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MOVING GRID METHOD WITHOUT INTERPOLATIONS

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A B S T R A C T

In their method, to solve a one-dimensional moving boundary problem, Crank and Gupta suggest a grid system which moves with the Interface. The method requires some interpolations to be carried out which they perform by using a cubic spline or an ordinary polynomial. In the present paper these interpolations are avoided by employing a Taylor's expansion in space and time dimensions. A practical diffusion problem is solved and the results are compared with those obtained from other methods.

MOVING GRID METHOD WITHOUT INTERPOLATIONS

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1. Introduction

As there is no exact analytical solution available for a general moving boundary problem, various methods have been put forward from time to time. Goodman [1] suggests what he calls an 'Integral Method' to get an approximate analytical solution, analagous to the momentum integral used in the field of boundary layer theory in fluid mechanics. Boley [2] gives a 'self-embedding' technique and arrives at two integro-differential equations to be solved usually by numerical methods. landau [3] fixes the moving boundary by making suitable change in the space variable. Chernous'ko [V] introduced an idea of 'isotherms' moving in space with respeot to time while Dix and Cizek [5] have studied this aspect more thoroughly.They transform the basic differential equation so as to make the space variable as a dependent variable and call this method an 'Isotherm Migration Method'. Purely numerical techniques have been given by several authors. Crank [6] suggested the use of Lagrange's formula to deal with unequal intervals near the interface which has been recently employed with some modifications in [7]. Ehrlich [8] uses the implicit scheme by expanding the dependent variable by Taylor's formula in space and time directions near the moving boundary. The details of the various methods to deal with the moving boundary problems in heat flow and diffusion have been given by Muehlbauer and Sunderland [9] and Bankoff [10].

It has to be noted that in all the numerical computations carried out in the foregoing methods the step sizes are kept constant throughout.

There are however a few methods which make use of variable grids. Douglas and Gallie [11] divide the whole region into a fixed number of intervals and this size of space mesh is kept fixed for all times but choose each time step such that the boundary moves one space mesh during that time. Murray and Landis [12] use a variable space mesh choosing a fixed time step* They keep the number of space intervals fixed in the region(s) at all times and thus the site of the space mesh either increases or decreases with time. A new technique has been given by Crank and Gupta [13] in which the whole grid is pushed along with the moving boundary, so that the unequal interval is transferred from the neighbourhood of the moving boundary to the fixed surface. It has been shown that this technique improves the degree of smoothness in the motion of the boundary, calculated by the usual method [7] which makes use of a unequal interval near the moving boundary.

In an earlier paper [13] the concept of a Moving Grid System was introduced and two methods were presented which required interpolations to be carried at each time step. We now adopt another approach baaed on the moving grid system but which avoids interpolations. A problem of biomechanics has been solved and the results have been compared with those obtained from the method of Murray and Land is [12] and with those of [7].

2. Comparative Grid Systems

Three different grid systems are shown in figures 1,2 and 3 for a general two-phase problem. The dotted line shows the position of the moving boundary or Interface which divides the two separate regions. In the fixed grid system fig.1 the use of Lagrange formula is made at the mesh point nearest to the moving boundary. We will call this method of computation in future discussions as Fixed Grid Lagrange (FGL) method. As soon as the moving boundary comes too close to the neighbouring mesh

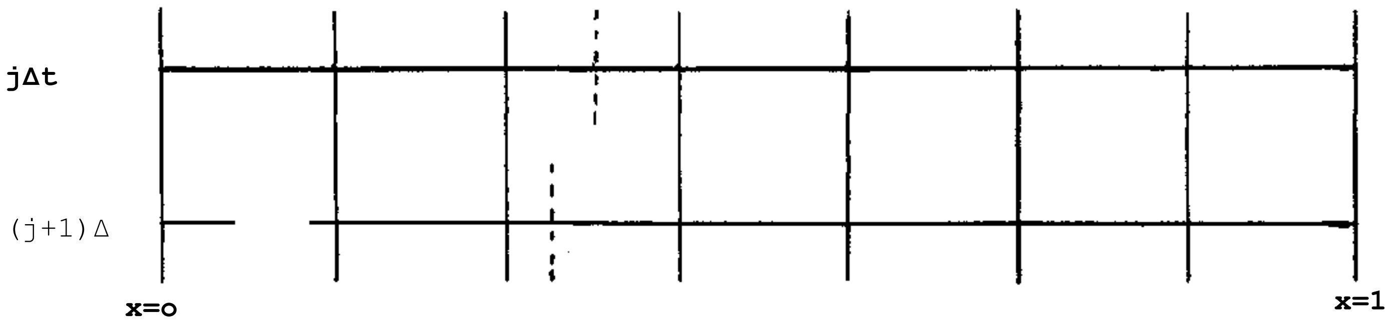

Fig.1 Fixed Grid System

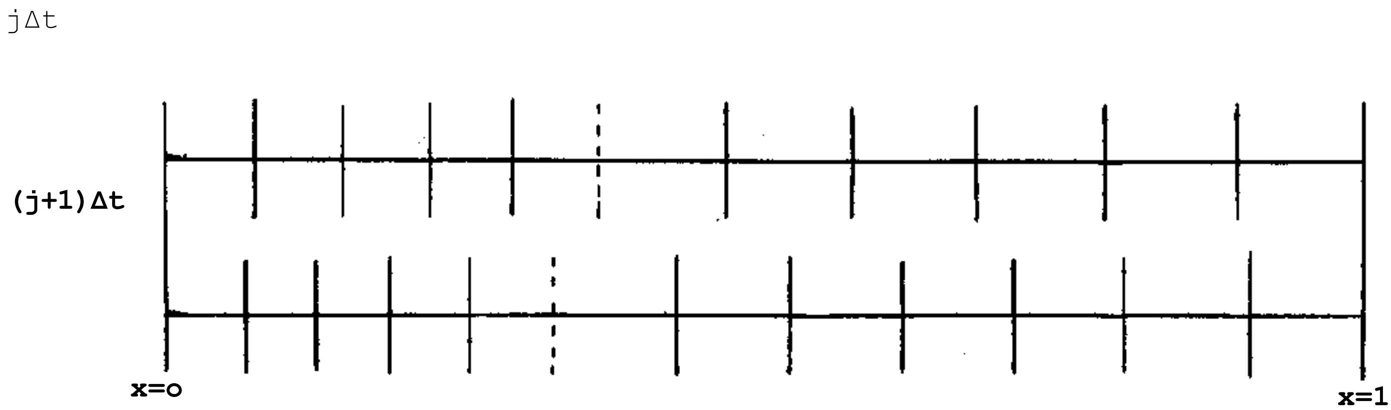

Fig.1 Murray and Landis System of Grid

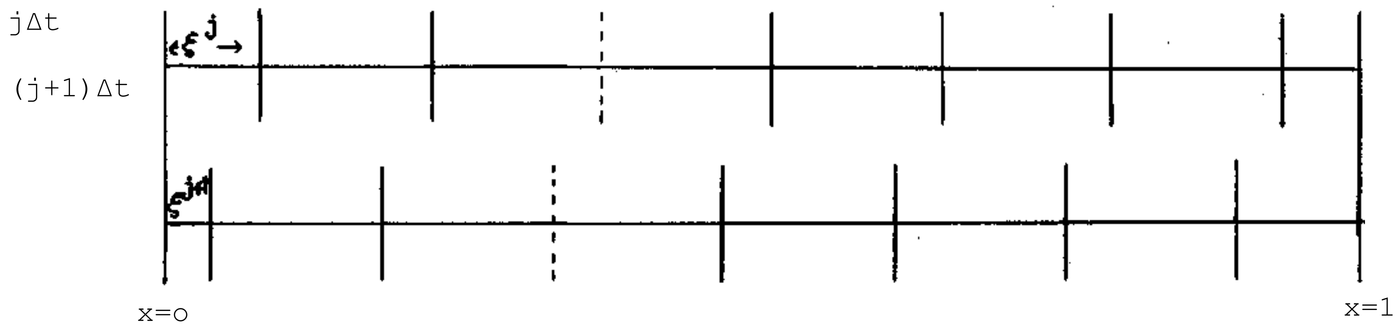

Fig.3 Moving Grid System

point the process is transferred to the next neighbouring point. At this change over a roughness in the boundary/time graph is observed which disappears in the method under discussion. In the method of Murray and Landis, henceforth called ML method, the space grid size decreases in one region and increases in the other (fig.2). In the system of moving grid the size of the space mesh remains fixed at all times except the intervals nearest to the fixed boundaries, (fig.3). The present method will be called Moving Grid or MG method. It will be seen that for the present problem we need to consider one region only.

3. Statement of the Problem

We shall introduce the new method by referring to a practical problem, arising from the diffusion of oxygen in absorbing tissue, which has been described in detail in [7]. Expressed in non- dimensional form we require the solution of the equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x} = 1, o \le x \le \delta(t) \ , \tag{3.1}$$

with the boundary conditions

$$\frac{\partial u}{\partial t} = o \quad , \quad x = o \quad , \quad t \ge o \quad , \tag{3.2}$$

$$u = \frac{\partial u}{\partial x} = o \ , \ x = \delta(t) \quad , \ t \ge o \quad , \tag{3.3}$$

and the initial condition

$$u = \tfrac{1}{2}(1-x)^2 \ , \ o \le x \le 1 \ , \ t = o \ , \tag{3.4.}$$

where S(t) denotes the position, of the moving boundary at time t.

4. Position of the Moving Boundary

By differentiating u with respect to t and using basic equation (3.1) it is easy to show that,

$$\frac{\partial^2 u}{\partial x^2} = 1 \ , \quad x = \delta(t). \tag{4.1}$$

Similarly it can also be shown that at the moving boundary x = δ

$$\frac{\partial^3 u}{\partial x^3} = -\dot{\delta} \quad , \quad \frac{\partial^4 u}{\partial x^4} = \dot{\delta}^2 \quad , \quad \frac{\partial^5 u}{\partial x^3} = -\ddot{\delta} - \dot{\delta}^3 \quad etc \ . \tag{4.2}$$

where $\dot{\delta}$ and $\ddot{\delta}$ denotes the first and second derivatives of 8 with respect to t. Further if h is the distance of a point where u = u_r from the moving boundary, in its neighbourhood, where we haveu $u = \frac{\partial u}{\partial x} = 0$ it is easy to show by using (4.1) and (4.2) in the Taylor's series for u_r that if the boundary is not moving too quickly then,

$$h \simeq (2ur)^{\frac{1}{2}}. \tag{4.3}$$

5. Description of Method

We subdivide the whole region o ≤ x ≤ 1 into n intervals each of width Ax such that $x_i = i\Delta x$; i = , 1 ,..., n (nΔx = 1) at t = o. As the boundary moves a distance at the next time step At the whole grid system is moved a distance ∊ towards the fixed surface x = o. The size of the first interval will then reduce to Δx - ∊ = ξ^1 (say) and the new mesh points at t = Δt will be x_1 - ∊, x_2 - ∊, ... etc. In general let us suppose that the position of the i^{th} mesh point at

$t = j\Delta t$ is denoted by x^j_i then following relations hold,

$$x^j_i = \xi^J + (i-1)\ \Delta x\ ,\ i = 1,\ 2,\ ..\ , \qquad (5.1)$$

$$x^{j+1} - x^j_i - \in^{j+1}\ , \qquad (5.2)$$

$$\xi^{j+1} = \xi^j - \in^{j+1} \qquad (5.3)$$

where $\in^{j+1}$ is the distance traversed by the moving point from time $j\Delta t$ to $(j+1)\ \Delta t$, If h^{j+1} denotes the value of h at the $(j+1)^{th}$ time step then,

$$\in^{j+1} = \Delta x - h^{j+1}. \qquad (5.4)$$

Let us now write the Taylor's series for u in two variables (space and time) as follows.

$$u(x - \in, t + \Delta t) = u(x,t) + \left\{\Delta t \frac{\partial u}{\partial t} - \in \frac{\partial u}{\partial t}\right\}$$

$$+ \frac{1}{2}\left\{\Delta t^2 \frac{\partial^2 u}{\partial t^2} - 2 \in \Delta t \frac{\partial^2 u}{\partial x \partial t} + \in^2 \frac{\partial^2 u}{\partial x^2}\right\} + \ldots\ldots$$

where $\in$ and At are as defined before.

putting $\frac{\partial u}{\partial t}$ *and* $\frac{\partial^2 u}{\partial x \partial t}$ from (3.1) the above formula may be written as given below,

$$u(x - \in, t + \Delta t) = u(x,t) - \in \frac{\partial u}{\partial x} + \Delta t\left(\frac{\partial^2 u}{\partial x^2} - 1\right) + \frac{\in^2}{2}\frac{\partial^2 u}{\partial x^2} - \in \Delta t \frac{\partial^3 u}{\partial x^3} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} +$$

(5.5)

Again neglecting $\frac{\partial^3 u}{\partial x^3}$ *and* $\frac{\partial^2 u}{\partial t^2}$ we werite (5.5) in the following truncated form,

$$u(x-\in,\ t+\Delta t) = u(x,t) - \in \frac{\partial u}{\partial x} + \Delta t\left(\frac{\partial^2 u}{\partial x^2} - 1\right) + \frac{\in^2}{2}\frac{\partial^2 u}{\partial x^2} \quad . \tag{5.6}$$

Formula (5.6) may give the value of u at the next time step of Δt at a grid point which has moved a distance ∈ during that time.

By comparison the governing differential equation in the ML method is written as follows,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} \tag{5.7}$$

where

$$\frac{dx}{dt} = \frac{x}{\delta}\dot{\delta} \quad . \tag{5.8}$$

Equation (5.7) may be rewritten using (5.8) and the basic equation (3.1) to give,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}, \frac{x}{\delta}\dot{\delta} + \left(\frac{\partial^2 u}{\partial x^2} - 1\right) \quad , \tag{5.9}$$

where δ(t) is the position of the moving boundary at time t.

In order to compare results obtained from the MG method and with those from the ML method we ignore the last term in (5.6) and write it in the following finite-difference form,

$$u_{i,j+1} = u_{i,j} - \in^{j+1} u_{i,j} \Delta t \ (u_{i,j}^{-1}) . \tag{5.10}$$

where dashes show the order of differentiation with respect to x. It should be remembered that $u_{i,j}$ is different in the present paper than used in the ordinary sense as x_i is not fixed and varies with time,

The values of u_i and u_i, at anr time, may be written using the usual finite difference formulae,

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2\Delta x} \tag{5.11}$$

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} . \tag{5.12}$$

i = *2, 3,* ...

At the grid point $x = x_1$ Lagrange type formulae have to be used allowing for the unequal interval ξ nearest to the surface $x = o$. The respective equations would be,

$$u_1 = \frac{\xi u_2}{\Delta x(\xi + \Delta x)} + \frac{(\Delta x - \xi)}{\xi \Delta x} - \frac{\Delta x u_o}{(\xi + \Delta x)\xi} \text{ and} \tag{5.13}$$

$$u_1' = 2\left\{ \frac{u_2}{\Delta x \ (\xi + \Delta x)} - \frac{u_1}{\xi \Delta x} + \frac{u_o}{\xi(\xi + \Delta x)} \right\} . \tag{5.14}$$

Let us assume that the function values $u_{i,j}$, i=o, 1, ... r, (r+1) are known at time $j\Delta t$ when the distance of the moving boundary from the surface x=o is $\xi^J + r\Delta x$. The value of $u_{r,j+1}$ is found using an explicit finite difference formula from the basic equation (3.1). Once u_r is known the value of h is known from (4.3) and hence ϵ^{j+1} is known from (5.4). The values of $u_{i,j+1}$, 1 = 1 , 2 , ... r are calculated from (5.10) using (5.11) through (5.14).

The relations similar to (5.10), (5.11) and (5.12) maybe written for the ML method and the values of u may be computed in exactly the same manner as described above.

The essential difference between the ML and the MG method is that the movement of each grid point in the present method is the same as that of the moving boundary while in the ML method it is proportional to its distance from the fixed surface x=o. The width of the space meshes except the one nearest to the surface remains constant in the MG method whereas it goes on decreasing in the ML method.

6. Results and Discussion

We notice that there is a discontinuity in the surface-gradient at t=o. Because of this the numerical methods based on finite differences are liable to give inaccurate solutions in the neighbourhood of the surface for short times. In an earlier paper [7], however, an analytical solution satisfactory for small times was obtained which is given by

$$u(x,t)=\frac{1}{2}(1-x)^2-2\sqrt{\left(\frac{t}{x}\right)}\exp\left\{-\left(\frac{x}{2\sqrt{t}}\right)^2\right\}+\text{xerfo}\left(\frac{x}{2\sqrt{t}}\right) \tag{6.1}$$

$o \le x \le 1$ and t small.

We start the present solutions from the values taken from (6.1) at t=0. 025 when the boundary $\delta = 1$, has not moved to an accuracy of six significant figures based on the FGL method.

The positions of the moving boundary and the surface values of u are computed in Tables I and II respectively for the following methods:

(i) Forward Difference Lagrange (FGL) method,

(ii) Moving Grid (MG) method and

(iii) Murray and Landis (ML) method.

The agreement between various results seem to be very good. The results obtained in Table I from the MG method are very much nearer to the results obtained from the FGL method than those obtained from the ML method. In Table II all the results are almost identical.

Table III gives the positions of the moving boundary at and around the times when the process for calculating u, in the neighbourhood of the moving point, in the FGL method, is transferred one space interval towards the surface x=o. The corresponding figures are given for the MG method. The irregularities produced in the former method are clearly visible, whereas their counterparts show a smooth behaviour throughout.

Table IV gives the surface values of u computed from the present method at and around such times when the first space interval ξ is increased to $\xi+\Delta x$ for succeeding computations. It is seen that the differences in the values of u show no sign of irregularities. The comparative figures for the FGL method are also given.

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TABLE I

Comparison of 10^4 δ at different times. The numerical solutions start from the analytical solution at t = 0.025.

Time / Method	0.040	0.060	0.100	0.120	0.140	0.160	0.180	0.185
FGL Ax=0.10	9988	9905	9312	8747	7912	6756	4849	4014
MG Δx=0.10	9988	9903	9301	8719	7885	6682	4766	4048
ML	9988	9904	9309	8740	7930	6776	4974	4308

NOTE: The ML method has been started from Δx = 0,10.

TABLE II

Comparison of 10^4u at the surface at different times.

All solutions start from analytical solution at t = 0.025

TIME / Method	0.040	0.060	0.100	0.120	0.140	0.160	0.180	0.185
FGL Ax=0.10	2745	2238	1434	1094	781	490	220	156
MG Δx=0. 10	2745	2238	1434	1093	780	490	219	155
ML	2745	2238	1434	1093	780	489	218	154

NOTE: The ML method has been started from Δx = 0.10.

TABLE III

Table showing the irregularities in the position of the moving boundary, calculated by the PGL method. Comparatively smooth figures are shown for the MG method ($\Delta x = 0.10$)

Time	FGL Method			MG Method		
	$10^4\delta$	$-\Delta$	$-\Delta^2$	$10^4\,\delta$	$-\Delta$	$-\Delta^z$
0.110	9099	29	1	9094	28	1
	9070	30	0	9066	29	0
	9040	30	-4	9037	29	2
	9010	26		9008	30	
	8984			8978		
0,137	8141	52	3	8122	46	0
	8089	55	-15	8076	46	2
	8034	40	0	8030	48	0
	7994	40		7982	48	
	7954			7934		
0.154	7277	73	7	7220	63	0
	7204	80	7	7157	63	3
	7124	87	-35	7094	66	0
	7037	52		7028	66	
	6985			6962		
0.167	6396	90	13	6294	83	2
	6306	103	55	6211	85	3
	6203	158	-92	6126	88	1
	6045	66		6038	89	
	5979			5949		
0.176	5499	106	19	5461	106	4
	5393	125	123	5355	110	3
	5268	248	-165	5245	113	5
	5020	83		5132	118	
	4937			5014		
0.184	4652	114	18	4498	143	7
	4538	132	260	4355	150	7
	4406	*392*	-290	4205	157	8
	4014	102		4048	165	
	3912			3883		

NOTE: The data are tabulated at an interval of time $\Delta t = 0.001$, The underlined values correspond to the times, shown in column 1, when the interpolation process near the moving boundary in FGL method is transferred one step towards the fixed surface $x = 0$.

TABLE IV

Table showing the smoothness of the function values at the fixed surfaoe calculated by the MG method. Comparative figures are shown for the FGL method (Δx = 0.10)

Time	FGL Method		MG Method	
	10^4u	- A	10^4u	- A
.093	1598	18	1598	
	1580	19	1580	18
	1561	18	1561	19
	1543	19	1543	18
	1524		1525	18
.127	1013	16	1012	
	997	16	996	16
	981	16	981	15
	965	15	965	16
	950		949	16
.148	691	14	691	
	677	15	676	15
	662	15	661	15
	647	14	647	14
	633		632	15
	476	14	476	
	462	14	462	14
.163	448	13	448	14
	435	14	434	14
	421		420	14
	326	14	325	
	312	13	312	13
.174	299	13	298	14
	286	14	285	13
	272		272	13
	220	13	219	
	207	14	206	13
182	194.	13	194	12
	181	13	181	13
	168		168	13
.188	143			
	130	13	142	
	117	13	130	12
	105	12	117	13
	92	12	105	12
			92	13

NOTE: The data are tabulated at an interval of time Δt = 0,001. The underlined values correspond to the times, shown in column 1 . when the first space interval, in MG method, is increased by Δx.

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