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Nonlinear Gamow vectors in nonlocal optical propagation

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Summary. — Shock waves dominate in a wide variety of fields in physics dealing with nonlinear phenomena, nevertheless the description of their evolution is not resolved for the entire dynamics. Here we propose an analytical method based on Gamow vectors, which belong to irreversible quantum mechanics. We theoretically and experimentally show the appearance of these decaying states during shock evolution allowing to describe the whole wave propagation. These results open new ways to the control of extreme nonlinear regimes such as supercontinuum generation or in the analogies of fundamental physical theories.

1. – Introduction

Dispersive shock waves emerge in a wide variety of fields in physics as fluidodynamics, astrophysics and Bose-Einstein condensation [1]. The reason why they are ubiquitous in nature is that they are singular solutions of the hyperbolic partial differential equations, a class of equations which describes a wide variety of wave-like physical phenomena ranging from water waves to plasmas and optics. A singular solution is a function which develops a discontinuity in its derivative as a consequence of an abrupt jump in some of the quantities involved in the wave propagation [2]. The point at which such a singularity arises is called shock point, and can be mathematically predicted together with its speed [3]. To do this, techniques like the Whitham approach are often used, but these are usually limited to integrable systems [4]. Hydrodynamic models support singular solutions and allow the prediction of the position and velocity of the shock occurrence. Studies have been reported on the variation of the position of the shock point as a function of the

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beam input power [3]. Unfortunately the hydrodynamic regime is only valid before the shock point and does not allow a complete description of the phenomenon.

The theoretical method here proposed to solve the complete dynamics of shock waves has been recently reported on [5], and its experimental proofs can be found in [6].

In optics the equation used to describe the nonlinear regime is the nonlinear Schrödinger equation (NLS). Let us start with an optical beam with amplitude A and wavelength λ propagating in a medium with refractive index n_0, and linear loss length L_{loss}. Its paraxial propagation equation reads as

$$2ik\frac{\partial A}{\partial Z} + \frac{\partial^2 A}{\partial X^2} + 2k^2\frac{\Delta n[|A|^2](X)}{n_0}A = -i\frac{k}{L_{loss}}A, \tag{1}$$

where Z is the propagation direction and X is the polarization direction. A is normalized such that the intensity is $I = |A|^2$ and $P_{MKS} = \int I\mathrm{d}X$ is the power, $k = 2\pi n_0/\lambda$ is the wavenumber. Δn is the nonlocal nonlinear perturbation to the refractive index defined as

$$\Delta n[I](X) = n_2\int G(X-X')I(X')\mathrm{d}X'. \tag{2}$$

where $G(X) = \exp(-|X|/L_{nloc})/2L_{nloc}$ is the kernel function, normalized such that $\int G\mathrm{d}X = 1$.

Letting W_0 be the beam waist and Z_d the diffraction length, we write eq. (1) in terms of the normalized variables $x = X/W_0$ and $z = Z/Z_d$ with $Z_d = kW_0^2$

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - PK(x) * |\psi(x)|^2\psi = -i\frac{\alpha}{2}A, \tag{3}$$

where $\alpha = Z_d/L_{loss}$ and $\psi = AW_0/\sqrt{P_{MKS}}$, where $\langle\psi|\psi\rangle = 1$ denotes the Hilbert space scalar product. P is defined as P_{MKS}/P_{REF}, with $P_{REF} = \lambda^2/4\pi^2 n_0|n_2|$. n_2 is the nonlinear refractive coefficient and $K(x) = W_0G(xW_0)$ is the nonlocality function.

Starting from eq. (3), if the perturbative refractive index is much larger than the beam dimension, *i.e.*, $L_{nloc} \gg W_0$, the highly nonlocal approximation (HNA) holds true. Hence, the convolution product $K(x) * |\psi(x)|^2$ can be written as a function $\kappa(x)$ and the NLS equation becomes a linear equation which has the same form of the linear Schrödinger equation of quantum mechanics

$$i\partial_z\psi = \hat{H}\psi, \qquad \text{with} \qquad \hat{H} = \frac{1}{2}\hat{p}^2 + V(x), \tag{4}$$

being $\hat{p} = -i\partial_x$ and $V(x) = P\kappa(x)$. Expanding the latter at the intensity maximum value

$$\kappa(x) \simeq \kappa_0^2 - \frac{1}{2}\kappa_2^2 x^2, \tag{5}$$

with $\kappa_0^2 = 1/2\sigma$ and $\kappa_2^2 = 1/\sqrt{\pi}\sigma$ for the exponential nonlocality, we obtain

$$\hat{H} = P\kappa_0^2 + \hat{H}_{rho}. \tag{6}$$

Writing $\psi = e^{i\kappa_0^2 Pz}\phi$, eq. (4) becomes

$$i\partial_z \phi = \hat{H}_{rho}\phi. \tag{7}$$

$\hat{H}_{rho}$ is the Hamiltonian of a reversed harmonic oscillator (RHO), which corresponds to a system in proximity to a maximum in its potential energy V

$$\hat{H}_{rho} = \frac{\hat{p}^2}{2} - \frac{\gamma^2 x^2}{2} \qquad \text{with} \qquad \gamma^2 = P\kappa_2^2, \tag{8}$$

where γ is the decay rate of the corresponding classical system. This system was first introduced by Glauber [7], and then revised by Chruscinski and others [8-10] to formulate the theory of irreversible quantum mechanics. Its eigenfunctions can be deduced as an extension in the complex plane ($x \to \sqrt{\mp i}x$) of the ones of the harmonic oscillator

$$\mathfrak{f}_n^{\pm}(x) = e^{\pm i\pi/8}\left(\frac{\sqrt{\pm i\gamma}}{2^n n!\sqrt{\pi}}\right)^{1/2} e^{\mp i\frac{\gamma}{2}x^2} H_n(\sqrt{\pm i\gamma}x), \tag{9}$$

where $H_n(x)$ are the n-th order Hermite polynomials. Their eigenvalues are purely imaginary numbers $E_n^{\pm} = \pm i\gamma(n+1/2)$.

It is worthwhile to notice that $\mathfrak{f}_n^{\pm}$ are unnormalizable functions belonging to a rigged Hilbert space (RHS) $\mathcal{H}^{\times}$, an enlarged Hilbert space $\mathcal{H}$ which includes unnormalizable functions. These are the so called Gamow vectors (Gvs), introduced by Gamow in 1928 to describe the exponential decays in particle physics [11]. For each n, two Gvs exist: for $z > 0$, $\mathfrak{f}_n^-$ decreases exponentially, while $\mathfrak{f}_n^+$ increases. In fig. 1 we report $|\mathfrak{f}_n^-|^2$ and their phase derivative (tilt), $\partial_x \arg(\mathfrak{f}_n^-)$, for even n.

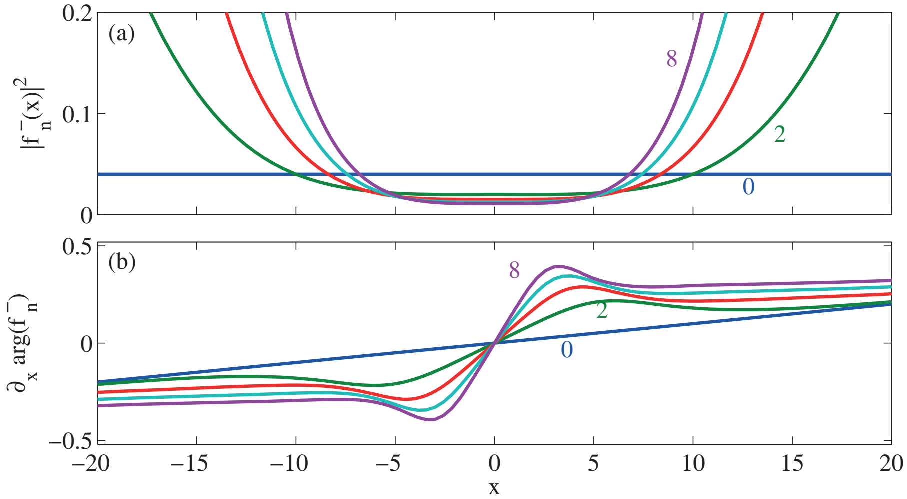

Fig. 1. – (Color online) (a) Square modulus $|\mathfrak{f}_n^-|^2$ of Gamow vectors for different $n = 0, 2, 4, 6, 8$; (b) Tilt, $\partial_x \arg(\mathfrak{f}_n^-)$, for values of n as in (a). (See [5]).

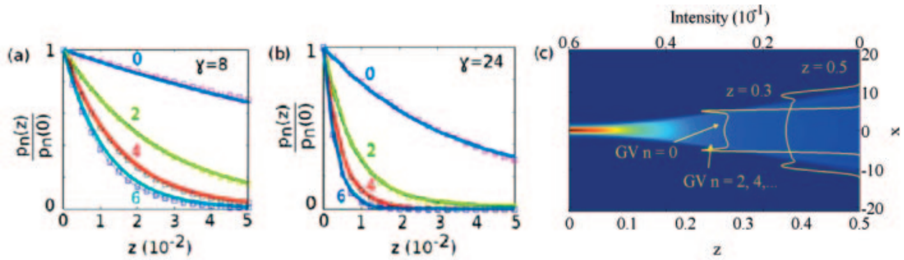

Fig. 2. – (a) Projection of the simulated solution (eq. (3)) on Gamow states for $n = 0, 2, 4, 6$ with $\alpha = 0.3$ and $\gamma = 8$, continuous lines are from eq. (3), dots are from eq. (11); (b) the same as in (a) with $\gamma = 24$; (c) numerical solution of eq. (3) with $P = 10^2$ and $\sigma^2 = 10$. (See [5]).

Since Gvs form a basis in $\mathcal{H}^\times$ for normalizable wavepackets, ϕ can be written as

$$\phi(x) = \sum_{n=0}^{N} \sqrt{\Gamma_n} \mathfrak{f}_n^- \langle \mathfrak{f}_n^+ | \psi(x,0)\rangle, \tag{10}$$

where $\psi(x, 0)$ is the physical state at $z = 0$ and $\Gamma_n = \gamma(2n + 1)$ are the quantized decay rates of Gvs, which are one of the characteristic signatures of their presence.

2. – Numerical simulations

In order to validate our theoretical analysis, we show that projecting eq. (10) over the state $\sqrt{\Gamma_n}\mathfrak{f}_n^-$ one can compute the probability $p_n(z)$ of finding the system in a Gamow state. The n-th order Gv probability $p_n(z)$ can be written as

$$p_n(z) = \Gamma_n |\langle \mathfrak{f}_n^+ | \psi(x,0)\rangle|^2 e^{-\Gamma_n z}. \tag{11}$$

for $z > 0$. The initial profile determines which Gv is excited. Hence, for a Gaussian beam

$$\psi(x,0) = \frac{e^{-x^2/2}}{\sqrt[4]{\pi}}, \tag{12}$$

all the odd terms in eq. (11) vanish due to the x-parity.

It is worthwhile to say that the decays of Gvs are not due to the coupling of the physical state with the environment (*i.e.*, extrinsic), but belong to a time-reversible Hamiltonian (intrinsic origin).

To exhibit the presence of Gvs in an optical beam evolving with eq. (3), we solve the latter equation numerically for a Gaussian beam profile (see eq. (12)). At low power we do not find any exponential decay, while, at high power, shock occurs and the intensity during propagation clearly reveals exponential decays (figs. 2(a), (b)). Comparing fig. 1(a) with fig. 2(c) we observe that the beam shape resembles the excitation of the Gamow states: the ground state appears in the central plateau while higher-order states cause the lateral peaks.

In order to provide quantitative evidence of $\mathfrak{f}_n^-$ in the shock wave, we project the numerical solution $\psi(x, z)$ of eq. (3) over $\mathfrak{f}_n^+$ and compute the n weights $p_n(z)$ in eq. (11).

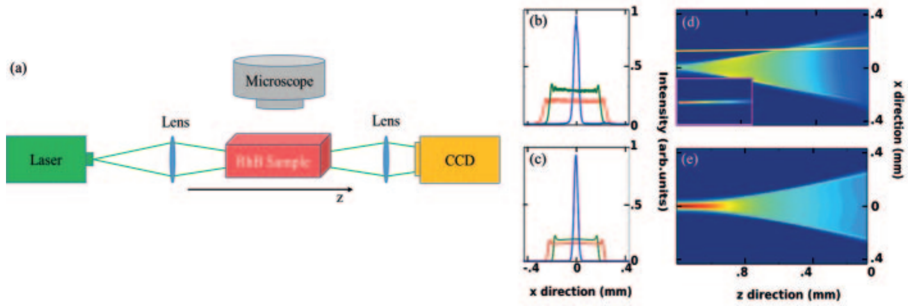

Fig. 3. – (a) Experimental apparatus to collect top-fluorescence and transmitted images of shock waves excited focusing a CW laser beam at 532 nm wavelength in a solution of Rhodamine B and water; (b) experimental intensity-profile section at different Z; (c) numerical intensity-profile section at different Z; (d) experimental top-fluorescence image of the laser beam propagating in RhB solution at $P = 380\,\mathrm{mW}$; (e) numerical solution in the condition of panel (d). (See [6]).

In fig. 2(a), (b) we report the exponential decays with quantized rates $\Gamma_n = \gamma(2n+1)$ of $p_n(z)$. This behavior has been tested for different values of γ^2 and initial conditions, and confirms the theoretical predictions. We remark that the evidence of the quantization of decay rates is the most direct signature of the presence of Gvs.

3. – Experimental results

A complete report on the experimental results can be found in [6]. The experimental setup used is shown in fig. 3(a). A shock wave is excited, letting a laser beam at 532 nm wavelength propagate in a solution 0.1 mM of Rhodamine B (RhB) and water. RhB is a dye with an high nonlinear index of refraction $|n_2| = 2 \times 10^{-12}\,\mathrm{m}^2/\mathrm{W}$. In order to access both the intensity profiles along the propagation direction Z and the angular spreading of the transmitted intensity from the exit face of the holder cell, we realized an experimental setup into two different configurations: the propagating profiles was visualized by collecting the top-fluorescence emission by a microscope placed above the sample top surface, while the transverse section of the beam profile was monitored with a Charged coupled device (CCD) camera.

Figures 3(d), (e) respectively show the top-fluorescence profiles along the propagation direction and the numerical simulation of the beam propagating according to eq. (3). As said before in the numerical analysis, at the high power when the shock is excited, the beam exhibits the characteristic double-peaked profile (see figs. 3(b), (c)) and presents fast and power-dependent decays along the propagation Z direction. At low power, the beam propagation is not affected by strong divergence, but it is dominated by diffraction, as is shown in the inset of fig. 3(d).

Figures 3(b), (c) report three experimental and numerical transverse sections of the intensity profile at different Z ($Z = 0.2$, 0.6 and 0.9 mm).

To observe the occurrence of the characteristic quantized decay rates, we analyzed a slice of the beam intensity profile along the propagation z-direction (see yellow line in fig. 3(d)) for different beam input powers. These results are shown in fig. 4. Figures 4(a), (b) show clearly the exponential decays. Quantization is observed in the data analysis which unveils the quantized-spectrum scaling $\Gamma_2/\Gamma_0 = 5$ at all the powers range explored. This gives an experimental evidence of the excitation of the ground state

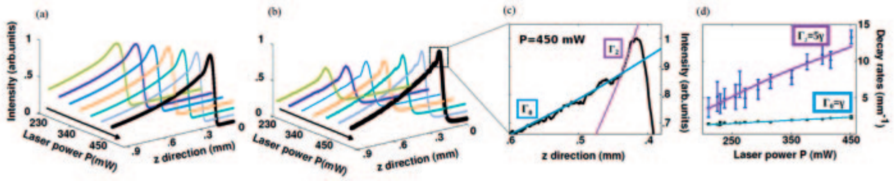

Fig. 4. – (a) Numerical intensity decay at different laser powers; (b) experimental decays obtained by slicing along the propagation Z direction the top-view intensity distribution (see the yellow line in fig. 3(e)); (c) enlargement of the experimental curve at $P = 450\,\mathrm{mW}$. The double exponential decay exhibited unveil the existence of two Gamow states, the fundamental one with $n = 0$ (slow decay) and the excited one with $n = 2$ (fast decay); (d) decay rates *vs.* power for the fundamental state Γ_0 (filled circles), and the excited state Γ_2 (triangles). (See [6]).

(decay rate Γ_0) and the second excited Gamow state (decay rate $\Gamma_2 = 5\Gamma_0$). As expected from the theory there is no evidence of the presence of the first excited Gamow state ($n = 1$), which has been predicted to be null due to the x-parity symmetry of the system. In fig. 4(d), we show that the observed decay rates exhibit the expected square-root on power behavior, highlighting the prevalence of the nonlinear nature of the system on linear losses due to absorption or scattering.

4. – Conclusions

We proposed a method based on nonlinear Gamow vectors in order to describe shock waves dynamics. We theoretically showed that these states are present in nonlinear waves and proved numerically their occurrence in nonlinear nonlocal laser-beam propagation. We reported on their main characteristic of having quantized decay rates and observed these features in the experiments in which a laser beam propagates in a thermal liquid. These decay rates depend on power, as expected from the intrinsic nonlinear nature of the system. This study sheds light on the possibility of controlling of extreme nonlinear regimes (supercontinuum generation) and opens ways to analogies with fundamental physical theories.

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