$, we add a set S_i of new vertices with

$$|S_i| = \frac{(n - |V(\mathcal{H})|)}{|E(\mathcal{H})|}$$

(note for $i \neq j$, we take $S_i \cap S_j = \emptyset$). Now we construct a graph G as follows: For each i with $1 \leq i \leq m$, consider the sets h_i , S_i and add all possible edges between h_i and S_i . That is, $E(G) = \{uv \mid u \in h_i, v \in S_i \text{ for some } 1 \leq i \leq m\}$. It is easy to check that G is C_{2k} -free. Note that G is a bipartite graph with parts $U := V(\mathcal{H})$ and $D := \cup_{i=1}^m S_i$. Moreover, the degree of every vertex of G in U is d times the size of a set S_i , so it is

$$\frac{d(n - |V(\mathcal{H})|)}{|E(\mathcal{H})|}.$$

And the degree of every vertex in D is the size of a set h_i , so it is $k - 1$. Therefore, the number of copies of P_l in G is at least

$$\left(\frac{d(n - |V(\mathcal{H})|)}{|E(\mathcal{H})|} \right)_{\lceil \frac{l}{2} \rceil} (k - 1)_{\lfloor \frac{l}{2} \rfloor} = (1 + o(1)) \left(\frac{dn}{|E(\mathcal{H})|} \right)^{\lceil \frac{l}{2} \rceil} (k - 1)_{\lfloor \frac{l}{2} \rfloor}.$$

Using (9), this is at least

$$(1 + o(1)) \left(\frac{(k - 1)n}{n_{k-1}(k + 1, d)} \right)^{\lceil \frac{l}{2} \rceil} (k - 1)_{\lfloor \frac{l}{2} \rfloor}.$$

Choosing $d = 2$ and using Theorem 31, we have

$$n_{k-1}(k + 1, d) = (k - 2) \left(1 + 2(k - 2) \frac{(k - 2)^{k+1} - 1}{(k - 2) - 1} \right) < 4(k - 2)^{k+2}.$$

So, $ex(n, P_l, C_{2k})$ is at least

$$(1 + o(1)) \left(\frac{(k-1)n}{n_{k-1}(k+1, 2)} \right)^{\lceil \frac{l}{2} \rceil} (k-1)_{\lfloor \frac{l}{2} \rfloor} > (1 + o(1)) \left(\frac{(k-1)}{4(k-2)^{k+2}} \right)^{\lceil \frac{l}{2} \rceil} (k-1)_{\lfloor \frac{l}{2} \rfloor} n^{\lceil \frac{l}{2} \rceil}.$$

□

7.2 Bounds on $ex(n, P_l, C_{2k+1})$

For the upper bound we will again use a spectral bound. We will use the following theorem of Nikiforov [34].

Theorem 32 (Nikiforov). *Let G be a C_{2k+1} -free graph on n vertices. Then for any $k \geq 1$ and $n > 320(2k+1)$, we have*

$$\mu(G) \leq \sqrt{n^2/4}.$$

Now we prove Theorem 23, restated below.

Theorem. *We have*

$$ex(n, P_l, C_{2k+1}) = (1 + o(1)) \left(\frac{n}{2} \right)^l.$$

Proof. For the lower bound, consider a complete bipartite graph B with $n/2$ vertices on each side. Then clearly, B does not contain any odd cycle and it contains at least

$$(1 + o(1)) \left(\frac{n}{2} \right)^l$$

copies of P_l .

The proof of the upper bound is similar to that of the proof of Theorem 21. Let A be the adjacency matrix of a C_{2k+1} -free graph G . Then, for n large enough, using Theorem 32 we get,

$$\frac{2\mathcal{N}(P_l, G)}{n} \leq \frac{\mathcal{N}(W_l, G)}{n} = \frac{\mathbf{1}^t A^{l-1} \mathbf{1}}{\mathbf{1}^t \mathbf{1}} \leq \mu(A^{l-1}) = (\mu(A))^{l-1} = (\mu(G))^{l-1} \leq \left(\sqrt{\frac{n^2}{4}} \right)^{l-1} = \left(\frac{n}{2} \right)^{l-1}.$$

Thus,

$$\mathcal{N}(P_l, G) \leq \left(\frac{n}{2} \right)^l,$$

completing the proof of the theorem. □

Remark 3. Note that the number of copies of P_{2l} in a graph G is at least $2l$ times the number of copies of C_{2l} in G . Indeed, every copy of C_{2l} contains $2l$ copies of P_{2l} . Moreover, a copy of P_{2l} belongs to at most one copy of C_{2l} . Thus Theorem 23 implies Theorem 24.

8 Concluding remarks and questions

We finish our article by posing some questions.

- Naturally, it would be interesting to prove asymptotic or exact results corresponding to our results where we only know the order of magnitude. For example it would be nice to close the gap between the lower and upper bounds in Theorem 10.

We also pose some conjectures when a family of cycles are forbidden.

- We proved in Theorem 14, that for any $k > l$ and $m \geq 2$ such that $2k \neq ml$ we have

$$ex(n, C_{ml}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

We conjecture that it is true for longer cycles as well.

Conjecture 33. *For any $k > l$, $m \geq 2$ and $1 \leq j < l$ with $ml + j \neq 2k$ we have*

$$ex(n, C_{ml+j}, \mathcal{C}_{2l-1} \cup \{C_{2k}\}) = \Theta(n^m).$$

- We prove in Theorem 18 that for $l > k \geq 2$ we have

$$ex(n, C_{2k+1}, \mathcal{C}_{2k} \cup \{C_{2l+1}\}) = O(n^2).$$

However, we conjecture that the truth is smaller.

Conjecture 34. *For any integers $k < l$, there is an $\epsilon > 0$ such that*

$$ex(n, C_{2k+1}, \mathcal{C}_{2k} \cup \{C_{2l+1}\}) = O(n^{2-\epsilon}).$$

The following theorem supports Conjecture 34.

Theorem 35. *We have*

$$ex(n, C_5, \mathcal{C}_4 \cup \{C_9\}) = O(n^{11/12}).$$

Proof. Let us consider a $\mathcal{C}_4 \cup \{C_9\} = \{C_3, C_4, C_9\}$ -free graph G . First we delete every edge that is contained in less than 17 C_5 's, then repeat this until every edge is contained in at least 17 C_5 's. We have deleted at most $17|E(G)| = O(n^{3/2})$ C_5 's this way (note that $|E(G)| = O(n^{3/2})$ follows from the fact that G is C_4 -free). Let G' be the graph obtained this way.

Observe that if a five-cycle $C := v_1v_2v_3v_4v_5v_1$ shares the edge v_1v_2 with another C_5 , then they either share also the edge v_2v_3 or v_5v_1 and no other vertices, or they share only the edge v_1v_2 . If there are at least six five-cycles sharing only v_1v_2 with C , we say v_1v_2 is an *unfriendly* edge for C , otherwise it is called a *friendly* edge for C . Our plan is to show first that a C_5 cannot contain both friendly and unfriendly edges, then using this we will show that a C_5 cannot contain friendly edges. Thus every edge is unfriendly for every C_5 , and this will imply that G' is C_6 -free.

Assume C contains both friendly and unfriendly edges. Then it is easy to see that it contains an unfriendly edge, say v_1v_2 , and a path P of two edges not containing v_1v_2 such that C shares P with a set \mathcal{S} of at least 6 other C_5 's. (Note that the cycles in \mathcal{S} only share P .) Now there is a cycle $v_1v_2w_3w_4w_5v_1$ by the unfriendliness of v_1v_2 that contains three new vertices w_3, w_4, w_5 . Then we replace v_1v_2 in C with $v_1w_3w_4w_5v_2$ to obtain a C_8 . Afterwards, there is a cycle in \mathcal{S} that does not contain any of w_3, w_4 and w_5 as the elements of \mathcal{S} are vertex disjoint outside C . Thus we can replace P in this C_8 with a path of three edges to obtain a C_9 , a contradiction.

Assume now C contains only friendly edges. The edge v_1v_2 is contained in at least 6 other C_5 's together with one of its two neighboring edges, say v_2v_3 . At least of these 6 C_5 's does not contain the vertices v_4 and v_5 , let it be $v_1v_2v_3w_1w_2v_1$. Thus replacing the two-edge path $v_1v_2v_3$ with the three-edge path $v_3w_1w_2v_1$ to obtain the six-cycle $v_4v_5v_1w_2w_1v_3v_4$. The edge w_1w_2 is friendly for $v_1v_2v_3w_1w_2v_1$ (because otherwise, it would contain both friendly and unfriendly edges). Thus w_1w_2 is in at least 6 other C_5 's together with either v_3w_1 or w_2v_1 . In the same way as before, we can replace this two-edge path with a three-edge path to obtain a seven-cycle. Repeating this procedure we can obtain an eight-cycle and then a nine-cycle, a contradiction. Indeed, at each step, we are given a cycle C' of length between 5 and 8, and we add two new vertices to it in place of one of its vertices by replacing a two-edge path P with a three-edge path to increase the length of C' . We have to make sure that the two new vertices are disjoint from the other vertices of C' . Since there are 6 C_5 's containing P which are vertex-disjoint outside P , it is easy to find a C_5 that avoids the at most 5 vertices of C' outside P .

Hence every edge is unfriendly to every C_5 in G' . Then we claim that there is no C_6 in G' . Indeed, otherwise we consider an arbitrary edge uv of that C_6 , there is a set \mathcal{S}' of at least 17 C_5 's that each contain uv . Because of the unfriendliness of uv to each of the cycles in \mathcal{S}' , they do not share any other vertices except u and v , so at least one of them is disjoint from the other vertices of the C_6 , thus we can exchange e to a 4-edge-path in the C_6 , obtaining a C_9 , a contradiction.

We obtained that after deleting $O(n^{3/2})$ edges, the resulting graph G' is $\{C_3, C_4, C_6\}$ -free, thus it contains at most $O(n^{11/12})$ C_5 's by Theorem 17. □

Remarks about $ex(n, C_l, \mathcal{C}_A)$ for a given set A of cycle lengths

After the investigation carried out in this article it is natural to ask to determine $ex(n, C_l, \mathcal{C}_A)$ for any set A .

Let us note that the behavior of $ex(n, C_l, \mathcal{C}_A)$ is more complicated if l is not 4 or 6. A simple construction of a \mathcal{C}_A -free graph G is the following. Let $2r$ be the shortest length of an even cycle which is allowed (note that if no even cycle is allowed, then the total number of cycles is $O(n)$ by a theorem in [16]). Let $p = \lfloor l/r \rfloor$. If r divides l , then the theta- (n, C_p, r) graph contains $\Omega(n^p)$ copies of C_l , some C_{2r} 's and no other cycles. If r does not divide l , it is easy to see that we can add a path with $l - pr$ new vertices between the two end vertices of a theta- $(n - (l - pr), P_{p+1}, r)$ graph to obtain a graph with $\Omega(n^p)$ many C_l 's, some C_{2r} 's and no other cycles.

Observe that in these cases, we still have an integer in the exponent. However, Theorem 9 (by Solymosi and Wong) shows that if $l \geq 4$ is even, then $ex(n, C_{2l}, \mathcal{C}_6) = \Theta(n^{l/3})$ since it is known that Erdős's Girth Conjecture holds for $m = 3$. This shows an example where the exponent is not an integer.

The situation is even more complicated when l is odd. Let us examine the simplest case $l = 5$, i.e. $ex(n, C_5, \mathcal{C}_A)$. If A contains only one element, Gishboliner and Shapira [20] determined the order of magnitude (it is 0 or n^2 or $n^{5/2}$). If there are at least two elements in A but $4 \notin A$, then the construction described above gives $ex(n, C_5, \mathcal{C}_A) = \Omega(n^2)$, while the result of Gishboliner and Shapira [20] implies $ex(n, C_5, \mathcal{C}_A) = O(n^2)$. If $A = \{C_3, C_4\}$, then Lemma 26 shows $ex(n, C_5, \{C_3, C_4\}) = \Theta(ex(n, C_5, C_4))$, which is $\Theta(n^{5/2})$ by Theorem 4. What remains is the case A contains 4 and another number. In this case Theorem 17 and Theorem 18 give some bounds that are not sharp.

Acknowledgements

We are grateful to an anonymous referee for carefully reading our paper and for their helpful remarks.

Research of Gerbner was supported by the János Bolyai Research Fellowship of the Hungarian Academy of Sciences and by the National Research, Development and Innovation Office – NKFIH, grant K 116769.

Research of Győri and Methuku was supported by the National Research, Development and Innovation Office – NKFIH, grant K 116769 and SNN 117879.

Research of Vizer was supported by the National Research, Development and Innovation Office – NKFIH, grant SNN 116095 and K 116769.

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