Constructing and embedding mutually orthogonal Latin squares: re-viewing both new and existing results.

Journal Item

Constructing and embedding mutually orthogonal Latin squares: reviewing both new and existing results

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July 23, 2020

Abstract

We review results for the embedding of orthogonal partial Latin squares in orthogonal Latin squares, comparing and contrasting these with results for embedding partial Latin squares in Latin squares. We also present a new construction that uses the existence of a set of t mutually orthogonal Latin squares of order n to construct a set of $2t$ mutually orthogonal Latin squares of order n^t.

1 Introduction

In combinatorial theory the seemingly straightforward question – “When is it possible to embed a partial combinatorial design in a complete design with related properties?” – has generated much research, including many challenging conjectures that have been answered to varying degrees. The Handbook of Combinatorial Designs [9] provides an excellent overview of this research.

In the current article we seek to collate the more recent research on the embedding of orthogonal partial Latin squares in orthogonal Latin squares (definitions provided below). The genesis of this research can be found in the study of embeddings for partial Latin squares, and so we begin with a brief overview of these earlier studies. This allows us to compare and contrast the impact of imposing the additional orthogonality condition on the size of the embedding.

In writing this review it is important to emphasize that there are a number of equivalent representations for a Latin square, and we will review results that arise in the associated algebraic and graph theory settings.

The different combinatorial representations for a partial Latin square will be discussed in Section 2. In Section 3 we give a brief overview of embedding results for partial Latin squares, extending these to orthogonal partial Latin squares in Section 4. Section 5 documents new results on the embedding of orthogonal partial Latin squares and a new construction for mutually orthogonal Latin squares. We show that the existence of a set of t mutually orthogonal Latin squares of order n can be used to verify the existence of a set of $2t$ mutually orthogonal Latin squares of order n^t. These results have not appeared in the earlier literature. We conclude the review article with open questions in Section 6.

2 Background and Definitions

In discrete mathematics, the study of combinatorial designs using different representations allows us to define and study the same discrete structure from different perspectives. These distinct representations provide valuable insights depending on the problem at hand. For instance, there are many equivalent representations for Latin squares. In this article there are four equivalent representations that feature strongly. On the set $N = \{0, 1, \ldots, n-1\}$, these representations are:

- A *Latin square* of order n, denoted LS(n), is an $n \times n$ array $L = [L(i,j)]$, where for all $i, j \in N$, $L(i,j) \in N$ is chosen in such a way that each element of N occurs once in every row and once in every column.

- A *quasigroup* of order n, denoted $(N, \circ)$, is defined by a binary operation $\circ$ closed on the set N and such that, for all $1 \leqslant i, j, i', j' \leqslant n$, if $i \circ j = i' \circ j$, then $i = i'$ and if $i \circ j = i \circ j'$, then $j = j'$.

- A *triangulation* of the complete tripartite graph $K_{n,n,n}$, where the triangles form a partition of the edge set of $K_{n,n,n}$.

- A *transversal design*, denoted TD$(3, n)$, comprises a set of $3n$ points partitioned into three n-subsets, called groups, and a set of n^2 triples such that each pair of points from different groups appears in precisely one triple and no triple contains more than one point from each group.

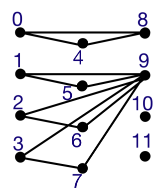
It will also be useful to use the ordered triple notation for an LS; that is, the LS $L = [L(i, j)]$ can be represented as a set of triples of the form $(i, j, L(i, j))$.

As stated, the focus here is on determining the "smallest complete structure" that contains a given partial structure. More specifically we begin with partial Latin squares:

- A *partial Latin square* of order n, denoted PLS(n), is an $n \times n$ array $P = [P(i, j)]$ with cells either empty or containing $P(i, j) \in N$ in such a way that each element of N occurs at most once in every row and at most once in every column. The *volume* of the partial Latin square is the number of filled cells.

Likewise we may define partial (incomplete) quasigroups, partial triangulations of $K_{n,n,n}$ and partial transversal designs.

Example 2.1. *Let $N = \{0, 1, 2, 3\}$. The following are equivalent partial systems.*

PLS(4), volume 4	*Partial $(N, \circ)$*	*Subgraph of $K_{4,4,4}$*	*PTD$(3,4)$*
$\begin{array}{\|c\|c\|c\|c\|}\hline 0 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array}$	$0 \circ 0 = 0$ $1 \circ 1 = 1$ $2 \circ 2 = 1$ $3 \circ 3 = 1$		$V = \bigcup_{i=0}^{2} G_i,\ i = 0, 1, 2,$ $G_i = \{4i + x : x = 0, \ldots, 3\},$ $B = \{\{0, 4, 8\}, \{1, 5, 9\},$ $\{2, 6, 9\}, \{3, 7, 9\}\}$

It is clear that the partial designs presented in Example 2.1 are not contained in complete designs of the same type and order, but what happens when we allow the order to be increased? This notion is formalized in the following definition.

- A PLS(n), $P = [P(i, j)]$, is said to *complete* to an LS(n), $L = [L(i, j)]$, if the empty cells of P can be filled with elements from N to obtain the LS(n) L. The PLS(m), $P = [P(i, j)]$, is said to be *embedded* in the LS(n), $L = [L(i, j)]$, $m < n$, if for all non-empty cells (i, j) of P, P agrees with L, that is, $P(i, j) = L(i, j)$.

Specifically, we are interested in embedding orthogonal PLS. Here we begin with the definition of orthogonal LS.

- Two LS(n), $A = [A(i, j)]$ and $B = [B(i, j)]$, are said to be *orthogonal*, if, for all $i, i', j, j' \in N$, $A(i, j) = A(i', j')$ implies $B(i, j) \neq B(i', j')$. A set of t LS(n) that are pairwise orthogonal are said to be *mutually orthogonal*. Such a collection of t mutually orthogonal LS(n), $A_1, \ldots, A_t$, will be denoted t-MOLS(n) and sometimes referred to as MOLS.

In this paper, it is assumed that $t > 1$. Now we may define orthogonal partial Latin squares.

- Two PLS(n), $P = [P(i,j)]$ and $Q = [Q(i,j)]$, are said to be *orthogonal*, if they have the same non-empty cells and, for all $i, i', j, j' \in N$, $P(i,j) = P(i',j')$ implies $Q(i,j) \neq Q(i',j')$. A set of t PLS(n) that are pairwise orthogonal are said to be *mutually orthogonal* and will be denoted t-MOPLS(n). Assume that the two MOLS(n) (A, B), agree in the MOPLS(m) (P, Q). If $m = n$, we say (P, Q) can be completed to (A, B), and if $n > m$ we say (P, Q) can be embedded in (A, B).

These definitions are illustrated in Example 2.2.

Example 2.2. *A pair of MOPLS(4), P and Q, and a pair of MOLS(5), A and B. The pair of MOPLS(4), (P, Q) can not be completed, but they can be embedded in the given pair of MOLS(5), (A, B).*

P

0	1	2	
1	2		0
	3		1
		1	3

Q

0	1	2	
2	3		1
	0		3
		0	2

A

0	1	2	4	3
1	2	3	0	4
2	3	4	1	0
4	0	1	3	2
3	4	0	2	1

B

0	1	2	4	3
2	3	4	1	0
4	0	1	3	2
3	4	0	2	1
1	2	3	0	4

Similarly, two quasigroups $(N, \circ)$ and $(N, *)$ are said to be orthogonal if the equations $x \circ y = z \circ w$ and $x * y = z * w$ together imply $x = z$ and $y = w$.

A *group divisible design* comprises a set of points $\mathcal{V}$ partitioned in groups $\mathcal{G}$ and a set of blocks $\mathcal{B}$ satisfying the property that each pair of points from different groups occurs in one block and no block contains more than one point from each group. The set $K = \{|B| : B \in \mathcal{B}\}$ gives the possible sizes of the blocks and if $K = \{k\}$ then the design is generally referred to as a k-GDD. A TD(k, n) is a k-GDD that contains k groups of n points. A TD$(k+2, n)$ is also equivalent to a collection of k-MOLS(n), $A_1 = [A_1(i,j)], \ldots, A_k = [A_k(i,j)]$, with the $k+2$ groups of the TD each associated with the set N and the set $\{(i, j, A_1(i,j), \ldots, A_k(i,j)) : 0 \leqslant i, j \leqslant n-1\}$ forming the set of n^2 blocks. Similarly, we have the equivalence between a PTD$(k+2, n)$ (partial transversal design) and k-MOPLS(n). MOLS can also be readily generalized to orthogonal arrays (see Section 3.6 in [9]). It is also worth noting that MOLS have also been studied as permutations and complete mappings, where each row (column) defines an orthomorphism on N; a representation first studied by Mann in 1942, see Section 6 of [9], [18] and [41].

Further the completion or embedding of a pair of MOPLS to a pair of MOLS, $L = [L(i,j)]$ and $M = [M(i,j)]$, results in a structure where all ordered pairs $(L(i,j), M(i,j))$ are distinct. However, one can also study the completion or embedding of a pair of MOPLS in complete structures $L = [L(i,j)]$ and $M = [M(i,j)]$, where the number of distinct ordered pairs $(L(i,j), M(i,j))$ is fixed, say r. In this context, it is said that, a pair of LS(n) $L = [L(i,j)]$ and $M = [M(i,j)]$ on N are *r-orthogonal* if $r = |\{(L(i,j), M(i,j)) : 0 \leqslant i, j \leqslant n-1\}|$. For further discussion see Subsection 3.8 of [9]. It is known that:

Theorem 2.3. *[10, 58] For n, a positive integer, a pair of r-orthogonal LS(n) exist if and only if $r \in \{n, n^2\}$ or $n+2 \leqslant r \leqslant n^2-2$, except when*

1. $n = 2$ *and* $r = 4$*;*

2. $n = 3$ *and* $r \in \{5, 6, 7\}$*;*

3. $n = 4$ *and* $r \in \{7, 10, 11, 13, 14\}$*;*

4. $n = 5$ *and* $r \in \{8, 9, 20, 22, 23\}$*;*

5. $n = 6$ *and* $r \in \{33, 36\}$.

Interestingly, in [5], Belyavskaya and Lumpov study these structures in terms of r-orthogonal quasigroups and document a product construction which is a generalization of the direct product construction (see Section 5). Belyavskaya and Lumpov give conditions under which this construction can be applied and employ the method for the construction of r-orthogonal quasigroups of composite order. They list two theorems, the first establishing the existence of quasigroups with r-orthogonal mates and the second establishing the existence of sets of mutually r-orthogonal quasigroups.

Theorem 2.4. *[5] If* $m, n \neq 2$*, then there exists an LS(mn) which has an* r*-orthogonal mate, for* $r = km^2 + (n - k + pt)m + (n^2 - n - t)p$ *with arbitrary* k, p, t *satisfying* $0 \leqslant k \leqslant n$*,* $0 \leqslant p \leqslant m$ *and* $0 \leqslant t \leqslant k(n - 1)$.

Theorem 2.5. *[5] If there exists a set of* $s + 1$ *mutually orthogonal quasigroups of order* n *and* s *mutually orthogonal quasigroups of order* m*, then there exists* s *mutually* r*-orthogonal quasigroups of order* mn*, for* $r = km^2 + (n-k)m + (n^2 - n - t)p + (2m - p)tp$ *with arbitrary* k, p, t *satisfying* $0 \leqslant k \leqslant n$*,* $0 \leqslant p \leqslant m$ *and* $0 \leqslant t \leqslant k(n - 1)$.

We will revisit r-orthogonal LS when we present a number of open questions in Section 6.

Determining which PLS are completable is a hard problem. Colbourn [8] (1984) has shown that the decision problem: *"Can a partial Latin square of order n be completed to a Latin square?"* is an NP-complete problem, even if there are no more than 3 empty cells in any row or column. Adding the additional orthogonality condition, that is, determining if MOPLS can be completed to MOLS does not diminish the complexity of this question. However, allowing the PLS to be embedded in LS of increased order does change the problem making it possible to apply a wider range of theoretical arguments. See for instance, Theorem 3.3 below where a linear order embedding of any PLS in an LS is established.

3 Completing and embedding PLS

In 1945, M. Hall [24] showed that every $(n - r) \times n$ array, satisfying the property that each element of the set N occurs once in every row and at most once in any column, could be extended to an LS(n).

Theorem 3.1. *[24] Given a rectangle of* r *rows and* n *columns such that each of the elements of* $\{0, 1, 2, \ldots, n-1\}$ *occurs once in every row and no element occurs more than once in any column, then there exist* $n - r$ *rows which may be added to the given rectangle to form an LS.*

M. Hall achieved this result by showing that the $(n-r) \times n$ array could be extended to an $(n-r+1) \times n$ array satisfying the same property. The elements to be added to each column are determined by a system of distinct representatives for the collection of subsets corresponding to the elements not appearing in the given columns. This process can then be repeated until an $n \times n$ array is obtained. This use of systems of distinct representatives can be traced back to the work of P. Hall [25] and earlier results by König, [33]. Later in 1951, Ryser [47] extended these arguments to show that under certain initial conditions it is always possible to complete an $r \times s$ Latin rectangle (i.e. a PLS where the filled cells define a complete $r \times s$ subarray) to an LS(n).

Theorem 3.2. *[47] Let T be an $r \times s$ Latin rectangle on the set N, and let $N(i)$ denote the number of times element i occurs in the cells of T. A necessary and sufficient condition for T to be extended to an LS(n) is that, for each $i = 1, \ldots, n$,*

$$N(i) \geqslant r + s - n.$$

In 1960 Evans, [19], was motivated by Ryser's work, and asked:

> *For each n, what is the minimum v such that there exists a PLS(n) of volume v which is* **not** *contained in any LS(n)?*

If we denote the minimum volume by $m_v(n)$, then the PLS(4) in Example 2.2 points to the conclusion that $m_v(n) \leqslant n$. However proving that the minimum $m_v(n)$ is n is non-trivial, with a number of papers appearing on this topic. Nosov, Sachkov and Tarakanov provide a brief review of these articles in [43], see also [2]. In 1970 Lindner, [37], solved the problem when the filled cells occur in less than $n/2$ rows and in 1981 Smetaniuk, see [48, 13], gave a construction for the case where the filled cells intersect more than $n/2$ rows. The intricacies of this question and its solution led Evans and others to the problem of establishing a finite embedding.

In this context Evans, [19], asked:

> *For each t, what is the minimum n such that any PLS(t) can be embedded in an LS(n)?*

Evans settled this question in [19], showing that:

Theorem 3.3. *[19] For each t, a PLS(t) P can be embedded in an LS(n) L, for any $n \geqslant 2t$.*

In proving this result Evans constructed a proxy LS(t), M, on a set N' disjoint from $\{0, 1, \ldots, t-1\}$ and used the corresponding elements in M to fill the empty cells of P to obtain a complete $t \times t$ array P^*. He then showed that the initial conditions for Ryser's Theorem (Theorem 3.2) were satisfied. Thus he verified that P^* may always be embedded in an LS(n), where $n \geqslant 2t$. Further, Evans proved that this embedding was the best possible.

In many articles the authors have highlighted the allied problem of embedding quasigroups that satisfy additional conditions. For instance, in [19] Evans specifically remarks that *"An incomplete loop containing n elements can be embedded in a loop containing $2n$ elements"*. A *loop* $(N, \circ)$ is a quasigroup where the addition algebraic identity $x \circ 0 = 0 \circ x = x$ is satisfied for all $x \in N$. Quasigroups that satisfy a specific collection of additional identities, are termed *varieties*. We will say that a partial quasigroup $(N, \circ)$ belongs to variety $\mathcal{V}$ if the given identities, associated with $\mathcal{V}$ are satisfied.

The embedding of partial quasigroups in the varieties defined by subsets of the set of identities $I = \{x^2 = x, x \circ y = y \circ x, (y \circ x) \circ x = y, x \circ (x \circ y) = y, x \circ (y \circ x) = y\}$ have been studied extensively, with embedding results summarized in the table compiled by Bennett and Lindner in Subsection 2.6 of [9], and reproduced below.

Variety of partial quasigroup of order t defined by $I =$	Best possible embedding of size n	Best embedding of size n, known to date
$\emptyset$	all $n \geqslant 2t$	all $n \geqslant 2t$ [19]
Commutative, $x \circ y = y \circ x$	all even $n \geqslant 2t$	all even $n \geqslant 2t$ [11]
$x \circ (x \circ y) = y$		
$(y \circ x) \circ x = y$		
Idempotent, $x^2 = x$	all $n \geqslant 2t + 1$	all $n \geqslant 2t + 1$ [3]
$x^2 = x$, $x \circ y = y \circ x$	all odd $n \geqslant 2t + 1$	all odd $n \geqslant 2t + 1$ [11]
$x^2 = x$, $x \circ (x \circ y) = y$		
$x^2 = x$, $(y \circ x) \circ x = y$		
Semisymmetric, $x \circ (y \circ x) = y$	all $n \geqslant 2t$	all $n \geqslant 6t$ s.t. $n \equiv 0, 3 (\text{mod } 6)$ [12]
Totally symmetric, $x \circ (x \circ y) = y$, $(y \circ x) \circ x = y$	all even $n \geqslant 2t + 4$	all even $n \geqslant 2t + 4$ [6]
Mendelsohn quasigroup, $x^2 = x$, $x \circ (y \circ x) = y$	all $n \geqslant 2t + 1$ s.t. $n \equiv 0, 1 (\text{mod } 3)$	all $n \geqslant 4t$ s.t. $n \equiv 0, 1 (\text{mod } 3)$ [46]
Steiner quasigroup, $x^2 = x$, $x \circ (x \circ y) = y$, $(y \circ x) \circ x = y$	all $n \geqslant 2t + 1$ s.t. $n \equiv 1, 3 (\text{mod } 6)$	all $n \geqslant 2t + 1$ s.t. $n \equiv 1, 3 (\text{mod } 6)$ [7]

In addition to the table given above, embeddings of other types of quasigroups are also studied. A loop L is said to be an inverse property loop (IP loop) if for all $x \in L$ there is a unique element x^{-1} of L such that $x^{-1}(xy) = y = (yx)x^{-1}$. Embeddings of IP-loops are discussed by Treash in [50] and, more recently by Vodička and Zlatoš in [52].

The early work by Evans [19], Treash [51], Lindner [36] and others has shed new light on the embedding of many combinatorial designs, including graph decompositions. For instance, in 1974 Lindner [40] observed that Evans' paper became a *"starting point for a fascinating collection of problems in the study of Latin squares"*. Further, Lindner exploited the connection between Latin squares and quasigroups to extend Evans' embedding result to Steiner quasigroups that are idempotent, commutative totally symmetric quasigroups as defined above. Steiner quasigroups are in one to one correspondence with Steiner triple systems (STS). An STS is a decomposition of the complete graph K_n into triangles. Lindner [38, 39] achieved this by representing a partial STS as a partial Steiner quasigroup, embedding this in a complete Steiner quasigroup which was then translated back to an STS. In this way the partial STS is finitely embedded in an STS. Other authors have extended this work to obtain embedding for Steiner quasigroups satisfying additional identities. A summary of this work can be found in Section 2.6 of [9].

These results have been further applied to finitely embed cycle systems where the length of the cycle is greater than 3. In this context a cycle system is a decomposition of the complete graph K_n into cycles of length k. A good starting point for interested readers is

[9], as well as Rodger's 1992 article [45]. Readers may also be interested in the recent work in [55] and [14].

But what about MOPLS? In 1960, Evans ([19]) raised the pivotal question:

> *Can a pair of MOPLS(t) be embedded in a pair of MOLS(n) and if so what is the smallest such n for each t?*

Evans suggests that the paper by Mann and Ryser [42] on system of distinct representatives, contained *"the ideas probably needed to attack this problem"*. Certainly, arguments using systems of distinct representatives have provided insights into this problem, see for instance [27], however the breadth of attack has been quite wide as we will see in the next section.

4 Completing and Embedding MOPLS

In Section 3 we documented results that show the minimum volume $m_v(n) = n$, that is, every PLS(n) of volume less than n is completable. In addition, the best possible embedding for any PLS(t) was documented in Theorem 3.3. In this section we extend these results and consider related questions for MOPLS:

> *For each n, what is the minimum $\mu_v(n)$ such that there exists a pair of MOPLS(n) of volume $\mu_v(n)$ which is* **not** *contained in any pair of MOLS(n)?*

Example 4.1. *The pair of MOPLS(4), P and Q, can not be completed to any pair of MOLS(4). Similarly the pair of MOPLS(5), R and S, can not be completed to any pair of MOLS(5).*

P

0	1		
1			
			3

Q

0	1		
2			
			3

R

0	1			
1	0			

S

0	1			
2	3			

The pair of MOPLS(4) P and Q given in Example 4.1 lead to the conclusion that $\mu_v(4) \leqslant 4$, but is $\mu_v(4) = 4$? For $n = 3$ it is easy to see $\mu_v(3) = 3$ and there are no MOLS(6) rendering the question of the size of $\mu_v(6)$ redundant. However, the pair of MOPLS(5) R and S in Example 4.1 indicate that $\mu_v(5) \leqslant 4 = n - 1$. This is easy to see as any LS(5) containing a 2×2 subsquare can not have an orthogonal mate. But what about $\mu_v(n)$, for $n \geqslant 7$? An extrapolation of the PLS(4) given in Example 2.1 suggests that $\mu_v(n) \leqslant n$, for all $n \geqslant 7$.

One may study sets of MOPLS(n) that are not contained in sets of MOLS(n). In this case, the MOPLS(n) do NOT have a completion to a set of MOLS(n). Stevens and Mendelsohn [49] investigated $(k-2)$-MOPLS(n) of volume v as packing arrays $\Pi A(v; k, n)$. A packing array $\Pi A(v; k, n)$ is a $v \times k$ array with entries from an n-set, so that every $v \times 2$ subarray contains every ordered pair of symbols at most once. Stevens and Mendelsohn asked what is the largest volume, denoted $\Pi A(k, n)$, for which there exists a $\Pi A(v; k, n)$. They obtained a number of bounds and investigated $\Pi A(k, n)$ for small values of n and k, see Subsection 3.8 of [9].

In studying the completion or embedding of pair of MOPLS(t), $P = [P(i, j)]$ and $Q = [Q(i, j)]$, one approach is to resolve two distinct issues; first the necessity of completing or embedding each PLS P and Q in LS(n), $A = [A(i, j)]$ and $B = [B(i, j)]$ and then the verification of the orthogonality condition for the pair A, B, that is, $N \times N = \{(A(i, j), B(i, j)) : 0 \leqslant i, j \leqslant n - 1\}$. The combination of the two issues makes this a complex problem. As a way of decoupling the two questions one might start with a complete LS(t). However, not all LS have orthogonal mates (such squares are termed bachelor LS). So for such squares we ask what is the smallest n such that any LS(t) can be embedded in a pair of MOLS(n)? A related question, studied in the early 1970's, is the existence of a pair of MOLS(n) that contain a pair of MOLS(t) ($t < n$) as subsquares occupying the same set of cells. Through a series of papers [17, 29, 54, 56, 57] it was established that a pair of MOLS(t) can be embedded in a pair of MOLS(n) if $n \geqslant 3t$, where the bound of $3t$ is best possible. In [26] Heinrich and Zhu completed the proof of this result by drawing on existence results for group divisible designs. This approach of first embedding a complete LS in a set of MOLS and then relaxing the result to embed PLS has yielded a number of results, as we will see later in this section.

In 1991, Gustavsson wrote his ground-breaking Ph.D. thesis [23] where he also studied MOPLS as PTD(m, n) and as subgraphs of m-partite graphs $K_{n,\ldots,n}$. Among other things, he showed that there exists a constant $\epsilon_m > 0$ such that if n is large enough (n is greater than some integer N_m) and the number of occurrence of any point in the PTD(m, n) is less than $\epsilon_m n$ then the given PTD(m, n) is completable. In terms of MOPLS this condition translates to the existence of a constant $\epsilon_m > 0$ such that if n is large enough and the occurrence of each row, column and element is less than $\epsilon_m n$ in each square (in ordered triple notation), then the set of $(m-2)$-MOPLS(n) is completable to $(m-2)$-MOLS(n). In his thesis Gustavsson states that $\epsilon_m \geqslant (2m)^{-29}10^{-7}$, but does not specify how big n needs to be. Gustavsson then uses this result to obtain H-decompositions (graph decomposition) of large graphs that satisfy the necessary condition that each vertex has high degree. This is a remarkable existence result, but can not be used to determine the best possible embedding, giving little insight into the structure of the resulting transversal design or equivalently the corresponding set of MOLS(n).

Recently, Barber et al. in [4] made some progress on this problem. By restricting the occurrence of elements in the MOPLS, they were able to prove:

Theorem 4.2. *[4] For every $r \geqslant 3$ and every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geqslant n_0$. Let*

$$c_r = \begin{cases} \frac{1}{25}, & \text{if } r = 3, \\ \frac{9}{10^7 r^3}, & \text{if } r \geqslant 4. \end{cases}$$

Let $T_1, \ldots, T_{r-2}$ be a set of $(r-2)$-MOPLS(n) (drawn in the same $n \times n$ array). Suppose that each row and each column of the underlying array contains at most $(c_r - \epsilon)n$ non-empty cells and that in each of $r-2$ arrays each element of N occurs at most $(c_r - \epsilon)n$ times. Then $T_1, \ldots, T_{r-2}$ can be completed to a set of MOLS(n).

We will discuss this result further in the latter part of this section.

In 1976 Lindner, [39] gave the first finite embedding result for a set of k-MOPLS(n), $P_1, P_2, \ldots, P_k$. Lindner proved:

Theorem 4.3. *[39] Any pair of MOPLS can be finitely embedded in a pair of MOLS.*

Lindner's approach was to take all k PLS and fill the empty cells with distinct elements, relabelling elements in filled cells to ensure that any 2 of the k $n \times n$ arrays contained distinct elements. He then represented these arrays as PTD(k', n'), where $n' \geqslant n$, and $k' \geqslant k + 2$ is a power of a prime. Here the blocks of the transversal design take the form $(i, j, P_1(i, j), P_2(i, j), P_3(i, j), \ldots, P_k(i, j))$ for each cell (i, j). This presentation allowed Lindner to apply an earlier result due to Quackenbush [44] who made use of the following result due to Ganter, see [22]. Here a balanced incomplete block design is a decomposition of the complete graph K_n into complete subgraphs K_q.

Theorem 4.4. *[22] Every finite partial balanced incomplete block design with block size q, where q is a power of a prime, can be embedded in a finite balanced incomplete block design of the same block size.*

However there is no indication of the size of the embedding only that it is finite.

Further investigations were made by Hilton, Rodgers and Wojciechowski's [3], in 1992, when they formulated necessary conditions for a pair of orthogonal Latin rectangles to be embedded in a pair of MOLS.

Since not every LS has an orthogonal mate it is reasonable to return to the investigation of the embedding of a single LS in a pair of MOLS. It is this problem that Jenkins [31] addressed in 2006 proving:

Theorem 4.5. *[31] Let L be an LS(n), $n \geqslant 3$ and $n \neq 6$. Then L can be embedded in an LS(n^2) for which there exists an orthogonal mate.*

Jenkins took $S = [S(i, j)]$ and $T = [T(i, j)]$, a pair of MOLS(n), such that $S(0, 0) = 0$, and strategically replaced the elements in these squares by carefully chosen $n \times n$ arrays. To this end, the element $S(i, j) = 0$ is replaced by a copy of L. In all other cells of S an element $S(i, j)$ is replaced by a copy of the LS corresponding to the cyclic group, C_n, of order n on the set of elements $\{nx, nx + 1, \ldots, nx + n - 1\}$. Then the elements of T are replaced by permuted copies of an $n \times n$ array, A, containing all elements of $\{0, 1, \ldots, n^2 - 1\}$ in lexicographical order. Thus S and T are "inflated" to a pair of $n^2 \times n^2$ arrays, denoted U and V. By using the orthogonality condition (i.e. the ordered pairs $(S(i, j), T(i, j))$ are all distinct) to determine the permutations applied to A, it is possible to show that U and V are a pair of MOLS(n^2). An example of the construction has been included, see Example 4.6.

Example 4.6. *Let $n = 4$, L be an LS(4), S and T be MOLS(4), A be a 4×4 array containing the elements of the set $\{0, \ldots, 15\}$ and C_4 be the cyclic group of order 4. Then L is embedded in the top left corner of U, an LS(4^2) which has an orthogonal mate V.*

L

0	3	1	2
3	0	2	1
1	2	3	0
2	1	0	3

S

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

T

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

A

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

C_4

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

U

0	3	1	2	4	5	6	7	8	9	10	11	12	13	14	15
3	0	2	1	5	6	7	4	9	10	11	8	13	14	15	12
1	2	3	0	6	7	4	5	10	11	8	9	14	15	12	13
2	1	0	3	7	4	5	6	11	8	9	10	15	12	13	14
4	5	6	7	0	3	1	2	12	13	14	15	8	9	10	11
5	6	7	4	3	0	2	1	13	14	15	12	9	10	11	8
6	7	4	5	1	2	3	0	14	15	12	13	10	11	8	9
7	4	5	6	2	1	0	3	15	12	13	14	11	8	9	10
8	9	10	11	12	13	14	15	0	3	1	2	4	5	6	7
9	10	11	8	13	14	15	12	3	0	2	1	5	6	7	4
10	11	8	9	14	15	12	13	1	2	3	0	6	7	4	5
11	8	9	10	15	12	13	14	2	1	0	3	7	4	5	6
12	13	14	15	8	9	10	11	4	5	6	7	0	3	1	2
13	14	15	12	9	10	11	8	5	6	7	4	3	0	2	1
14	15	12	13	10	11	8	9	6	7	4	5	1	2	3	0
15	12	13	14	11	8	9	10	7	4	5	6	2	1	0	3

V

0	1	2	3	15	12	13	14	10	11	8	9	5	6	7	4
4	5	6	7	3	0	1	2	14	15	12	13	9	10	11	8
8	9	10	11	7	4	5	6	2	3	0	1	13	14	15	12
12	13	14	15	11	8	9	10	6	7	4	5	1	2	3	0
14	15	12	13	1	2	3	0	4	5	6	7	11	8	9	10
2	3	0	1	5	6	7	4	8	9	10	11	15	12	13	14
6	7	4	5	9	10	11	8	12	13	14	15	3	0	1	2
10	11	8	9	13	14	15	12	0	1	2	3	7	4	5	6
9	10	11	8	6	7	4	5	3	0	1	2	12	13	14	15
13	14	15	12	10	11	8	9	7	4	5	6	0	1	2	3
1	2	3	0	14	15	12	13	11	8	9	10	4	5	6	7
5	6	7	4	2	3	0	1	15	12	13	14	8	9	10	11
7	4	5	6	8	9	10	11	13	14	15	12	2	3	0	1
11	8	9	10	12	13	14	15	1	2	3	0	6	7	4	5
15	12	13	14	0	1	2	3	5	6	7	4	10	11	8	9
3	0	1	2	4	5	6	7	9	10	11	8	14	15	12	13

Once this embedding was established, Jenkins was able to relax the initial conditions and work with PLS. Jenkins returned to PLS and used Evans' result (Theorem 3.3), to embed a PLS(t) in an LS(n), where $n \geqslant 2t$, and subsequently applied Theorem 4.5 to prove:

Theorem 4.7. *[31, 32] If* $t \geqslant 4$*, then a PLS(t) can be embedded in an LS(*$4t^2$*) which has an orthogonal mate.*

Jenkins result naturally extends to idempotent MOPLS:

Theorem 4.8. *[31, 32] An idempotent PLS(t),* $t \geqslant 3$*, can be embedded in an idempotent LS(*$(2t+1)^2$*), which has an idempotent orthogonal mate.*

Further, these ideas proved to be valuable for embeddings of a class of block designs with block size 4: a K_4-design (X, B) is a decomposition of the edge set of the complete graph K_n on vertex set X into a set B of copies of K_4. Jenkins began by defining a *free vertex* of a partial K_4-design (X, P), to be $x \in X$ such that point x occurs in exactly one block of P. In [30], Jenkins used the existence of group divisible designs with block size 4 to obtain a cubic embedding of any partial K_4-design with the property that every block in the partial design contains at least two free vertices.

In 2014, Donovan and Yazıcı [15] revisited Jenkins' work, extending it to obtain a polynomial order embedding of a pair of MOPLS. Their approach was to begin with pair of MOPLS, P and Q, such that all the elements in P are distinct. From there they used techniques similar to Jenkins to prove:

Theorem 4.9. *[15] Suppose* $2^m \geqslant 2n$*. Let* P *and* Q *be a pair of MOPLS(n), such that each element of* N *occurs in at most one cell of* P*. Then* P *and* Q *can be embedded in a pair of MOLS(*2^{2m}*).*

Their proof begins by employing Evans' result (Theorem 3.3) to embed Q in an LS(2^m) denoted B, where $2^m \geqslant 2n$. Then the PLS P is completed to an $2^m \times 2^m$ array, denoted A, containing all elements in the set $\{0, 1, \ldots, 2^{2m} - 1\}$. The significance of 2^m is that the cells of Cayley table of the elementary Abelian 2-group are "inflated" with permuted copies of A and B. The nature of the permutations is determined by the binary operation of this underlying elementary Abelian 2-group of order 2^m. In this way Donovan and Yazici avoid the necessity for the pair of MOLS S and T in Jenkins' construction.

The use of the elementary Abelian 2-group also allows Donovan and Yazıcı [15] to remove the restriction that all the elements in P are distinct to obtain a more general embedding than that given in Theorem 4.9, but at the price of increasing the order of the embedding.

Theorem 4.10. *[15, 16] Let P and Q be a pair of MOPLS(n). Then P and Q can be embedded in a pair of MOLS(k^4) and any order greater than or equal to $3k^4$ where $2^a = k \geqslant 2n > 2^{a-1}$ for some integer a.*

More recently, Donovan, Grannell and Yazıcı [16] have capitalized on these techniques to develop a construction for embedding a PLS(n) in an Latin square which has many orthogonal mates, as well as embedding a pair of MOPLS(n) in a set of many MOLS. While we state the results here, we will leave a fuller description of the methods to Section 5 where we give new generalizations of these constructions.

Theorem 4.11. *[16] Let P be a PLS(t), $t \geqslant 3$. Then P can be embedded in B, an LS(n) with $n \leqslant 16t^2$, which belongs to a set of at least $2t$ MOLS(n^2). Furthermore if P is idempotent, then B can be constructed to be idempotent.*

Theorem 4.12. *[16] For any $t \geqslant 2$, a pair of MOPLS(n) can be embedded in a set of t MOLS of polynomial order with respect to n.*

In [16] Donovan, Grannell and Yazıcı compared these results to the result given by Barber et.al. in [4] further interpreting Theorem 4.2 which states that, for any $s \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, any set of s-MOPLS(n) can be embedded in a set of s-MOLS(m), for every $m \geqslant k_0 n$. That there is such a k_0 is an important existence result because it gives a linear order embedding. However, the proof given in [4] does not yield an estimate for the best (i.e., lowest) value of k_0. For $s = 1$, Evans' result shows that $k_0 = 2$ is the best possible value. For $s \geqslant 2$, the proof given in [4] requires that $k_0 > 10^7(s + 2)^3/9$ and, being an existence result, there is little information about the structure of the resulting set of MOLS. For $s = 2$ and small n, certainly $n \leqslant 113$ and possibly much larger, [16] gives a tighter embedding than that of [4], and it more closely specifies the structure of the resulting pair of MOLS.

Other results which advance our understanding of embedding of MOPLS can be found in papers on the enumeration of sets of MOLS. Specifically, in 2019 Boyadzhiyska, Das and Szabó remarked that dividing the number of s-MOLS(n) by the number of $(s+1)$-MOLS(n) gives a lower bound on the average number of extensions of an s-MOLS(n) to an $(s + 1)$-MOLS(n). This computation is made possible by earlier enumeration results of Luria [35] and Keevash [34], namely:

Theorem 4.13. *[34, 35] For every fixed $k \in \mathbb{N}$, the number of k-MOLS of order n is*

$$\left((1 + \mathrm{o}(1)) \frac{n^k}{e^{\binom{k+2}{2} - 1}} \right)^{n^2}.$$

Boyadzhiyska, Das and Szabó calculated that the average number of extensions of an s-MOLS to an $(s+1)$-MOLS(n) is at least

$$\left((1 + \mathrm{o}(1)) \frac{n}{e^{s+2}} \right)^{n^2}.$$

This result then gives the average number of embeddings of a set of s-MOLS in a set of $(s+1)$-MOLS(n).

5 Sets of Many MOLS

In this section we revisit the work of Donovan, Grannell and Yazıcı [16]. They build on the following well know fact:

Lemma 5.1. *Given a pair of MOLS(m), $A = [A(p_1, q_1)]$ and $A' = [A'(p_1, q_1)]$, and a pair of MOLS(n), $B = [B(p_2, q_2)]$ and $B' = [B'(p_1, q_2)]$, there exists a pair of MOLS(mn), $A \otimes B$ and $A' \otimes B'$, where the element in cell $((p_1, p_2), (q_1, q_2))$ is $(A(p_1, q_1), B(p_2, q_2))$ in $A \otimes B$ and $(A'(p_1, q_1), B'(p_2, q_2))$ in $A' \otimes B'$.*

But in addition to taking direct products Donovan, Grannell and Yazıcı also inflated the cells of A with copies of B where the elements in either the rows or the columns of B have been permuted. By carefully choosing the permutation they could ensure that the orthogonality of the $n^2 \times n^2$ arrays was maintained. A generalization of these results is presented in Theorem 5.2.

Let n and t be positive integers. To simplify the exposition, we will abuse notation and use $p_1 p_2 \dots p_t$ to represent $(p_1, p_2, \dots, p_t)$ and write $A_{\alpha i}$ to represent $A_{\alpha,i}$.

Theorem 5.2. *For $1 \leqslant \alpha \leqslant t$, let*

$$\mathcal{A}_\alpha = \{A_{\alpha 1}, \dots A_{\alpha t}\} \quad \textit{and} \quad \mathcal{C} = \{C_1, \dots, C_t\},$$

represent a collection of $t + 1$, not necessarily distinct, sets of t-MOLS(n).

Then, for $1 \leqslant u \leqslant t$ and $1 \leqslant v \leqslant t$ define $2t$; the $n^t \times n^t$ arrays

$$\begin{aligned} \mathcal{X}_u &= A_{1u} \otimes A_{2u} \otimes \cdots \otimes A_{tu}, \\ \mathcal{Y}_v &= \{(p_1 p_2 \dots p_t, q_1 q_2 \dots q_t, (C_v(p_2, A_{11}(p_1, q_1)), C_v(p_3, A_{22}(p_2, q_2)), \dots, \\ &\quad C_v(p_t, A_{(t-1)(t-1)}(p_{t-1}, q_{t-1})), C_v(q_1, A_{tt}(p_t, q_t))))\} \end{aligned}$$

form a set of $2t$-MOLS(n^t).

Proof. The proof that the arrays $\mathcal{X}_u$ and $\mathcal{Y}_v$ are LS, of order n^t, is fairly straightforward and omitted here. Further, since the arrays $\mathcal{X}_u$ have been obtained by taking products of MOLS, these t squares are pairwise mutually orthogonal.

Thus we are required to prove that the set of arrays $\mathcal{Y}_v$ form a set of t-MOLS(n^t) and pairwise $\mathcal{Y}_v$ and $\mathcal{X}_u$ are orthogonal.

For any $1 \leqslant u \leqslant t$ and any $1 \leqslant v, v' \leqslant t$ with $v \neq v'$, consider $\mathcal{Y}_v, \mathcal{Y}_{v'}$ or $\mathcal{X}_u, \mathcal{Y}_v$.

Assume that $p_1 \dots p_t \neq p'_1 \dots p'_t$ and $q_1 \dots q_t \neq q'_1 \dots q'_t$; that is, $(p_1 \dots p_t, q_1 \dots q_t)$ and $(p'_1 \dots p'_t, q'_1 \dots, q'_t)$ are distinct cells. Then assume that in $\mathcal{Y}_v$ the entries in these cells are equal as are the entries in $\mathcal{Y}_{v'}$.

It follows that

$$\begin{aligned} C_v(p_2, A_{11}(p_1, q_1)) &= C_v(p'_2, A_{11}(p'_1, q'_1)), && (1)\\ C_v(p_3, A_{22}(p_2, q_2)) &= C_v(p'_3, A_{22}(p'_2, q'_2)), && (2)\\ &\vdots \\ C_v(p_t, A_{(t-1)(t-1)}(p_{t-1}, q_{t-1})) &= C_v(p'_t, A_{(t-1)(t-1)}(p'_{t-1}, q'_{t-1}), && (3)\\ C_v(q_1, A_{tt}(p_t, q_t)) &= C_v(q'_1, A_{tt}(p'_t, q'_t)), \text{ and,} && (4) \end{aligned}$$

$$\begin{aligned} C_{v'}(p_2, A_{11}(p_1, q_1)) &= C_{v'}(p'_2, A_{11}(p'_1, q'_1)), && (5)\\ C_{v'}(p_3, A_{22}(p_2, q_2)) &= C_{v'}(p'_3, A_{22}(p'_2, q'_2)), && (6)\\ &\vdots \\ C_{v'}(p_t, A_{(t-1)(t-1)}(p_{t-1}, q_{t-1})) &= C_{v'}(p'_t, A_{(t-1)(t-1)}(p'_{t-1}, q'_{t-1}), && (7)\\ C_{v'}(q_1, A_{tt}(p_t, q_t)) &= C_{v'}(q'_1, A_{tt}(p'_t, q'_t)). && (8) \end{aligned}$$

Since $v \neq v'$, by assumption C_v is orthogonal to $C_{v'}$ and Equations (1) and (5) imply

$$\begin{aligned} p_2 &= p'_2, && (9)\\ A_{11}(p_1, q_1) &= A_{11}(p'_1, q'_1). && (10) \end{aligned}$$

Continuing in this manner we get:

$$\begin{aligned} p_3 &= p'_3, && (11)\\ A_{22}(p_2, q_2) &= A_{22}(p'_2, q'_2), && (12)\\ p_4 &= p'_4, && (13)\\ A_{33}(p_3, q_3) &= A_{33}(p'_3, q'_3), && (14)\\ &\vdots \\ p_t &= p'_t, && (15)\\ A_{(t-1)(t-1)}(p_{t-1}, q_{t-1}) &= A_{(t-1)(t-1)}(p'_{t-1}, q'_{t-1}), && (16)\\ q_1 &= q'_1, && (17)\\ A_{tt}(p_t, q_t) &= A_{tt}(p'_t, q'_t). && (18) \end{aligned}$$

Further Equation (9) substituted into Equation (12), implies $q_2 = q'_2$, with a similar argument verifying that $q_3 = q'_3, \ \dots, \ q_t = q'_t$ and then Equation (17) substituted into Equation (10) implies $p_1 = p'_1$.

Thus we have shown that $\mathcal{Y}_v$ and $\mathcal{Y}_{v'}$, where $v \neq v'$, are orthogonal LS(n^t).

Next assume that the entries in cells $(p_1 \dots p_t, q_1 \dots q_t)$ and $(p'_1 \dots p'_t, q'_1 \dots, q'_t)$ of $\mathcal{X}_u$ are equal, as are the entries of $\mathcal{Y}_v$. Thus for $i = 1, \dots, t$ and $j = 1, \dots, t-1$

$$
\begin{aligned}
A_{iu}(p_i, q_i) &= A_{iu}(p'_i, q'_i), \text{ and,} && (19)\\
C_v(p_{j+1}, A_{jj}(p_j, q_j)) &= C_v(p'_{j+1}, A_{jj}(p'_j, q'_j)), && (20)\\
C_v(q_1, A_{tt}(p_t, q_t)) &= C_v(q'_1, A_{tt}(p'_t, q'_t)). && (21)
\end{aligned}
$$

In Equation (19) set $i = u$ and substitute into Equation (20) where $j = u$ to get $p_{u+1} = p'_{u+1}$. Then returning to Equation (19) with $i = u + 1$ gives $q_{u+1} = q'_{u+1}$ and so $A_{(u+1)(u+1)}(p_{u+1}, q_{u+1}) = A_{(u+1)(u+1)}(p'_{u+1}, q'_{u+1})$.

Using the same argument when substituting into Equation (20) with $j = u+1, \dots, t-1$ gives $p_{u+2} = p'_{u+2}$ up to $p_t = p'_t$, $q_{u+2} = q'_{u+2}$ up to $q_t = q'_t$ and $A_{(u+2)(u+2)}(p_{u+2}, q_{u+2}) = A_{(u+2)(u+2)}(p'_{u+2}, q'_{u+2})$ up to $A_{tt}(p_t, q_t) = A_{tt}(p'_t, q'_t)$. When this is substituted into Equation (21) we obtain $q_1 = q'_1$ which when substituted into Equation (19) with $i = 1$ gives $p_1 = p'_1$. So $A_{11}(p_1, q_1) = A_{11}(p'_1, q'_1)$.

Finally picking up the above argument at the substitution into Equation (20) with $j = 1, \dots, u-2$ gives $p_2 = p'_2$ up to $p_{u-1} = p'_{u-1}$ and $q_2 = q'_2$ up to $q_{u-1} = q'_{u-1}$.

Hence for $1 \leqslant u, v \leqslant t$, $\mathcal{X}_u$ and $\mathcal{Y}_v$ are orthogonal. □

6 Conclusions and Open Questions

It is natural to extend transversal designs TD$(3, n)$ to transversal designs TD(k, n) with block size $k \geqslant 3$, or equivalently the decomposition of the complete tripartite graph $K_{n,n,n}$ into triangles to decompositions of the k-partite graph $K_{n,\dots,n}$ into K_k. In the same way it is natural to extend LS(n) to sets of $(k-2)$-MOLS(n). However, imposing this orthogonality condition significantly increases the complexity, making it harder to construct and determine the properties of MOLS, for instance in determining the existence question for 3-MOLS(10) or the study of the smallest possible embedding for MOPLS. This leaves us with many open questions, some of which we state or restate below.

Q1. For each n, what is the minimum volume $\mu_v(n)$ such that all pairs of MOPLS(n) of volume less than $\mu_v(n)$ can be completed to a pair of MOLS(n)?

Q2. For each t, what is the smallest n such that any pair of MOPLS(t) can be embedded in a pair of MOLS(n)?

Q3. For each t, what is the smallest n such that any pair of MOPLS(t) can be embedded in k-MOLS(n), for $k \geqslant 3$?

Q4. For each t, what is the smallest n such that any k-MOPLS(t), $k \geqslant 3$, can be embedded in k-MOLS(n)?

Q5. What are the constraints on n and r such that a pair of MOPLS(t) of volume r can be embedded in a pair of r-orthogonal LS(n)?

Q6. For each t and each admissible r, what is the smallest n such that an r-orthogonal LS(t) can be embedded in a pair of r-orthogonal LS(n)?

Recently, in [20] Falcón, Falcón and Núñez gave results on the existence of orthogonal partial quasigroups $(N, \circ)$ that are totally conjugate orthogonal, in that the six conjugates are distinct and pairwise orthogonal. The six conjugates are the partial quasigroups defined by the binary operations $\circ, \circ_2, \circ_3, \circ_4, \circ_5, \circ_6$ on N, where given $x \circ y = z$, $y \circ_2 x = z, x \circ_3 z = y, z \circ_4 x = y, z \circ_5 y = x, y \circ_6 z = x$. This work leads to the following question.

Q7. What is the smallest size of the embedding for the totally conjugate orthogonal partial quasigroups of small orders given in [20] and what is the smallest n such that totally conjugate orthogonal partial quasigroup, of order t, can be embedded in a totally conjugate orthogonal quasigroup of order n?

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